# Model Predictive Control:
## Theory and Design

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# Model Predictive Control:
## Theory and Design

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*To Cheryl and Josephine,*

*for their love, encouragement, and patience.*

# Preface

Our goal in this text is to provide a comprehensive and foundational treatment of the theory and design of model predictive control (MPC). By now several excellent monographs emphasizing various aspects of MPC have appeared (a list appears at the beginning of Chapter 1), and the reader may naturally wonder what is offered here that is new and different. By providing a comprehensive treatment of the MPC foundation, we hope that this text enables researchers to learn and *teach* the fundamentals of MPC without continuously searching the diverse control research literature for omitted arguments and requisite background material. When teaching the subject, it is essential to have a collection of exercises that enables the students to assess their level of comprehension and mastery of the topics. To support the teaching and learning of MPC, we have included more than 200 end-of-chapter exercises. A complete solution manual (more than 300 pages) is available for course instructors.

Chapter 1 is introductory. It is intended for graduate students in engineering who have not yet had a systems course. But it serves a second purpose for those who have already taken the first graduate systems course. It derives all the results of the linear quadratic regulator and optimal Kalman filter using only those arguments that extend to the nonlinear and constrained cases to be covered in the later chapters. Instructors may find that this tailored treatment of the introductory systems material serves both as a review and a preview of arguments to come in the later chapters.

Chapters 2–4 are foundational and should probably be covered in any graduate level MPC course. Chapter 2 covers regulation to the origin for nonlinear and constrained systems. This material presents in a unified fashion many of the major research advances in MPC that took place during the last 20 years. It also includes more recent topics such as regulation to an unreachable setpoint that are only now appearing in the research literature. Chapter 3 addresses MPC design for robustness, with a focus on MPC using tubes or bundles of trajectories in place of the single nominal trajectory. This chapter again unifies a large body of research literature concerned with robust MPC. Chapter 4 covers state estimation with an emphasis on moving horizon estimation, but also

covers extended and unscented Kalman filtering, and particle filtering.

Chapters 5–7 present more specialized topics. Chapter 5 addresses the special requirements of MPC based on output measurement instead of state measurement. Chapter 6 discusses how to design distributed MPC controllers for large-scale systems that are decomposed into many smaller, interacting subsystems. Chapter 7 covers the explicit optimal control of constrained linear systems. The choice of coverage of these three chapters may vary depending on the instructor's or student's own research interests.

Three appendices are included, again, so that the reader is not sent off to search a large research literature for the fundamental arguments used in the text. Appendix A covers the required mathematical background. Appendix B summarizes the results used for stability analysis including the various types of stability and Lyapunov function theory. Since MPC is an optimization-based controller, Appendix C covers the relevant results from optimization theory. In order to reduce the size and expense of the text, the three appendices are available on the web: `www.che.wisc.edu/~jbraw/mpc`. Note, however, that all material in the appendices is included in the book's printed table of contents, and subject and author indices. The website also includes sample exams, and homework assignments for a one-semester graduate course in MPC. All of the examples and exercises in the text were solved with Octave. Octave is freely available from `www.octave.org`.

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# Notation

## Mathematical notation

| | |
|---|---|
| $\exists$ | there exists |
| $\in$ | is an element of |
| $\forall$ | for all |
| $\Longrightarrow \quad \Longleftarrow$ | implies; is implied by |
| $\not\Longrightarrow \quad \not\Longleftarrow$ | does not imply; is not implied by |
| $:=$ | equal by definition or is defined by |
| $\approx$ | approximately equal |
| $V(\cdot)$ | function $V$ |
| $V : \mathbb{A} \to \mathbb{B}$ | $V$ is a function mapping set $\mathbb{A}$ into set $\mathbb{B}$ |
| $x \mapsto V(x)$ | function $V$ maps variable $x$ to value $V(x)$ |
| $x^+$ | value of $x$ at next sample time (discrete time system) |
| $\dot{x}$ | time derivative of $x$ (continuous time system) |
| $f_x$ | partial derivative of $f(x)$ with respect to $x$ |
| $\nabla$ | nabla or del operator |
| $\delta$ | unit impulse or delta function |
| $\lvert x \rvert$ | absolute value of scalar; norm of vector (two-norm unless stated otherwise); induced norm of matrix |
| $\mathbf{x}$ | sequence of vector-valued variable $x$, $\{x(0), x(1), \ldots, \}$ |
| $\lVert \mathbf{x} \rVert$ | sup norm over a sequence, $\sup_{i \geq 0} \lvert x(i) \rvert$ |
| $\lVert \mathbf{x} \rVert_{a:b}$ | $\max_{a \leq i \leq b} \lvert x(i) \rvert$ |
| $\mathrm{tr}(A)$ | trace of matrix $A$ |
| $\det(A)$ | determinant of matrix $A$ |
| $\mathrm{eig}(A)$ | set of eigenvalues of matrix $A$ |
| $\rho(A)$ | spectral radius of matrix $A$, $\max_i \lvert \lambda_i \rvert$ for $\lambda_i \in \mathrm{eig}(A)$ |
| $A^{-1}$ | inverse of matrix $A$ |
| $A^\dagger$ | pseudo-inverse of matrix $A$ |
| $A'$ | transpose of matrix $A$ |
| inf | infimum or greatest lower bound |
| min | minimum |
| sup | supremum or least upper bound |
| max | maximum |
| arg | argument or solution of an optimization |

| | |
|---|---|
| $\mathbb{I}$ | integers |
| $\mathbb{I}_{\geq 0}$ | nonnegative integers |
| $\mathbb{I}_{n:m}$ | integers in the interval $[n, m]$ |
| $\mathbb{R}$ | real numbers |
| $\mathbb{R}_{\geq 0}$ | nonnegative real numbers |
| $\mathbb{R}^n$ | real-valued $n$-vectors |
| $\mathbb{R}^{m\times n}$ | real-valued $m \times n$ matrices |
| $\mathbb{C}$ | complex numbers |
| $\mathcal{B}$ | ball in $\mathbb{R}^n$ of unit radius |
| $x \sim p_x$ | random variable $x$ is distributed with probability density $p_x$ |
| $\mathcal{E}(x)$ | expectation of random variable $x$ |
| $\mathrm{var}(x)$ | variance of random variable $x$ |
| $\mathrm{cov}(x, y)$ | covariance of random variables $x$ and $y$ |
| $N(m, P)$ | normal distribution with mean $m$ and covariance $P$, $x \sim N(m, P)$ |
| $n(x, m, P)$ | normal probability density, $p_x(x) = n(x, m, P)$ |
| $\emptyset$ | the empty set |
| $\mathrm{aff}(\mathbb{A})$ | affine hull of set $\mathbb{A}$ |
| $\mathrm{int}(\mathbb{A})$ | interior of set $\mathbb{A}$ |
| $\mathrm{co}(\mathbb{A})$ | convex hull of the set $\mathbb{A}$ |
| $\overline{\mathbb{A}}$ | closure of set $\mathbb{A}$ |
| $\mathrm{epi}(f)$ | epigraph of function $f$ |
| $\mathrm{lev}_a V$ | sublevel set of function $V$, $\{x \mid V(x) \leq a\}$ |
| $\mathbb{A} \oplus \mathbb{B}$ | set addition of sets $\mathbb{A}$ and $\mathbb{B}$ |
| $\mathbb{A} \ominus \mathbb{B}$ | set subtraction of set $\mathbb{B}$ from set $\mathbb{A}$ |
| $\mathbb{A} \setminus \mathbb{B}$ | elements of set $\mathbb{A}$ not in set $\mathbb{B}$ |
| $\mathbb{A} \cup \mathbb{B}$ | union of sets $\mathbb{A}$ and $\mathbb{B}$ |
| $\mathbb{A} \cap \mathbb{B}$ | intersection of sets $\mathbb{A}$ and $\mathbb{B}$ |
| $\mathbb{A} \subseteq \mathbb{B}$ | set $\mathbb{A}$ is a subset of set $\mathbb{B}$ |
| $\mathbb{A} \supseteq \mathbb{B}$ | set $\mathbb{A}$ is a superset of set $\mathbb{B}$ |
| $\mathbb{A} \subset \mathbb{B}$ | set $\mathbb{A}$ is a proper (or strict) subset of set $\mathbb{B}$ |
| $\mathbb{A} \supset \mathbb{B}$ | set $\mathbb{A}$ is a proper (or strict) superset of set $\mathbb{B}$ |
| $d(a, \mathbb{B})$ | Distance between element $a$ and set $\mathbb{B}$ |
| $d_H(\mathbb{A}, \mathbb{B})$ | Hausdorff distance between sets $\mathbb{A}$ and $\mathbb{B}$ |
| $x \searrow y$ | $x$ converges to $y$ from above |
| $x \nearrow y$ | $x$ converges to $y$ from below |
| $\mathrm{sat}(x)$ | saturation, $\mathrm{sat}(x) = x$ if $\lvert x\rvert \leq 1, -1$ if $x \leq -1, 1$ if $x \geq 1$ |

## Symbols

| | |
|---|---|
| $A, B, C$ | system matrices, discrete time, $x^+ = Ax + Bu$, $y = Cx$ |
| $A_c, B_c$ | system matrices, continuous time, $\dot{x} = A_c x + B_c u$ |
| $A_{ij}$ | state transition matrix for player $i$ to player $j$ |
| $A_i$ | state transition matrix for player $i$ |
| $A_{Li}$ | estimate error transition matrix $A_i - L_i C_i$ |
| $B_d$ | input disturbance matrix |
| $B_{ij}$ | input matrix of player $i$ for player $j$'s inputs |
| $B_i$ | input matrix of player $i$ |
| $C_{ij}$ | output matrix of player $i$ for player $j$'s interaction states |
| $C_i$ | output matrix of player $i$ |
| $C_d$ | output disturbance matrix |
| $\mathcal{C}$ | controllability matrix |
| $C^*$ | polar cone of cone $C$ |
| $d$ | integrating disturbance |
| $E, F$ | constraint matrices, $Fx + Eu \leq e$ |
| $f, h$ | system functions, discrete time, $x^+ = f(x,u)$, $y = h(x)$ |
| $f_c(x,u)$ | system function, continuous time, $\dot{x} = f_c(x,u)$ |
| $F(x,u)$ | difference inclusion, $x^+ \in F(x,u)$, $F$ is set valued |
| $G$ | input noise-shaping matrix |
| $G_{ij}$ | steady-state gain of player $i$ to player $j$ |
| $H$ | controlled variable matrix |
| $I(x,u)$ | index set of constraints active at $(x,u)$ |
| $I^0(x)$ | index set of constraints active at $(x, u^0(x))$ |
| $k$ | sample time |
| $K$ | optimal controller gain |
| $\ell(x,u)$ | stage cost |
| $\ell_N(x,u)$ | final stage cost |
| $L$ | optimal estimator gain |
| $m$ | input dimension |
| $M$ | cross-term penalty matrix $x'Mu$ |
| $M$ | number of players, Chapter 6 |
| $\mathcal{M}$ | class of admissible input policies, $\boldsymbol{\mu} \in \mathcal{M}$ |
| $n$ | state dimension |
| $N$ | horizon length |
| $\mathcal{O}$ | observability matrix, Chapters 1 and 4 |
| $\mathcal{O}$ | compact robust control invariant set containing the origin, Chapter 3 |
| $p$ | output dimension |

| | |
|---|---|
| $p$ | optimization iterate, Chapter 6 |
| $p_\xi$ | probability density of random variable $\xi$ |
| $p_s(x)$ | sampled probability density, $p_s(x) = \sum_i w_i \delta(x - x_i)$ |
| $P$ | covariance matrix in the estimator |
| $P_f$ | terminal penalty matrix |
| $\mathcal{P}$ | polytopic partition |
| $\mathbb{P}_N(x)$ | MPC optimization problem; horizon $N$ and initial state $x$ |
| $q$ | importance function in importance sampling |
| $Q$ | state penalty matrix |
| $r$ | controlled variable, $r = Hy$ |
| $R$ | input penalty matrix |
| $s$ | number of samples in a sampled probability density |
| $S$ | input rate of change penalty matrix |
| $S(x,u)$ | index set of active polytopes at $(x,u)$ |
| $S^0(x)$ | index set of active polytopes at $(x,u^0(x))$ |
| $t$ | time |
| $T$ | current time in estimation problem |
| $u$ | input (manipulated variable) vector |
| $\tilde{\mathbf{u}}^+$ | warm start for input sequence |
| $\mathbf{u}^+$ | improved input sequence |
| $\mathcal{U}_N(x)$ | control constraint set |
| $\mathbb{U}$ | input constraint set |
| $v$ | output disturbance, Chapters 1 and 4 |
| $v$ | nominal control input, Chapters 3 and 5 |
| $V_N(x,\mathbf{u})$ | MPC objective function |
| $V_N^0(x)$ | MPC optimal value function |
| $V_T(\chi,\boldsymbol{\omega})$ | Full information state estimation objective function at time $T$ with initial state $\chi$ and disturbance sequence $\boldsymbol{\omega}$ |
| $\hat{V}_T(\chi,\boldsymbol{\omega})$ | MHE objective function at time $T$ with initial state $\chi$ and disturbance sequence $\boldsymbol{\omega}$ |
| $V_f(x)$ | terminal penalty |
| $\mathcal{V}_N(z)$ | nominal control input constraint set |
| $\mathbb{V}$ | output disturbance constraint set |
| $w$ | disturbance to the state evolution |
| $w_i$ | weights in a sampled probability density, Chapter 4 |
| $w_i$ | convex weight for player $i$, Chapter 6 |
| $\overline{w}_i$ | normalized weights in a sampled probability density |
| $\mathcal{W}$ | class of admissible disturbance sequences, $\mathbf{w} \in \mathcal{W}$ |
| $\mathbb{W}$ | state disturbance constraint set |

| | |
|---|---|
| $x$ | state vector |
| $x_i$ | sample values in a sampled probability density |
| $x_{ij}$ | state interaction vector from player $i$ to player $j$ |
| $\overline{x}(0)$ | mean of initial state density |
| $X(k; x, \boldsymbol{\mu})$ | state tube at time $k$ with initial state $x$ and control policy $\boldsymbol{\mu}$ |
| $\mathcal{X}_j$ | set of feasible states for optimal control problem at stage $j$ |
| $\mathbb{X}$ | state constraint set |
| $\mathbb{X}_f$ | terminal region |
| $y$ | output (measurement) vector |
| $\mathbb{Y}$ | output constraint set |
| $z$ | nominal state, Chapters 3 and 5 |
| $Z_T(x)$ | full information arrival cost |
| $\hat{Z}_T(x)$ | MHE arrival cost |
| $\widetilde{Z}_T(x)$ | MHE smoothing arrival cost |
| $\mathbb{Z}$ | system constraint region, $(x, u) \in \mathbb{Z}$ |
| $\mathbb{Z}_f$ | terminal constraint region, $(x, u) \in \mathbb{Z}_f$ |
| $\mathbb{Z}_N(x, \mathbf{u})$ | constraint set for state and input sequence |

Greek letters

| | |
|---|---|
| $\Gamma_T(\chi)$ | MHE prior weighting on state at time $T$ |
| $\Delta$ | sample time |
| $\kappa$ | control law |
| $\kappa_j$ | control law at stage $j$ |
| $\kappa_f$ | control law applied in terminal region $\mathbb{X}_f$ |
| $\mu_i(x)$ | control law at stage $i$ |
| $\boldsymbol{\mu}(x)$ | control policy or sequence of control laws |
| $\nu$ | output disturbance decision variable in estimation problem |
| $\Pi$ | cost to go matrix in regulator, Chapter 1 |
| $\Pi$ | covariance matrix in the estimator, Chapter 5 |
| $\rho_i$ | objective function weight for player $i$ |
| $\Sigma_i$ | Solution to Lyapunov equation for player $i$ |
| $\phi(k; x, \mathbf{u})$ | state at time $k$ given initial state $x$ and input sequence $\mathbf{u}$ |
| $\phi(k; x, i, \mathbf{u})$ | state at time $k$ given state at time $i$ is $x$ and input sequence $\mathbf{u}$ |
| $\phi(k; x, \mathbf{u}, \mathbf{w})$ | state at time $k$ given initial state is $x$, input sequence is $\mathbf{u}$, and disturbance sequence is $\mathbf{w}$ |
| $\chi$ | state decision variable in estimation problem |
| $\omega$ | state disturbance decision variable in estimation problem |

## Subscripts, superscripts and accents

- $\hat{x}$ estimate
- $\hat{x}^-$ estimate before measurement
- $\tilde{x}$ estimate error
- $x_s$ steady state
- $x_i$ subsystem $i$ in a decomposed large-scale system
- $x_{\text{sp}}$ setpoint
- $V^0$ optimal
- $V^{\text{uc}}$ unconstrained
- $V^{\text{sp}}$ unreachable setpoint

# Acronyms

| | |
|---|---|
| CLF | control-Lyapunov function |
| DARE | discrete algebraic Riccati equation |
| DP | dynamic programming |
| FLOP | floating point operation |
| FSO | final state observability |
| GAS | global asymptotic stability |
| GES | global exponential stability |
| GPC | generalized predictive control |
| EKF | extended Kalman filter |
| i-IOSS | incrementally input/output-to-state stable |
| IOSS | input/output-to-state stable |
| ISS | input-to-state stable |
| KF | Kalman filter |
| KKT | Karush-Kuhn-Tucker |
| LAR | linear absolute regulator |
| LP | linear program |
| LQ | linear quadratic |
| LQG | linear quadratic Gaussian |
| LQR | linear quadratic regulator |
| MHE | moving horizon estimation |
| MPC | model predictive control |
| OSS | output-to-state stable |
| PF | particle filter |
| PID | proportional-integral-derivative |
| QP | quadratic program |
| RGA | relative gain array |
| RGAS | robust global asymptotic stability |
| RHC | receding horizon control |
| UKF | unscented Kalman filter |

# 1

# Getting Started with Model Predictive Control

## 1.1 Introduction

The main purpose of this chapter is to provide a compact and accessible overview of the essential elements of model predictive control (MPC). We introduce deterministic and stochastic models, regulation, state estimation, dynamic programming (DP), tracking, disturbances, and some important performance properties such as closed-loop stability and zero offset to disturbances. The reader with background in MPC and linear systems theory may wish to skim this chapter briefly and proceed to Chapter 2. Other introductory texts covering the basics of MPC include Maciejowski (2002); Camacho and Bordons (2004); Rossiter (2004); Goodwin, Seron, and De Doná (2005); Kwon (2005); Wang (2009).

## 1.2 Models and Modeling

Model predictive control has its roots in optimal control. The basic concept of MPC is to use a dynamic model to forecast system behavior, and optimize the forecast to produce the best decision — the control move at the current time. Models are therefore central to every form of MPC. Because the optimal control move depends on the initial state of the dynamic system, a second basic concept in MPC is to use the past record of measurements to determine the most likely initial state of the system. The state estimation problem is to examine the record of past data, and reconcile these measurements with the model to determine the most likely value of the state at the current time. Both the regulation problem, in which a model forecast is used to produce the optimal control action, and the estimation problem, in which the past record

of measurements is used to produce an optimal state estimate, involve dynamic models and optimization.

We first discuss the dynamic models used in this text. We start with the familiar differential equation models

$$\begin{aligned} \frac{dx}{dt} &= f(x,u,t) \\ y &= h(x,u,t) \\ x(t_0) &= x_0 \end{aligned}$$

in which $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, $y \in \mathbb{R}^p$ is the output, and $t \in \mathbb{R}$ is time. We use $\mathbb{R}^n$ to denote the set of real-valued $n$-vectors. The initial condition specifies the value of the state $x$ at time $t = t_0$, and we seek a solution to the differential equation for time greater than $t_0$, $t \in \mathbb{R}_{\geq t_0}$. Often we define the initial time to be zero, with a corresponding initial condition, in which case $t \in \mathbb{R}_{\geq 0}$.

### 1.2.1 Linear Dynamic Models

**Time-varying model.** The most general *linear* state space model is the time-varying model

$$\begin{aligned} \frac{dx}{dt} &= A(t)x + B(t)u \\ y &= C(t)x + D(t)u \\ x(0) &= x_0 \end{aligned}$$

in which $A(t) \in \mathbb{R}^{n\times n}$ is the state transition matrix, $B(t) \in \mathbb{R}^{n\times m}$ is the input matrix, $C(t) \in \mathbb{R}^{p\times n}$ is the output matrix, and $D(t) \in \mathbb{R}^{p\times m}$ allows a direct coupling between $u$ and $y$. In many applications $D = 0$.

**Time-invariant model.** If $A$, $B$, $C$, and $D$ are time invariant, the linear model reduces to

$$\begin{aligned} \frac{dx}{dt} &= Ax + Bu \\ y &= Cx + Du \\ x(0) &= x_0 \end{aligned} \tag{1.1}$$

One of the main motivations for using linear models to approximate physical systems is the ease of solution and analysis of linear models.

Equation (1.1) can be solved to yield

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \tag{1.2}$$

in which $e^{At} \in \mathbb{R}^{n\times n}$ is the matrix exponential.[1] Notice the solution is a convolution integral of the entire $u(t)$ behavior weighted by the matrix exponential of $At$. We will see later that the eigenvalues of $A$ determine whether the past $u(t)$ has more effect or less effect on the current $x(t)$ as time increases.

### 1.2.2 Input-Output Models

If we know little about the internal structure of a system, it may be convenient to take another approach in which we suppress the state variable, and focus attention only on the manipulatable inputs and measurable outputs. As shown in Figure 1.1, we consider the system to be the connection between $u$ and $y$. In this viewpoint, we usually perform system identification experiments in which we manipulate $u$ and measure $y$, and develop simple linear models for $G$. To take advantage of the usual block diagram manipulation of simple series and feedback connections, it is convenient to consider the Laplace transform of the signals rather than the time functions,

$$\overline{y}(s) := \int_0^\infty e^{-st}y(t)dt$$

in which $s \in \mathbb{C}$ is the complex-valued Laplace transform variable, in contrast to $t$, which is the real-valued time variable. The symbol := means "equal by definition" or "is defined by." The transfer function matrix is then identified from the data, and the block diagram represents the following mathematical relationship between input and output

$$\overline{y}(s) = G(s)\overline{u}(s)$$

$G(s) \in \mathbb{C}^{p\times m}$ is the transfer function matrix. Notice the state does not appear in this input-output description. If we are obtaining $G(s)$ instead from a state space model, then $G(s) = C(sI - A)^{-1}B$, and we assume $x(0) = 0$ as the system initial condition.

[1] We can define the exponential of matrix $X$ in terms of its Taylor series,

$$e^X := \frac{1}{0!}I + \frac{1}{1!}X + \frac{1}{2!}X^2 + \frac{1}{3!}X^3 + \cdots$$

This series converges for all $X$.

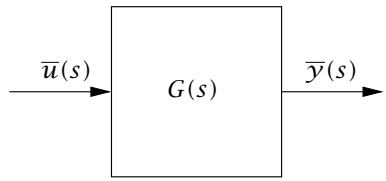


**Figure 1.1:** System with input $\overline{u}$, output $\overline{y}$ and transfer function matrix $G$ connecting them; the model is $\overline{y} = G\overline{u}$.

### 1.2.3 Distributed Models

Distributed models arise whenever we consider systems that are not spatially uniform. Consider, for example, a multicomponent, chemical mixture undergoing convection and chemical reaction. The microscopic mass balance for species $A$ is

$$\frac{\partial c_A}{\partial t} + \nabla \cdot (c_A v_A) - R_A = 0$$

in which $c_A$ is the molar concentration of species $A$, $v_A$ is the velocity of species $A$, and $R_A$ is the production rate of species $A$ due to chemical reaction, in which

$$\nabla := \delta_x \frac{\partial}{\partial x} + \delta_y \frac{\partial}{\partial y} + \delta_z \frac{\partial}{\partial z}$$

and the $\delta_{x,y,z}$ are the respective unit vectors in the $(x, y, z)$ spatial coordinates.

We also should note that the distribution does not have to be "spatial." Consider a particle size distribution $f(r,t)$ in which $f(r,t)dr$ represents the number of particles of size $r$ to $r+dr$ in a particle reactor at time $t$. The reactor volume is considered well mixed and spatially homogeneous. If the particles nucleate at zero size with nucleation rate $B(t)$ and grow with growth rate, $G(t)$, the evolution of the particle size distribution is given by

$$\begin{aligned} \frac{\partial f}{\partial t} &= -G\frac{\partial f}{\partial r} \\ f(r,t) &= B/G \qquad r = 0 \qquad t \geq 0 \\ f(r,t) &= f_0(r) \qquad r \geq 0 \qquad t = 0 \end{aligned}$$

Again we have partial differential equation descriptions even though the particle reactor is well mixed and spatially uniform.

### 1.2.4 Discrete Time Models

Discrete time models are often convenient if the system of interest is sampled at discrete times. If the sampling rate is chosen appropriately, the behavior between the samples can be safely ignored and the model describes exclusively the behavior at the sample times. The finite dimensional, linear, time-invariant, discrete time model is

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k) \\ x(0) &= x_0 \end{aligned} \tag{1.3}$$

in which $k \in \mathbb{I}_{\geq 0}$ is a nonnegative integer denoting the sample number, which is connected to time by $t = k\Delta$ in which $\Delta$ is the sample time. We use $\mathbb{I}$ to denote the set of integers and $\mathbb{I}_{\geq 0}$ to denote the set of nonnegative integers. The linear discrete time model is a linear difference equation.

It is sometimes convenient to write the time index with a subscript

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k + Du_k \\ x_0 &\quad \text{given} \end{aligned}$$

but we avoid this notation in this text. To reduce the notational complexity we usually express (1.3) as

$$\begin{aligned} x^+ &= Ax + Bu \\ y &= Cx + Du \\ x(0) &= x_0 \end{aligned}$$

in which the superscript $^+$ means the state at the next sample time. The linear discrete time model is convenient for presenting the ideas and concepts of MPC in the simplest possible mathematical setting. Because the model is linear, analytical solutions are readily derived. The solution to (1.3) is

$$x(k) = A^k x_0 + \sum_{j=0}^{k-1} A^{k-j-1} Bu(j) \tag{1.4}$$

Notice that a convolution sum corresponds to the convolution integral of (1.2) and powers of $A$ correspond to the matrix exponential. Because (1.4) involves only multiplication and addition, it is convenient to program for computation.

The discrete time analog of the continuous time input-output model is obtained by defining the Z-transform of the signals

$$\overline{y}(z) := \sum_{k=0}^{\infty} z^k y(k)$$

The discrete transfer function matrix $G(z)$ then represents the discrete input-output model

$$\overline{y}(z) = G(z)\overline{u}(z)$$

and $G(z) \in \mathbb{C}^{p\times m}$ is the transfer function matrix. Notice the state does not appear in this input-output description. We make only passing reference to transfer function models in this text.

### 1.2.5 Constraints

The manipulated inputs (valve positions, voltages, torques, etc.) to most physical systems are bounded. We include these constraints by linear inequalities

$$Eu(k) \le e \qquad k \in \mathbb{I}_{\ge 0}$$

in which

$$E = \begin{bmatrix} I \\ -I \end{bmatrix} \qquad e = \begin{bmatrix} \overline{u} \\ -\underline{u} \end{bmatrix}$$

are chosen to describe simple bounds such as

$$\underline{u} \le u(k) \le \overline{u} \qquad k \in \mathbb{I}_{\ge 0}$$

We sometimes wish to impose constraints on states or outputs for reasons of safety, operability, product quality, etc. These can be stated as

$$Fx(k) \le f \qquad k \in \mathbb{I}_{\ge 0}$$

Practitioners find it convenient in some applications to limit the rate of change of the input, $u(k) - u(k-1)$. To maintain the state space form of the model, we may augment the state as

$$\widetilde{x}(k) = \begin{bmatrix} x(k) \\ u(k-1) \end{bmatrix}$$

and the augmented system model becomes

$$\widetilde{x}^+ = \widetilde{A}\widetilde{x} + \widetilde{B}u$$
$$y = \widetilde{C}\widetilde{x}$$

in which

$$\widetilde{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \qquad \widetilde{B} = \begin{bmatrix} B \\ I \end{bmatrix} \qquad \widetilde{C} = \begin{bmatrix} C & 0 \end{bmatrix}$$

A rate of change constraint such as

$$\underline{\Delta} \leq u(k) - u(k-1) \leq \overline{\Delta} \qquad k \in \mathbb{I}_{\geq 0}$$

is then stated as

$$F\widetilde{x}(k) + Eu(k) \leq e \qquad F = \begin{bmatrix} 0 & -I \\ 0 & I \end{bmatrix} \qquad E = \begin{bmatrix} I \\ -I \end{bmatrix} \qquad e = \begin{bmatrix} \overline{\Delta} \\ -\underline{\Delta} \end{bmatrix}$$

To simplify analysis, it pays to maintain linear constraints when using linear dynamic models. So if we want to consider fairly general constraints for a linear system, we choose the form

$$Fx(k) + Eu(k) \leq e \qquad k \in \mathbb{I}_{\geq 0}$$

which subsumes all the forms listed previously.

When we consider nonlinear systems, analysis of the controller is not significantly simplified by maintaining linear inequalities, and we generalize the constraints to set membership

$$x(k) \in \mathbb{X} \qquad u(k) \in \mathbb{U} \qquad k \in \mathbb{I}_{\geq 0}$$

or, more generally,

$$(x(k), u(k)) \in \mathbb{Z} \qquad k \in \mathbb{I}_{\geq 0}$$

We should bear in mind one general distinction between input constraints, and output or state constraints. The input constraints often represent *physical limits*. In these cases, if the controller does not respect the input constraints, the physical system enforces them. In contrast, the output or state constraints are usually *desirables*. They may not be achievable depending on the disturbances affecting the system. It is often the function of an MPC controller to determine in real time that the output or state constraints are not achievable, and relax them in some satisfactory manner. As we discuss in Chapter 2, these considerations lead implementers of MPC often to set up the optimization problem using hard constraints for the input constraints and some form of soft constraints for the output or state constraints.

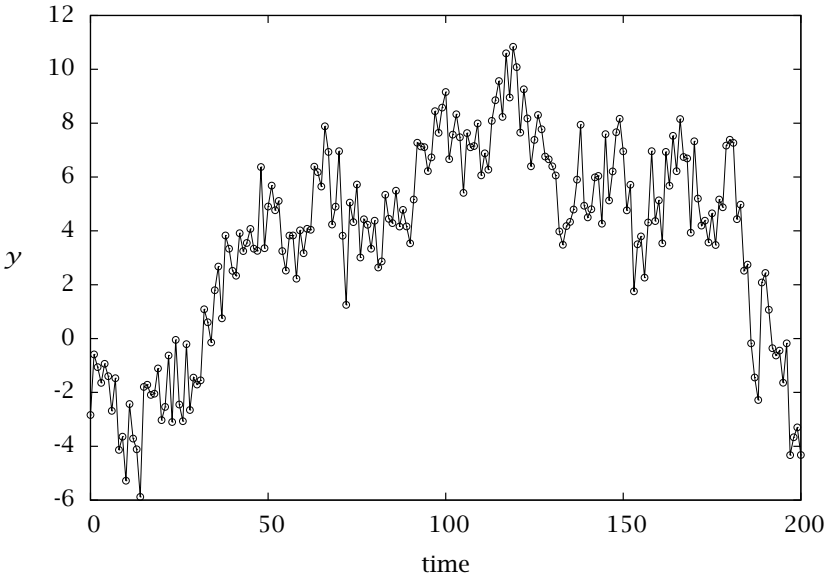


**Figure 1.2:** Output of a stochastic system versus time.

### 1.2.6 Deterministic and Stochastic

If one examines measurements coming from any complex, physical process, fluctuations in the data as depicted in Figure 1.2 are invariably present. For applications at small length scales, the fluctuations may be caused by the random behavior of small numbers of molecules. This type of application is becoming increasingly prevalent as scientists and engineers study applications in nanotechnology. This type of system also arises in life science applications when modeling the interactions of a few virus particles or protein molecules with living cells. In these applications there is no deterministic simulation model; the only system model available is stochastic.

**Linear time-invariant models.** In mainstream, classical process control problems, we are usually concerned with modeling, monitoring and controlling macroscopic systems, i.e., we are not considering systems composed of small numbers of molecules. So one may naturally ask (many do) what is the motivation for stochastic models in this arena? The motivation for stochastic models is to account for the unmodeled effects of the environment (disturbances) on the system under study. If

we examine the measurement from any process control system of interest, no matter how "macroscopic," we are confronted with the physical reality that the measurement still looks a lot like Figure 1.2. If it is important to model the observed measurement fluctuations, we turn to stochastic models.

Some of the observed fluctuation in the data is assignable to the measurement device. This source of fluctuation is known as measurement "noise." Some of the observed fluctuation in the data is assignable to unmodeled disturbances from the environment affecting the state of the system. The simplest stochastic model for representing these two possible sources of disturbances is a linear model with added random variables

$$\begin{aligned} x^+ &= Ax + Bu + Gw \\ y &= Cx + Du + v \end{aligned}$$

with initial condition $x(0) = x_0$. The variable $w \in \mathbb{R}^g$ is the random variable acting on the state transition, $v \in \mathbb{R}^p$ is a random variable acting on the measured output, and $x_0$ is a random variable specifying the initial state. The random variable $v$ is used to model the measurement noise and $w$ models the process disturbance. The matrix $G \in \mathbb{R}^{n\times g}$ allows further refinement of the modeling between the source of the disturbance and its effect on the state. Often $G$ is chosen to be the identity matrix with $g = n$.

## 1.3 Introductory MPC Regulator

### 1.3.1 Linear Quadratic Problem

We start by designing a controller to take the state of a deterministic, linear system to the origin. If the setpoint is not the origin, or we wish to track a time-varying setpoint trajectory, we will subsequently make modifications of the zero setpoint problem to account for that. The system model is

$$\begin{aligned} x^+ &= Ax + Bu \\ y &= Cx \end{aligned} \tag{1.5}$$

In this first problem, we assume that the state is measured, or $C = I$. We will handle the output measurement problem with state estimation in the next section. Using the model we can predict how the state evolves

given any set of inputs we are considering. Consider $N$ time steps into the future and collect the input sequence into $\mathbf{u}$,

$$\mathbf{u} = \{u(0), u(1), \ldots, u(N-1)\}$$

Constraints on the $\mathbf{u}$ sequence (i.e., valve saturations, etc.) are covered extensively in Chapter 2. The constraints are the main feature that distinguishes MPC from the standard linear quadratic (LQ) control.

We first define an objective function $V(\cdot)$ to measure the deviation of the trajectory of $x(k), u(k)$ from zero by summing the weighted squares

$$V(x(0), \mathbf{u}) = \frac{1}{2} \sum_{k=0}^{N-1} \left[ x(k)'Qx(k) + u(k)'Ru(k) \right] + \frac{1}{2} x(N)'P_f x(N)$$

subject to

$$x^+ = Ax + Bu$$

The objective function depends on the input sequence and state sequence. The initial state is available from the measurement. The remainder of the state trajectory, $x(k), k = 1, \ldots, N$, is determined by the model and the input sequence $\mathbf{u}$. So we show the objective function's explicit dependence on the input sequence and initial state. The tuning parameters in the controller are the matrices $Q$ and $R$. We allow the final state penalty to have a different weighting matrix, $P_f$, for generality. Large values of $Q$ in comparison to $R$ reflect the designer's intent to drive the state to the origin quickly at the expense of large control action. Penalizing the control action through large values of $R$ relative to $Q$ is the way to reduce the control action and slow down the rate at which the state approaches the origin. Choosing appropriate values of $Q$ and $R$ (i.e., tuning) is not always obvious, and this difficulty is one of the challenges faced by industrial practitioners of LQ control. Notice that MPC inherits this tuning challenge.

We then formulate the following optimal LQ control problem

$$\min_{\mathbf{u}} V(x(0), \mathbf{u}) \tag{1.6}$$

The $Q$, $P_f$ and $R$ matrices often are chosen to be diagonal, but we do not assume that here. We assume, however, that $Q$, $P_f$, and $R$ are *real and symmetric*; $Q$ and $P_f$ are *positive semidefinite*; and $R$ is *positive definite*. These assumptions guarantee that the solution to the optimal control problem exists and is unique.

### 1.3.2 Optimizing Multistage Functions

We next provide a brief introduction to methods for solving multistage optimization problems like (1.6). Consider the set of variables $w$, $x$, $y$, and $z$, and the following function to be optimized

$$f(w,x) + g(x,y) + h(y,z)$$

Notice that the objective function has a special structure in which each stage's cost function in the sum depends only on adjacent variable pairs. For the first version of this problem, we consider $w$ to be a fixed parameter, and we would like to solve the problem

$$\min_{x,y,z} f(w,x) + g(x,y) + h(y,z) \qquad w \text{ fixed}$$

One option is to optimize simultaneously over all three decision variables. Because of the objective function's special structure, however, we can obtain the solution by optimizing a sequence of three single-variable problems defined as follows

$$\min_x \left[ f(w,x) + \min_y \left[ g(x,y) + \min_z h(y,z) \right] \right]$$

We solve the inner problem over $z$ first, and denote the optimal value and solution as follows

$$\underline{h}^0(y) = \min_z h(y,z) \qquad \underline{z}^0(y) = \arg\min_z h(y,z)$$

Notice that the optimal $z$ and value function for this problem are both expressed as a function of the $y$ variable. We then move to the next optimization problem and solve for the $y$ variable

$$\min_y g(x,y) + \underline{h}^0(y)$$

and denote the solution and value function as

$$\underline{g}^0(x) = \min_y g(x,y) + \underline{h}^0(y) \qquad \underline{y}^0(x) = \arg\min_y g(x,y) + \underline{h}^0(y)$$

The optimal solution for $y$ is a function of $x$, the remaining variable to be optimized. The third and final optimization is

$$\min_x f(w,x) + \underline{g}^0(x)$$

with solution and value function

$$\underline{f}^0(w) = \min_x f(w,x) + \underline{g}^0(x) \qquad \underline{x}^0(w) = \arg\min_x f(w,x) + \underline{g}^0(x)$$

We summarize the recursion with the following annotated equation

$$\underbrace{\min_x \Big[ f(w,x) + \overbrace{\min_y \big[ g(x,y) + \underbrace{\min_z h(y,z)}_{\underline{h}^0(y),\, \underline{z}^0(y)} \big]}^{\underline{g}^0(x),\, \underline{y}^0(x)} \Big]}_{\underline{f}^0(w),\, \underline{x}^0(w)}$$

If we are mainly interested in the first variable $x$, then the function $\underline{x}^0(w)$ is of primary interest and we have obtained this function quite efficiently. This nested solution approach is an example of a class of techniques known as dynamic programming (DP). DP was developed by Bellman (Bellman, 1957; Bellman and Dreyfus, 1962) as an efficient means for solving these kinds of multistage optimization problems. Bertsekas (1987) provides an overview of DP.

The version of the method we just used is called *backward* DP because we find the variables in reverse order: first $z$, then $y$, and finally $x$. Notice we find the optimal solutions as *functions* of the variables to be optimized at the next stage. If we wish to find the other variables $y$ and $z$ as a function of the known parameter $w$, then we nest the optimal solutions found by the backward DP recursion

$$\underset{\sim}{y}^0(w) = \underline{y}^0(\underline{x}^0(w)) \qquad \underset{\sim}{z}^0(w) = \underline{z}^0(\underset{\sim}{y}^0(w)) = \underline{z}^0(\underline{y}^0(\underline{x}^0(w)))$$

As we see shortly, backward DP is the method of choice for the regulator problem.

In the state estimation problem to be considered later in this chapter, $w$ becomes a variable to be optimized, and $z$ plays the role of a parameter. We wish to solve the problem

$$\min_{w,x,y} f(w,x) + g(x,y) + h(y,z) \qquad z \text{ fixed}$$

We can still break the problem into three smaller nested problems, but

the order is reversed

$$\underbrace{\min_{y}\left[h(y,z)+\overbrace{\min_{x}\left[g(x,y)+\underbrace{\min_{w} f(w,x)}_{\overline{f}^0(x),\,\overline{w}^0(x)}\right]}^{\overline{g}^0(y),\,\overline{x}^0(y)}\right]}_{\overline{h}^0(z),\,\overline{y}^0(z)} \tag{1.7}$$

This form is called *forward* DP because we find the variables in the order given: first $w$, then $x$, and finally $y$. The optimal value functions and optimal solutions at each of the three stages are shown in (1.7). This version is preferable if we are primarily interested in finding the final variable $y$ as a function of the parameter $z$. As before, if we need the other optimized variables $x$ and $w$ as a function of the parameter $z$, we must insert the optimal functions found by the forward DP recursion

$$\tilde{x}^0(z)=\overline{x}^0(\overline{y}^0(z)) \qquad \tilde{w}^0(z)=\overline{w}^0(\tilde{x}^0(z))=\overline{w}^0(\overline{x}^0(\overline{y}^0(z)))$$

For the reader interested in trying some exercises to reinforce the concepts of DP, Exercise 1.15 considers finding the function $\tilde{w}^0(z)$ with *backward* DP instead of forward DP as we just did here. Exercise C.1 discusses showing that the nested optimizations indeed give the same answer as simultaneous optimization over all decision variables.

Finally, if we optimize over all four variables, including the one considered as a fixed parameter in the two versions of DP we used, then we have two equivalent ways to express the value of the complete optimization

$$\min_{w,x,y,z} f(w,x)+g(x,y)+h(y,z)=\min_{w}\underline{f}^0(w)=\min_{z}\overline{h}^0(z)$$

The result in the next example proves useful in combining quadratic functions to solve the LQ problem.

**Example 1.1: Sum of quadratic functions**

Consider the two quadratic functions given by

$$V_1(x)=(1/2)(x-a)'A(x-a) \qquad V_2(x)=(1/2)(x-b)'B(x-b)$$

in which $A,B>0$ are positive definite matrices and $a$ and $b$ are $n$-vectors locating the minimum of each function. Figure 1.3 displays the

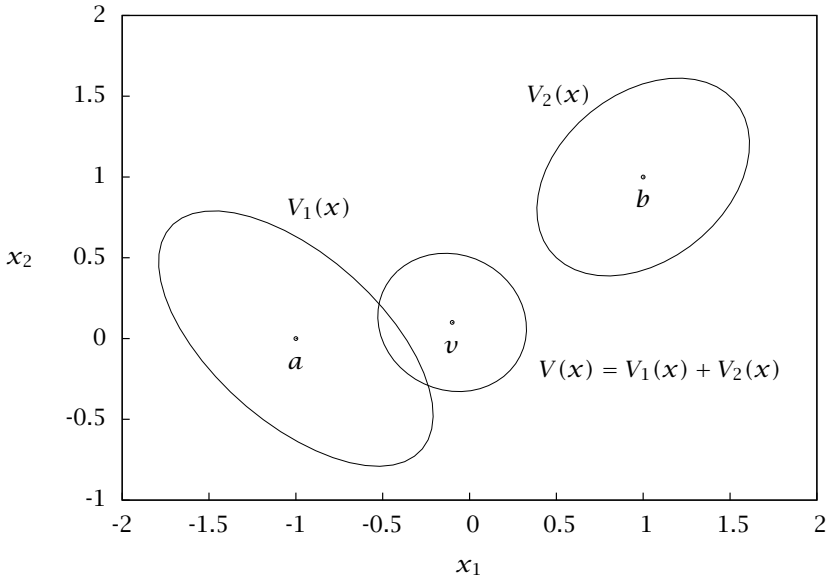


**Figure 1.3:** Two quadratic functions and their sum; $V(x) = V_1(x) + V_2(x)$.

ellipses defined by the level sets $V_1(x) = 1/4$ and $V_2(x) = 1/4$ for the following data

$$A = \begin{bmatrix} 1.25 & 0.75 \\ 0.75 & 1.25 \end{bmatrix} \qquad a = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$
$$B = \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 1.5 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(a) Show that the sum $V(x) = V_1(x) + V_2(x)$ is also quadratic

$$V(x) = (1/2)(x - v)'H(x - v) + \text{constant}$$

in which

$$H = A + B \qquad v = H^{-1}(Aa + Bb)$$

and verify the three ellipses given in Figure 1.3.

(b) Consider a generalization useful in the discussion of the upcoming state estimation problem. Let

$$V_1(x) = (1/2)(x-a)'A(x-a) \qquad V_2(x) = (1/2)(Cx-b)'B(Cx-b)$$

Derive the formulas for $H$ and $v$ for this case.

(c) Use the matrix inversion lemma (see Exercise 1.12) and show that $V(x)$ can be expressed also in an inverse form, which is useful in state estimation problems

$$\begin{aligned} V(x) &= (1/2)(x-v)'\tilde{H}^{-1}(x-v) + \text{constant} \\ \tilde{H} &= A^{-1} - A^{-1}C'(CA^{-1}C' + B^{-1})^{-1}CA^{-1} \\ v &= a + A^{-1}C'(CA^{-1}C' + B^{-1})^{-1}\ (b - Ca) \end{aligned}$$

**Solution**

(a) The sum of two quadratics is also quadratic, so we parameterize the sum as

$$V(x) = (1/2)(x-v)'H(x-v) + d$$

and solve for $v$, $H$, and $d$. Comparing zeroth, first and second derivatives gives

$$\begin{aligned} V(v) = d \qquad &= V_1(v) + V_2(v) \\ V_x(x) = H(x-v) &= A(x-a) + B(x-b) \\ V_{xx} = H \qquad &= A + B \end{aligned}$$

Solving these gives

$$\begin{aligned} H &= A + B \\ v &= H^{-1}(Aa + Bb) \\ d &= V_1(v) + V_2(v) \end{aligned} \tag{1.8}$$

Notice that $H$ is positive definite since $A$ and $B$ are positive definite. Substituting the values of $a$, $A$, $b$, and $B$, and setting $d = 0$ gives

$$V(x) = (1/2)(x-v)'H(x-v)$$

$$H = \begin{bmatrix} 2.75 & 0.25 \\ 0.25 & 2.75 \end{bmatrix} \qquad v = \begin{bmatrix} -0.1 \\ 0.1 \end{bmatrix}$$

$V(x) = 1/4$ is plotted for the choice of constant $d = 0$.

(b) Comparing zeroth, first and second derivatives gives

$$\begin{aligned} V(v) = d \qquad &= V_1(v) + V_2(v) \\ V_x(x) = H(x - v) &= A(x - a) + C'B(Cx - b) \\ V_{xx} = H \qquad &= A + C'BC \end{aligned}$$

Solving these gives

$$\begin{aligned} H &= A + C'BC \\ v &= H^{-1}(Aa + C'Bb) \\ d &= V_1(v) + V_2(v) \end{aligned}$$

Notice that $H$ is positive definite since $A$ is positive definite and $C'BC$ is positive semidefinite for any $C$.

(c) Define $\overline{x} = x - a$ and $\overline{b} = b - Ca$, and express the problem as

$$\begin{aligned} V(\overline{x}) &= (1/2)\overline{x}'A\overline{x} + (1/2)(C(\overline{x} + a) - b)'B(C(\overline{x} + a) - b) \\ &= (1/2)\overline{x}'A\overline{x} + (1/2)(C\overline{x} - \overline{b})'B(C\overline{x} - \overline{b}) \end{aligned}$$

Apply the solution of the previous part and set the constant to zero to obtain

$$\begin{aligned} V(\overline{x}) &= (1/2)(\overline{x} - \overline{v})'H(\overline{x} - \overline{v}) \\ H &= A + C'BC \\ \overline{v} &= H^{-1}C'B\overline{b} \end{aligned}$$

Use the matrix inversion lemma's (1.55) on $H$ and (1.56) on $\overline{v}$ to obtain

$$\begin{aligned} \widetilde{H} &= A^{-1} - A^{-1}C'(CA^{-1}C' + B^{-1})^{-1}CA^{-1} \\ \overline{v} &= A^{-1}C'(CA^{-1}C' + B^{-1})^{-1}\overline{b} \end{aligned}$$

The function $V$ is then given by

$$\begin{aligned} V &= (1/2)(\overline{x} - \overline{v})'\widetilde{H}^{-1}(\overline{x} - \overline{v}) \\ V &= (1/2)(x - (a + \overline{v}))'\widetilde{H}^{-1}(x - (a + \overline{v})) \\ V &= (1/2)(x - v)'\widetilde{H}^{-1}(x - v) \end{aligned}$$

in which

$$v = a + A^{-1}C'(CA^{-1}C' + B^{-1})^{-1}\,(b - Ca) \qquad \square$$

### 1.3.3 Dynamic Programming Solution

After this brief introduction to DP, we apply it to solve the LQ control problem. We first rewrite (1.6) in the following form to see the structure clearly

$$V(x(0), \mathbf{u}) = \sum_{k=0}^{N-1} \ell(x(k), u(k)) + \ell_N(x(N)) \qquad \text{s.t. } x^+ = Ax + Bu$$

in which the *stage cost* $\ell(x,u) = (1/2)(x'Qx + u'Ru), k = 0,\ldots,N-1$ and the terminal stage cost $\ell_N(x) = (1/2)x'P_f x$. Since $x(0)$ is known, we choose *backward* DP as the convenient method to solve this problem. We first rearrange the overall objective function so we can optimize over input $u(N-1)$ and state $x(N)$

$$\min_{u(0),x(1),\ldots u(N-2),x(N-1)} \ell(x(0),u(0)) + \ell(x(1),u(1)) + \cdots + \min_{u(N-1),x(N)} \ell(x(N-1),u(N-1)) + \ell_N(x(N))$$

subject to

$$x(k+1) = Ax(k) + Bu(k) \qquad k = 0,\ldots N-1$$

The problem to be solved at the last stage is

$$\min_{u(N-1),x(N)} \ell(x(N-1),u(N-1)) + \ell_N(x(N)) \tag{1.9}$$

subject to

$$x(N) = Ax(N-1) + Bu(N-1)$$

in which $x(N-1)$ appears in this stage as a parameter. We denote the optimal cost by $V^0_{N-1}(x(N-1))$ and the optimal decision variables by $u^0_{N-1}(x(N-1))$ and $x^0_N(x(N-1))$. The optimal cost and decisions at the last stage are parameterized by the state at the previous stage as we expect in backward DP. We next solve this optimization. First we substitute the state equation for $x(N)$ and combine the two quadratic terms using (1.8)

$$\begin{aligned}
&\ell(x(N-1),u(N-1)) + \ell_N(x(N)) \\
&\quad = (1/2)\left(|x(N-1)|^2_Q + |u(N-1)|^2_R + |Ax(N-1) + Bu(N-1)|^2_{P_f}\right) \\
&\quad = (1/2)\left(|x(N-1)|^2_Q + |(u(N-1) - v)|^2_H\right) + d
\end{aligned}$$

in which

$$\begin{aligned} H &= R + B'P_fB \\ v &= K(N-1)x(N-1) \\ d &= (1/2)x(N-1)'\Big(K(N-1)'RK(N-1)+ \\ &\qquad (A+BK(N-1))'P_f(A+BK(N-1))\Big)x(N-1) \\ K(N-1) &= -(B'P_fB+R)^{-1}B'P_fA \end{aligned} \tag{1.10}$$

Given this form of the cost function, we see by inspection that the optimal input for $u(N-1)$ is $v$ defining the optimal control law at stage $N-1$ to be a linear function of the state $x(N-1)$. Then using the model equation, the optimal final state is also a linear function of state $x(N-1)$. The optimal cost is $d$, which makes the optimal cost a quadratic function of $x(N-1)$. Summarizing, for all $x$

$$\begin{aligned} u^0_{N-1}(x) &= K(N-1)\,x \\ x^0_N(x) &= (A+BK(N-1))\,x \\ V^0_{N-1}(x) &= (1/2)x'\,\Pi(N-1)\,x \\ \Pi(N-1) &= Q + A'P_fA+ \\ &\qquad K(N-1)'(B'P_fB+R)K(N-1) + 2K(N-1)'B'P_fA \end{aligned}$$

We can rewrite $\Pi(N-1)$ using the result

$$\begin{aligned} K(N-1)'(B'P_fB+R)K(N-1) + 2K(N-1)'B'P_fA = \\ -A'P_fB(B'P_fB+R)^{-1}B'P_fA \end{aligned}$$

which is obtained by substituting (1.10) for $K$ into the equation for $\Pi(N-1)$ and simplifying. Substituting this result into the previous equation gives

$$\Pi(N-1) = Q + A'P_fA - A'P_fB(B'P_fB+R)^{-1}B'P_fA$$

The function $V^0_{N-1}(x)$ defines the optimal *cost to go* from state $x$ for the last stage under the optimal control law $u^0_{N-1}(x)$. Having this function allows us to move to the next stage of the DP recursion. For the next stage we solve the optimization

$$\min_{u(N-2),x(N-1)} \ell(x(N-2),u(N-2)) + V^0_{N-1}(x(N-1))$$

subject to

$$x(N-1) = Ax(N-2) + Bu(N-2)$$

Notice that this problem is identical in structure to the stage we just solved, (1.9), and we can write out the solution by simply renaming variables

$$\begin{aligned}
u^0_{N-2}(x) &= K(N-2)\,x \\
x^0_{N-1}(x) &= (A + BK(N-2))\,x \\
V^0_{N-2}(x) &= (1/2)x'\,\Pi(N-2)\,x \\
K(N-2) &= -(B'\Pi(N-1)B + R)^{-1}B'\Pi(N-1)A \\
\Pi(N-2) &= Q + A'\Pi(N-1)A - \\
&\qquad A'\Pi(N-1)B(B'\Pi(N-1)B + R)^{-1}B'\Pi(N-1)A
\end{aligned}$$

The recursion from $\Pi(N-1)$ to $\Pi(N-2)$ is known as a backward Riccati iteration. To summarize, the backward Riccati iteration is defined as follows

$$\Pi(k-1) = Q + A'\Pi(k)A - A'\Pi(k)B\left(B'\Pi(k)B + R\right)^{-1}B'\Pi(k)A \qquad k = N, N-1, \ldots, 1 \quad (1.11)$$

with terminal condition

$$\Pi(N) = P_f \quad (1.12)$$

The terminal condition replaces the typical initial condition because the iteration is running backward. The optimal control policy at each stage is

$$u^0_k(x) = K(k)x \qquad k = N-1, N-2, \ldots, 0 \quad (1.13)$$

The optimal gain at time $k$ is computed from the Riccati matrix at time $k+1$

$$K(k) = -\left(B'\Pi(k+1)B + R\right)^{-1}B'\Pi(k+1)A \qquad k = N-1, N-2, \ldots, 0 \quad (1.14)$$

and the optimal cost to go from time $k$ to time $N$ is

$$V^0_k(x) = (1/2)x'\Pi(k)x \qquad k = N, N-1, \ldots, 0 \quad (1.15)$$

### 1.3.4 The Infinite Horizon LQ Problem

Let us motivate the infinite horizon problem by showing a weakness of the finite horizon problem. Kalman (1960b, p.113) pointed out in his classic 1960 paper that optimality does not ensure stability.

> In the engineering literature it is often assumed (tacitly and incorrectly) that a system with optimal control law (6.8) is necessarily stable.

Assume that we use as our control law the first feedback gain of the finite horizon problem, $K(0)$,

$$u(k) = K(0)x(k)$$

Then the stability of the closed-loop system is determined by the eigenvalues of $A+BK(0)$. We now construct an example that shows choosing $Q > 0$, $R > 0$, and $N \geq 1$ does not ensure stability. In fact, we can find reasonable values of these parameters such that the controller destabilizes a stable system.[2] Let

$$A = \begin{bmatrix} 4/3 & -2/3 \\ 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad C = [-2/3 \;\; 1]$$

This system is chosen so that $G(z)$ has a *zero* at $z = 3/2$, i.e., an unstable zero. We now construct an LQ controller that inverts this zero and hence produces an unstable system. We would like to choose $Q = C'C$ so that $y$ itself is penalized, but that $Q$ is only semidefinite. We add a small positive definite piece to $C'C$ so that $Q$ is positive definite, and choose a *small* positive $R$ penalty (to encourage the controller to misbehave), and $N = 5$,

$$Q = C'C + 0.001I = \begin{bmatrix} 4/9 + .001 & -2/3 \\ -2/3 & 1.001 \end{bmatrix} \qquad R = 0.001$$

We now iterate the Riccati equation four times starting from $\Pi = P_f = Q$ and compute $K(0)$ for $N = 5$; then we compute the eigenvalues of $A + BK(0)$ and achieve[3]

$$\text{eig}(A + BK_5(0)) = \{1.307, 0.001\}$$

[2] In Chapter 2, we present several controller design methods that prevent this kind of instability.

[3] Please check this answer with Octave or MATLAB.

Using this controller the closed-loop system evolution is $x(k) = (A + BK_5(0))^k x_0$. Since an eigenvalue of $A + BK_5(0)$ is greater than unity, $x(k) \to \infty$ as $k \to \infty$. In other words the closed-loop system is unstable.

If we continue to iterate the Riccati equation, which corresponds to increasing the horizon in the controller, we obtain for $N = 7$

$$\text{eig}(A + BK_7(0)) = \{0.989, 0.001\}$$

and the controller is stabilizing. If we continue iterating the Riccati equation, we converge to the following steady-state closed-loop eigenvalues

$$\text{eig}(A + BK_\infty(0)) = \{0.664, 0.001\}$$

This controller corresponds to an infinite horizon control law. Notice that it is stabilizing and has a reasonable stability margin. Nominal stability is a guaranteed property of infinite horizon controllers as we prove in the next section.

With this motivation, we are led to consider directly the infinite horizon case

$$V(x(0), \mathbf{u}) = \frac{1}{2} \sum_{k=0}^{\infty} x(k)'Qx(k) + u(k)'Ru(k) \tag{1.16}$$

in which $x(k)$ is the solution at time $k$ of $x^+ = Ax+Bu$ if the initial state is $x(0)$ and the input sequence is $\mathbf{u}$. If we are interested in a continuous process (i.e., no final time), then the natural cost function is an infinite horizon cost. If we were truly interested in a batch process (i.e., the process does stop at $k = N$), then stability is not a relevant property, and we naturally would use the finite horizon LQ controller and the *time-varying* controller, $u(k) = K(k)x(k), k = 0, 1, \ldots, N$.

In considering the infinite horizon problem, we first restrict attention to systems for which there exist input sequences that give bounded cost. Consider the case $A = I$ and $B = 0$, for example. Regardless of the choice of input sequence, (1.16) is unbounded for $x(0) \neq 0$. It seems clear that we are not going to stabilize an unstable system ($A = I$) without any input ($B = 0$). This is an example of an *uncontrollable* system. In order to state the sharpest results on stabilization, we require the concepts of controllability, stabilizability, observability, and detectability. We shall define these concepts subsequently.

### 1.3.5 Controllability

A system is *controllable* if, for any pair of states $x, z$ in the state space, $z$ can be reached in finite time from $x$ (or $x$ controlled to $z$) (Sontag, 1998, p.83). A *linear discrete time* system $x^+ = Ax + Bu$ is therefore controllable if there exists a finite time $N$ and a sequence of inputs

$$\{u(0), u(1), \ldots u(N-1)\}$$

that can transfer the system from any $x$ to any $z$ in which

$$z = A^N x + \begin{bmatrix} B & AB & \cdots & A^{N-1}B \end{bmatrix} \begin{bmatrix} u(N-1) \\ u(N-2) \\ \vdots \\ u(0) \end{bmatrix}$$

We can simplify this condition by noting that the matrix powers $A^k$ for $k \geq n$ are expressible as linear combinations of the powers 0 to $n-1$. This result is a consequence of the Cayley-Hamilton theorem (Horn and Johnson, 1985, pp. 86–87). Therefore the range of the matrix $\begin{bmatrix} B & AB & \cdots & A^{N-1}B \end{bmatrix}$ for $N \geq n$ is the same as $\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$. In other words, for an unconstrained linear system, if we cannot reach $z$ in $n$ moves, we cannot reach $z$ in any number of moves. The question of *controllability* of a linear time-invariant system is therefore a question of *existence* of solutions to linear equations for an arbitrary right-hand side

$$\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \begin{bmatrix} u(n-1) \\ u(n-2) \\ \vdots \\ u(0) \end{bmatrix} = z - A^n x$$

The matrix appearing in this equation is known as the *controllability matrix* $C$

$$C = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \tag{1.17}$$

From the fundamental theorem of linear algebra, we know a solution exists for all right-hand sides if and only if the *rows* of the $n \times nm$ controllability matrix are linearly independent.[4] Therefore, the system $(A, B)$ is controllable if and only if

$$\operatorname{rank}(C) = n$$

[4] See Section A.4 of Appendix A or (Strang, 1980, pp.87–88) for a review of this result.

The following result for checking controllability also proves useful (Hautus, 1972).

**Lemma 1.2** (Hautus Lemma for controllability)**.** *A system is controllable if and only if*

$$\text{rank}\begin{bmatrix} \lambda I - A & B \end{bmatrix} = n \qquad \text{for all } \lambda \in \mathbb{C} \tag{1.18}$$

*in which $\mathbb{C}$ is the set of complex numbers.*

Notice that the first $n$ columns of the matrix in (1.18) are linearly independent if $\lambda$ is not an eigenvalue of $A$, so (1.18) is equivalent to checking the rank at just the eigenvalues of $A$

$$\text{rank}\begin{bmatrix} \lambda I - A & B \end{bmatrix} = n \qquad \text{for all } \lambda \in \text{eig}(A)$$

### 1.3.6 Convergence of the Linear Quadratic Regulator

We now show that the infinite horizon regulator asymptotically stabilizes the origin for the closed-loop system. Define the infinite horizon objective function

$$V(x, \mathbf{u}) = \frac{1}{2} \sum_{k=0}^{\infty} x(k)'Qx(k) + u(k)'Ru(k)$$

subject to

$$\begin{aligned} x^+ &= Ax + Bu \\ x(0) &= x \end{aligned}$$

with $Q, R > 0$. If $(A, B)$ is controllable, the solution to the optimization problem

$$\min_{\mathbf{u}} V(x, \mathbf{u})$$

exists and is unique for all $x$. We denote the optimal solution by $\mathbf{u}^0(x)$, and the first input in the optimal sequence by $u^0(x)$. The feedback control law $\kappa_\infty(\cdot)$ for this infinite horizon case is then defined as $u = \kappa_\infty(x)$ in which $\kappa_\infty(x) = u^0(x) = \mathbf{u}^0(0; x)$. As stated in the following lemma, this infinite horizon linear quadratic regulator (LQR) is stabilizing.

**Lemma 1.3** (LQR convergence)**.** *For $(A, B)$ controllable, the infinite horizon LQR with $Q, R > 0$ gives a convergent closed-loop system*

$$x^+ = Ax + B\kappa_\infty(x)$$

*Proof.* The cost of the infinite horizon objective is bounded above for all $x(0)$ because $(A, B)$ is controllable. Controllability implies that there exists a sequence of $n$ inputs $\{u(0), u(1), \ldots, u(n-1)\}$ that transfers the state from any $x(0)$ to $x(n) = 0$. A zero control sequence after $k = n$ for $\{u(n+1), u(n+2), \ldots\}$ generates zero cost for all terms in $V$ after $k = n$, and the objective function for this infinite control sequence is therefore finite. The cost function is strictly convex in $\mathbf{u}$ because $R > 0$ so the solution to the optimization is unique.

If we consider the sequence of costs to go along the closed-loop trajectory, we have

$$V_{k+1} = V_k - (1/2)\left(x(k)'Qx(k) + u(k)'Ru(k)\right)$$

in which $V_k = V^0(x(k))$ is the cost at time $k$ for state value $x(k)$ and $u(k) = u^0(x(k))$ is the optimal control for state $x(k)$. The cost along the closed-loop trajectory is nonincreasing and bounded below (by zero). Therefore, the sequence $\{V_k\}$ converges and

$$x(k)'Qx(k) \to 0 \qquad u(k)'Ru(k) \to 0 \qquad \text{as } k \to \infty$$

Since $Q, R > 0$, we have

$$x(k) \to 0 \qquad u(k) \to 0 \qquad \text{as } k \to \infty$$

and closed-loop convergence is established. ■

In fact we know more. From the previous sections, we know the optimal solution is found by iterating the Riccati equation, and the optimal infinite horizon control law and optimal cost are given by

$$u^0(x) = Kx \qquad V^0(x) = (1/2)x'\Pi x$$

in which

$$\begin{aligned} K &= -(B'\Pi B + R)^{-1}B'\Pi A \\ \Pi &= Q + A'\Pi A - A'\Pi B(B'\Pi B + R)^{-1}B'\Pi A \end{aligned} \tag{1.19}$$

Proving Lemma 1.3 has shown also that for $(A, B)$ controllable and $Q, R > 0$, a positive definite solution to the discrete algebraic Riccati equation (DARE), (1.19), exists and the eigenvalues of $(A + BK)$ are asymptotically stable for the $K$ corresponding to this solution (Bertsekas, 1987, pp.58–64).

This basic approach to establishing regulator stability will be generalized in Chapter 2 to handle constrained and nonlinear systems, so it

is helpful for the new student to first become familiar with these ideas in the unconstrained, linear setting. For linear systems, asymptotic convergence is equivalent to asymptotic stability, and we delay the discussion of stability until Chapter 2. In Chapter 2 the optimal cost is shown to be a Lyapunov function for the closed-loop system. We also can strengthen the stability for linear systems from asymptotic stability to exponential stability based on the form of the Lyapunov function.

The LQR convergence result in Lemma 1.3 is the simplest to establish, but we can enlarge the class of systems and penalties for which closed-loop stability is guaranteed. The system restriction can be weakened from controllability to *stabilizability*, which is discussed in Exercises 1.19 and 1.20. The restriction on the allowable state penalty $Q$ can be weakened from $Q > 0$ to $Q \geq 0$ and $(A, Q)$ *detectable*, which is also discussed in Exercise 1.20. The restriction $R > 0$ is retained to ensure uniqueness of the control law. In applications, if one cares little about the cost of the control, then $R$ is chosen to be small, but positive definite.

## 1.4 Introductory State Estimation

The next topic is state estimation. In most applications, the variables that are conveniently or economically measurable ($y$) are a small subset of the variables required to model the system ($x$). Moreover, the measurement is corrupted with sensor noise and the state evolution is corrupted with process noise. Determining a good state estimate for use in the regulator in the face of a noisy and incomplete output measurement is a challenging task. That is the challenge of state estimation.

To fully appreciate the fundamentals of state estimation, we must address the fluctuations in the data. Probability theory has proven itself as the most successful and versatile approach to modeling these fluctuations. In this section we introduce the probability fundamentals necessary to develop an optimal state estimator in the simplest possible setting: a linear discrete time model subject to normally distributed process and measurement noise. This optimal state estimator is known as the Kalman filter (Kalman, 1960a). In Chapter 4 we revisit the state estimation problem in a much wider setting, and consider nonlinear models and constraints on the system that preclude an analytical solution such as the Kalman filter. The probability theory presented here is also preparation for understanding that chapter.

### 1.4.1 Linear Systems and Normal Distributions

This section summarizes the probability and random variable results required for deriving a linear optimal estimator such as the Kalman filter. We assume that the reader is familiar with the concepts of a random variable, probability density and distribution, the multivariate normal distribution, mean and variance, statistical independence, and conditional probability. Readers unfamiliar with these terms should study the material in Appendix A before reading this and the next sections.

In the following discussion let $x$, $y$, and $z$ be vectors of random variables. We use the notation

$$x \sim N(m, P)$$
$$p_x(x) = n(x, m, P)$$

to denote random variable $x$ is normally distributed with mean $m$ and covariance (or simply variance) $P$, in which

$$n(x, m, P) = \frac{1}{(2\pi)^{n/2}(\det P)^{1/2}} \exp\left[-\frac{1}{2}(x-m)'P^{-1}(x-m)\right] \tag{1.20}$$

and $\det P$ denotes the determinant of matrix $P$. Note that if $x \in \mathbb{R}^n$, then $m \in \mathbb{R}^n$ and $P \in \mathbb{R}^{n\times n}$ is a positive definite matrix. We require three main results. The simplest version can be stated as follows.

**Joint independent normals.** If $x$ and $y$ are normally distributed and (statistically) independent[5]

$$x \sim N(m_x, P_x) \qquad y \sim N(m_y, P_y)$$

then their joint density is given by

$$p_{x,y}(x, y) = n(x, m_x, P_x)\; n(y, m_y, P_y)$$
$$\begin{bmatrix} x \\ y \end{bmatrix} \sim N\left(\begin{bmatrix} m_x \\ m_y \end{bmatrix}, \begin{bmatrix} P_x & 0 \\ 0 & P_y \end{bmatrix}\right) \tag{1.21}$$

Note that, depending on convenience, we use both $(x, y)$ and the vector $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ to denote the pair of random variables.

**Linear transformation of a normal.** If $x$ is normally distributed with mean $m$ and variance $P$, and $y$ is a linear transformation of $x$, $y = Ax$, then $y$ is distributed with mean $Am$ and variance $APA'$

$$x \sim N(m, P) \qquad y = Ax \qquad y \sim N(Am, APA') \tag{1.22}$$

[5] We may emphasize that two vectors of random variables are independent using *statistically independent* to distinguish this concept from linear independence of vectors.

**Conditional of a joint normal.** If $x$ and $y$ are jointly normally distributed as

$$\begin{bmatrix} x \\ y \end{bmatrix} \sim N\left( \begin{bmatrix} m_x \\ m_y \end{bmatrix} \begin{bmatrix} P_x & P_{xy} \\ P_{yx} & P_y \end{bmatrix} \right)$$

then the conditional density of $x$ given $y$ is also normal

$$p_{x|y}(x|y) = n(x, m, P) \tag{1.23}$$

in which the mean is

$$m = m_x + P_{xy} P_y^{-1} (y - m_y)$$

and the covariance is

$$P = P_x - P_{xy} P_y^{-1} P_{yx}$$

Note that the conditional mean $m$ is itself a random variable because it depends on the random variable $y$.

To derive the optimal estimator, we actually require these three main results conditioned on additional random variables. The analogous results are the following.

**Joint independent normals.** If $p_{x|z}(x|z)$ is normal, and $y$ is statistically independent of $x$ and $z$ and normally distributed

$$p_{x|z}(x|z) = n(x, m_x, P_x)$$
$$y \sim N(m_y, P_y) \qquad y \text{ independent of } x \text{ and } z$$

then the conditional joint density of $(x, y)$ given $z$ is

$$p_{x,y|z}(x, y|z) = n(x, m_x, P_x)\, n(y, m_y, P_y)$$
$$p_{x,y|z}\left( \begin{bmatrix} x \\ y \end{bmatrix} \middle| z \right) = n\left( \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} m_x \\ m_y \end{bmatrix}, \begin{bmatrix} P_x & 0 \\ 0 & P_y \end{bmatrix} \right) \tag{1.24}$$

**Linear transformation of a normal.**

$$p_{x|z}(x|z) = n(x, m, P) \qquad y = Ax$$
$$p_{y|z}(y|z) = n(y, Am, APA') \tag{1.25}$$

**Conditional of a joint normal.** If $x$ and $y$ are jointly normally distributed as

$$p_{x,y|z}\left( \begin{bmatrix} x \\ y \end{bmatrix} \middle| z \right) = n\left( \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} m_x \\ m_y \end{bmatrix}, \begin{bmatrix} P_x & P_{xy} \\ P_{yx} & P_y \end{bmatrix} \right)$$

then the conditional density of $x$ given $y, z$ is also normal

$$p_{x|y,z}(x|y,z) = n(x,m,P) \tag{1.26}$$

in which

$$\begin{aligned} m &= m_x + P_{xy}P_y^{-1}(y - m_y) \\ P &= P_x - P_{xy}P_y^{-1}P_{yx} \end{aligned}$$

### 1.4.2 Linear Optimal State Estimation

We start by assuming the initial state $x(0)$ is normally distributed with some mean and covariance

$$x(0) \sim N(\overline{x}(0), Q(0))$$

In applications, we often do not know $\overline{x}(0)$ or $Q(0)$. In such cases we often set $\overline{x}(0) = 0$ and choose a large value for $Q(0)$ to indicate our lack of prior knowledge. This choice is referred to in the statistics literature as a *noninformative prior*. The choice of noninformative prior forces the upcoming $y(k)$ measurements to determine the state estimate $\hat{x}(k)$.

**Combining the measurement.** We obtain noisy measurement $y(0)$ satisfying

$$y(0) = Cx(0) + v(0)$$

in which $v(0) \sim N(0,R)$ is the measurement noise. If the measurement process is quite noisy, then $R$ is large. If the measurements are highly accurate, then $R$ is small. We choose a zero mean for $v$ because all of the deterministic effects with nonzero mean are considered part of the model, and the measurement noise reflects what is left after all these other effects have been considered. Given the measurement $y(0)$, we want to obtain the conditional density $p_{x(0)|y(0)}(x(0)|y(0))$. This conditional density describes the change in our knowledge about $x(0)$ after we obtain measurement $y(0)$. This step is the essence of state estimation. To derive this conditional density, first consider the pair of variables $(x(0), y(0))$ given as

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} I & 0 \\ C & I \end{bmatrix} \begin{bmatrix} x(0) \\ v(0) \end{bmatrix}$$

We assume that the noise $v(0)$ is statistically independent of $x(0)$, and use the independent joint normal result (1.21) to express the joint

density of $(x(0), v(0))$

$$\begin{bmatrix} x(0) \\ v(0) \end{bmatrix} \sim N\left( \begin{bmatrix} \overline{x}(0) \\ 0 \end{bmatrix}, \begin{bmatrix} Q(0) & 0 \\ 0 & R \end{bmatrix} \right)$$

From the previous equation, the pair $(x(0), y(0))$ is a linear transformation of the pair $(x(0), v(0))$. Therefore, using the linear transformation of normal result (1.22), and the density of $(x(0), v(0))$ gives the density of $(x(0), y(0))$

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} \sim N\left( \begin{bmatrix} \overline{x}(0) \\ C\overline{x}(0) \end{bmatrix}, \begin{bmatrix} Q(0) & Q(0)C' \\ CQ(0) & CQ(0)C' + R \end{bmatrix} \right)$$

Given this joint density, we then use the conditional of a joint normal result (1.23) to obtain

$$p_{x(0)|y(0)}\left(x(0)|y(0)\right) = n\left(x(0), m, P\right)$$

in which

$$\begin{aligned} m &= \overline{x}(0) + L(0)\left(y(0) - C\overline{x}(0)\right) \\ L(0) &= Q(0)C'(CQ(0)C' + R)^{-1} \\ P &= Q(0) - Q(0)C'(CQ(0)C' + R)^{-1}CQ(0) \end{aligned}$$

We see that the conditional density $p_{x(0)|y(0)}$ is normal. The *optimal* state estimate is the value of $x(0)$ that maximizes this conditional density. For a normal, that is the mean, and we choose $\hat{x}(0) = m$. We also denote the variance in this conditional after measurement $y(0)$ by $P(0) = P$ with $P$ given in the previous equation. The change in variance after measurement ($Q(0)$ to $P(0)$) quantifies the information increase by obtaining measurement $y(0)$. The variance after measurement, $P(0)$, is always less than or equal to $Q(0)$, which implies that we can only gain information by measurement; but the information gain may be small if the measurement device is poor and the measurement noise variance $R$ is large.

**Forecasting the state evolution.** Next we consider the state evolution from $k = 0$ to $k = 1$, which satisfies

$$x(1) = \begin{bmatrix} A & I \end{bmatrix} \begin{bmatrix} x(0) \\ w(0) \end{bmatrix}$$

in which $w(0) \sim N(0, Q)$ is the process noise. If the state is subjected to large disturbances, then $Q$ is large, and if the disturbances are small, $Q$

is small. Again we choose zero mean for $w$ because the nonzero mean disturbances should have been accounted for in the system model. We next calculate the conditional density $p_{x(1)|y(0)}$. Now we require the conditional version of the joint density $(x(0), w(0))$. We assume that the process noise $w(0)$ is statistically independent of both $x(0)$ and $v(0)$, hence it is also independent of $y(0)$, which is a linear combination of $x(0)$ and $v(0)$. Therefore we use (1.24) to obtain

$$\begin{bmatrix} x(0) \\ w(0) \end{bmatrix} \sim N\left( \begin{bmatrix} \hat{x}(0) \\ 0 \end{bmatrix}, \begin{bmatrix} P(0) & 0 \\ 0 & Q \end{bmatrix} \right)$$

We then use the conditional version of the linear transformation of a normal (1.25) to obtain

$$p_{x(1)|y(0)}(x(1)|y(0)) = n(x(1), \hat{x}^-(1), P^-(1))$$

in which the mean and variance are

$$\hat{x}^-(1) = A\hat{x}(0) \qquad P^-(1) = AP(0)A' + Q$$

We see that forecasting forward one time step may increase or decrease the conditional variance of the state. If the eigenvalues of $A$ are less than unity, for example, the term $AP(0)A'$ *may* be smaller than $P(0)$, but the process noise $Q$ adds a positive contribution. If the system is unstable, $AP(0)A'$ *may* be larger than $P(0)$, and then the conditional variance definitely increases upon forecasting. See also Exercise 1.27 for further discussion of this point.

Given that $p_{x(1)|y(0)}$ is also a normal, we are situated to add measurement $y(1)$ and continue the process of adding measurements followed by forecasting forward one time step until we have processed all the available data. Because this process is recursive, the storage requirements are small. We need to store only the current state estimate and variance, and can discard the measurements as they are processed. The required online calculation is minor. These features make the optimal linear estimator an ideal candidate for rapid online application. We next summarize the state estimation recursion.

**Summary.** Denote the measurement trajectory by

$$\mathbf{y}(k) := \{y(0), y(1), \ldots y(k)\}$$

At time $k$ the conditional density with data $\mathbf{y}(k-1)$ is normal

$$p_{x(k)|\mathbf{y}(k-1)}(x(k)|\mathbf{y}(k-1)) = n(x(k), \hat{x}^-(k), P^-(k))$$

and we denote the mean and variance with a superscript minus to indicate these are the statistics *before* measurement $y(k)$. At $k = 0$, the recursion starts with $\hat{x}^-(0) = \overline{x}(0)$ and $P^-(0) = Q(0)$ as discussed previously. We obtain measurement $y(k)$ which satisfies

$$\begin{bmatrix} x(k) \\ y(k) \end{bmatrix} = \begin{bmatrix} I & 0 \\ C & I \end{bmatrix} \begin{bmatrix} x(k) \\ v(k) \end{bmatrix}$$

The density of $(x(k), v(k))$ follows from (1.24) since measurement noise $v(k)$ is independent of $x(k)$ and $\mathbf{y}(k-1)$

$$\begin{bmatrix} x(k) \\ v(k) \end{bmatrix} \sim N\left( \begin{bmatrix} \hat{x}^-(k) \\ 0 \end{bmatrix}, \begin{bmatrix} P^-(k) & 0 \\ 0 & R \end{bmatrix} \right)$$

Equation (1.25) then gives the joint density

$$\begin{bmatrix} x(k) \\ y(k) \end{bmatrix} \sim N\left( \begin{bmatrix} \hat{x}^-(k) \\ C\hat{x}^-(k) \end{bmatrix}, \begin{bmatrix} P^-(k) & P^-(k)C' \\ CP^-(k) & CP^-(k)C' + R \end{bmatrix} \right)$$

We note $\{\mathbf{y}(k-1), y(k)\} = \mathbf{y}(k)$, and using the conditional density result (1.26) gives

$$p_{x(k)|\mathbf{y}(k)}(x(k)|\mathbf{y}(k)) = n(x(k), \hat{x}(k), P(k))$$

in which

$$\begin{aligned} \hat{x}(k) &= \hat{x}^-(k) + L(k)\left(y(k) - C\hat{x}^-(k)\right) \\ L(k) &= P^-(k)C'(CP^-(k)C' + R)^{-1} \\ P(k) &= P^-(k) - P^-(k)C'(CP^-(k)C' + R)^{-1}CP^-(k) \end{aligned}$$

We forecast from $k$ to $k+1$ using the model

$$x(k+1) = \begin{bmatrix} A & I \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix}$$

Because $w(k)$ is independent of $x(k)$ and $\mathbf{y}(k)$, the joint density of $(x(k), w(k))$ follows from a second use of (1.24)

$$\begin{bmatrix} x(k) \\ w(k) \end{bmatrix} \sim N\left( \begin{bmatrix} \hat{x}(k) \\ 0 \end{bmatrix}, \begin{bmatrix} P(k) & 0 \\ 0 & Q \end{bmatrix} \right)$$

and a second use of the linear transformation result (1.25) gives

$$p_{x(k+1)|\mathbf{y}(k)}(x(k+1)|\mathbf{y}(k)) = n(x(k+1), \hat{x}^-(k+1), P^-(k+1))$$

in which

$$\hat{x}^-(k+1) = A\hat{x}(k)$$
$$P^-(k+1) = AP(k)A' + Q$$

and the recursion is complete.

### 1.4.3 Least Squares Estimation

We next consider the state estimation problem as a deterministic optimization problem rather than an exercise in maximizing conditional density. This viewpoint proves valuable in Chapter 4 when we wish to add constraints to the state estimator. Consider a time horizon with measurements $y(k), k = 0, 1, \ldots, T$. We consider the prior information to be our best initial guess of the initial state $x(0)$, denoted $\overline{x}(0)$, and weighting matrices $P^-(0)$, $Q$, and $R$ for the initial state, process disturbance, and measurement disturbance. A reasonably flexible choice for objective function is

$$V_T(\mathbf{x}(T)) = \frac{1}{2}\Bigg( |x(0) - \overline{x}(0)|^2_{(P^-(0))^{-1}} + \sum_{k=0}^{T-1} |x(k+1) - Ax(k)|^2_{Q^{-1}} + \sum_{k=0}^{T} \left| y(k) - Cx(k) \right|^2_{R^{-1}} \Bigg) \quad (1.27)$$

in which $\mathbf{x}(T) := \{x(0), x(1), \ldots, x(T)\}$. We claim and then show that the following (deterministic) least squares optimization problem produces the same result as the conditional density function maximization of the Kalman filter

$$\min_{\mathbf{x}(T)} V_T(\mathbf{x}(T)) \quad (1.28)$$

**Game plan.** Using forward DP, we can decompose and solve recursively the least squares state estimation problem. To see clearly how the procedure works, first we write out the terms in the state estimation least squares problem (1.28)

$$\min_{x(0),\ldots,x(T)} \frac{1}{2}\Bigg( |x(0) - \overline{x}(0)|^2_{(P^-(0))^{-1}} + \left| y(0) - Cx(0) \right|^2_{R^{-1}} + |x(1) - Ax(0)|^2_{Q^{-1}} + \left| y(1) - Cx(1) \right|^2_{R^{-1}} + |x(2) - Ax(1)|^2_{Q^{-1}} + \cdots + |x(T) - Ax(T-1)|^2_{Q^{-1}} + \left| y(T) - Cx(T) \right|^2_{R^{-1}} \Bigg) \quad (1.29)$$

We decompose this $T$-stage optimization problem with forward DP. First we combine the prior and the measurement $y(0)$ into the quadratic function $V_0(x(0))$ as shown in the following equation

$$\min_{x(T),\ldots,x(1)} \overbrace{\min_{x(0)} \underbrace{\frac{1}{2}\Big( |x(0)-\overline{x}(0)|^2_{(P^-(0))^{-1}} + |y(0)-Cx(0)|^2_{R^{-1}}}_{\text{combine } V_0(x(0))} + |x(1)-Ax(0)|^2_{Q^{-1}} + }^{\text{arrival cost } V_1^-(x(1))}$$
$$|y(1)-Cx(1)|^2_{R^{-1}} + |x(2)-Ax(1)|^2_{Q^{-1}} + \cdots +$$
$$|x(T)-Ax(T-1)|^2_{Q^{-1}} + |y(T)-Cx(T)|^2_{R^{-1}}\Big)$$

Then we optimize over the first state, $x(0)$. This produces the arrival cost for the first stage, $V_1^-(x(1))$, which we will show is also quadratic

$$V_1^-(x(1)) = \frac{1}{2}\,|x(1)-\hat{x}^-(1)|^2_{(P^-(1))^{-1}}$$

Next we combine the arrival cost of the first stage with the next measurement $y(1)$ to obtain $V_1(x(1))$

$$\min_{x(T),\ldots,x(2)} \overbrace{\min_{x(1)} \underbrace{\frac{1}{2}\Big( |x(1)-\hat{x}^-(1)|^2_{(P^-(1))^{-1}} + |y(1)-Cx(1)|^2_{R^{-1}}}_{\text{combine } V_1(x(1))} + |x(2)-Ax(1)|^2_{Q^{-1}} + }^{\text{arrival cost } V_2^-(x(2))}$$
$$|y(2)-Cx(2)|^2_{R^{-1}} + |x(3)-Ax(2)|^2_{Q^{-1}} + \cdots +$$
$$|x(T)-Ax(T-1)|^2_{Q^{-1}} + |y(T)-Cx(T)|^2_{R^{-1}}\Big) \quad (1.30)$$

We optimize over the second state, $x(1)$, which defines arrival cost for the first two stages, $V_2^-(x(2))$. We continue in this fashion until we have optimized finally over $x(T)$ and have solved (1.29). Now that we have in mind an overall game plan for solving the problem, we look at each step in detail and develop the recursion formulas of forward DP.

**Combine prior and measurement.** Combining the prior and measurement defines $V_0$

$$V_0(x(0)) = \frac{1}{2}\Big( \underbrace{|x(0)-\overline{x}(0)|^2_{(P^-(0))^{-1}}}_{\text{prior}} + \underbrace{|y(0)-Cx(0)|^2_{R^{-1}}}_{\text{measurement}} \Big) \quad (1.31)$$

which can be expressed also as

$$V_0(x(0)) = \frac{1}{2}\Big( |x(0) - \overline{x}(0)|^2_{(P^-(0))^{-1}} + \big| (y(0) - C\overline{x}(0)) - C(x(0) - \overline{x}(0)) \big|^2_{R^{-1}} \Big)$$

Using the third form in Example 1.1 we can combine these two terms into a single quadratic function

$$V_0(x(0)) = (1/2)\,(x(0) - \overline{x}(0) - v)'\tilde{H}^{-1}(x(0) - \overline{x}(0) - v) + \text{constant}$$

in which

$$\begin{aligned} v &= P^-(0)C'(CP^-(0)C' + R)^{-1}\left(y(0) - C\overline{x}(0)\right) \\ \tilde{H} &= P^-(0) - P^-(0)C'(CP^-(0)C' + R)^{-1}CP^-(0) \end{aligned}$$

and we set the constant term to zero because it does not depend on $x(1)$. If we define

$$\begin{aligned} P(0) &= P^-(0) - P^-(0)C'(CP^-(0)C' + R)^{-1}CP^-(0) \\ L(0) &= P^-(0)C'(CP^-(0)C' + R)^{-1} \end{aligned}$$

and define the state estimate $\hat{x}(0)$ as follows

$$\begin{aligned} \hat{x}(0) &= \overline{x}(0) + v \\ \hat{x}(0) &= \overline{x}(0) + L(0)\left(y(0) - C\overline{x}(0)\right) \end{aligned}$$

and we have derived the following compact expression for the function $V_0$

$$V_0(x(0)) = (1/2)\,|x(0) - \hat{x}(0)|^2_{P(0)^{-1}}$$

**State evolution and arrival cost.** Now we add the next term in (1.29) to the function $V_0(\cdot)$ and denote the sum as $V(\cdot)$

$$\begin{aligned} V(x(0), x(1)) &= V_0(x(0)) + (1/2)\,|x(1) - Ax(0)|^2_{Q^{-1}} \\ V(x(0), x(1)) &= \frac{1}{2}\big( |x(0) - \hat{x}(0)|^2_{P(0)^{-1}} + |x(1) - Ax(0)|^2_{Q^{-1}} \big) \end{aligned}$$

Again using the third form in Example 1.1, we can add the two quadratics to obtain

$$V(x(0), x(1)) = (1/2)\,|x(0) - v|^2_{\tilde{H}^{-1}} + d$$

in which

$$v = \hat{x}(0) + P(0)A'\left(AP(0)A' + Q\right)^{-1}\left(x(1) - A\hat{x}(0)\right)$$
$$d = (1/2)\left(\,|v - \hat{x}(0)|^2_{P(0)^{-1}} + |x(1) - Av|^2_{Q^{-1}}\,\right)$$

This form is convenient for optimization over the first decision variable $x(0)$; by inspection the solution is $x(0) = v$ and the cost is $d$. We define the arrival cost to be the result of this optimization

$$V_1^-(x(1)) = \min_{x(0)} V(x(0), x(1))$$

Substituting $v$ into the expression for $d$ and simplifying gives

$$V_1^-(x(1)) = (1/2)\,|x(1) - A\hat{x}(0)|^2_{(P^-(1))^{-1}}$$

in which

$$P^-(1) = AP(0)A' + Q$$

We define $\hat{x}^-(1) = A\hat{x}(0)$ and express the arrival cost compactly as

$$V_1^-(x(1)) = (1/2)\,\left|x(1) - \hat{x}^-(1)\right|^2_{(P^-(1))^{-1}}$$

**Combine arrival cost and measurement.** We now combine the arrival cost and measurement for the next stage of the optimization to obtain

$$V_1(x(1)) = \underbrace{V_1^-(x(1))}_{\text{prior}} + \underbrace{(1/2)\,\left|(y(1) - Cx(1))\right|^2_{R^{-1}}}_{\text{measurement}}$$
$$V_1(x(1)) = \frac{1}{2}\left(\,\left|x(1) - \hat{x}^-(1)\right|^2_{(P^-(1))^{-1}} + \left|y(1) - Cx(1)\right|^2_{R^{-1}}\right)$$

We can see that this equation is exactly the form as (1.31) of the previous step, and, by simply changing the variable names, we have that

$$P(1) = P^-(1) - P^-(1)C'(CP^-(1)C' + R)^{-1}CP^-(1)$$
$$L(1) = P^-(1)C'(CP^-(1)C' + R)^{-1}$$
$$\hat{x}(1) = \hat{x}^-(1) + L(1)(y(1) - C\hat{x}^-(1))$$

and the cost function $V_1$ is defined as

$$V_1(x(1)) = (1/2)(x(1) - \hat{x}(1))'P(1)^{-1}(x(1) - \hat{x}(1))$$

in which

$$\hat{x}^-(1) = A\hat{x}(0)$$
$$P^-(1) = AP(0)A' + Q$$

**Recursion and termination.** The recursion can be summarized by two steps. Adding the measurement at time $k$ produces

$$\begin{aligned} P(k) &= P^-(k) - P^-(k)C'(CP^-(k)C' + R)^{-1}CP^-(k) \\ L(k) &= P^-(k)C'(CP^-(k)C' + R)^{-1} \\ \hat{x}(k) &= \hat{x}^-(k) + L(k)(y(k) - C\hat{x}^-(k)) \end{aligned}$$

Propagating the model to time $k+1$ produces

$$\begin{aligned} \hat{x}^-(k+1) &= A\hat{x}(k) \\ P^-(k+1) &= AP(k)A' + Q \end{aligned}$$

and the recursion starts with the prior information $\hat{x}^-(0) = \overline{x}(0)$ and $P^-(0)$. The arrival cost, $V_k^-$, and arrival cost plus measurement, $V_k$, for each stage are given by

$$\begin{aligned} V_k^-(x(k)) &= (1/2)\,|x(k) - \hat{x}^-(k)|^2_{(P^-(k))^{-1}} \\ V_k(x(k)) &= (1/2)\,|x(k) - \hat{x}(k)|^2_{(P(k))^{-1}} \end{aligned}$$

The process terminates with the final measurement $y(T)$, at which point we have recursively solved the original problem (1.29).

We see by inspection that the recursion formulas given by forward DP of (1.29) are the same as those found by calculating the conditional density function in Section 1.4.2. Moreover, the conditional densities before and after measurement are closely related to the least squares value functions as shown below

$$\begin{aligned} p(x(k)|\mathbf{y}(k-1)) &= \frac{1}{(2\pi)^{n/2}(\det P^-(k))^{1/2}} \exp(-V_k^-(x(k))) \qquad (1.32) \\ p(x(k)|\mathbf{y}(k)) &= \frac{1}{(2\pi)^{n/2}(\det P(k))^{1/2}} \exp(-V_k(x(k))) \end{aligned}$$

The discovery (and rediscovery) of the close connection between recursive least squares and optimal statistical estimation has not always been greeted happily by researchers:

> The recursive least squares approach was actually inspired by probabilistic results that automatically produce an equation of evolution for the estimate (the conditional mean). In fact, much of the recent least squares work did nothing more than rederive the probabilistic results (perhaps in an attempt to understand them). As a result, much of the least squares work contributes very little to estimation theory.
> — Jazwinski (1970, pp.152–153)

In contrast with this view, we find both approaches valuable in the subsequent development. The probabilistic approach, which views the state estimator as maximizing conditional density of the state given measurement, offers the most insight. It provides a rigorous basis for comparing different estimators based on the variance of their estimate error. It also specifies what information is required to define an optimal estimator, with variances $Q$ and $R$ of primary importance. In the probabilistic framework, these parameters should be found from modeling and data. The main deficiency in the least squares viewpoint is that the objective function, although reasonable, is ad hoc and not justified. The choice of weighting matrices $Q$ and $R$ is arbitrary. Practitioners generally choose these parameters based on a tradeoff between the competing goals of speed of estimator response and insensitivity to measurement noise. But a careful statement of this tradeoff often just leads back to the probabilistic viewpoint in which the process disturbance and measurement disturbance are modeled as normal distributions. If we restrict attention to unconstrained linear systems, the probabilistic viewpoint is clearly superior.

Approaching state estimation with the perspective of least squares pays off, however, when the models are significantly more complex. It is generally intractable to find and maximize the conditional density of the state given measurements for complex, nonlinear and constrained models. Although the state estimation problem can be stated in the language of probability, it cannot be solved with current methods. But reasonable objective functions can be chosen for even complex, nonlinear and constrained models. Moreover, knowing which least squares problems correspond to which statistically optimal estimation problems for the simple linear case, provides the engineer with valuable insight in choosing useful objective functions for nonlinear estimation. We explore these more complex and realistic estimation problems in Chapter 4. The perspective of least squares also leads to succinct arguments for establishing estimator stability, which we take up shortly. First we consider situations in which it is advantageous to use moving horizon estimation.

### 1.4.4 Moving Horizon Estimation

When using nonlinear models or considering constraints on the estimates, we cannot calculate the conditional density recursively in closed form as we did in Kalman filtering. Similarly, we cannot solve recursively the least squares problem. If we use least squares we must opti-

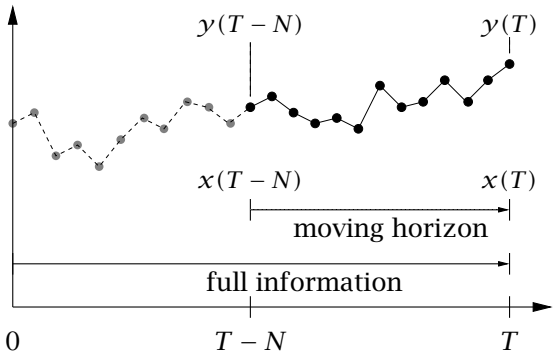


**Figure 1.4:** Schematic of the moving horizon estimation problem.

mize all the states in the trajectory $\mathbf{x}(T)$ simultaneously to obtain the state estimates. This optimization problem becomes computationally intractable as $T$ increases. Moving horizon estimation (MHE) removes this difficulty by considering only the most recent $N$ measurements and finds only the most recent $N$ values of the state trajectory as sketched in Figure 1.4. The states to be estimated are $\mathbf{x}_N(T) = \{x(T-N), \ldots, x(T)\}$ given measurements $\mathbf{y}_N(T) = \{y(T-N), \ldots, y(T)\}$. The data have been broken into two sections with $\{\mathbf{y}(T-N-1), \mathbf{y}_N(T)\} = \mathbf{y}(T)$. We assume here that $T \geq N-1$ to ignore the initial period in which the estimation window fills with measurements and assume that the window is always full.

The simplest form of MHE is the following least squares problem

$$\min_{\mathbf{x}_N(T)} \hat{V}_T(\mathbf{x}_N(T)) \tag{1.33}$$

in which the objective function is

$$\hat{V}_T(\mathbf{x}_N(T)) = \frac{1}{2}\Bigg( \sum_{k=T-N}^{T-1} |x(k+1) - Ax(k)|^2_{Q^{-1}} + \sum_{k=T-N}^{T} |\,y(k) - Cx(k)\,|^2_{R^{-1}} \Bigg) \tag{1.34}$$

We use the circumflex (hat) to indicate this is the MHE cost function considering data sequence from $T-N$ to $T$ rather than the full information or least squares cost considering the data from 0 to $T$.

**MHE in terms of least squares.** Notice that from our previous DP recursion in (1.30), we can write the full least squares problem as

$$V_T(\mathbf{x}_N(T)) = V_{T-N}^-(x(T-N)) + \frac{1}{2}\Bigg( \sum_{k=T-N}^{T-1} |x(k+1) - Ax(k)|^2_{Q^{-1}} + \sum_{k=T-N}^{T} |y(k) - Cx(k)|^2_{R^{-1}} \Bigg)$$

in which $V_{T-N}^-(\cdot)$ is the arrival cost at time $T-N$. Comparing these two objective functions, it is clear that the simplest form of MHE is equivalent to setting up a full least squares problem, but then setting the arrival cost function $V_{T-N}^-(\cdot)$ to zero.

**MHE in terms of conditional density.** Because we have established the close connection between least squares and conditional density in (1.32), we can write the full least squares problem also as an equivalent conditional density maximization

$$\max_{x(T)} p_{x(T)|\mathbf{y}_N(T)}(x(T)|\mathbf{y}_N(T))$$

with prior density

$$p_{x(T-N)|\mathbf{y}(T-N-1)}(x|\mathbf{y}(T-N-1)) = c\exp(-V_{T-N}^-(x)) \tag{1.35}$$

in which the constant $c$ can be found from (1.20) if desired, but its value does not change the solution to the optimization. We can see from (1.35) that setting $V_{T-N}^-(\cdot)$ to zero in the simplest form of MHE is equivalent to giving infinite variance to the conditional density of $x(T-N)|\mathbf{y}(T-N-1)$. This means we are using a noninformative prior for the state $x(T-N)$ and completely discounting the previous measurements $\mathbf{y}(T-N-1)$.

To provide a more flexible MHE problem, we therefore introduce a penalty on the first state to account for the neglected data $\mathbf{y}(T-N-1)$

$$\hat{V}_T(\mathbf{x}_N(T)) = \Gamma_{T-N}(x(T-N)) + \frac{1}{2}\Bigg( \sum_{k=T-N}^{T-1} |x(k+1) - Ax(k)|^2_{Q^{-1}} + \sum_{k=T-N}^{T} |y(k) - Cx(k)|^2_{R^{-1}} \Bigg)$$

For the linear Gaussian case, we can account for the neglected data exactly with no approximation by setting $\Gamma$ equal to the arrival cost, or, equivalently, the negative logarithm of the conditional density of the state given the prior measurements. Indeed, there is no need to use

MHE for the linear Gaussian problem at all because we can solve the full problem recursively. When addressing nonlinear and constrained problems in Chapter 4, however, we must approximate the conditional density of the state given the prior measurements in MHE to obtain a computationally tractable and high-quality estimator.

### 1.4.5 Observability

We next explore the convergence properties of the state estimators. For this we require the concept of system observability. The basic idea of observability is that any two distinct states can be *distinguished* by applying some input and observing the two system outputs over some finite time interval (Sontag, 1998, p.262–263). We discuss this general definition in more detail when treating nonlinear systems in Chapter 4, but observability for linear systems is much simpler. First of all, the applied input is irrelevant and we can set it to zero. Therefore consider the linear time-invariant system $(A, C)$ with zero input

$$\begin{aligned} x(k+1) &= Ax(k) \\ y(k) &= Cx(k) \end{aligned}$$

The system is observable if there exists a finite $N$, such that for every $x(0)$, $N$ measurements $\{y(0), y(1), \ldots, y(N-1)\}$ distinguish uniquely the initial state $x(0)$. Similarly to the case of controllability, if we cannot determine the initial state using $n$ measurements, we cannot determine it using $N > n$ measurements. Therefore we can develop a convenient test for observability as follows. For $n$ measurements, the system model gives

$$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(n-1) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} x(0) \tag{1.36}$$

The question of *observability* is therefore a question of *uniqueness* of solutions to these linear equations. The matrix appearing in this equation is known as the *observability matrix* $\mathcal{O}$

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{1.37}$$

From the fundamental theorem of linear algebra, we know the solution to (1.36) is unique if and only if the *columns* of the $np \times n$ observability matrix are linearly independent.[6] Therefore, we have that the system $(A, C)$ is observable if and only if

$$\text{rank}(\mathcal{O}) = n$$

The following result for checking observability also proves useful (Hautus, 1972).

**Lemma 1.4** (Hautus Lemma for observability)**.** *A system is observable if and only if*

$$\text{rank} \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n \qquad \textit{for all } \lambda \in \mathbb{C} \tag{1.38}$$

*in which* $\mathbb{C}$ *is the set of complex numbers.*

Notice that the first $n$ rows of the matrix in (1.38) are linearly independent if $\lambda \notin \text{eig}(A)$, so (1.38) is equivalent to checking the rank at just the eigenvalues of $A$

$$\text{rank} \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n \qquad \text{for all } \lambda \in \text{eig}(A)$$

### 1.4.6 Convergence of the State Estimator

Next we consider the question of convergence of the estimates of several of the estimators we have considered. The simplest convergence question to ask is the following. Given an initial estimate error, and zero state and measurement noises, does the state estimate converge to the state as time increases and more measurements become available? If the answer to this question is yes, we say the estimates converge; sometimes we say the estimator converges. As with the regulator, optimality of an estimator does not ensure its stability. Consider the case $A = I, C = 0$. The optimal estimate is $\hat{x}(k) = \overline{x}(0)$, which does not converge to the true state unless we have luckily chosen $\overline{x}(0) = x(0)$.[7] Obviously the lack of stability is caused by our choosing an unobservable (undetectable) system.

We treat first the Kalman filtering or full least squares problem. Recall that this estimator optimizes over the entire state trajectory

[6] See Section A.4 of Appendix A or (Strang, 1980, pp.87–88) for a review of this result.

[7] If we could count on that kind of luck, we would have no need for state estimation.

$\mathbf{x}(T) := \{x(0), \ldots, x(T)\}$ based on all measurements $\mathbf{y}(T) := \{y(0), \ldots, y(T)\}$. In order to establish convergence, the following result on the optimal estimator cost function proves useful.

**Lemma 1.5** (Convergence of estimator cost)**.** *Given noise-free measurements* $\mathbf{y}(T) = \{Cx(0), CAx(0), \ldots, CA^T x(0)\}$*, the optimal estimator cost* $V_T^0(\mathbf{y}(T))$ *converges as* $T \to \infty$*.*

*Proof.* Denote the optimal state sequence at time $T$ given measurement $\mathbf{y}(T)$ by

$$\{\hat{x}(0|T),\ \hat{x}(1|T),\ \ldots,\ \hat{x}(T|T)\}$$

We wish to compare the optimal costs at time $T$ and $T-1$. Therefore, consider using the first $T-1$ elements of the solution at time $T$ as decision variables in the state estimation problem at time $T-1$. The cost for those decision variables at time $T-1$ is given by

$$V_T^0 \ - \ \frac{1}{2}\Big( |\hat{x}(T|T) - A\hat{x}(T-1|T)|^2_{Q^{-1}} \ + \ |y(T) - C\hat{x}(T|T)|^2_{R^{-1}} \Big)$$

In other words, we have the full cost at time $T$ and we deduct the cost of the last stage, which is not present at $T-1$. Now this choice of decision variables is not necessarily optimal at time $T-1$, so we have the inequality

$$V_{T-1}^0 \le V_T^0 - \frac{1}{2}\Big( |\hat{x}(T|T) - A\hat{x}(T-1|T)|^2_{Q^{-1}} + |y(T) - C\hat{x}(T|T)|^2_{R^{-1}} \Big)$$

Because the quadratic terms are nonnegative, the sequence of optimal estimator costs is nondecreasing with increasing $T$. We can establish that the optimal cost is bounded above as follows: at any time $T$ we can choose the decision variables to be $\{x(0), Ax(0), \ldots, A^T x(0)\}$, which achieves cost $|x(0) - \overline{x}(0)|^2_{(P^-(0))^{-1}}$ independent of $T$. The optimal cost sequence is nondecreasing and bounded above and, therefore, converges. ■

The optimal estimator cost converges regardless of system observability. But if we want the optimal estimate to converge to the state, we have to restrict the system further. The following lemma provides an example of what is required.

**Lemma 1.6** (Estimator convergence)**.** *For* $(A, C)$ *observable,* $Q, R > 0$*, and noise-free measurements* $\mathbf{y}(T) = \{Cx(0), CAx(0), \ldots, CA^T x(0)\}$*, the optimal linear state estimate converges to the state*

$$\hat{x}(T) \to x(T) \quad \text{as } T \to \infty$$

*Proof.* To compress the notation somewhat, let $\hat{w}_T(j) = \hat{x}(T+j+1|T+n-1) - A\hat{x}(T+j|T+n-1)$. Using the optimal solution at time $T+n-1$ as decision variables at time $T-1$ allows us to write the following inequality

$$V^0_{T-1} \leq V^0_{T+n-1} - \frac{1}{2}\Big( \sum_{j=-1}^{n-2} |\hat{w}_T(j)|^2_{Q^{-1}} + \sum_{j=0}^{n-1} |y(T+j) - C\hat{x}(T+j|T+n-1)|^2_{R^{-1}} \Big)$$

Because the sequence of optimal costs converges with increasing $T$, and $Q^{-1}, R^{-1} > 0$, we have established that for increasing $T$

$$\begin{aligned} \hat{w}_T(j) &\to 0 \quad j = -1, \ldots, n-2 \\ y(T+j) - C\hat{x}(T+j|T+n-1) &\to 0 \quad j = 0, \ldots, n-1 \end{aligned} \tag{1.39}$$

From the system model we have the following relationship between the last $n$ stages in the optimization problem at time $T+n-1$ with data $\mathbf{y}(T+n-1)$

$$\begin{bmatrix} \hat{x}(T|T+n-1) \\ \hat{x}(T+1|T+n-1) \\ \vdots \\ \hat{x}(T+n-1|T+n-1) \end{bmatrix} = \begin{bmatrix} I \\ A \\ \vdots \\ A^{n-1} \end{bmatrix} \hat{x}(T|T+n-1) + \begin{bmatrix} 0 & & & \\ I & 0 & & \\ \vdots & \vdots & \ddots & \\ A^{n-2} & A^{n-3} & \cdots & I \end{bmatrix} \begin{bmatrix} \hat{w}_T(0) \\ \hat{w}_T(1) \\ \vdots \\ \hat{w}_T(n-2) \end{bmatrix} \tag{1.40}$$

We note the measurements satisfy

$$\begin{bmatrix} y(T) \\ y(T+1) \\ \vdots \\ y(T+n-1) \end{bmatrix} = \mathcal{O}x(T)$$

Multiplying (1.40) by $C$ and subtracting gives

$$\begin{bmatrix} y(T) - C\hat{x}(T|T+n-1) \\ y(T+1) - C\hat{x}(T+1|T+n-1) \\ \vdots \\ y(T+n-1) - C\hat{x}(T+n-1|T+n-1) \end{bmatrix} = \mathcal{O}\big(x(T) - \hat{x}(T|T+n-1)\big) - \begin{bmatrix} 0 & & & \\ C & 0 & & \\ \vdots & \vdots & \ddots & \\ CA^{n-2} & CA^{n-3} & \cdots & C \end{bmatrix} \begin{bmatrix} \hat{w}_T(0) \\ \hat{w}_T(1) \\ \vdots \\ \hat{w}_T(n-2) \end{bmatrix}$$

Applying (1.39) to this equation, we conclude $\mathcal{O}(x(T) - \hat{x}(T|T+n-1)) \to 0$ with increasing $T$. Because the observability matrix has independent columns, we conclude $x(T) - \hat{x}(T|T+n-1) \to 0$ as $T \to \infty$. Thus we conclude that the *smoothed* estimate $\hat{x}(T|T+n-1)$ converges to the state $x(T)$. Because the $\hat{w}_T(j)$ terms go to zero with increasing $T$, the last line of (1.40) gives $\hat{x}(T+n-1|T+n-1) \to A^{n-1}\hat{x}(T|T+n-1)$ as $T \to \infty$. From the system model $A^{n-1}x(T) = x(T+n-1)$ and, therefore, after replacing $T+n-1$ by $T$, we have

$$\hat{x}(T|T) \to x(T) \quad \text{as } T \to \infty$$

and asymptotic convergence of the estimator is established. ■

This convergence result also covers MHE with prior weighting set to the exact arrival cost because that is equivalent to Kalman filtering and full least squares. The simplest form of MHE, which discounts prior data completely, is also a convergent estimator, however, as discussed in Exercise 1.28.

The estimator convergence result in Lemma 1.6 is the simplest to establish, but, as in the case of the LQ regulator, we can enlarge the class of systems and weighting matrices (variances) for which estimator convergence is guaranteed. The system restriction can be weakened from observability to *detectability*, which is discussed in Exercises 1.31 and 1.32. The restriction on the process disturbance weight (variance) $Q$ can be weakened from $Q > 0$ to $Q \geq 0$ and $(A, Q)$ *stabilizable*, which is discussed in Exercise 1.33. The restriction $R > 0$ remains to ensure uniqueness of the estimator.

## 1.5 Tracking, Disturbances, and Zero Offset

In the last section of this chapter we show briefly how to use the MPC regulator and MHE estimator to handle different kinds of control problems, including setpoint tracking and rejecting nonzero disturbances.

### 1.5.1 Tracking

It is a standard objective in applications to use a feedback controller to move the measured outputs of a system to a specified and constant setpoint. This problem is known as setpoint tracking. In Section 2.9 we consider the case in which the system is nonlinear and constrained, but for simplicity here we consider the linear unconstrained system in which $y_{\text{sp}}$ is an arbitrary constant. In the regulation problem of Section 1.3 we assumed that the goal was to take the state of the system to the origin. Such a regulator can be used to treat the setpoint tracking problem with a coordinate transformation. Denote the desired output setpoint as $y_{\text{sp}}$. Denote a steady state of the system model as $(x_s, u_s)$. From (1.5), the steady state satisfies

$$\begin{bmatrix} I - A & -B \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = 0$$

For *unconstrained* systems, we also impose the requirement that the steady state satisfies $Cx_s = y_{\text{sp}}$ for the tracking problem, giving the set of equations

$$\begin{bmatrix} I - A & -B \\ C & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} 0 \\ y_{\text{sp}} \end{bmatrix} \tag{1.41}$$

If this set of equations has a solution, we can then define deviation variables

$$\begin{aligned} \tilde{x}(k) &= x(k) - x_s \\ \tilde{u}(k) &= u(k) - u_s \end{aligned}$$

that satisfy the dynamic model

$$\begin{aligned} \tilde{x}(k+1) &= x(k+1) - x_s \\ &= Ax(k) + Bu(k) - (Ax_s + Bu_s) \\ \tilde{x}(k+1) &= A\tilde{x}(k) + B\tilde{u}(k) \end{aligned}$$

so that the deviation variables satisfy the same model equation as the original variables. The zero regulation problem applied to the system in

deviation variables finds $\widetilde{u}(k)$ that takes $\widetilde{x}(k)$ to zero, or, equivalently, which takes $x(k)$ to $x_s$, so that at steady state, $Cx(k) = Cx_s = y_{\text{sp}}$, which is the goal of the setpoint tracking problem. After solving the regulation problem in deviation variables, the input applied to the system is $u(k) = \widetilde{u}(k) + u_s$.

We next discuss when we can solve (1.41). We also note that for *constrained* systems, we must impose the constraints on the steady state $(x_s, u_s)$. The matrix in (1.41) is a $(n + p) \times (n + m)$ matrix. For (1.41) to have a solution for all $y_{\text{sp}}$, it is sufficient that the rows of the matrix are linearly independent. That requires $p \leq m$: we require at least as many inputs as outputs with setpoints. But it is not uncommon in applications to have many more measured outputs than manipulated inputs. To handle these more general situations, we choose a matrix $H$ and denote a new variable $r = Hy$ as a selection of linear combinations of the measured outputs. The variable $r \in \mathbb{R}^{n_c}$ is known as the *controlled variable*. For cases in which $p > m$, we choose some set of outputs $n_c \leq m$, as controlled variables, and assign setpoints to $r$, denoted $r_{\text{sp}}$.

We also wish to treat systems with more inputs than outputs, $m > p$. For these cases, the solution to (1.41) may exist for some choice of $H$ and $r_{\text{sp}}$, but cannot be unique. If we wish to obtain a unique steady state, then we also must provide desired values for the steady inputs, $u_{\text{sp}}$. To handle constrained systems, we simply impose the constraints on $(x_s, u_s)$.

**Steady-state target problem.** Our candidate optimization problem is therefore

$$\min_{x_s, u_s} \frac{1}{2} \left( |u_s - u_{\text{sp}}|_{R_s}^2 + |Cx_s - y_{\text{sp}}|_{Q_s}^2 \right) \tag{1.42a}$$

subject to:

$$\begin{bmatrix} I - A & -B \\ HC & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} 0 \\ r_{\text{sp}} \end{bmatrix} \tag{1.42b}$$

$$Eu_s \leq e \tag{1.42c}$$

$$FCx_s \leq f \tag{1.42d}$$

We make the following assumptions:

**Assumption 1.7** (Target feasibility and uniqueness)**.**

(a) The target problem is feasible for the controlled variable setpoints of interest $r_{\text{sp}}$.

(b) The steady-state input penalty $R_s$ is positive definite.

Assumption 1.7 (a) ensures that the solution $(x_s, u_s)$ exists, and Assumption 1.7 (b) ensures that the solution is unique. If one chooses $n_c = 0$, then no controlled variables are required to be at setpoint, and the problem is feasible for any $(u_{\text{sp}}, y_{\text{sp}})$ because $(x_s, u_s) = (0, 0)$ is a feasible point. Exercises 1.56 and 1.57 explore the connection between feasibility of the equality constraints and the number of controlled variables relative to the number of inputs and outputs. One restriction is that the number of controlled variables chosen to be offset free must be less than or equal to the number of manipulated variables and the number of measurements, $n_c \leq m$ and $n_c \leq p$.

**Dynamic regulation problem.** Given the steady-state solution, we define the following multistage objective function

$$V(\tilde{x}(0), \tilde{\mathbf{u}}) = \frac{1}{2} \sum_{k=0}^{N-1} |\tilde{x}(k)|_Q^2 + |\tilde{u}(k)|_R^2 \qquad \text{s.t. } \tilde{x}^+ = A\tilde{x} + B\tilde{u}$$

in which $\tilde{x}(0) = \hat{x}(k) - x_s$, i.e., the initial condition for the regulation problem comes from the state estimate shifted by the steady-state $x_s$. The regulator solves the following dynamic, zero-state regulation problem

$$\min_{\tilde{\mathbf{u}}} V(\tilde{x}(0), \tilde{\mathbf{u}})$$

subject to

$$E\tilde{u} \leq e - Eu_s$$
$$FC\tilde{x} \leq f - FCx_s$$

in which the constraints also are shifted by the steady state $(x_s, u_s)$. The optimal cost and solution are $V^0(\tilde{x}(0))$ and $\tilde{\mathbf{u}}^0(\tilde{x}(0))$. The moving horizon control law uses the first move of this optimal sequence, $\tilde{u}^0(\tilde{x}(0)) = \tilde{\mathbf{u}}^0(0; \tilde{x}(0))$, so the controller output is $u(k) = \tilde{u}^0(\tilde{x}(0)) + u_s$.

### 1.5.2 Disturbances and Zero Offset

Another common objective in applications is to use a feedback controller to compensate for an unmeasured disturbance to the system with the input so the disturbance's effect on the controlled variable is mitigated. This problem is known as disturbance rejection. We may

wish to design a feedback controller that compensates for nonzero disturbances such that the selected controlled variables asymptotically approach their setpoints without offset. This property is known as zero offset. In this section we show a simple method for constructing an MPC controller to achieve zero offset.

In Chapter 5, we address the full problem. Here we must be content to limit our objective. We will ensure that *if the system is stabilized in the presence of the disturbance*, then there is zero offset. But we will not attempt to construct the controller that ensures stabilization over an interesting class of disturbances. That topic is treated in Chapter 5.

This more limited objective is similar to what one achieves when using the integral mode in proportional-integral-derivative (PID) control of an unconstrained system: either there is zero steady offset, or the system trajectory is unbounded. In a constrained system, the statement is amended to: either there is zero steady offset, or the system trajectory is unbounded, or the system constraints are active at steady state. In both constrained and unconstrained systems, the zero-offset property *precludes* one undesirable possibility: the system settles at an unconstrained steady state, and the steady state displays offset in the controlled variables.

A simple method to compensate for an unmeasured disturbance is to (i) model the disturbance, (ii) use the measurements and model to estimate the disturbance, and (iii) find the inputs that minimize the effect of the disturbance on the controlled variables. The choice of disturbance model is motivated by the zero-offset goal. To achieve offset-free performance we augment the system state with an *integrating* disturbance $d$ driven by a white noise $w_d$

$$d^+ = d + w_d \tag{1.43}$$

This choice is motivated by the works of Davison and Smith (1971, 1974); Qiu and Davison (1993) and the Internal Model Principle of Francis and Wonham (1976). To remove offset, one designs a control system that can remove asymptotically constant, nonzero disturbances (Davison and Smith, 1971), (Kwakernaak and Sivan, 1972, p.278). To accomplish this end, the original system is augmented with a replicate of the constant, nonzero disturbance model, (1.43). Thus the states of the original system are moved onto the manifold that cancels the effect of the disturbance on the controlled variables. The augmented system

model used for the state estimator is given by

$$\begin{bmatrix} x \\ d \end{bmatrix}^+ = \begin{bmatrix} A & B_d \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ d \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + w \tag{1.44a}$$

$$y = \begin{bmatrix} C & C_d \end{bmatrix} \begin{bmatrix} x \\ d \end{bmatrix} + v \tag{1.44b}$$

and we are free to choose how the integrating disturbance affects the states and measured outputs through the choice of $B_d$ and $C_d$. The only restriction is that the augmented system is detectable. That restriction can be easily checked using the following result.

**Lemma 1.8** (Detectability of the augmented system)**.** *The augmented system* (1.44) *is detectable if and only if the nonaugmented system* $(A, C)$ *is detectable, and the following condition holds:*

$$\operatorname{rank} \begin{bmatrix} I - A & -B_d \\ C & C_d \end{bmatrix} = n + n_d \tag{1.45}$$

**Corollary 1.9** (Dimension of the disturbance)**.** *The maximal dimension of the disturbance* $d$ *in* (1.44) *such that the augmented system is detectable is equal to the number of measurements, that is*

$$n_d \leq p$$

A pair of matrices $(B_d, C_d)$ such that (1.45) is satisfied always exists. In fact, since $(A, C)$ is detectable, the submatrix $\left[\begin{smallmatrix} I-A \\ C \end{smallmatrix}\right] \in \mathbb{R}^{(p+n)\times n}$ has rank $n$. Thus, we can choose any $n_d \leq p$ columns in $\mathbb{R}^{p+n}$ independent of $\left[\begin{smallmatrix} I-A \\ C \end{smallmatrix}\right]$ for $\left[\begin{smallmatrix} -B_d \\ C_d \end{smallmatrix}\right]$.

The state and the additional integrating disturbance are estimated from the plant measurement using a Kalman filter designed for the augmented system. The variances of the stochastic disturbances $w$ and $v$ may be treated as adjustable parameters or found from input-output measurements (Odelson, Rajamani, and Rawlings, 2006). The estimator provides $\hat{x}(k)$ and $\hat{d}(k)$ at each time $k$. The best forecast of the steady-state disturbance using (1.43) is simply

$$\hat{d}_s = \hat{d}(k)$$

The steady-state target problem is therefore modified to account for the nonzero disturbance $\hat{d}_s$

$$\min_{x_s, u_s} \frac{1}{2} \left( |u_s - u_{\text{sp}}|^2_{R_s} + \left| Cx_s + C_d \hat{d}_s - y_{\text{sp}} \right|^2_{Q_s} \right) \tag{1.46a}$$

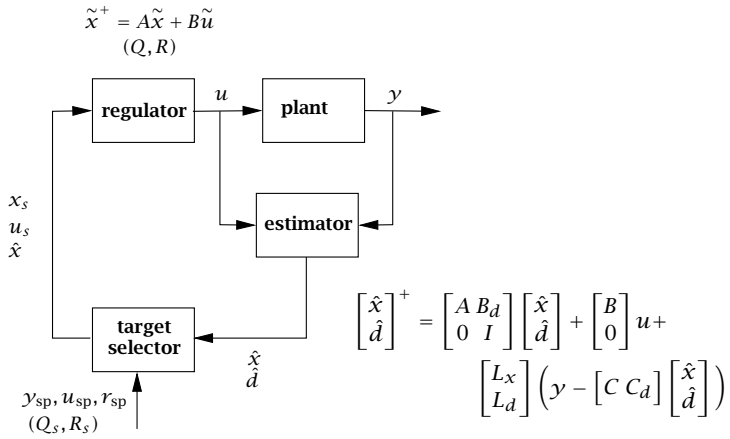


**Figure 1.5:** MPC controller consisting of: receding horizon regulator, state estimator, and target selector.

subject to:

$$\begin{bmatrix} I - A & -B \\ HC & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} B_d \hat{d}_s \\ r_{\text{sp}} - HC_d \hat{d}_s \end{bmatrix} \tag{1.46b}$$

$$Eu_s \le e \tag{1.46c}$$

$$FCx_s \le f - FC_d \hat{d}_s \tag{1.46d}$$

Comparing (1.42) to (1.46), we see the disturbance model affects the steady-state target determination in four ways.

1. The output target is modified in (1.46a) to account for the effect of the disturbance on the measured output ($y_{\text{sp}} \to y_{\text{sp}} - C_d \hat{d}_s$).
2. The output constraint in (1.46d) is similarly modified ($f \to f - FC_d \hat{d}_s$).
3. The system steady-state relation in (1.46b) is modified to account for the effect of the disturbance on the state evolution ($0 \to B_d \hat{d}_s$).
4. The controlled variable target in (1.46b) is modified to account for the effect of the disturbance on the controlled variable ($r_{\text{sp}} \to r_{\text{sp}} - HC_d \hat{d}_s$).

Given the steady-state target, the same dynamic regulation problem as presented in the tracking section, Section 1.5, is used for the regulator. In other words, the regulator is based on the deterministic system

$(A, B)$ in which the current state is $\hat{x}(k) - x_s$ and the goal is to take the system to the origin.

The following lemma summarizes the offset-free control property of the combined control system.

**Lemma 1.10** (Offset-free control)**.** *Consider a system controlled by the MPC algorithm as shown in Figure 1.5. The target problem* (1.46) *is assumed feasible. Augment the system model with a number of integrating disturbances equal to the number of measurements* ($n_d = p$)*; choose any* $B_d \in \mathbb{R}^{n\times p}$, $C_d \in \mathbb{R}^{p\times p}$ *such that*

$$\operatorname{rank}\begin{bmatrix} I - A & -B_d \\ C & C_d \end{bmatrix} = n + p$$

*If the plant output* $y(k)$ *goes to steady state* $y_s$*, the closed-loop system is stable, and constraints are not active at steady state, then there is zero offset in the controlled variables, that is*

$$Hy_s = r_{\text{sp}}$$

The proof of this lemma is given in Pannocchia and Rawlings (2003). It may seem surprising that the number of integrating disturbances must be equal to the number of *measurements* used for feedback rather than the number of *controlled variables* to guarantee offset-free control. To gain insight into the reason, consider the disturbance part (bottom half) of the Kalman filter equations shown in Figure 1.5

$$\hat{d}^+ = \hat{d} + L_d\left(y - \begin{bmatrix} C & C_d \end{bmatrix}\begin{bmatrix} \hat{x} \\ \hat{d} \end{bmatrix}\right)$$

Because of the integrator, the disturbance estimate cannot converge until

$$L_d\left(y - \begin{bmatrix} C & C_d \end{bmatrix}\begin{bmatrix} \hat{x} \\ \hat{d} \end{bmatrix}\right) = 0$$

But notice this condition merely restricts the output prediction error to lie in the nullspace of the matrix $L_d$, which is an $n_d \times p$ matrix. If we choose $n_d = n_c < p$, then the number of columns of $L_d$ is greater than the number of rows and $L_d$ has a nonzero nullspace.[8] In general, we require the output prediction error to be *zero* to achieve zero offset independently of the regulator tuning. For $L_d$ to have only the zero vector in its nullspace, we require $n_d \geq p$. Since we also know $n_d \leq p$ from Corollary 1.9, we conclude $n_d = p$.

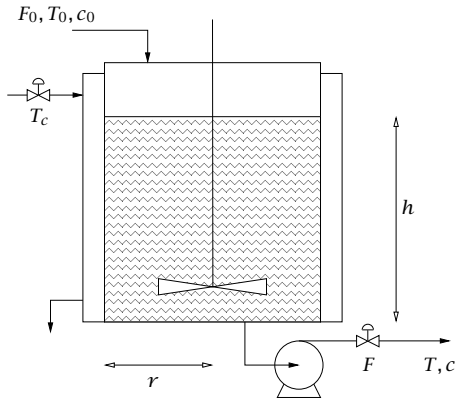


**Figure 1.6:** Schematic of the well-stirred reactor.

| Parameter | Nominal value | Units |
|---|---|---|
| $F_0$ | 0.1 | $m^3/min$ |
| $T_0$ | 350 | K |
| $c_0$ | 1 | $kmol/m^3$ |
| $r$ | 0.219 | m |
| $k_0$ | $7.2 \times 10^{10}$ | $min^{-1}$ |
| $E/R$ | 8750 | K |
| $U$ | 54.94 | $kJ/min \cdot m^2 \cdot K$ |
| $\rho$ | 1000 | $kg/m^3$ |
| $C_p$ | 0.239 | $kJ/kg \cdot K$ |
| $\Delta H$ | $-5 \times 10^4$ | kJ/kmol |

**Table 1.1:** Parameters of the well-stirred reactor.

Notice also that Lemma 1.10 does not require that the plant output be generated by the model. The theorem applies regardless of what generates the plant output. *If the plant is identical to the system plus disturbance model assumed in the estimator*, then the conclusion can be strengthened. In the nominal case without measurement or process noise ($w = 0$, $v = 0$), *for a set of plant initial states*, the closed-loop system *converges to a steady state* and the feasible steady-state target is achieved leading to zero offset in the controlled variables. Characterizing the set of initial states in the region of convergence, and stabilizing

[8] This is another consequence of the fundamental theorem of linear algebra. The result is depicted in Figure A.1.

the system when the plant and the model differ, are treated in Chapters 3 and 5. We conclude the chapter with a nonlinear example that demonstrates the use of Lemma 1.10.

**Example 1.11: More measured outputs than inputs and zero offset**

We consider a well-stirred chemical reactor depicted in Figure 1.6, as in Pannocchia and Rawlings (2003). An irreversible, first-order reaction A⟶ B occurs in the liquid phase and the reactor temperature is regulated with external cooling. Mass and energy balances lead to the following nonlinear state space model:

$$\begin{aligned}
\frac{dc}{dt} &= \frac{F_0(c_0 - c)}{\pi r^2 h} - k_0 \exp\left(-\frac{E}{RT}\right) c \\
\frac{dT}{dt} &= \frac{F_0(T_0 - T)}{\pi r^2 h} + \frac{-\Delta H}{\rho C_p} k_0 \exp\left(-\frac{E}{RT}\right) c + \frac{2U}{r\rho C_p}(T_c - T) \\
\frac{dh}{dt} &= \frac{F_0 - F}{\pi r^2}
\end{aligned}$$

The controlled variables are $h$, the level of the tank, and $c$, the molar concentration of species $A$. The additional state variable is $T$, the reactor temperature; while the manipulated variables are $T_c$, the coolant liquid temperature, and $F$, the outlet flowrate. Moreover, it is assumed that the inlet flowrate acts as an unmeasured disturbance. The model parameters in nominal conditions are reported in Table 1.1. The open-loop stable steady-state operating conditions are the following:

$$c^s = 0.878\,\text{kmol/m}^3 \qquad T^s = 324.5\,\text{K} \qquad h^s = 0.659\,\text{m}$$
$$T_c^s = 300\,\text{K} \qquad F^s = 0.1\,\text{m}^3/\text{min}$$

Using a sampling time of 1 min, a linearized discrete state space model is obtained and, assuming that all the states are measured, the state space variables are:

$$x = \begin{bmatrix} c - c^s \\ T - T^s \\ h - h^s \end{bmatrix} \qquad u = \begin{bmatrix} T_c - T_c^s \\ F - F^s \end{bmatrix} \qquad y = \begin{bmatrix} c - c^s \\ T - T^s \\ h - h^s \end{bmatrix} \qquad p = F_0 - F_0^s$$

The corresponding linear model is:

$$\begin{aligned}
x(k+1) &= Ax(k) + Bu(k) + B_p p \\
y(k) &= Cx(k)
\end{aligned}$$

in which

$$A = \begin{bmatrix} 0.2681 & -0.00338 & -0.00728 \\ 9.703 & 0.3279 & -25.44 \\ 0 & 0 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} -0.00537 & 0.1655 \\ 1.297 & 97.91 \\ 0 & -6.637 \end{bmatrix} \qquad B_p = \begin{bmatrix} -0.1175 \\ 69.74 \\ 6.637 \end{bmatrix}$$

(a) Since we have two inputs, $T_c$ and $F$, we try to remove offset in two controlled variables, $c$ and $h$. Model the disturbance with *two* integrating output disturbances on the two controlled variables. Assume that the covariances of the state noises are zero except for the two integrating states. Assume that the covariances of the three measurements' noises are also zero.

Notice that although there are only two controlled variables, this choice of *two* integrating disturbances does not follow the prescription of Lemma 1.10 for zero offset.

Simulate the response of the controlled system after a 10% increase in the inlet flowrate $F_0$ at time $t = 10$ min. Use the nonlinear differential equations for the plant model. Do you have steady offset in any of the outputs? Which ones?

(b) Follow the prescription of Lemma 1.10 and choose a disturbance model with *three* integrating modes. Can you choose three integrating output disturbances for this plant? If so, prove it. If not, state why not.

(c) Again choose a disturbance model with three integrating modes; choose two integrating output disturbances on the two controlled variables. Choose one integrating input disturbance on the outlet flowrate $F$. Is the augmented system detectable?

Simulate again the response of the controlled system after a 10% increase in the inlet flowrate $F_0$ at time $t = 10$ min. Again use the nonlinear differential equations for the plant model. Do you have steady offset in any of the outputs? Which ones?

Compare and contrast the closed-loop performance for the design with two integrating disturbances and the design with three integrating disturbances. Which control system do you recommend and why?

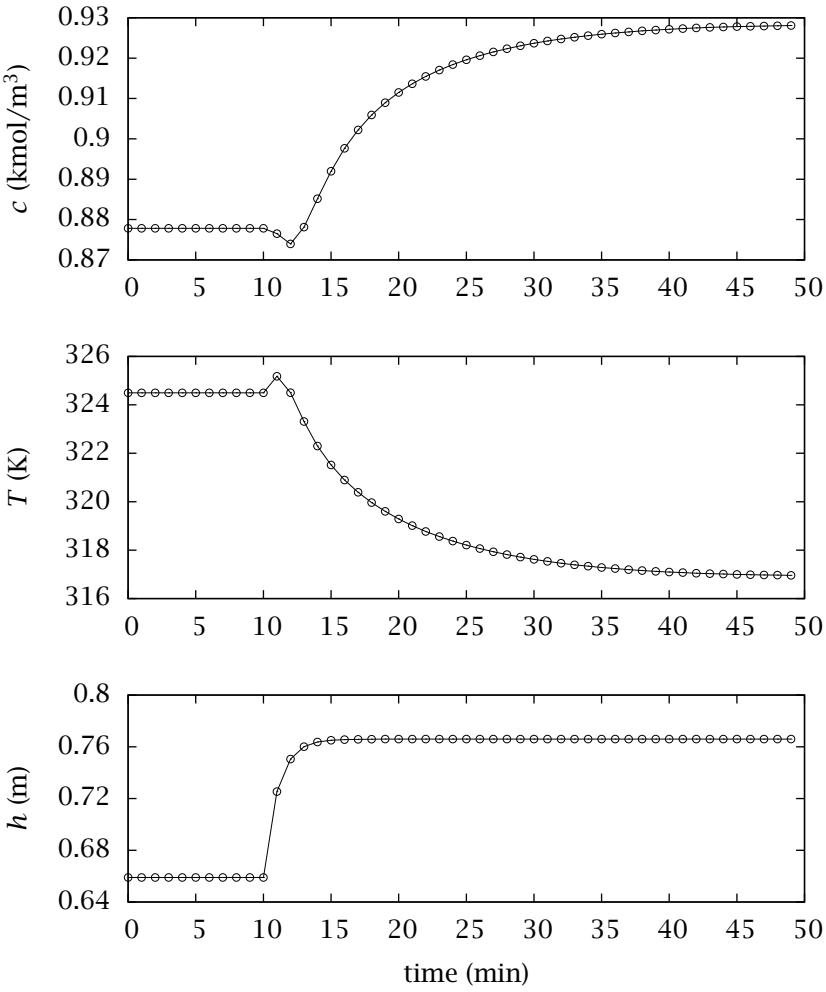


**Figure 1.7:** Three measured outputs versus time after a step change in inlet flowrate at 10 minutes; $n_d = 2$.

**Solution**

(a) Integrating disturbances are added to the two controlled variables (first and third outputs) by choosing

$$C_d = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad B_d = 0$$

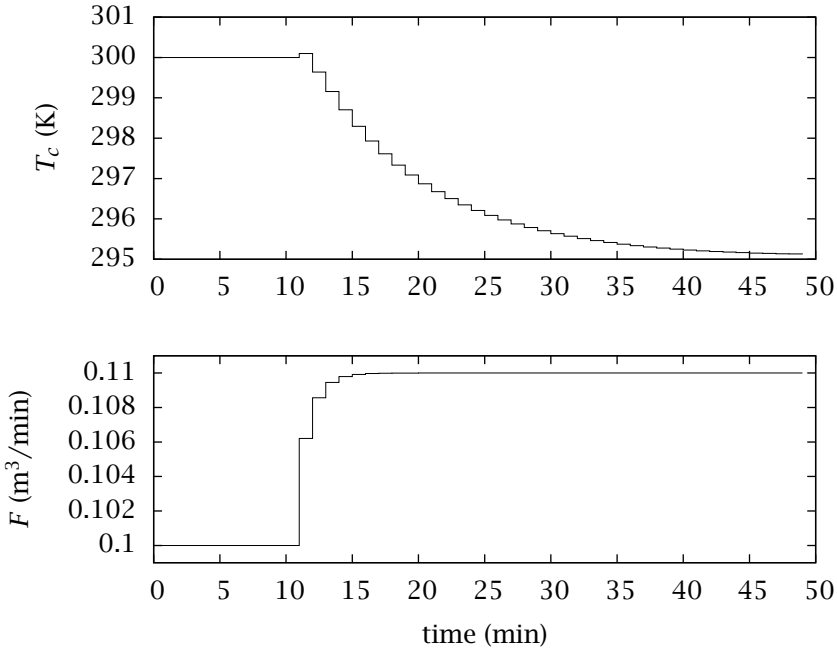


**Figure 1.8:** Two manipulated inputs versus time after a step change in inlet flowrate at 10 minutes; $n_d = 2$.

The results with two integrating disturbances are shown in Figures 1.7 and 1.8. Notice that despite adding integrating disturbances to the two controlled variables, $c$ and $h$, both of these controlled variables as well as the third output, $T$, all display nonzero offset at steady state.

(b) A third integrating disturbance is added to the second output giving

$$C_d = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad B_d = 0$$

The augmented system is not detectable with this disturbance model. The rank of $\begin{bmatrix} I-A & -B_d \\ C & C_d \end{bmatrix}$ is only 5 instead of 6. The problem here is that the system level is itself an integrator, and we cannot distinguish $h$ from the integrating disturbance added to $h$.

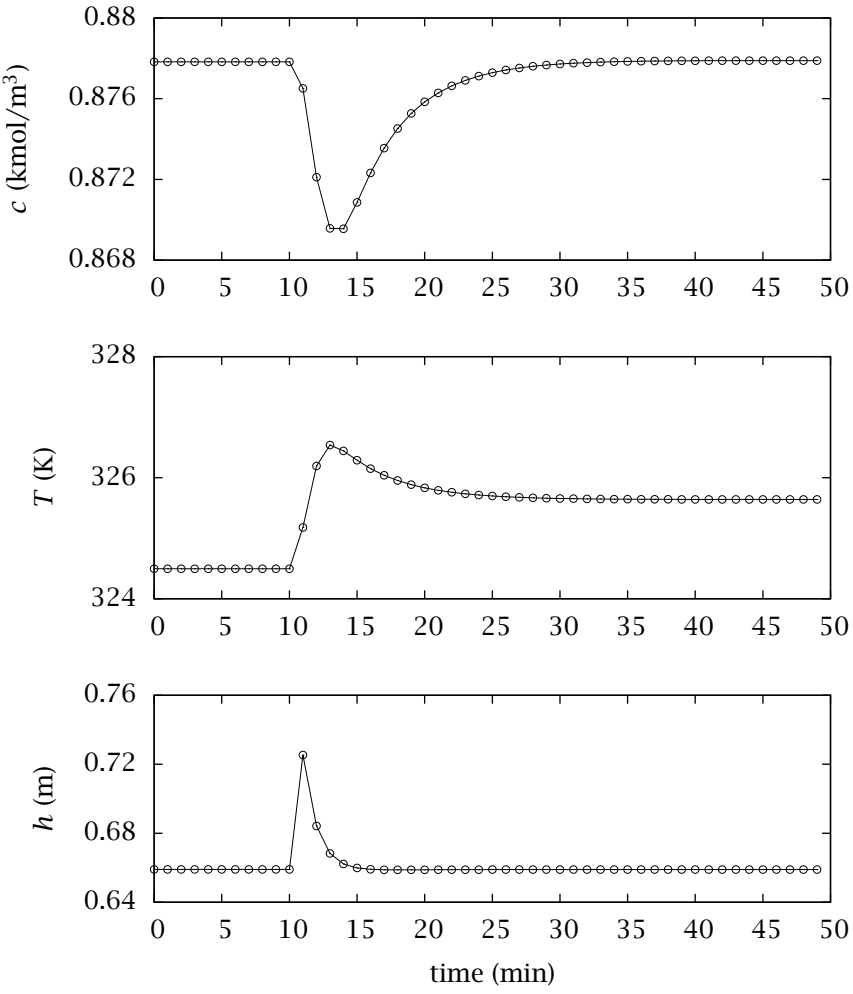


**Figure 1.9:** Three measured outputs versus time after a step change in inlet flowrate at 10 minutes; $n_d = 3$.

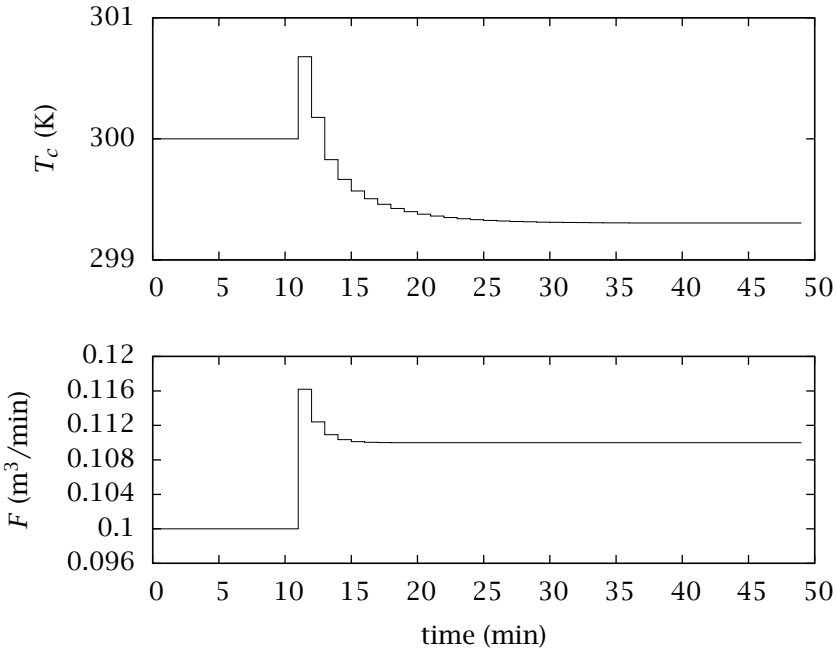


**Figure 1.10:** Two manipulated inputs versus time after a step change in inlet flowrate at 10 minutes; $n_d = 3$.

(c) Next we try three integrating disturbances: two added to the two controlled variables, and one added to the second manipulated variable

$$C_d = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad B_d = \begin{bmatrix} 0 & 0 & 0.1655 \\ 0 & 0 & 97.91 \\ 0 & 0 & -6.637 \end{bmatrix}$$

The augmented system is detectable for this disturbance model.

The results for this choice of three integrating disturbances are shown in Figures 1.9 and 1.10. Notice that we have zero offset in the two controlled variables, $c$ and $h$, and have successfully forced the steady-state effect of the inlet flowrate disturbance entirely into the second output, $T$.

Notice also that the dynamic behavior of all three outputs is supe-

rior to that achieved with the model using two integrating disturbances. The true disturbance, which is a step at the inlet flowrate, is better represented by including the integrator in the outlet flowrate. With a more accurate disturbance model, better overall control is achieved. The controller uses smaller manipulated variable action and also achieves better output variable behavior. An added bonus is that steady offset is removed in the maximum possible number of outputs. □

## Further notation

| | |
|---|---|
| $G$ | transfer function matrix |
| $m$ | mean of normally distributed random variable |
| $T$ | reactor temperature |
| $\tilde{u}$ | input deviation variable |
| $x, y, z$ | spatial coordinates for a distributed system |
| $\tilde{x}$ | state deviation variable |

## 1.6 Exercises

### Exercise 1.1: State space form for chemical reaction model

Consider the following chemical reaction kinetics for a two-step series reaction

$$\mathrm{A} \xrightarrow{k_1} \mathrm{B} \qquad \mathrm{B} \xrightarrow{k_2} \mathrm{C}$$

We wish to follow the reaction in a constant volume, well-mixed, batch reactor. As taught in the undergraduate chemical engineering curriculum, we proceed by writing material balances for the three species giving

$$\frac{dc_A}{dt} = -r_1 \qquad \frac{dc_B}{dt} = r_1 - r_2 \qquad \frac{dc_C}{dt} = r_2$$

in which $c_j$ is the concentration of species $j$, and $r_1$ and $r_2$ are the rates (mol/(time·vol)) at which the two reactions occur. We then assume some rate law for the reaction kinetics, such as

$$r_1 = k_1 c_A \qquad r_2 = k_2 c_B$$

We substitute the rate laws into the material balances and specify the starting concentrations to produce three differential equations for the three species concentrations.

(a) Write the linear state space model for the deterministic series chemical reaction model. Assume we can measure the component A concentration. What are $x$, $y$, $A$, $B$, $C$, and $D$ for this model?

(b) Simulate this model with initial conditions and parameters given by

$$c_{A0} = 1 \quad c_{B0} = c_{C0} = 0 \qquad k_1 = 2 \quad k_2 = 1$$

### Exercise 1.2: Distributed systems and time delay

We assume familiarity with the transfer function of a time delay from an undergraduate systems course

$$\overline{y}(s) = e^{-\theta s}\overline{u}(s)$$

Let's see the connection between the delay and the distributed systems, which give rise to it. A simple physical example of a time delay is the delay caused by transport in a flowing system. Consider plug flow in a tube depicted in Figure 1.11.

(a) Write down the equation of change for moles of component $j$ for an arbitrary volume element and show that

$$\frac{\partial c_j}{\partial t} = -\nabla \cdot (c_j v_j) + R_j$$




**Figure 1.11:** Plug-flow reactor.

in which $c_j$ is the molar concentration of component $j$, $v_j$ is the velocity of component $j$, and $R_j$ is the production rate of component $j$ due to chemical reaction.[9]

Plug flow means the fluid velocity of all components is purely in the $z$ direction, and is independent of $r$ and $\theta$ and, we assume here, $z$

$$v_j = v\delta_z$$

(b) Assuming plug flow and neglecting chemical reaction in the tube, show that the equation of change reduces to

$$\frac{\partial c_j}{\partial t} = -v\frac{\partial c_j}{\partial z} \tag{1.47}$$

This equation is known as a hyperbolic, first-order partial differential equation. Assume the boundary and initial conditions are

$$c_j(z,t) = u(t) \qquad 0 = z \qquad t \geq 0 \tag{1.48}$$

$$c_j(z,t) = c_{j0}(z) \qquad 0 \leq z \leq L \quad t = 0 \tag{1.49}$$

In other words, we are using the feed concentration as the manipulated variable, $u(t)$, and the tube starts out with some initial concentration profile of component $j$, $c_{j0}(z)$.

(c) Show that the solution to (1.47) with these boundary conditions is

$$c_j(z,t) = \begin{cases} u(t - z/v) & vt > z \\ c_{j0}(z - vt) & vt < z \end{cases} \tag{1.50}$$

(d) If the reactor starts out empty of component $j$, show that the transfer function between the outlet concentration, $y = c_j(L,t)$, and the inlet concentration, $c_j(0,t) = u(t)$, is a time delay. What is the value of $\theta$?

### Exercise 1.3: Pendulum in state space

Consider the pendulum suspended at the end of a rigid link depicted in Figure 1.12. Let $r$ and $\theta$ denote the polar coordinates of the center of the pendulum, and let $p = r\delta_r$ be the position vector of the pendulum, in which $\delta_r$ and $\delta_\theta$ are the unit vectors in polar coordinates. We wish to determine a state space description of the system. We are able to apply a torque $T$ to the pendulum as our manipulated variable. The pendulum has mass $m$, the only other external force acting on the pendulum is gravity, and we neglect friction. The link provides force $-t\delta_r$ necessary to maintain the pendulum at distance $r = R$ from the axis of rotation, and we measure this force $t$.

(a) Provide expressions for the four partial derivatives for changes in the unit vectors with $r$ and $\theta$

$$\frac{\partial \delta_r}{\partial r} \qquad \frac{\partial \delta_r}{\partial \theta} \qquad \frac{\partial \delta_\theta}{\partial r} \qquad \frac{\partial \delta_\theta}{\partial \theta}$$

(b) Use the chain rule to find the velocity of the pendulum in terms of the time derivatives of $r$ and $\theta$. Do not simplify yet by assuming $r$ is constant. We want the general result.

[9] You will need the Gauss divergence theorem and 3D Leibniz formula to go from a mass balance on a volume element to the equation of continuity.

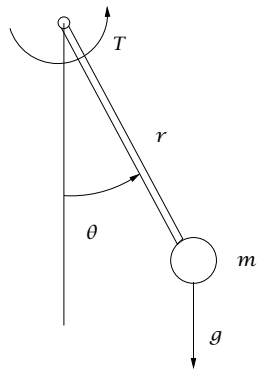


**Figure 1.12:** Pendulum with applied torque.

(c) Differentiate again to show that the acceleration of the pendulum is

$$\ddot{p} = (\ddot{r} - r\dot{\theta}^2)\delta_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\delta_\theta$$

(d) Use a momentum balance on the pendulum mass (you may assume it is a point mass) to determine both the force exerted by the link

$$t = mR\dot{\theta}^2 + mg\cos\theta$$

and an equation for the acceleration of the pendulum due to gravity and the applied torque

$$mR\ddot{\theta} - T/R + mg\sin\theta = 0$$

(e) Define a state vector and give a state space description of your system. What is the physical significance of your state. Assume you measure the force exerted by the link.

One answer is

$$\begin{aligned}\frac{dx_1}{dt} &= x_2\\ \frac{dx_2}{dt} &= -(g/R)\sin x_1 + u\\ y &= mRx_2^2 + mg\cos x_1\end{aligned}$$

in which $u = T/(mR^2)$

### Exercise 1.4: Time to Laplace domain

Take the Laplace transform of the following set of differential equations and find the transfer function, $G(s)$, connecting $\overline{u}(s)$ and $\overline{y}(s)$, $\overline{y} = G\overline{u}$

$$\begin{aligned}\frac{dx}{dt} &= Ax + Bu\\ y &= Cx + Du\end{aligned} \tag{1.51}$$

For $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, and $u \in \mathbb{R}^m$, what is the dimension of the $G$ matrix? What happens to the initial condition, $x(0) = x_0$?

### Exercise 1.5: Converting between continuous and discrete time models

Given a prescribed $u(t)$, derive and check the solution to (1.51). Given a prescribed $u(k)$ sequence, what is the solution to the discrete time model

$$x(k+1) = \widetilde{A}x(k) + \widetilde{B}u(k)$$
$$y(k) = \widetilde{C}x(k) + \widetilde{D}u(k)$$

(a) Compute $\widetilde{A}, \widetilde{B}, \widetilde{C}$, and $\widetilde{D}$ so that the two solutions agree at the sample times for a zero-order hold input, i.e., $y(k) = y(t_k)$ for $u(t) = u(k),\ t \in (t_k, t_{k+1})$ in which $t_k = k\Delta$ for sample time $\Delta$.

(b) Is your result valid for $A$ singular? If not, how can you find $\widetilde{A}, \widetilde{B}, \widetilde{C}$, and $\widetilde{D}$ for this case?

### Exercise 1.6: Continuous to discrete time conversion for nonlinear models

Consider the autonomous nonlinear differential equation model

$$\frac{dx}{dt} = f(x,u)$$
$$x(0) = x_0 \tag{1.52}$$

Given a zero-order hold on the input, let $s(t,u,x_0), 0 \le t \le \Delta$, be the solution to (1.52) given initial condition $x_0$ at time $t = 0$, and constant input $u$ is applied for $t$ in the interval $0 \le t \le \Delta$. Consider also the nonlinear discrete time model

$$x(k+1) = F(x(k), u(k))$$

(a) What is the relationship between $F$ and $s$ so that the solution of the discrete time model agrees at the sample times with the continuous time model with a zero-order hold?

(b) Assume $f$ is linear and apply this result to check the result of Exercise 1.5.

### Exercise 1.7: Commuting functions of a matrix

Although matrix multiplication does not commute in general

$$AB \neq BA$$

multiplication of functions of the same matrix do commute. You may have used the following fact in Exercise 1.5

$$A^{-1}\exp(At) = \exp(At)A^{-1} \tag{1.53}$$

(a) Prove that (1.53) is true assuming $A$ has distinct eigenvalues and can therefore be represented as

$$A = Q\Lambda Q^{-1} \qquad \Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

in which $\Lambda$ is a diagonal matrix containing the eigenvalues of $A$, and $Q$ is the matrix of eigenvectors such that

$$Aq_i = \lambda_i q_i, \qquad i = 1, \ldots, n$$

in which $q_i$ is the $i$th column of matrix $Q$.

(b) Prove the more general relationship

$$f(A)g(A) = g(A)f(A) \tag{1.54}$$

in which $f$ and $g$ are any functions definable by Taylor series.

(c) Prove that (1.54) is true without assuming the eigenvalues are distinct.

Hint: use the Taylor series defining the functions and apply the Cayley-Hamilton theorem (Horn and Johnson, 1985, pp. 86–87).

### Exercise 1.8: Finite difference formula and approximating the exponential

Instead of computing the exact conversion of a continuous time to a discrete time system as in Exercise 1.5, assume instead one simply approximates the time derivative with a first-order finite difference formula

$$\frac{dx}{dt} \approx \frac{x(t_{k+1}) - x(t_k)}{\Delta}$$

with step size equal to the sample time, $\Delta$. For this approximation of the continuous time system, compute $\widetilde{A}$ and $\widetilde{B}$ so that the discrete time system agrees with the approximate continuous time system at the sample times. Comparing these answers to the exact solution, what approximation of $e^{A\Delta}$ results from the finite difference approximation? When is this a good approximation of $e^{A\Delta}$?

### Exercise 1.9: Mapping eigenvalues of continuous time systems to discrete time systems

Consider the continuous time differential equation and discrete time difference equation

$$\frac{dx}{dt} = Ax$$
$$x^+ = \widetilde{A}x$$

and the transformation

$$\widetilde{A} = e^{A\Delta}$$

Consider the scalar $A$ case.

(a) What $A$ represents an integrator in continuous time? What is the corresponding $\widetilde{A}$ value for the integrator in discrete time?

(b) What $A$ give purely oscillatory solutions? What are the corresponding $\widetilde{A}$?

(c) For what $A$ is the solution of the ODE stable? Unstable? What are the corresponding $\widetilde{A}$?

(d) Sketch and label these $A$ and $\widetilde{A}$ regions in two complex-plane diagrams.

### Exercise 1.10: State space realization

Define a state vector and realize the following models as state space models **by hand**. One should do a few by hand to understand what the Octave or MATLAB calls are doing. Answer the following questions. What is the connection between the poles of $G$ and the state space description? For what kinds of $G(s)$ does one obtain a nonzero $D$ matrix? What is the order and gain of these systems? Is there a connection between order and the numbers of inputs and outputs?

(a) $G(s) = \dfrac{1}{2s+1}$

(b) $G(s) = \dfrac{1}{(2s+1)(3s+1)}$

(c) $G(s) = \dfrac{2s+1}{3s+1}$

(d) $y(k+1) = y(k) + 2u(k)$

(e) $y(k+1) = a_1 y(k) + a_2 y(k-1) + b_1 u(k) + b_2 u(k-1)$

### Exercise 1.11: Minimal realization

Find minimal realizations of the state space models you found by hand in Exercise 1.10. Use Octave or MATLAB for computing minimal realizations. Were any of your hand realizations nonminimal?

### Exercise 1.12: Partitioned matrix inversion lemma

Let matrix $Z$ be partitioned into

$$Z = \begin{bmatrix} B & C \\ D & E \end{bmatrix}$$

and assume $Z^{-1}, B^{-1}$ and $E^{-1}$ exist.

(a) Perform row elimination and show that

$$Z^{-1} = \begin{bmatrix} B^{-1} + B^{-1}C(E - DB^{-1}C)^{-1}DB^{-1} & -B^{-1}C(E - DB^{-1}C)^{-1} \\ -(E - DB^{-1}C)^{-1}DB^{-1} & (E - DB^{-1}C)^{-1} \end{bmatrix}$$

Note that this result is still valid if $E$ is singular.

(b) Perform column elimination and show that

$$Z^{-1} = \begin{bmatrix} (B - CE^{-1}D)^{-1} & -(B - CE^{-1}D)^{-1}CE^{-1} \\ -E^{-1}D(B - CE^{-1}D)^{-1} & E^{-1} + E^{-1}D(B - CE^{-1}D)^{-1}CE^{-1} \end{bmatrix}$$

Note that this result is still valid if $B$ is singular.

(c) A host of other useful control-related inversion formulas follow from these results. Equate the (1,1) or (2,2) entries of $Z^{-1}$ and derive the identity

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(DA^{-1}B + C^{-1})^{-1}DA^{-1} \tag{1.55}$$

A useful special case of this result is

$$(I + X^{-1})^{-1} = I - (I + X)^{-1}$$

(d) Equate the (1,2) or (2,1) entries of $Z^{-1}$ and derive the identity

$$(A + BCD)^{-1}BC = A^{-1}B(DA^{-1}B + C^{-1})^{-1} \tag{1.56}$$

Equations (1.55) and (1.56) prove especially useful in rearranging formulas in least squares estimation.

### Exercise 1.13: Perturbation to an asymptotically stable linear system

Given the system

$$x^+ = Ax + Bu$$

If $A$ is an asymptotically stable matrix, prove that if $u(k) \to 0$, then $x(k) \to 0$.

### Exercise 1.14: Exponential stability of a perturbed linear system

Given the system

$$x^+ = Ax + Bu$$

If $A$ is an asymptotically stable matrix, prove that if $u(k)$ decreases exponentially to zero, then $x(k)$ decreases exponentially to zero.

### Exercise 1.15: Are we going forward or backward today?

In the chapter we derived the solution to

$$\min_{w,x,y} f(w,x) + g(x,y) + h(y,z)$$

in which $z$ is a fixed parameter using forward dynamic programming (DP)

$$\begin{aligned}
&\overline{y}^0(z) \\
&\tilde{x}^0(z) = \overline{x}^0(\overline{y}^0(z)) \\
&\tilde{w}^0(z) = \overline{w}^0(\overline{x}^0(\overline{y}^0(z)))
\end{aligned}$$

(a) Solve for optimal $w$ as a function of $z$ using backward DP.

(b) Is forward or backward DP more efficient if you want optimal $w$ as a function of $z$?

### Exercise 1.16: Method of Lagrange multipliers

Consider the objective function $V(x) = (1/2)x'Hx + h'x$ and optimization problem

$$\min_x V(x) \tag{1.57}$$

subject to

$$Dx = d$$

in which $H > 0$, $x \in \mathbb{R}^n$, $d \in \mathbb{R}^m$, $m < n$, i.e., fewer constraints than decisions. Rather than partially solving for $x$ using the constraint and eliminating it, we make use of the method of Lagrange multipliers for treating the equality constraints (Fletcher, 1987; Nocedal and Wright, 1999).

In the method of Lagrange multipliers, we augment the objective function with the constraints to form the Lagrangian function, $L$

$$L(x,\lambda) = (1/2)x'Hx + h'x - \lambda'(Dx - d)$$

in which $\lambda \in \mathbb{R}^m$ is the vector of Lagrange multipliers. The necessary and sufficient conditions for a global minimizer are that the partial derivatives of $L$ with respect to $x$ and $\lambda$ vanish (Nocedal and Wright, 1999, p. 444), (Fletcher, 1987, p.198,236)

(a) Show that the necessary and sufficient conditions are equivalent to the matrix equation

$$\begin{bmatrix} H & -D' \\ -D & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = - \begin{bmatrix} h \\ d \end{bmatrix} \tag{1.58}$$

The solution to (1.58) then provides the solution to the original problem (1.57).

(b) We note one other important feature of the Lagrange multipliers, their relationship to the optimal cost of the purely quadratic case. For $h = 0$, the cost is given by

$$V^0 = (1/2)(x^0)'Hx^0$$

Show that this can also be expressed in terms of $\lambda^0$ by the following

$$V^0 = (1/2)d'\lambda^0$$

### Exercise 1.17: Minimizing a constrained, quadratic function

Consider optimizing the positive definite quadratic function subject to a linear constraint

$$\min_x (1/2)x'Hx \qquad \text{s.t. } Ax = b$$

Using the method of Lagrange multipliers presented in Exercise 1.16, show that the optimal solution, multiplier, and cost are given by

$$\begin{aligned} x^0 &= H^{-1}A'(AH^{-1}A')^{-1}b \\ \lambda^0 &= (AH^{-1}A')^{-1}b \\ V^0 &= (1/2)b'(AH^{-1}A')^{-1}b \end{aligned}$$

### Exercise 1.18: Minimizing a partitioned quadratic function

Consider the partitioned constrained minimization

$$\min_{x_1,x_2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' \begin{bmatrix} H_1 & \\ & H_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

subject to

$$\begin{bmatrix} D & I \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = d$$

The solution to this optimization is required in two different forms, depending on whether one is solving an estimation or regulation problem. Show that the solution can be expressed in the following two forms if both $H_1$ and $H_2$ are full rank.

- Regulator form

$$\begin{aligned} V^0(d) &= d'(H_2 - H_2D(D'H_2D + H_1)^{-1}D'H_2)d \\ x_1^0(d) &= \widetilde{K}d \qquad \widetilde{K} = (D'H_2D + H_1)^{-1}D'H_2 \\ x_2^0(d) &= (I - D\widetilde{K})d \end{aligned}$$

- Estimator form

$$\begin{aligned} V^0(d) &= d'(DH_1^{-1}D' + H_2^{-1})^{-1}d \\ x_1^0(d) &= \widetilde{L}d \qquad \widetilde{L} = H_1^{-1}D'(DH_1^{-1}D' + H_2^{-1})^{-1} \\ x_2^0(d) &= (I - D\widetilde{L})d \end{aligned}$$

### Exercise 1.19: Stabilizability and controllability canonical forms

Consider the partitioned system

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^+ = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u$$

with $(A_{11}, B_1)$ controllable. This form is known as controllability canonical form.

(a) Show that the system is *not* controllable by checking the rank of the controllability matrix.

(b) Show that the modes $x_1$ can be controlled from any $x_1(0)$ to any $x_1(n)$ with a sequence of inputs $u(0), \ldots, u(n-1)$, but the modes $x_2$ *cannot* be controlled from any $x_2(0)$ to any $x_2(n)$. The states $x_2$ are termed the uncontrollable modes.

(c) If $A_{22}$ is stable the system is termed *stabilizable*. Although not all modes can be controlled, the uncontrollable modes are stable and decay to steady state.

The following lemma gives an equivalent condition for stabilizability.

**Lemma 1.12** (Hautus Lemma for stabilizability)**.** *A system is stabilizable if and only if*

$$\operatorname{rank} \begin{bmatrix} \lambda I - A & B \end{bmatrix} = n \qquad \text{for all } |\lambda| \geq 1$$

Prove this lemma using Lemma 1.2 as the condition for controllability.

### Exercise 1.20: Regulator stability, stabilizable systems, and semidefinite state penalty

(a) Show that the infinite horizon LQR is stabilizing for $(A, B)$ *stabilizable* with $R, Q > 0$.

(b) Show that the infinite horizon LQR is stabilizing for $(A, B)$ stabilizable and $R > 0$, $Q \geq 0$, and $(A, Q)$ detectable. Discuss what happens to the controller's stabilizing property if $Q$ is not positive semidefinite or $(A, Q)$ is not detectable.

### Exercise 1.21: Time-varying linear quadratic problem

Consider the time-varying version of the LQ problem solved in the chapter. The system model is

$$x(k+1) = A(k)x(k) + B(k)u(k)$$

The objective function also contains time-varying penalties

$$\min_{\mathbf{u}} V(x(0), \mathbf{u}) = \frac{1}{2} \left( \sum_{k=0}^{N-1} \left( x(k)'Q(k)x(k) + u(k)'R(k)u(k) \right) + x(N)'Q(N)x(N) \right)$$

subject to the model. Notice the penalty on the final state is now simply $Q(N)$ instead of $P_f$.

Apply the DP argument to this problem and determine the optimal input sequence and cost. Can this problem also be solved in closed form like the time-invariant case?

### Exercise 1.22: Steady-state Riccati equation

Generate a random $A$ and $B$ for a system model for whatever $n(\geq 3)$ and $m(\geq 3)$ you wish. Choose a positive semidefinite $Q$ and positive definite $R$ of the appropriate sizes.

(a) Iterate the DARE by hand with Octave or MATLAB until $\Pi$ stops changing. Save this result. Now call the MATLAB or Octave function to solve the steady-state DARE. Do the solutions agree? Where in the complex plane are the eigenvalues of $A+BK$? Increase the size of $Q$ relative to $R$. Where do the eigenvalues move?

(b) Repeat for a singular $A$ matrix. What happens to the two solution techniques?

(c) Repeat for an unstable $A$ matrix.

### Exercise 1.23: Positive definite Riccati iteration

If $\Pi(k), Q, R > 0$ in (1.11), show that $\Pi(k-1) > 0$.
Hint: apply (1.55) to the term $(B'\Pi(k)B + R)^{-1}$.

### Exercise 1.24: Existence and uniqueness of the solution to constrained least squares

Consider the least squares problem subject to linear constraint

$$\min_x (1/2)x'Qx \qquad \text{subject to} \quad Ax = b$$

in which $x \in \mathbb{R}^n$, $b \in \mathbb{R}^p$, $Q \in \mathbb{R}^{n\times n}$, $Q \geq 0$, $A \in \mathbb{R}^{p\times n}$. Show that this problem has a solution for every $b$ and the solution is unique if and only if

$$\text{rank}(A) = p \qquad \text{rank}\begin{bmatrix} Q \\ A \end{bmatrix} = n$$

### Exercise 1.25: Rate-of-change penalty

Consider the generalized LQR problem with the cross term between $x(k)$ and $u(k)$

$$V(x(0), \mathbf{u}) = \frac{1}{2}\sum_{k=0}^{N-1} \left(x(k)'Qx(k) + u(k)'Ru(k) + 2x(k)'Mu(k)\right) + (1/2)x(N)'P_f x(N)$$

(a) Solve this problem with backward DP and write out the Riccati iteration and feedback gain.

(b) Control engineers often wish to tune a regulator by penalizing the rate of change of the input rather than the absolute size of the input. Consider the additional positive definite penalty matrix $S$ and the modified objective function

$$V(x(0), \mathbf{u}) = \frac{1}{2}\sum_{k=0}^{N-1} \left(x(k)'Qx(k) + u(k)'Ru(k) + \Delta u(k)'S\Delta u(k)\right) + (1/2)x(N)'P_f x(N)$$

in which $\Delta u(k) = u(k) - u(k-1)$. Show that you can augment the state to include $u(k-1)$ via

$$\widetilde{x}(k) = \begin{bmatrix} x(k) \\ u(k-1) \end{bmatrix}$$

and reduce this new problem to the standard LQR with the cross term. What are $\widetilde{A}$, $\widetilde{B}$, $\widetilde{Q}$, $\widetilde{R}$, and $\widetilde{M}$ for the augmented problem (Rao and Rawlings, 1999)?

### Exercise 1.26: Existence, uniqueness and stability with the cross term

Consider the linear quadratic problem with system

$$x^+ = Ax + Bu \tag{1.59}$$

and infinite horizon cost function

$$V(x(0), \mathbf{u}) = (1/2) \sum_{k=0}^{\infty} x(k)'Qx(k) + u(k)'Ru(k)$$

The existence, uniqueness and stability conditions for this problem are: $(A, B)$ stabilizable, $Q \geq 0$, $(A, Q)$ detectable, and $R > 0$. Consider the modified objective function with the cross term

$$V = (1/2) \sum_{k=0}^{\infty} x(k)'Qx(k) + u(k)'Ru(k) + 2x(k)'Mu(k) \tag{1.60}$$

(a) Consider reparameterizing the input as

$$v(k) = u(k) + Tx(k) \tag{1.61}$$

Choose $T$ such that the cost function in $x$ and $v$ does not have a cross term, and express the existence, uniqueness and stability conditions for the transformed system. Goodwin and Sin (1984, p.251) discuss this procedure in the state estimation problem with nonzero covariance between state and output measurement noises.

(b) Translate and simplify these to obtain the existence, uniqueness and stability conditions for the original system with cross term.

### Exercise 1.27: Forecasting and variance increase or decrease

Given positive definite initial state variance $P(0)$ and process disturbance variance $Q$, the variance after forecasting one sample time was shown to be

$$P^-(1) = AP(0)A' + Q$$

(a) If $A$ is stable, is it true that $AP(0)A' < P(0)$? If so, prove it. If not, provide a counterexample.

(b) If $A$ is unstable, is it true that $AP(0)A' > P(0)$? If so, prove it. If not, provide a counterexample.

(c) If the magnitudes of *all* the eigenvalues of $A$ are unstable, is it true that $AP(0)A' > P(0)$? If so, prove it. If not, provide a counterexample.

### Exercise 1.28: Convergence of MHE with noninformative prior

Show that the simplest form of MHE defined in (1.33) and (1.34) is also a convergent estimator for an observable system. What restrictions on the horizon length $N$ do you require for this result to hold?

Hint: you can solve the MHE optimization problem by inspection when there is no prior weighting of the data.

### Exercise 1.29: Symmetry in regulation and estimation

In this exercise we display the symmetry of the backward DP recursion for regulation, and the forward DP recursion for estimation. In the regulation problem we solve at stage $k$

$$\min_{x,u} \ell(z,u) + V_k^0(x) \qquad \text{s.t. } x = Az + Bu$$

In backward DP, $x$ is the state at the current stage and $z$ is the state at the previous stage. The stage cost and cost to go are given by

$$\ell(z,u) = (1/2)(z'Qz + u'Ru) \qquad V_k^0(x) = (1/2)x'\Pi(k)x$$

and the optimal cost is $V_{k-1}^0(z)$ since $z$ is the state at the previous stage.

In estimation we solve at stage $k$

$$\min_{x,w} \ell(z,w) + V_k^0(x) \qquad \text{s.t. } z = Ax + w$$

In forward DP, $x$ is the state at the current stage, $z$ is the state at the next stage. The stage cost and arrival cost are given by

$$\ell(z,w) = (1/2)(\,|y(k+1) - Cz|_{R^{-1}}^2 + w'Q^{-1}w) \qquad V_k^0(x) = (1/2)\,|x - \hat{x}(k)|_{P(k)^{-1}}^2$$

and we wish to find $V_{k+1}^0(z)$ in the estimation problem.

(a) In the estimation problem, take the $z$ term outside the optimization and solve

$$\min_{x,w} \frac{1}{2}\Big(w'Q^{-1}w + (x - \hat{x}(k))'P(k)^{-1}(x - \hat{x}(k))\Big) \qquad \text{s.t. } z = Ax + w$$

using the inverse form in Exercise 1.18, and show that the optimal cost is given by

$$V^0(z) = (1/2)(z - A\hat{x}(k))'(P^-(k+1))^{-1}(z - A\hat{x}(k))$$
$$P^-(k+1) = AP(k)A' + Q$$

Add the $z$ term to this cost using the third part of Example 1.1 and show that

$$V_{k+1}^0(z) = (1/2)(z - \hat{x}(k+1))'P^{-1}(k+1)(z - \hat{x}(k+1))$$
$$P(k+1) = P^-(k+1) - P^-(k+1)C'(CP^-(k+1)C' + R)^{-1}CP^-(k+1)$$
$$\hat{x}(k+1) = A\hat{x}(k) + L(k+1)(y(k+1) - CA\hat{x}(k))$$
$$L(k+1) = P^-(k+1)C'(CP^-(k+1)C' + R)^{-1}$$

(b) In the regulator problem, take the $z$ term outside the optimization and solve the remaining two-term problem using the regulator form of Exercise 1.18. Then

add the $z$ term and show that

$$\begin{aligned} V_{k-1}^0(z) &= (1/2)z'\Pi(k-1)z \\ \Pi(k-1) &= Q + A'\Pi(k)A - A'\Pi(k)B(B'\Pi(k)B+R)^{-1}B'\Pi(k)A \\ u^0(z) &= K(k-1)z \\ x^0(z) &= (A+BK(k-1))z \\ K(k-1) &= -(B'\Pi(k)B+R)^{-1}B'\Pi(k)A \end{aligned}$$

This symmetry can be developed further if we pose an output tracking problem rather than zero state regulation problem in the regulator.

### Exercise 1.30: Symmetry in the Riccati iteration

Show that the covariance before measurement $P^-(k+1)$ in estimation satisfies an identical iteration to the cost to go $\Pi(k-1)$ in regulation under the change of variables $P^- \longrightarrow \Pi$, $A \longrightarrow A'$, $C \longrightarrow B'$.

### Exercise 1.31: Detectability and observability canonical forms

Consider the partitioned system

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^+ = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$y = \begin{bmatrix} C_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

with $(A_{11}, C_1)$ observable. This form is known as observability canonical form.

(a) Show that the system is *not* observable by checking the rank of the observability matrix.

(b) Show that the modes $x_1$ can be uniquely determined from a sequence of measurements, but the modes $x_2$ *cannot* be uniquely determined from the measurements. The states $x_2$ are termed the unobservable modes.

(c) If $A_{22}$ is stable the system is termed *detectable*. Although not all modes can be observed, the unobservable modes are stable and decay to steady state.

The following lemma gives an equivalent condition for detectability.

**Lemma 1.13** (Hautus Lemma for detectability)**.** *A system is detectable if and only if*

$$\text{rank} \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n \qquad \textit{for all } |\lambda| \geq 1$$

Prove this lemma using Lemma 1.4 as the condition for observability.

### Exercise 1.32: Estimator stability and detectable systems

Show that the least squares estimator given in (1.28) is stable for $(A, C)$ *detectable* with $Q > 0$.

### Exercise 1.33: Estimator stability and semidefinite state noise penalty

We wish to show that the least squares estimator is stable for $(A, C)$ detectable and $Q \geq 0$, $(A, Q)$ stabilizable.

(a) Because $Q^{-1}$ is not defined in this problem, the objective function defined in (1.27) requires modification. Show that the objective function with semidefinite $Q \geq 0$ can be converted into the following form

$$V(x(0), \mathbf{w}(T)) = \frac{1}{2}\Bigg( |x(0) - \overline{x}(0)|^2_{(P^-(0))^{-1}} + \sum_{k=0}^{T-1} |w(k)|^2_{\widetilde{Q}^{-1}} + \sum_{k=0}^{T} |y(k) - Cx(k)|^2_{R^{-1}} \Bigg)$$

in which

$$x^+ = Ax + Gw \qquad \widetilde{Q} > 0$$

Find expressions for $\widetilde{Q}$ and $G$ in terms of the original semidefinite $Q$. How are the dimension of $\widetilde{Q}$ and $G$ related to the rank of $Q$?

(b) What is the probabilistic interpretation of the state estimation problem with semidefinite $Q$?

(c) Show that $(A, Q)$ stabilizable implies $(A, G)$ stabilizable in the converted form.

(d) Show that this estimator is stable for $(A, C)$ detectable and $(A, G)$ stabilizable with $\widetilde{Q}, R > 0$.

(e) Discuss what happens to the estimator's stability if $Q$ is not positive semidefinite or $(A, Q)$ is not stabilizable.

### Exercise 1.34: Calculating mean and variance from data

We are sampling a real-valued scalar random variable $x(k) \in \mathbb{R}$ at time $k$. Assume the random variable comes from a distribution with mean $\overline{x}$ and variance $P$, and the samples at different times are statistically independent.

A colleague has suggested the following formulas for estimating the mean and variance from $N$ samples

$$\hat{x}_N = \frac{1}{N}\sum_{j=1}^{N} x(j) \qquad \hat{P}_N = \frac{1}{N}\sum_{j=1}^{N} (x(j) - \hat{x}_N)^2$$

(a) Prove that the estimate of the mean is unbiased for all $N$, i.e., show that for all $N$

$$\mathcal{E}(\hat{x}_N) = \overline{x}$$

(b) Prove that the estimate of the variance is not unbiased for any $N$, i.e., show that for all $N$

$$\mathcal{E}(\hat{P}_N) \neq P$$

(c) Using the result above, provide an alternative formula for the variance estimate that is unbiased for all $N$. How large does $N$ have to be before these two estimates of $P$ are within 1%?

### Exercise 1.35: Expected sum of squares

Given that a random variable $x$ has mean $m$ and covariance $P$, show that the expected sum of squares is given by the formula (Selby, 1973, p.138)

$$\mathcal{E}(x'Qx) = m'Qm + \mathrm{tr}(QP)$$

The trace of a square matrix $A$, written $\mathrm{tr}(A)$, is defined to be the sum of the diagonal elements

$$\mathrm{tr}(A) := \sum_i A_{ii}$$

### Exercise 1.36: Normal distribution

Given a normal distribution with scalar parameters $m$ and $\sigma$,

$$p_\xi(x) = \sqrt{\frac{1}{2\pi\sigma^2}} \exp\left[-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right] \tag{1.62}$$

By direct calculation, show that

(a)

$$\mathcal{E}(\xi) = m$$
$$\mathrm{var}(\xi) = \sigma^2$$

(b) Show that the mean and the maximum likelihood are equal for the normal distribution. Draw a sketch of this result. The maximum likelihood estimate, $\hat{x}$, is defined as

$$\hat{x} := \arg\max_x p_\xi(x)$$

in which arg returns the solution to the optimization problem.

### Exercise 1.37: Conditional densities are positive definite

We show in Example A.44 that if $\xi$ and $\eta$ are jointly normally distributed as

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} \sim N(m, P)$$
$$\sim N\left(\begin{bmatrix} m_x \\ m_y \end{bmatrix}, \begin{bmatrix} P_x & P_{xy} \\ P_{yx} & P_y \end{bmatrix}\right)$$

then the conditional density of $\xi$ given $\eta$ is also normal

$$(\xi|\eta) \sim N(m_{x|y}, P_{x|y})$$

in which the conditional mean is

$$m_{x|y} = m_x + P_{xy}P_y^{-1}(y - m_y)$$

and the conditional covariance is

$$P_{x|y} = P_x - P_{xy}P_y^{-1}P_{yx}$$

Given that the joint density is well defined, prove the marginal densities and the conditional densities also are well defined, i.e., given $P > 0$, prove $P_x > 0$, $P_y > 0$, $P_{x|y} > 0$, $P_{y|x} > 0$.

### Exercise 1.38: Expectation and covariance under linear transformations

Consider the random variable $x \in \mathbb{R}^n$ with density $p_x$ and mean and covariance

$$\mathcal{E}(x) = m_x \qquad \text{cov}(x) = P_x$$

Consider the random variable $y \in \mathbb{R}^p$ defined by the linear transformation

$$y = Cx$$

(a) Show that the mean and covariance for $y$ are given by

$$\mathcal{E}(y) = Cm_x \qquad \text{cov}(y) = CP_xC'$$

Does this result hold for all $C$? If yes, prove it; if no, provide a counterexample.

(b) Apply this result to solve Exercise A.35.

### Exercise 1.39: Normal distributions under linear transformations

Given the normally distributed random variable, $\xi \in \mathbb{R}^n$, consider the random variable, $\eta \in \mathbb{R}^n$, obtained by the linear transformation

$$\eta = A\xi$$

in which $A$ is a nonsingular matrix. Using the result on transforming probability densities, show that if $\xi \sim N(m, P)$, then $\eta \sim N(Am, APA')$. This result basically says that linear transformations of normal random variables are normal.

### Exercise 1.40: More on normals and linear transformations

Consider a normally distributed random variable $x \in \mathbb{R}^n$, $x \sim N(m_x, P_x)$. You showed in Exercise 1.39 for $C \in \mathbb{R}^{n\times n}$ invertible, that the random variable $y$ defined by the linear transformation $y = Cx$ is also normal and is distributed as

$$y \sim N(Cm_x, CP_xC')$$

Does this result hold for all $C$? If yes, prove it; if no, provide a counterexample.

### Exercise 1.41: Signal processing in the good old days — recursive least squares

Imagine we are sent back in time to 1960 and the only computers available have extremely small memories. Say we have a large amount of data coming from a process and we want to compute the least squares estimate of model parameters from these data. Our immediate challenge is that we cannot load all of these data into memory to make the standard least squares calculation.

Alternatively, go 150 years further back in time and consider the situation from Gauss's perspective,

> It occasionally happens that after we have completed all parts of an extended calculation on a sequence of observations, we learn of a new observation that we would like to include. In many cases we will not want to have to redo the entire elimination but instead to find the modifications due to the new observation in the most reliable values of the unknowns and in their weights.
> C.F. Gauss, 1823
> G.W. Stewart Translation, 1995, p. 191.

Given the linear model

$$y_i = X_i'\theta$$

in which scalar $y_i$ is the measurement at sample $i$, $X_i'$ is the independent model variable (row vector, $1 \times p$) at sample $i$, and $\theta$ is the parameter vector ($p \times 1$) to be estimated from these data. Given the weighted least squares objective and $n$ measurements, we wish to compute the usual estimate

$$\hat{\theta} = (X'X)^{-1}X'y \tag{1.63}$$

in which

$$y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \qquad X = \begin{bmatrix} X_1' \\ \vdots \\ X_n' \end{bmatrix}$$

We do not wish to store the large matrices $X(n \times p)$ and $y(n \times 1)$ required for this calculation. Because we are planning to process the data one at a time, we first modify our usual least squares problem to deal with small $n$. For example, we wish to estimate the parameters when $n < p$ and the inverse in (1.63) does not exist. In such cases, we may choose to regularize the problem by modifying the objective function as follows

$$\Phi(\theta) = (\theta - \overline{\theta})' P_0^{-1} (\theta - \overline{\theta}) + \sum_{i=1}^{n} (y_i - X_i'\theta)^2$$

in which $\overline{\theta}$ and $P_0$ are chosen by the user. In Bayesian estimation, we call $\overline{\theta}$ and $P_0$ the prior information, and often assume that the prior density of $\theta$ (without measurements) is normal

$$\theta \sim N(\overline{\theta}, P_0)$$

The solution to this modified least squares estimation problem is

$$\hat{\theta} = \overline{\theta} + (X'X + P_0^{-1})^{-1} X'(y - X\overline{\theta}) \tag{1.64}$$

Devise a means to *recursively* estimate $\theta$ so that:

1. We never store more than one measurement at a time in memory.
2. After processing all the measurements, we obtain the same least squares estimate given in (1.64).

### Exercise 1.42: Least squares parameter estimation and Bayesian estimation

Consider a model linear in the parameters

$$y = X\theta + e \tag{1.65}$$

in which $y \in \mathbb{R}^p$ is a vector of measurements, $\theta \in \mathbb{R}^m$ is a vector of parameters, $X \in \mathbb{R}^{p \times m}$ is a matrix of known constants, and $e \in \mathbb{R}^p$ is a random variable modeling the measurement error. The standard parameter estimation problem is to find the best estimate of $\theta$ given the measurements $y$ corrupted with measurement error $e$, which we assume is distributed as

$$e \sim N(0, R)$$

(a) Consider the case in which the errors in the measurements are independently and identically distributed with variance $\sigma^2$, $R = \sigma^2 I$. For this case, the classic least squares problem and solution are

$$\min_{\theta} |y - X\theta|^2 \qquad \hat{\theta} = (X'X)^{-1} X'y$$

Consider the measurements to be sampled from (1.65) with true parameter value $\theta_0$. Show that using the least squares formula, the parameter estimate is distributed as

$$\hat{\theta} \sim N(\theta_0, P_{\hat{\theta}}) \qquad P_{\hat{\theta}} = \sigma^2 \left(X'X\right)^{-1}$$

(b) Now consider again the model of (1.65) and a Bayesian estimation problem. Assume a prior distribution for the random variable $\theta$

$$\theta \sim N(\overline{\theta}, \overline{P})$$

Compute the conditional density of $\theta$ given measurement $y$, show that this density is normal, and find its mean and covariance

$$p_{\theta|y}(\theta|y) = n(\theta, m, P)$$

Show that Bayesian estimation and least squares estimation give the same result in the limit of a noninformative prior. In other words, if the covariance of the prior is large compared to the covariance of the measurement error, show that

$$m \approx (X'X)^{-1}X'y \qquad P \approx P_{\hat{\theta}}$$

(c) What (weighted) least squares minimization problem is solved for the general measurement error covariance

$$e \sim N(0, R)$$

Derive the least squares estimate formula for this case.

(d) Again consider the measurements to be sampled from (1.65) with true parameter value $\theta_0$. Show that the weighted least squares formula gives parameter estimates that are distributed as

$$\hat{\theta} \sim N(\theta_0, P_{\hat{\theta}})$$

and find $P_{\hat{\theta}}$ for this case.

(e) Show again that Bayesian estimation and least squares estimation give the same result in the limit of a noninformative prior.

### Exercise 1.43: Least squares and minimum variance estimation

Consider again the model linear in the parameters and the least squares estimator from Exercise 1.42

$$y = X\theta + e \qquad e \sim N(0, R)$$
$$\hat{\theta} = \left(X'R^{-1}X\right)^{-1} X'R^{-1}y$$

Show that the covariance of the least squares estimator is the smallest covariance of all linear unbiased estimators.

### Exercise 1.44: Two stages are not better than one

We often can decompose an estimation problem into stages. Consider the following case in which we wish to estimate $x$ from measurements of $z$, but we have the model between $x$ and an intermediate variable, $y$, and the model between $y$ and $z$

$$y = Ax + e_1 \qquad \text{cov}(e_1) = Q_1$$
$$z = By + e_2 \qquad \text{cov}(e_2) = Q_2$$

(a) Write down the optimal least squares problem to solve for $\hat{y}$ given the $z$ measurements and the second model. Given $\hat{y}$, write down the optimal least squares problem for $\hat{x}$ in terms of $\hat{y}$. Combine these two results together and write the resulting estimate of $\hat{x}$ given measurements of $z$. Call this the two-stage estimate of $x$.

(b) Combine the two models together into a single model and show that the relationship between $z$ and $x$ is

$$z = BAx + e_3 \qquad \text{cov}(e_3) = Q_3$$

Express $Q_3$ in terms of $Q_1, Q_2$ and the models $A, B$. What is the optimal least squares estimate of $\hat{x}$ given measurements of $z$ and the one-stage model? Call this the one-stage estimate of $x$.

(c) Are the one-stage and two-stage estimates of $x$ the same? If yes, prove it. If no, provide a counterexample. Do you have to make any assumptions about the models $A, B$?

### Exercise 1.45: Time-varying Kalman filter

Derive formulas for the conditional densities of $x(k)|\mathbf{y}(k-1)$ and $x(k)|\mathbf{y}(k)$ for the time-varying linear system

$$\begin{aligned} x(k+1) &= A(k)x(k) + G(k)w(k) \\ y(k) &= C(k)x(k) + v(k) \end{aligned}$$

in which the initial state, state noise and measurement noise are independently distributed as

$$x(0) \sim N(\overline{x}_0, Q_0) \qquad w(k) \sim N(0, Q) \qquad v(k) \sim N(0, R)$$

### Exercise 1.46: More on conditional densities

In deriving the discrete time Kalman filter, we have $p_{x|\mathbf{y}}(x(k)|\mathbf{y}(k))$ and we wish to calculate recursively $p_{x|\mathbf{y}}(x(k+1)|\mathbf{y}(k+1))$ after we collect the output measurement at time $k+1$. It is straightforward to calculate $p_{x,y|\mathbf{y}}(x(k+1), y(k+1)|\mathbf{y}(k))$ from our established results on normal densities and knowledge of $p_{x|\mathbf{y}}(x(k)|\mathbf{y}(k))$, but we still need to establish a formula for pushing the $y(k+1)$ to the other side of the conditional density bar. Consider the following statement as a possible lemma to aid in this operation.

$$p_{a|b,c}(a|b,c) = \frac{p_{a,b|c}(a,b|c)}{p_{b|c}(b|c)}$$

If this statement is true, prove it. If it is false, give a counterexample.

### Exercise 1.47: Other useful conditional densities

Using the definitions of marginal and conditional density, establish the following useful conditional density relations

1. $p_{A|B}(a|b) = \int p_{A|B,C}(a|b,c)p_{C|B}(c|b)dc$
2. $p_{A|B,C}(a|b,c) = p_{C|A,B}(c|a,b)\dfrac{p_{A|B}(a|b)}{p_{C|B}(c|b)}$

### Exercise 1.48: Optimal filtering and deterministic least squares

Given the data sequence $\{y(0), \ldots, y(k)\}$ and the system model

$$x^+ = Ax + w$$
$$y = Cx + v$$

(a) Write down a least squares problem whose solution would provide a good state estimate for $x(k)$ in this situation. What probabilistic interpretation can you assign to the estimate calculated from this least squares problem?

(b) Now consider the nonlinear model

$$x^+ = f(x) + w$$
$$y = g(x) + v$$

What is the corresponding nonlinear least squares problem for estimating $x(k)$ in this situation? What probabilistic interpretation, if any, can you assign to this estimate in the nonlinear model context?

(c) What is the motivation for changing from these least squares estimators to the moving horizon estimators we discussed in the chapter?

### Exercise 1.49: A nonlinear transformation and conditional density

Consider the following relationship between the random variable $y$, and $x$ and $v$

$$y = f(x) + v$$

The author of a famous textbook wants us to believe that

$$p_{y|x}(y|x) = p_v(y - f(x))$$

Derive this result and state what additional assumptions on the random variables $x$ and $v$ are required for this result to be correct.

### Exercise 1.50: Some smoothing

One of the problems with asking you to derive the Kalman filter is that the derivation is in so many textbooks that it is difficult to tell if you are thinking independently. So here's a variation on the theme that should help you evaluate your level of understanding of these ideas. Let's calculate a smoothed rather than filtered estimate and covariance. Here's the problem.

We have the usual setup with a prior on $x(0)$

$$x(0) \sim N(\overline{x}(0), Q_0)$$

and we receive data from the following system

$$x(k+1) = Ax(k) + w(k)$$
$$y(k) = Cx(k) + v(k)$$

in which the random variables $w(k)$ and $v(k)$ are independent, identically distributed normals, $w(k) \sim N(0, Q), v(k) \sim N(0, R)$.

(a) Calculate the standard density for the filtering problem, $p_{x(0)|y(0)}(x(0)|y(0))$.

(b) Now calculate the density for the smoothing problem

$$p_{x(0)|y(0),y(1)}(x(0)|y(0),y(1))$$

that is, *not* the usual $p_{x(1)|y(0),y(1)}(x(1)|y(0),y(1))$.

### Exercise 1.51: Alive on arrival

The following two optimization problems are helpful in understanding the arrival cost decomposition in state estimation.

(a) Let $V(x,y,z)$ be a positive, strictly convex function consisting of the sum of two functions, one of which depends on both $x$ and $y$, and the other of which depends on $y$ and $z$

$$V(x,y,z) = g(x,y) + h(y,z) \qquad V : \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}_{\geq 0}$$

Consider the optimization problem

$$P1 : \min_{x,y,z} V(x,y,z)$$

The arrival cost decomposes this three-variable optimization problem into two, smaller dimensional optimization problems. Define the "arrival cost" $\widetilde{g}$ for this problem as the solution to the following single-variable optimization problem

$$\widetilde{g}(y) = \min_x g(x,y)$$

and define optimization problem $P2$ as follows

$$P2 : \min_{y,z} \widetilde{g}(y) + h(y,z)$$

Let $(x',y',z')$ denote the solution to $P1$ and $(x^0,y^0,z^0)$ denote the solution to $P2$, in which

$$x^0 = \arg\min_x g(x,y^0)$$

Prove that the two solutions are equal

$$(x',y',z') = (x^0,y^0,z^0)$$

(b) Repeat the previous part for the following optimization problems

$$V(x,y,z) = g(x) + h(y,z)$$

Here the $y$ variables do not appear in $g$ but restrict the $x$ variables through a linear constraint. The two optimization problems are:

$$P1 : \min_{x,y,z} V(x,y,z) \qquad \text{subject to } Ex = y$$

$$P2 : \min_{y,z} \widetilde{g}(y) + h(y,z)$$

in which

$$\widetilde{g}(y) = \min_x g(x) \qquad \text{subject to } Ex = y$$

### Exercise 1.52: On-time arrival

Consider the deterministic, full information state estimation optimization problem

$$\min_{x(0),\mathbf{w},\mathbf{v}} \frac{1}{2}\Big( |x(0)-\overline{x}(0)|^2_{(P^-(0))^{-1}} + \sum_{i=0}^{T-1} |w(i)|^2_{Q^{-1}} + |v(i)|^2_{R^{-1}} \Big) \tag{1.66}$$

subject to

$$\begin{aligned} x^+ &= Ax + w \\ y &= Cx + v \end{aligned} \tag{1.67}$$

in which the sequence of measurements $\mathbf{y}(T)$ are known values. Notice we assume the noise-shaping matrix, $G$, is an identity matrix here. See Exercise 1.53 for the general case. Using the result of the first part of Exercise 1.51, show that this problem is equivalent to the following problem

$$\min_{x(T-N),\mathbf{w},\mathbf{v}} V^-_{T-N}(x(T-N)) + \frac{1}{2}\sum_{i=T-N}^{T-1} |w(i)|^2_{Q^{-1}} + |v(i)|^2_{R^{-1}}$$

subject to (1.67). The arrival cost is defined as

$$V^-_N(a) := \min_{x(0),\mathbf{w},\mathbf{v}} \frac{1}{2}\Big( |x(0)-\overline{x}(0)|^2_{(P^-(0))^{-1}} + \sum_{i=0}^{N-1} |w(i)|^2_{Q^{-1}} + |v(i)|^2_{R^{-1}} \Big)$$

subject to (1.67) and $x(N) = a$. Notice that any value of $N$, $0 \le N \le T$, can be used to split the cost function using the arrival cost.

### Exercise 1.53: Arrival cost with noise-shaping matrix $G$

Consider the deterministic, full information state estimation optimization problem

$$\min_{x(0),\mathbf{w},\mathbf{v}} \frac{1}{2}\Big( |x(0)-\overline{x}(0)|^2_{(P^-(0))^{-1}} + \sum_{i=0}^{T-1} |w(i)|^2_{Q^{-1}} + |v(i)|^2_{R^{-1}} \Big)$$

subject to

$$\begin{aligned} x^+ &= Ax + Gw \\ y &= Cx + v \end{aligned} \tag{1.68}$$

in which the sequence of measurements $\mathbf{y}$ are known values. Using the result of the second part of Exercise 1.51, show that this problem also is equivalent to the following problem

$$\min_{x(T-N),\mathbf{w},\mathbf{v}} V^-_{T-N}(x(T-N)) + \frac{1}{2}\Big( \sum_{i=T-N}^{T-1} |w(i)|^2_{Q^{-1}} + |v(i)|^2_{R^{-1}} \Big)$$

subject to (1.68). The arrival cost is defined as

$$V^-_N(a) := \min_{x(0),\mathbf{w},\mathbf{v}} \frac{1}{2}\Big( |x(0)-\overline{x}(0)|^2_{(P^-(0))^{-1}} + \sum_{i=0}^{N-1} |w(i)|^2_{Q^{-1}} + |v(i)|^2_{R^{-1}} \Big)$$

subject to (1.68) and $x(N) = a$. Notice that any value of $N$, $0 \le N \le T$, can be used to split the cost function using the arrival cost.

### Exercise 1.54: Where is the steady state?

Consider the two-input, two-output system

$$A = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.6 \end{bmatrix} \quad B = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.4 \\ 0.25 & 0 \\ 0 & 0.6 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

(a) The output setpoint is $y_{\text{sp}} = \begin{bmatrix} 1 & -1 \end{bmatrix}'$ and the input setpoint is $u_{\text{sp}} = \begin{bmatrix} 0 & 0 \end{bmatrix}'$. Calculate the target triple $(x_s, u_s, y_s)$. Is the output setpoint feasible, i.e., does $y_s = y_{\text{sp}}$?

(b) Assume only input one $u_1$ is available for control. Is the output setpoint feasible? What is the target in this case using $Q_s = I$?

(c) Assume both inputs are available for control but only the first output has a setpoint, $y_{1t} = 1$. What is the solution to the target problem for $R_s = I$?

### Exercise 1.55: Detectability of integrating disturbance models

(a) Prove Lemma 1.8; the augmented system is detectable if and only if the system $(A, C)$ is detectable and

$$\operatorname{rank} \begin{bmatrix} I - A & -B_d \\ C & C_d \end{bmatrix} = n + n_d$$

(b) Prove Corollary 1.9; the augmented system is detectable only if $n_d \le p$.

### Exercise 1.56: Unconstrained tracking problem

(a) For an *unconstrained* system, show that the following condition is *sufficient* for feasibility of the target problem for any $r_{\text{sp}}$.

$$\operatorname{rank} \begin{bmatrix} I - A & -B \\ HC & 0 \end{bmatrix} = n + n_c \tag{1.69}$$

(b) Show that (1.69) implies that the number of controlled variables without offset is less than or equal to the number of manipulated variables and the number of measurements, $n_c \le m$ and $n_c \le p$.

(c) Show that (1.69) implies the rows of $H$ are independent.

(d) Does (1.69) imply that the rows of $C$ are independent? If so, prove it; if not, provide a counterexample.

(e) By choosing $H$, how can one satisfy (1.69) if one has installed redundant sensors so several rows of $C$ are identical?

### Exercise 1.57: Unconstrained tracking problem for stabilizable systems

If we restrict attention to stabilizable systems, the sufficient condition of Exercise 1.56 becomes a necessary and sufficient condition. Prove the following lemma.

**Lemma 1.14** (Stabilizable systems and feasible targets)**.** *Consider an unconstrained, stabilizable system* $(A, B)$*. The target is feasible for any* $r_{\text{sp}}$ *if and only if*

$$\operatorname{rank}\begin{bmatrix} I-A & -B \\ HC & 0 \end{bmatrix} = n + n_c$$

### Exercise 1.58: Existence and uniqueness of the unconstrained target

Assume a system having $p$ controlled variables $z = Hx$, with setpoints $r_{\text{sp}}$, and $m$ manipulated variables $u$, with setpoints $u_{\text{sp}}$. Consider the steady-state target problem

$$\min_{x,u}(1/2)(u-u_{\text{sp}})'R(u-u_{\text{sp}}) \qquad R>0$$

subject to

$$\begin{bmatrix} I-A & -B \\ H & 0 \end{bmatrix}\begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ r_{\text{sp}} \end{bmatrix}$$

Show that the steady-state solution $(x, u)$ exists for any $(r_{\text{sp}}, u_{\text{sp}})$ and is unique if

$$\operatorname{rank}\begin{bmatrix} I-A & -B \\ H & 0 \end{bmatrix} = n+p \qquad \operatorname{rank}\begin{bmatrix} I-A \\ H \end{bmatrix} = n$$

### Exercise 1.59: Choose a sample time

Consider the unstable continuous time system

$$\frac{dx}{dt} = Ax + Bu \qquad y = Cx$$

in which

$$A = \begin{bmatrix} -0.281 & 0.935 & 0.035 & 0.008 \\ 0.047 & -0.116 & 0.053 & 0.383 \\ 0.679 & 0.519 & 0.030 & 0.067 \\ 0.679 & 0.831 & 0.671 & -0.083 \end{bmatrix} \qquad B = \begin{bmatrix} 0.687 \\ 0.589 \\ 0.930 \\ 0.846 \end{bmatrix} \qquad C = I$$

Consider regulator tuning parameters and constraints

$$Q = \operatorname{diag}(1,2,1,2) \qquad R = 1 \qquad N = 10 \qquad |x| \le \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}$$

(a) Compute the eigenvalues of $A$. Choose a sample time of $\Delta = 0.04$ and simulate the MPC regulator response given $x(0) = \begin{bmatrix} -0.9 & -1.8 & 0.7 & 2 \end{bmatrix}'$ until $t = 20$. Use an ODE solver to simulate the continuous time plant response. Plot all states and the input versus time.

Now add an input disturbance to the regulator so the control applied to the plant is $u_d$ instead of $u$ in which

$$u_d(k) = (1 + 0.1w_1)u(k) + 0.1w_2$$

and $w_1$ and $w_2$ are zero mean, normally distributed random variables with unit variance. Simulate the regulator's performance given this disturbance. Plot all states and $u_d(k)$ versus time.

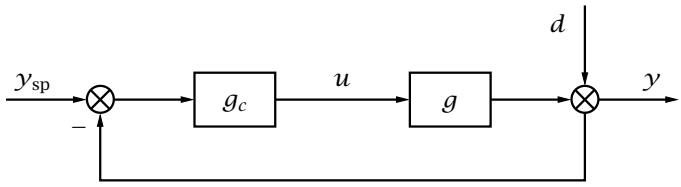


**Figure 1.13:** Feedback control system with output disturbance $d$, and setpoint $y_{sp}$.

(b) Repeat the simulations with and without disturbance for $\Delta = 0.4$ and $\Delta = 2$.

(c) Compare the simulations for the different sample times. What happens if the sample time is too large? Choose an appropriate sample time for this system and justify your choice.

### Exercise 1.60: Disturbance models and offset

Consider the following two-input, three-output plant discussed in Example 1.11

$$x^+ = Ax + Bu + B_p p$$
$$y = Cx$$

in which

$$A = \begin{bmatrix} 0.2681 & -0.00338 & -0.00728 \\ 9.703 & 0.3279 & -25.44 \\ 0 & 0 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$B = \begin{bmatrix} -0.00537 & 0.1655 \\ 1.297 & 97.91 \\ 0 & -6.637 \end{bmatrix} \qquad B_p = \begin{bmatrix} -0.1175 \\ 69.74 \\ 6.637 \end{bmatrix}$$

The input disturbance $p$ results from a reactor inlet flowrate disturbance.

(a) Since there are two inputs, choose two outputs in which to remove steady-state offset. Build an output disturbance model with two integrators. Is your augmented model detectable?

(b) Implement your controller using $p = 0.01$ as a step disturbance at $k = 0$. Do you remove offset in your chosen outputs? Do you remove offset in any outputs?

(c) Can you find any two-integrator disturbance model that removes offset in two outputs? If so, which disturbance model do you use? If not, why not?

### Exercise 1.61: MPC, PID and time delay

Consider the following first-order system with time delay shown in Figure 1.13

$$g(s) = \frac{k}{\tau s + 1} e^{-\theta s}, \qquad k = 1, \tau = 1, \theta = 5$$

Consider a unit step change in setpoint $y_{sp}$, at $t = 0$.

(a) Choose a reasonable sample time, $\Delta$, and disturbance model, and simulate an offset-free discrete time MPC controller for this setpoint change. List all of your chosen parameters.

(b) Choose PID tuning parameters to achieve "good performance" for this system. List your PID tuning parameters. Compare the performances of the two controllers.

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# 2

# Model Predictive Control — Regulation

## 2.1 Introduction

In Chapter 1 we investigated a special, but useful, form of model predictive control (MPC); an important feature of this form of MPC is that there exists a set of initial states for which it is actually optimal for an *infinite horizon* optimal control problem and therefore inherits the associated advantages. Just as there are many methods other than infinite horizon linear quadratic control for stabilizing linear systems, however, there are alternative forms of MPC that can stabilize linear and even nonlinear systems. We explore these alternatives in the remainder of this chapter. But first we place MPC in a more general setting to facilitate comparison with other control methods.

MPC is, as we have seen earlier, a form of control in which the control action is obtained by solving *online*, at each sampling instant, a *finite horizon* optimal control problem in which the initial state is the current state of the plant. Optimization yields a finite control sequence, and the first control action in this sequence is applied to the plant. MPC differs, therefore, from conventional control in which the control law is precomputed offline. But this is not an essential difference; MPC implicitly implements a control law that can, in principle, be computed offline as we shall soon see. Specifically, if the current state of the system being controlled is $x$, MPC obtains, by solving an open-loop optimal control problem for this initial state, a specific control action $u$ to apply to the plant.

Dynamic programming (DP) may be used to solve a feedback version of the same optimal control problem, however, yielding a receding horizon control *law* $\kappa(\cdot)$. The important fact is that if $x$ is the current state, the optimal control $u$ obtained by MPC (by solving an open-loop optimal control problem) satisfies $u = \kappa(x)$. For example, MPC computes the

*value* $\kappa(x)$ of the optimal receding horizon control law for the current state $x$, while DP yields the control *law* $\kappa(\cdot)$ that can be used for *any* state. DP would appear to be preferable since it provides a control law that can be implemented simply (as a look-up table). Obtaining a DP solution is difficult, if not impossible, however, for most optimal control problems if the state dimension is reasonably high — unless the system is linear, the cost quadratic and there are no control or state constraints. The great advantage of MPC is that open-loop optimal control problems often can be solved rapidly enough, using standard mathematical programming algorithms, to permit the use of MPC even though the system being controlled is nonlinear, and hard constraints on states and controls must be satisfied. Thus MPC permits the application of a DP solution, even though explicit determination of the optimal control law is intractable. MPC is an effective *implementation* of the DP solution and not a new method of control.

In this chapter we study MPC for the case when the state is known. This case is particularly important, even though it rarely arises in practice, because important properties, such as stability and performance, may be relatively easily established. The relative simplicity of this case arises from the fact that if the state is known and if there are no disturbances or model error, the problem is *deterministic*, i.e., there is no uncertainty making feedback unnecessary in principle. As we pointed out previously, for deterministic systems the MPC action for a given state is identical to the receding horizon control law, determined using DP, and evaluated at the given state. When the state is *not* known, it has to be estimated and state estimation error, together with model error and disturbances, makes the system uncertain in that future trajectories cannot be precisely predicted. The simple connection between MPC and the DP solution is lost because there does not exist an open-loop optimal control problem whose solution yields a control action that is the same as that obtained by the DP solution. A practical consequence is that special techniques are required to ensure robustness against these various forms of uncertainty. So the results of this chapter hold when there is no uncertainty. We prove, in particular, that the optimal control problem that defines the model predictive control can always be solved if the initial optimal control problem can be solved, and that the optimal cost can always be reduced allowing us to prove asymptotic or exponential stability of the target state. We refer to stability in the absence of uncertainty as *nominal stability*.

When uncertainty is present, however, neither of these two asser-

tions is necessarily true; uncertainty may cause the state to wander outside the region where the optimal control problem can be solved and may lead to instability. Procedures for overcoming the problems arising from uncertainty are presented in Chapters 3 and 5. In most of the control algorithms presented in this chapter, the decrease in the optimal cost, on which the proof of stability is founded, is based on the assumption that the next state is exactly as predicted and that the global solution to the optimal control problem can be computed. In the suboptimal control algorithm in Section 2.8, where global optimality is not required, the decrease in the optimal cost is still based on the assumption that the current state is exactly the state as predicted one step back in time.

## 2.2 Model Predictive Control

As discussed briefly in Chapter 1, most nonlinear system descriptions derived from physical arguments are continuous time descriptions in the form of nonlinear differential equations

$$\frac{dx}{dt} = f(x,u)$$

For this class of systems, the control law with arguably the best closed-loop properties is the solution to the following infinite horizon, constrained optimal control problem. The cost is defined to be

$$V_\infty(x,u(\cdot)) = \int_0^\infty \ell(x(t),u(t))dt$$

in which $x(t)$ and $u(t)$ satisfy $\dot{x} = f(x,u)$. The optimal control problem $\mathbb{P}(x)$ is defined by

$$\min_{u(\cdot)} V_\infty(x,u(\cdot))$$

subject to:

$$\begin{aligned} \dot{x} = f(x,u) \qquad & x(0) = x_0 \\ u(t) \in \mathbb{U} \qquad x(t) \in \mathbb{X} \qquad & \text{for all } t \in (0,\infty) \end{aligned}$$

If $\ell(\cdot)$ is positive definite, the goal of the regulator is to steer the state of the system to the origin.

We denote the solution to this problem (when it exists) and the optimal value function by

$$V_\infty^0(x) \qquad u_\infty^0(\cdot;x)$$

The closed-loop system under this optimal control law evolves as

$$\frac{dx(t)}{dt} = f(x(t), u_\infty^0(t;x))$$

We can demonstrate that the origin is an asymptotically stable solution for the closed-loop system as follows. If $f(\cdot)$ and $\ell(\cdot)$ satisfy certain differentiability and growth assumptions and there are no state constraints, then a solution to $\mathbb{P}(x)$ exists for all $x$; $V_\infty^0(\cdot)$ is differentiable and satisfies

$$\dot{V}_\infty^0(x) = -\ell(x, u_\infty^0(0;x))$$

Using this and upper and lower bounds on $V_\infty^0(\cdot)$ enables global asymptotic stability of the origin to be established.

Although the control law $u_\infty^0(0;\cdot)$ provides excellent closed-loop properties, there are several impediments to its use. A feedback, rather than an open-loop, control is usually necessary because of uncertainty. Solution of the optimal control problem $\mathbb{P}(x)$ yields the optimal control $u_\infty^0(0;x)$ for the state $x$ but does not provide a control law. Dynamic programming may, in principle, be employed, but is generally impractical if the state dimension and the horizon are not small.

If we turn instead to an MPC approach in which we generate online only the value of $u_\infty^0(\cdot;x)$ for the currently measured value of $x$, rather than for all $x$, the problem remains formidable for the following reasons. First, we are optimizing over a time *function*, $u(\cdot)$, and functions are infinite dimensional. Secondly, the time interval of interest, $[0, \infty)$, is a semi-infinite interval, which poses other numerical challenges. Finally, the cost function $V(x, u(\cdot))$ is usually not a convex function of $u(\cdot)$, which presents significant optimization difficulties, especially in an online setting. Even proving existence of the optimal control in this general setting is a challenge.

Our task in this chapter may therefore be viewed as restricting the system and control parameterization to replace problem $\mathbb{P}(x)$ with a more easily computed approximate problem. We show how to pose various approximate problems for which we can establish existence of the optimal solution and asymptotic closed-loop stability of the resulting controller. For these approximate problems, we almost always replace the continuous time differential equation with a discrete time difference equation. We often replace the semi-infinite time interval with a finite time interval and append a terminal region such that we can approximate the cost to go for the semi-infinite interval once the system enters the terminal region. Although the solution of problem

$\mathbb{P}(x)$ in its full generality is out of reach with today's computational methods, its value lies in distinguishing what is *desirable* in the control problem formulation and what is *achievable* with available computing technology.

We develop here MPC for the control of constrained nonlinear time-invariant systems. The nonlinear system is described by the nonlinear difference equation

$$x^+ = f(x,u) \tag{2.1}$$

in which $x \in \mathbb{R}^n$ is the current state, $u$ is the current control, and $x^+$ the successor state; $x^+ = f(x,u)$ is the discrete time analog of the continuous time differential equation $\dot{x} = f(x,u)$. The function $f(\cdot)$ is assumed to be continuous and to satisfy $f(0,0) = 0$, i.e., 0 is an equilibrium point. Any solution $x(\cdot)$ of (2.1), if the initial state is $x(0) = x_0$ and the input (control) is $u(\cdot)$, satisfies

$$x(k+1) = f(x(k), u(k)) \qquad k = 0, 1, \ldots$$

and the initial condition $x(0) = x_0$.

We introduce here some notation that we employ in the sequel. The set $\mathbb{I}$ denotes the set of integers, $\mathbb{I}_{\geq 0} := \{0, 1, 2, \ldots\}$ and, for any two integers $m$ and $n$ satisfying $m \leq n$, $\mathbb{I}_{m:n} := \{m, m+1, \ldots, n\}$. We refer to the pair $(x,i)$ as an event; an event $(x,i)$ denotes that the state at time $i$ is $x$. We use $\mathbf{u}$ to denote the possibly infinite control sequence $\{u(k) \mid k \in \mathbb{I}_{\geq 0}\} = \{u(0), u(1), u(2), \ldots\}$. In the context of MPC, $\mathbf{u}$ frequently denotes the finite sequence $\{u(0), u(1), \ldots, u(N-1)\}$ in which $N$ is the control *horizon*. For any integer $j \in \mathbb{I}_{\geq 0}$, we employ $\mathbf{u}_j$ to denote the finite sequence $\{u(0), u(1), \ldots, u(j-1)\}$. Similarly $\mathbf{x}$ denotes the possibly infinite state sequence $\{x(0), x(1), x(2), \ldots\}$ and $\mathbf{x}_j$ the finite sequence $\{x(0), x(1), \ldots, x(j)\}$. When no confusion can arise we often employ, for simplicity in notation, $\mathbf{u}$ in place of $\mathbf{u}_N$ and $\mathbf{x}$ in place of $\mathbf{x}_N$. Also for simplicity in notation, $\mathbf{u}$, when used in algebraic expressions, denotes the column vector $(u(0)', u(1)', \ldots, u(N-1)')'$; similarly $\mathbf{x}$ in algebraic expressions denotes the column vector $(x(0)', x(1)', \ldots, x(N)')'$.

The solution of (2.1) at time $k$, if the initial state at time 0 is $x$ and the control sequence is $\mathbf{u}$, is denoted by $\phi(k; x, \mathbf{u})$; the solution at time $k$ depends only on $u(0), u(1), \ldots, u(k-1)$. Similarly, the solution of the system (2.1) at time $k$, if the initial state at time $i$ is $x$ and the control sequence is $\mathbf{u}$, is denoted by $\phi(k; (x,i), \mathbf{u})$. Because the system is time invariant, the solution does not depend on the initial time; if the initial state is $x$ at time $i$, the solution at time $j \geq i$ is $\phi(j-i; x, \mathbf{u})$.

Thus the solution at time $k$ if the initial event is $(x, i)$ is identical to the solution at time $k - i$ if the initial event is $(x, 0)$. For each $k$, the function $(x, \mathbf{u}) \mapsto \phi(k; x, \mathbf{u})$ is continuous as we show next.

**Proposition 2.1** (Continuous system solution)**.** *Suppose the function $f(\cdot)$ is continuous. Then, for each integer $k \in \mathbb{I}$, the function $(x, \mathbf{u}) \mapsto \phi(k; x, \mathbf{u})$ is continuous.*

*Proof.*
Since $\phi(1; x, u(0)) = f(x, u(0))$, the function $(x, u(0)) \mapsto \phi(1; x, u(0))$ is continuous. Suppose the function $(x, \mathbf{u}_{j-1}) \mapsto \phi(j; x, \mathbf{u}_{j-1})$ is continuous and consider the function $(x, \mathbf{u}_j) \mapsto \phi(j+1; x, \mathbf{u}_j)$. Since

$$\phi(j+1; x, \mathbf{u}_j) = f(\phi(j; x, \mathbf{u}_{j-1}), u(j))$$

where $f(\cdot)$ and $\phi(j; \cdot)$ are continuous and since $\phi(j+1; \cdot)$ is the composition of two continuous functions $f(\cdot)$ and $\phi(j; \cdot)$, it follows that $\phi(j+1; \cdot)$ is continuous. By induction $\phi(k; \cdot)$ is continuous for any positive integer $k$. ■

The system (2.1) is subject to hard constraints which may take the form

$$u(k) \in \mathbb{U} \quad x(k) \in \mathbb{X} \qquad \text{for all } k \in \mathbb{I}_{\geq 0} \tag{2.2}$$

The constraint (2.2) does not couple $u(k)$ or $x(k)$ at *different* times; constraints that involve the control at several times are avoided by introducing extra states. Thus the common rate constraint $|u(k) - u(k-1)| \leq c$ may be expressed as $|u(k) - z(k)| \leq c$ where $z$ is an extra state variable satisfying the difference equation $z^+ = u$ so that $z(k) = u(k-1)$. The constraint $|u - z| \leq c$ is an example of a *mixed* constraint, i.e., a constraint that involves both states and controls. Hence, a more general constraint formulation of the form

$$y(k) \in \mathbb{Y} \qquad \text{for all } k \in \mathbb{I}_{\geq 0} \tag{2.3}$$

in which the output $y$ satisfies

$$y = h(x, u)$$

is sometimes required. A mixed constraint often is expressed in the form $Fx + Eu \leq e$, and may be regarded as a state dependent control constraint. Because the constraint (2.3) is more general, the constraint (2.2) may be expressed as $y(k) \in \mathbb{Y}$ by an appropriate choice of the output function $h(\cdot)$ and the output constraint set $\mathbb{Y}$ ($y = (x, u)$ and

$\mathbb{Y} = \mathbb{X} \times \mathbb{U}$). We assume in this chapter that the state $x$ is known; if the state $x$ is estimated, uncertainty (state estimation error) is introduced and *robust* MPC, discussed in Chapter 3, is required.

The next ingredient of the optimal control problem is the cost function. Practical considerations require that the cost be defined over a finite horizon $N$ — to ensure the resultant optimal control problem can be solved sufficiently rapidly to permit effective control. We consider initially the regulation problem where the target state is the origin. If $x$ is the current state and $i$ the current time, then the optimal control problem may be posed as minimizing a cost defined over the interval from time $i$ to time $N + i$. The optimal control problem $\mathbb{P}_N(x, i)$ at event $(x, i)$ is the problem of minimizing the cost

$$\sum_{k=i}^{i+N-1} \ell(x(k), u(k)) + V_f(x(N+i))$$

with respect to the sequences $\mathbf{x} := \{x(i), x(i+1), \ldots, x(i+N)\}$ and $\mathbf{u} := \{u(i), u(i+1), \ldots, u(i+N-1)\}$ subject to the constraints that $\mathbf{x}$ and $\mathbf{u}$ satisfy the difference equation (2.1), the initial condition $x(i) = x$, and the state and control constraints (2.2). We assume that $\ell(\cdot)$ is continuous and that $\ell(0, 0) = 0$. The optimal control and state sequences, obtained by solving $\mathbb{P}_N(x, i)$, are functions of the initial event $(x, i)$

$$\mathbf{u}^0(x, i) = \{u^0(i; (x, i)), u^0(i+1; (x, i)), \ldots, u^0(i+N-1; (x, i))\}$$
$$\mathbf{x}^0(x, i) = \{x^0(i; (x, i)), x^0(i+1; (x, i)), \ldots, x^0(i+N; (x, i))\}$$

where $x^0(i; (x, i)) = x$. In MPC, the first control action $u^0(i; (x, i))$ in the optimal control sequence $\mathbf{u}^0(x, i)$ is applied to the plant, i.e., $u(i) = u^0(i; (x, i))$. Because the system $x^+ = f(x, u)$, the stage cost $\ell(\cdot)$, and the terminal cost $V_f(\cdot)$ are all time invariant, however, the solution of $\mathbb{P}_N(x, i)$, for any time $i \in \mathbb{I}_{\geq 0}$, is identical to the solution of $\mathbb{P}_N(x, 0)$ so that

$$\mathbf{u}^0(x, i) = \mathbf{u}^0(x, 0)$$
$$\mathbf{x}^0(x, i) = \mathbf{x}^0(x, 0)$$

In particular, $u^0(i; (x, i)) = u^0(0; (x, 0))$, i.e., the control $u^0(i; (x, i))$ applied to the plant is equal to $u^0(0; (x, 0))$, the first element in the sequence $\mathbf{u}^0(x, 0)$. Hence we may as well merely consider problem $\mathbb{P}_N(x, 0)$ which, since the initial time is irrelevant, we call $\mathbb{P}_N(x)$. Similarly, for simplicity in notation, we replace $\mathbf{u}^0(x, 0)$ and $\mathbf{x}^0(x, 0)$ by, respectively, $\mathbf{u}^0(x)$ and $\mathbf{x}^0(x)$.

The optimal control problem $\mathbb{P}_N(x)$ may then be expressed as minimization of

$$\sum_{k=0}^{N-1} \ell(x(k), u(k)) + V_f(x(N))$$

with respect to the *decision variables* $(\mathbf{x}, \mathbf{u})$ subject to the constraints that the state and control sequences $\mathbf{x}$ and $\mathbf{u}$ satisfy the difference equation (2.1), the initial condition $x(0) = x$, and the state and control constraints (2.2). Here $\mathbf{u}$ denotes the control sequence $\{u(0), u(1), \ldots, u(N-1)\}$ and $\mathbf{x}$ the state sequence $\{x(0), x(1), \ldots, x(N)\}$. Retaining the state sequence in the set of decision variables is discussed in Chapter 6. For the purpose of analysis, however, it is preferable to constrain the state sequence $\mathbf{x}$ *a priori* to be a solution of $x^+ = f(x, u)$ enabling us to express the problem in the equivalent form of minimizing, with respect to the decision variable $\mathbf{u}$, a cost that is purely a function of the initial state $x$ and the control sequence $\mathbf{u}$. This formulation is possible since the state sequence $\mathbf{x}$ may be expressed, via the difference equation $x^+ = f(x, u)$, as a function of $(x, \mathbf{u})$. The cost becomes $V_N(x, \mathbf{u})$ defined by

$$V_N(x, \mathbf{u}) := \sum_{k=0}^{N-1} \ell(x(k), u(k)) + V_f(x(N)) \tag{2.4}$$

where, now, $x(k) := \phi(k; x, \mathbf{u})$ for all $k \in \mathbb{I}_{0:N}$. Similarly the constraints (2.2), together with an additional terminal constraint

$$x(N) \in \mathbb{X}_f$$

where $\mathbb{X}_f \subseteq \mathbb{X}$, impose an implicit constraint on the control sequence of the form

$$\mathbf{u} \in \mathcal{U}_N(x) \tag{2.5}$$

in which the control constraint set $\mathcal{U}_N(x)$ is the set of control sequences $\mathbf{u} := \{u(0), u(1), \ldots, u(N-1)\}$ satisfying the state and control constraints. It is therefore defined by

$$\mathcal{U}_N(x) := \{\mathbf{u} \mid (x, \mathbf{u}) \in \mathbb{Z}_N\} \tag{2.6}$$

in which the set $\mathbb{Z}_N \subset \mathbb{R}^n \times \mathbb{R}^{Nm}$ is defined by

$$\mathbb{Z}_N := \{(x, \mathbf{u}) \mid u(k) \in \mathbb{U}, \quad \phi(k; x, \mathbf{u}) \in \mathbb{X}, \quad \forall k \in \mathbb{I}_{0:N-1}, \text{ and } \phi(N; x, \mathbf{u}) \in \mathbb{X}_f\} \tag{2.7}$$

The optimal control problem $\mathbb{P}_N(x)$, is, therefore

$$\mathbb{P}_N(x): \qquad V_N^0(x) := \min_{\mathbf{u}} \{V_N(x,\mathbf{u}) \mid \mathbf{u} \in \mathcal{U}_N(x)\} \tag{2.8}$$

Problem $\mathbb{P}_N(x)$ is a *parametric* optimization problem in which the decision variable is $\mathbf{u}$, and both the cost and the constraint set depend on the *parameter* $x$. The set $\mathbb{Z}_N$ is the set of admissible $(x,\mathbf{u})$, i.e., the set of $(x,\mathbf{u})$ for which $x \in \mathbb{X}$ and the constraints of $\mathbb{P}_N(x)$ are satisfied. Let $\mathcal{X}_N$ be the set of states in $\mathbb{X}$ for which $\mathbb{P}_N(x)$ has a solution

$$\mathcal{X}_N := \{x \in \mathbb{X} \mid \mathcal{U}_N(x) \neq \emptyset\} \tag{2.9}$$

It follows from (2.8) and (2.9) that

$$\mathcal{X}_N = \{x \in \mathbb{R}^n \mid \exists \mathbf{u} \in \mathbb{R}^{Nm} \text{ such that } (x,\mathbf{u}) \in \mathbb{Z}_N\}$$

which is the orthogonal projection of $\mathbb{Z}_N \subset \mathbb{R}^n \times \mathbb{R}^{Nm}$ onto $\mathbb{R}^n$. The domain of $V_N^0(\cdot)$, i.e., the set of states in $\mathbb{X}$ for which $\mathbb{P}_N(x)$ has a solution, is $\mathcal{X}_N$.

Not every optimization problem has a solution. For example, the problem $\min\{x \mid x \in (0,1)\}$ does not have a solution; $\inf\{x \mid x \in (0,1)\} = 0$ but $x = 0$ does not lie in the constraint set $(0,1)$. By Weierstrass's theorem, however, an optimization problem does have a solution if the cost is continuous (in the decision variable) and the constraint set compact (see Proposition A.7). This is the case for our problem as shown subsequently in Proposition 2.4. We assume, without further comment, that the following standing conditions are satisfied in the sequel.

**Assumption 2.2** (Continuity of system and cost)**.** The functions $f : \mathbb{X} \times \mathbb{U} \to \mathbb{R}^n$, $\ell : \mathbb{X} \times \mathbb{U} \to \mathbb{R}_{\geq 0}$ and $V_f : \mathbb{X}_f \to \mathbb{R}_{\geq 0}$ are continuous, $f(0,0) = 0$, $\ell(0,0) = 0$ and $V_f(0) = 0$.

**Assumption 2.3** (Properties of constraint sets)**.** The sets $\mathbb{X}$ and $\mathbb{X}_f$ are closed, $\mathbb{X}_f \subseteq \mathbb{X}$ and $\mathbb{U}$ are compact; each set contains the origin.

The sets $\mathbb{U}$, $\mathbb{X}$ and $\mathbb{X}_f$ are assumed to contain the origin because the first problem we tackle is regulation to the origin. This assumption is modified when we consider the tracking problem.

**Proposition 2.4** (Existence of solution to optimal control problem)**.** *Suppose Assumptions 2.2 and 2.3 hold. Then*

*(a) The function $V_N(\cdot)$ is continuous in $\mathbb{Z}_N$.*

*(b) For each $x \in \mathcal{X}_N$, the control constraint set $\mathcal{U}_N(x)$ is compact.*

*(c) For each $x \in \mathcal{X}_N$, a solution to $\mathbb{P}_N(x)$ exists.*

*Proof.*

(a) That $(x, \mathbf{u}) \mapsto V_N(x, \mathbf{u})$ is continuous follows from continuity of $\ell(\cdot)$ and $V_f(\cdot)$ in Assumption 2.2, and the continuity of $(x, \mathbf{u}) \mapsto \phi(j; x, \mathbf{u})$ for each $j \in \mathbb{I}_{0:N-1}$, established in Proposition 2.1.

(b) We have to show that for each $x \in \mathcal{X}_N$, the set $\mathcal{U}_N(x)$ is closed and bounded. It is clearly bounded since $\mathcal{U}_N(x) \subseteq \mathbb{U}^N$, which is compact (bounded and closed) by Assumption 2.3. By Proposition 2.1, the function $\phi(j; \cdot)$ is continuous for any $j \in \mathbb{I}_{0:N}$. Since $\mathbb{U}$, $\mathbb{X}$ and $\mathbb{X}_f$ are all closed, any sequence $\{(x_i, \mathbf{u}_i)\}$ in $\mathbb{Z}_N$, defined in (2.7), that converges to, say, $(\bar{x}, \bar{\mathbf{u}})$ satisfies $\phi(j; \bar{x}, \bar{\mathbf{u}}) \in \mathbb{X}$ for all $j \in \mathbb{I}_{0:N-1}$, $\phi(N; \bar{x}, \bar{\mathbf{u}}) \in \mathbb{X}_f$ and $\bar{\mathbf{u}} \in \mathbb{U}^N$. Hence $(\bar{x}, \bar{\mathbf{u}}) \in \mathbb{Z}_N$ so that $\mathbb{Z}_N$ is closed. It follows that $\mathcal{U}_N(x) = \{\mathbf{u} \mid (x, \mathbf{u}) \in \mathbb{Z}_N\}$ is closed and, therefore, compact for all $x \in \mathcal{X}_N$.

(c) Since $V_N(x, \cdot)$ is continuous and $\mathcal{U}_N(x)$ is compact, by Weierstrass's theorem (Proposition A.7) a solution to $\mathbb{P}_N(x)$ exists for each $x \in \mathcal{X}_N$. ■

Although the function $(x, \mathbf{u}) \mapsto V_N(x, \mathbf{u})$ is continuous, the function $x \mapsto V_N^0(x)$ is not necessarily continuous; we discuss this possibility and its implications later. For each $x \in \mathcal{X}_N$, the solution of $\mathbb{P}_N(x)$ is

$$\mathbf{u}^0(x) = \arg\min_{\mathbf{u}} \{V_N(x, \mathbf{u}) \mid \mathbf{u} \in \mathcal{U}_N(x)\}$$

If $\mathbf{u}^0(x) = \{u^0(0; x), u^0(1; x), \ldots, u^0(N-1; x)\}$ is unique for each $x \in \mathcal{X}_N$, then $\mathbf{u}^0 : \mathbb{R}^n \to \mathbb{R}^{Nm}$ is a function; otherwise it is a set-valued function.[1] In MPC, the control applied to the plant is the first element $u^0(0; x)$ of the optimal control sequence. At the next sampling instant, the procedure is repeated for the successor state. Although MPC computes $\mathbf{u}^0(x)$ only for specific values of the state $x$, it could, in principle, be used to compute $\mathbf{u}^0(x)$ and, hence, $u^0(0; x)$ for every $x$ for which $\mathbb{P}_N(x)$ is feasible, yielding the implicit MPC control law $\kappa_N(\cdot)$ defined by

$$\kappa_N(x) := u^0(0; x), \qquad x \in \mathcal{X}_N$$

[1] A set-valued function $\phi(\cdot)$ is a function whose value $\phi(x)$ for each $x$ in its domain is a set.

MPC does *not* require determination of the control law $\kappa_N(\cdot)$, a task that is usually intractable when constraints or nonlinearities are present; it is this fact that makes MPC so useful.

If, at a given state $x$, the solution of $\mathbb{P}_N(x)$ is not unique, then $\kappa_N(\cdot) = u^0(0;\cdot)$ is set valued and the model predictive controller selects one element from the set $\kappa_N(x)$.

**Example 2.5: Linear quadratic MPC**

Suppose the system is described by

$$x^+ = f(x,u) := x + u$$

with initial state $x$. The stage cost and terminal cost are

$$\ell(x,u) := (1/2)(x^2 + u^2) \qquad V_f(x) := (1/2)x^2$$

The control constraint is

$$u \in [-1,1]$$

and there are no state or terminal constraints. Suppose the horizon is $N = 2$. Under the first approach, the decision variables are $\mathbf{u}$ and $\mathbf{x}$, and the optimal control problem is minimization of

$$V_N(x(0),x(1),x(2),u(0),u(1)) = (1/2)\left(x(0)^2 + x(1)^2 + x(2)^2 + u(0)^2 + u(1)^2\right)$$

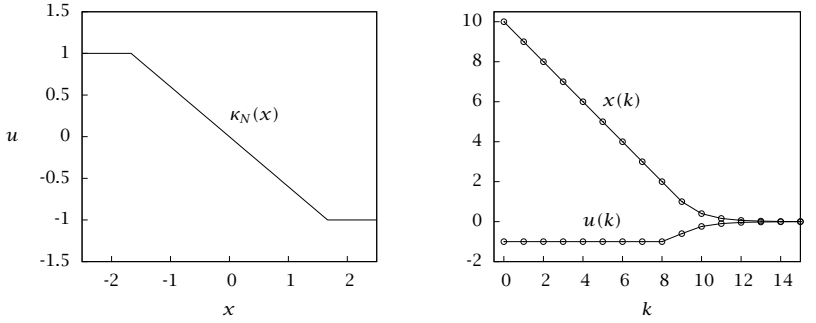
with respect to $(x(0),x(1),x(2))$, and $(u(0),u(1))$ subject to the following constraints

$$x(0) = x \qquad x(1) = x(0) + u(0) \qquad x(2) = x(1) + u(1)$$
$$u(0) \in [-1,1] \qquad u(1) \in [-1,1]$$

The constraint $u \in [-1,1]$ is equivalent to two inequality constraints, $u \le 1$ and $-u \le 1$. The first three constraints are equality constraints enforcing satisfaction of the difference equation.

In the second approach, the decision variable is merely $\mathbf{u}$ because the first three constraints are automatically enforced by requiring $\mathbf{x}$ to be a solution of the difference equation. Hence, the optimal control problem becomes minimization with respect to $\mathbf{u} = (u(0)', u(1))'$ of

$$\begin{aligned} V_N(x,\mathbf{u}) &= (1/2)(x^2 + (x + u(0))^2 + (x + u(0) + u(1))^2 + \\ &\qquad u(0)^2 + u(1)^2) \\ &= (3/2)x^2 + \begin{bmatrix} 2x & x \end{bmatrix}\mathbf{u} + (1/2)\mathbf{u}'H\mathbf{u} \end{aligned}$$




(a) Implicit MPC control law. (b) Trajectories of controlled system.

**Figure 2.1:** Example of MPC.

in which

$$H = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

subject to the constraint $\mathbf{u} \in \mathcal{U}_N(x)$ where

$$\mathcal{U}_N(x) = \{\mathbf{u} \mid |u(k)| \leq 1 \;\; k = 0, 1\}$$

Because there are no state or terminal constraints, the set $\mathcal{U}_N(x) = \mathcal{U}_N$ for this example does not depend on the parameter $x$; often it does. Both optimal control problems are quadratic programs.[2] The solution for $x = 10$ is $u^0(1;10) = u^0(2;10) = -1$ so the optimal state trajectory is $x^0(0;10) = 10$, $x^0(1;10) = 9$ and $x^0(2;10) = 8$. The value $V_N^0(10) = 124$. By solving $\mathbb{P}_N(x)$ for every $x \in [-10, 10]$, the optimal control law $\kappa_N(\cdot)$ on this set can be determined, and is shown in Figure 2.1(a). The implicit MPC control law is *time invariant* since the system being controlled, the cost, and the constraints are all time invariant. For our example, the controlled system (the system with MPC) satisfies the difference equation

$$x^+ = x + \kappa_N(x) \qquad \kappa_N(x) = -\text{sat}((3/5)x)$$

and the state and control trajectories for an initial state of $x = 10$ are shown in Figure 2.1(b). It turns out that the origin is exponentially stable for this simple case; often, however, the terminal cost and terminal constraint set have to be carefully chosen to ensure stability. □

[2] A quadratic program is an optimization problem in which the cost is quadratic and the constraint set is polyhedral, i.e., defined by linear inequalities.

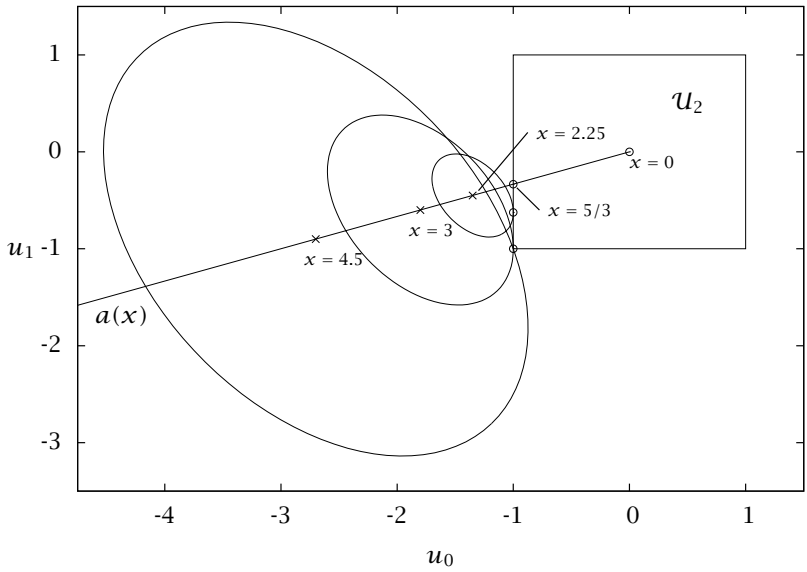



**Figure 2.2:** Feasible region $\mathcal{U}_2$, elliptical cost contours, and ellipse center, $a(x)$, and constrained minimizers for different values of $x$.

**Example 2.6: Closer inspection of linear quadratic MPC**

We revisit the MPC problem discussed in Example 2.5. The objective function is

$$V_N(x, \mathbf{u}) = (1/2)\mathbf{u}'H\mathbf{u} + c(x)'\mathbf{u} + d(x)$$

where $c(x)' = [2\ 1]x$ and $d(x) = (3/2)x^2$. The objective function may be written in the form

$$V_N(x, \mathbf{u}) = (1/2)(\mathbf{u} - a(x))'H(\mathbf{u} - a(x)) + e(x)$$

Expanding the second form shows the two forms are equal if

$$a(x) = -H^{-1}c(x) = K_1 x \qquad K_1 = -(1/5)\begin{bmatrix}3\\1\end{bmatrix}$$

and

$$e(x) + (1/2)a(x)'Ha(x) = d(x)$$

Since $H$ is positive definite, $a(x)$ is the unconstrained minimizer of the objective function; indeed $\nabla_{\mathbf{u}} V_N(x, a(x)) = 0$ since

$$\nabla_{\mathbf{u}} V_N(x, \mathbf{u}) = H\mathbf{u} + c(x)$$

The locus of $a(x)$ for $x \geq 0$ is shown in Figure 2.2. Clearly the unconstrained minimizer $a(x) = K_1 x$ is equal to the constrained minimizer $\mathbf{u}^0(x)$ for all $x$ such that $a(x) \in \mathcal{U}_2$ where $\mathcal{U}_2$ is the unit square illustrated in Figure 2.2; since $a(x) = K_1 x$, $a(x) \in \mathcal{U}_2$ for all $x \in \mathbb{X}_1 = [0, x_{c1}]$ where $x_{c1} = 5/3$. For $x > x_{c1}$, the unconstrained minimizer lies outside $\mathcal{U}_2$ as shown in Figure 2.2 for $x = 2.25$, $x = 3$ and $x = 5$. For such $x$, the constrained minimizer $\mathbf{u}^0(x)$ is a point that lies on the intersection of a level set of the objective function (which is an ellipse) and the boundary of $\mathcal{U}_2$. For $x \in [x_{c1}, x_{c2})$, $\mathbf{u}^0(x)$ lies on the left face of the box $\mathcal{U}_2$ and for $x \geq x_{c2} = 3$, $\mathbf{u}^0(x)$ remains at $(-1, -1)$, the bottom left vertex of $\mathcal{U}_2$.

When $u^0(x)$ lies on the left face of $\mathcal{U}_2$, the gradient $\nabla_{\mathbf{u}} V_N(x, \mathbf{u}^0(x))$ of the objective function is normal to the left face of $\mathcal{U}_2$, i.e., the level set of $V_N^0(\cdot)$ passing through $\mathbf{u}^0(x)$ is tangential to the left face of $\mathcal{U}_2$. The outward normal to $\mathcal{U}_2$ at a point on the left face is $-e_1 = (-1, 0)$ so that at $\mathbf{u} = \mathbf{u}^0(x)$

$$\nabla_{\mathbf{u}} V(x, \mathbf{u}^0(x)) + \lambda(-e_1) = 0$$

for some $\lambda > 0$; this is a standard condition of optimality. Since $\mathbf{u} = [-1 \ v]'$ for some $v \in [-1, 1]$ and since $\nabla_{\mathbf{u}} V(x, \mathbf{u}) = H(\mathbf{u} - a(x)) = H\mathbf{u} + c(x)$, the condition of optimality is

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ v \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} x - \begin{bmatrix} \lambda \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or

$$\begin{aligned} -3 + v + 2x - \lambda &= 0 \\ -1 + 2v + x &= 0 \end{aligned}$$

which, when solved, yields $v = (1/2) - (1/2)x$ and $\lambda = -(5/2) + (3/2)x$. Hence,

$$\mathbf{u}^0(x) = b_2 + K_2 x \qquad b_2 = \begin{bmatrix} -1 \\ (1/2) \end{bmatrix} \qquad K_2 = \begin{bmatrix} 0 \\ -(1/2) \end{bmatrix}$$

for all $x \in \mathbb{X}_2 = [x_{c1}, x_{c2}]$ where $x_{c2} = 3$ since $\mathbf{u}^0(x) \in \mathcal{U}_2$ for all $x$ in this range. For all $x \in \mathbb{X}_3 = [x_{c_2}, \infty)$, $\mathbf{u}^0(x) = (-1, -1)'$. Summarizing:

$$\begin{aligned} x \in [0, (5/3)] &\implies \mathbf{u}^0(x) = K_1 x \\ x \in [(5/3), 3] &\implies \mathbf{u}^0(x) = K_2 x + b_2 \\ x \in [3, \infty) &\implies \mathbf{u}^0(x) = b_3 \end{aligned}$$

in which

$$K_1 = \begin{bmatrix} -(3/5) \\ -(1/5) \end{bmatrix} \qquad K_2 = \begin{bmatrix} 0 \\ -(1/2) \end{bmatrix} \qquad b_2 = \begin{bmatrix} -1 \\ (1/2) \end{bmatrix} \qquad b_3 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

The optimal control for $x \leq 0$ may be obtained by symmetry; $\mathbf{u}^0(-x) = -\mathbf{u}^0(x)$ for all $x \geq 0$ so that:

$$\begin{aligned} x \in [0, -(5/3)] &\implies \mathbf{u}^0(x) = -K_1 x \\ x \in [-(5/3), -3] &\implies \mathbf{u}^0(x) = -K_2 x - b_2 \\ x \in [-3, -\infty) &\implies \mathbf{u}^0(x) = -b_3 \end{aligned}$$

It is easily checked that $\mathbf{u}^0(\cdot)$ is continuous and satisfies the constraint for all $x \in \mathbb{R}$. The MPC control law $\kappa_N(\cdot)$ is the first component of $\mathbf{u}^0(\cdot)$ and, therefore, is defined by:

$$\begin{aligned} \kappa_N(x) &= 1 && x \leq -3 \\ \kappa_N(x) &= 1 && x \in [-(5/3), -3] \\ \kappa_N(x) &= -(3/5)x && x \in [-(5/3), (5/3)] \\ \kappa_N(x) &= -1 && x \in [(5/3), 3] \\ \kappa_N(x) &= -1 && x \geq 3 \end{aligned}$$

i.e., $\kappa_N(x) = -\text{sat}((3/5)x)$ which is the saturating control law depicted in Figure 2.1(a). The control law is piecewise affine and the value function piecewise quadratic. The structure of the solution to constrained linear quadratic optimal control problems is explored more fully in Chapter 7. □

As we show in Chapter 3, continuity of the value function is desirable. Unfortunately, this is not true in general; the major difficulty is in establishing that the set-valued function $x \mapsto \mathcal{U}_N(x)$ has certain continuity properties. Continuity of the value function $V_N^0(\cdot)$ and of the implicit control law $\kappa_N(\cdot)$ may be established for a few important cases, however, as is shown by the next result, which assumes satisfaction of our standing assumptions: 2.2 and 2.3 so that the cost function $V_N(\cdot)$ is continuous in $(x, \mathbf{u})$.

**Theorem 2.7** (Continuity of value function and control law)**.** *Suppose that Assumptions 2.2 and 2.3 hold.*

*(a) Suppose that there are no state constraints so that* $\mathbb{X} = \mathbb{X}_f = \mathbb{R}^n$*. Then the value function* $V_N^0 : \mathcal{X}_N \to \mathbb{R}$ *is continuous and* $\mathcal{X}_N = \mathbb{R}^n$*.*

*(b) Suppose* $f(\cdot)$ *is linear (*$x^+ = Ax + Bu$*) and that the state and control constraints sets* $\mathbb{X}$ *and* $\mathbb{U}$ *are polyhedral.*[3] *Then the value function* $V_N^0 : \mathcal{X}_N \to \mathbb{R}$ *is continuous.*

*(c) If, in addition, the solution* $\mathbf{u}^0(x)$ *of* $\mathbb{P}_N(x)$ *is unique at each* $x \in \mathcal{X}_N$*, then the implicit MPC control law* $\kappa_N(\cdot)$ *is continuous.*

The proof of this theorem is given in Section C.3 of Appendix C. The following example, due to Meadows, Henson, Eaton, and Rawlings (1995), shows that there exist nonlinear examples where the value function and implicit control law are not continuous.

**Example 2.8: Discontinuous MPC control law**

Consider the nonlinear system defined by

$$\begin{aligned} x_1^+ &= x_1 + u \\ x_2^+ &= x_2 + u^3 \end{aligned}$$

The control horizon is $N = 3$ and the cost function $V_3(\cdot)$ is defined by

$$V_3(x, \mathbf{u}) := \sum_{k=0}^{2} \ell(x(k), u(k))$$

and the stage cost $\ell(\cdot)$ is defined by

$$\ell(x, u) := |x|^2 + u^2$$

The constraint sets are $\mathbb{X} = \mathbb{R}^2$, $\mathbb{U} = \mathbb{R}$, and $\mathbb{X}_f := \{0\}$, i.e., there are no state and control constraints, and the terminal state must satisfy the constraint $x(3) = 0$. Hence, although there are three control actions, $u(0)$, $u(1)$, and $u(2)$, two must be employed to satisfy the terminal constraint, leaving only one degree of freedom. Choosing $u(0)$ to be the free decision variable automatically constrains $u(1)$ and $u(2)$ to be functions of the initial state $x$ and the first control action $u(0)$. Solving

[3] A set $\mathbb{X}$ is polyhedral if it may be defined as set of linear inequalities, i.e., if it may be expressed in the form $\mathbb{X} = \{x \mid Mx \leq m\}$.

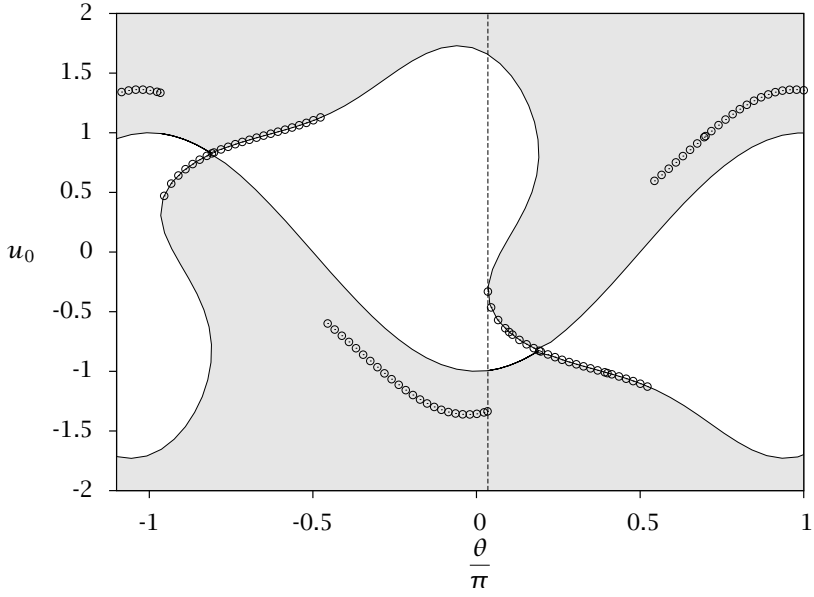



**Figure 2.3:** First element of control constraint set $\mathcal{U}_3(x)$ (shaded) and control law $\kappa_3(x)$ (circle) versus $x = (\cos(\theta), \sin(\theta))$, $\theta \in [-\pi, \pi]$ on the unit circle for a nonlinear system with terminal constraint.

the equation

$$\begin{aligned} x_1(3) &= x_1 + u(0) + u(1) + u(2) \quad = 0 \\ x_2(3) &= x_2 + u(0)^3 + u(1)^3 + u(2)^3 = 0 \end{aligned}$$

for $u(1)$ and $u(2)$ yields

$$\begin{aligned} u(1) &= -x_1/2 - u(0)/2 \pm \sqrt{b} \\ u(2) &= -x_1/2 - u(0)/2 \mp \sqrt{b} \end{aligned}$$

in which

$$b = \frac{3u(0)^3 - 3u(0)^2 x_1 - 3u(0)x_1^2 - x_1^3 + 4x_2}{12(u(0) + x_1)}$$

Clearly a real solution exists only if $b$ is positive, i.e., if both the numerator and denominator in the expression for $b$ have the same sign. The

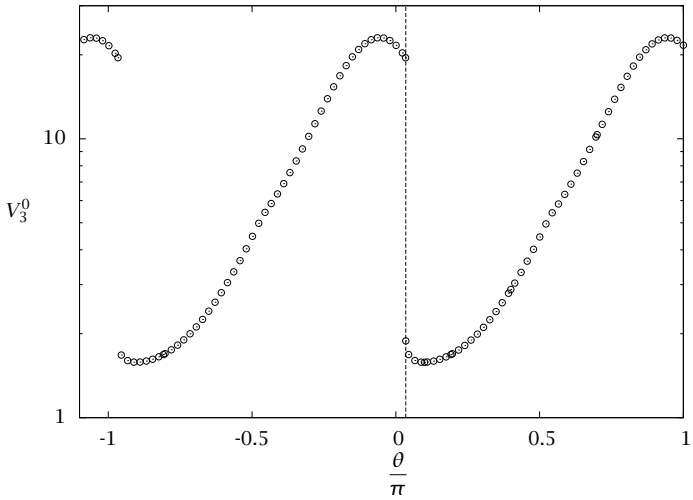



**Figure 2.4:** Optimal cost $V_3^0(x)$ versus $x = (\cos(\theta), \sin(\theta))$, $\theta \in [-\pi, \pi]$ on the unit circle; the discontinuity in $V_3^0$ is caused by the discontinuity in $\mathcal{U}_3$ as $\theta$ crosses the dashed line in Figure 2.3.

optimal control problem $\mathbb{P}_3(x)$ is defined by

$$V_3^0(x) = \min_{\mathbf{u}} \{V_3(x, \mathbf{u}) \mid \phi(3; x, \mathbf{u}) = 0\}$$

and the implicit MPC control law is $\kappa_3(\cdot)$ where $\kappa_3(x) = u^0(0; x)$, the first element in the minimizing sequence $\mathbf{u}^0(x)$. It can be shown, using analysis presented later in this chapter, that the origin is asymptotically stable for the controlled system $x^+ = f(x, \kappa_N(x))$. That this control law is necessarily discontinuous may be shown as follows. If the control is strictly positive, any trajectory originating in the first quadrant ($x_1, x_2 > 0$) moves away from the origin. If the control is strictly negative, any control originating in the third quadrant ($x_1, x_2 < 0$) also moves away from the origin. But the control cannot be zero at any nonzero point lying in the domain of attraction. If it were, this point would be a fixed point for the controlled system, contradicting the fact that it lies in the domain of attraction.

In fact, both the value function $V_3^0(\cdot)$ and the MPC control law $\kappa_3(\cdot)$ are discontinuous. Figures 2.3 and 2.4 show how $\mathcal{U}_3(x)$, $\kappa_3(x)$, and $V_3^0(x)$ vary as $x = (\cos(\theta), \sin(\theta))$ ranges over the unit circle. A further

conclusion that can be drawn from this example is that it is possible for the MPC control law to be discontinuous at points where the value function is continuous. □

## 2.3 Dynamic Programming Solution

We examine next the DP solution of the optimal control problem $\mathbb{P}_N(x)$, not because it provides a practical procedure but because of the insight it provides. DP can rarely be used for constrained and/or nonlinear control problems unless the state dimension $n$ is small. MPC is best regarded as a practical means of implementing the DP solution; for a given state $x$ it provides $V_N^0(x)$ and $\kappa_N(x)$, the value, respectively, of the value function and control law at a *point* $x$. DP, on the other hand, yields the value function $V_N^0(\cdot)$ and the implicit MPC control law $\kappa_N(\cdot)$.

The optimal control problem $\mathbb{P}_N(x)$ is defined, as before, by (2.8) with the cost function $V_N(\cdot)$ defined by (2.4) and the constraints by (2.5). DP yields an optimal policy $\boldsymbol{\mu}^0 = \{\mu_0^0(\cdot), \mu_1^0(\cdot), \ldots, \mu_{N-1}^0(\cdot)\}$, i.e., a sequence of control laws $\mu_i : \mathcal{X}_i \to \mathbb{U}$, $i = 0, 1, \ldots, N-1$. The domain $\mathcal{X}_i$ of each control law will be defined later. The optimal controlled system is time varying and satisfies

$$x^+ = f(x, \mu_i^0(x)),\ i = 0, 1, \ldots, N-1$$

in contrast with the system using MPC, which is time invariant and satisfies

$$x^+ = f(x, \kappa_N(x)),\ i = 0, 1, \ldots, N-1$$

where $\kappa_N(\cdot) = \mu_0^0(\cdot)$. The optimal control law at time $i$ is $\mu_i^0(\cdot)$ whereas receding horizon control (RHC) uses the time-invariant control law $\kappa_N(\cdot)$ obtained by assuming that at each time $t$, the terminal time or *horizon* is $t + N$ so that the horizon $t + N$ recedes as $t$ increases. One consequence is that the time-invariant control law $\kappa_N(\cdot)$ is *not* optimal for the problem of controlling $x^+ = f(x, u)$ over the fixed interval $[0, T]$ in such a way as to minimize $V_N$ and satisfy the constraints.

For all $j \in \mathbb{I}_{0:N-1}$, let $V_j(x, \mathbf{u})$, $\mathcal{U}_j(x)$, $\mathbb{P}_j(x)$, and $V_j^0(x)$ be defined, respectively, by (2.4), (2.5), (2.6), and (2.7), with $N$ replaced by $j$. As shown in Section C.1 of Appendix C, DP solves not only $\mathbb{P}_N(x)$ for all $x \in \mathcal{X}_N$, the domain of $V_N^0(\cdot)$, but also $\mathbb{P}_j(x)$ for all $x \in \mathcal{X}_j$, the domain

of $V_j^0(\cdot)$, all $j \in \mathbb{I}_{0:N-1}$. The DP equations are

$$V_j^0(x) = \min_{u\in\mathbb{U}}\{\ell(x,u) + V_{j-1}^0(f(x,u)) \mid f(x,u) \in \mathcal{X}_{j-1}\},\ \forall x \in \mathcal{X}_j \tag{2.10}$$

$$\kappa_j(x) = \arg\min_{u\in\mathbb{U}}\{\ell(x,u) + V_{j-1}^0(f(x,u)) \mid f(x,u) \in \mathcal{X}_{j-1}\},\ \forall x \in \mathcal{X}_j \tag{2.11}$$

$$\mathcal{X}_j = \{x \in \mathbb{X} \mid \exists u \in \mathbb{U} \text{ such that } f(x,u) \in \mathcal{X}_{j-1}\} \tag{2.12}$$

for $j = 1, 2, \ldots, N$ ($j$ is *time to go*), with terminal conditions

$$V_0^0(x) = V_f(x)\ \forall x \in \mathcal{X}_0 \qquad \mathcal{X}_0 = \mathbb{X}_f$$

For each $j$, $V_j^0(x)$ is the optimal cost for problem $\mathbb{P}_j(x)$ if the current state is $x$, current time is 0 (or $i$), and the terminal time is $j$ (or $i+j$), and $\mathcal{X}_j$ is its domain; $\mathcal{X}_j$ is also the set of states in $\mathbb{X}$ that can be steered to the terminal set $\mathbb{X}_f$ in $j$ steps by an *admissible* control sequence, i.e., a control sequence that satisfies the control, state, and terminal constraints and, therefore, lies in the set $\mathcal{U}_j(x)$. Hence, for each $j$

$$\mathcal{X}_j = \{x \in \mathbb{X} \mid \mathcal{U}_j(x) \neq \emptyset\}$$

**Definition 2.9** (Feasible preimage of the state)**.** Let $\mathbb{Z} := \mathbb{X} \times \mathbb{U}$. The set-valued function $f_{\mathbb{Z}}^{-1} : \mathbb{X} \to \mathbb{Z}$ is defined by

$$f_{\mathbb{Z}}^{-1}(x) := f^{-1}(x) \cap \mathbb{Z}$$

in which

$$f^{-1}(x) := \{z \in \mathbb{R}^n \times \mathbb{R}^m \mid f(z) = x\}$$

For all $j \geq 0$, let the set $\mathcal{Z}_j \subseteq \mathbb{R}^n \times \mathbb{R}^m$ be defined by

$$\mathcal{Z}_j := f_{\mathbb{Z}}^{-1}(\mathcal{X}_{j-1}) = \{(x,u) \mid f(x,u) \in \mathcal{X}_{j-1}\} \cap \mathbb{Z}$$

The set $\mathcal{X}_j$ may then be expressed as

$$\mathcal{X}_j = \{x \in \mathbb{R}^n \mid \exists u \in \mathbb{R}^m \text{ such that } (x,u) \in \mathcal{Z}_j\}$$

i.e., $\mathcal{X}_j$ is the orthogonal projection of $\mathcal{Z}_j \subseteq \mathbb{R}^n \times \mathbb{R}^m$ onto $\mathbb{R}^n$.

DP yields much more than an optimal control sequence for a given initial state; it yields an optimal feedback *policy* $\boldsymbol{\mu}^0$ or sequence of control laws where

$$\boldsymbol{\mu}^0 := \{\mu_0(\cdot), \mu_1(\cdot), \ldots, \mu_{N-1}(\cdot)\} = \{\kappa_N(\cdot), \kappa_{N-1}(\cdot), \ldots, \kappa_1(\cdot)\}$$

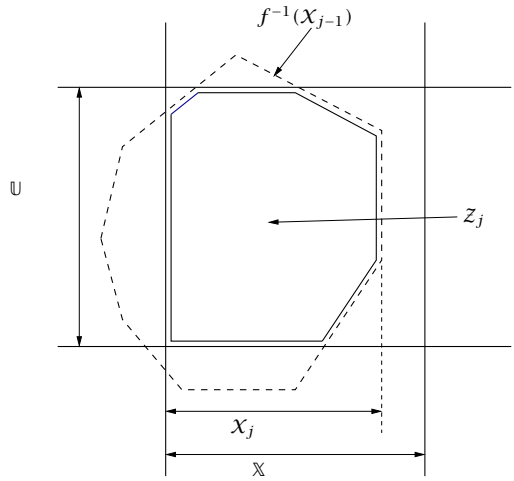



**Figure 2.5:** The sets $\mathcal{Z}_j$ and $\mathcal{X}_j$.

At event $(x, i)$, i.e., at state $x$ at time $i$, the time to go is $N - i$ and the optimal control is

$$\mu_i^0(x) = \kappa_{N-i}(x)$$

i.e., $\mu_i(\cdot)$ is the control law at time $i$. Consider an initial *event* $(x, 0)$, i.e., state $x$ at time 0. If the terminal time (horizon) is $N$, the optimal control for $(x, 0)$ is $\kappa_N(x)$. The successor state, at time 1, is

$$x^+ = f(x, \kappa_N(x))$$

At event $(x^+, 1)$, the time to go to the terminal time is $N - 1$ and the optimal control is $\kappa_{N-1}(x^+) = \kappa_{N-1}(f(x, \kappa_N(x)))$. For a given initial event $(x, 0)$, the optimal policy generates the optimal state and control trajectories $\mathbf{x}^0(x)$ and $\mathbf{u}^0(x)$ that satisfy the difference equations

$$x(0) = x \qquad u(0) = \kappa_N(x) \tag{2.13}$$

$$x(i+1) = f(x(i), u(i)) \qquad u(i) = \kappa_{N-i}(x(i)) \tag{2.14}$$

for $i = 0, 1, \ldots, N-1$. These state and control trajectories are identical to those obtained, as in MPC, by solving $\mathbb{P}_N(x)$ directly for the particular initial event $(x, 0)$ using a mathematical programming algorithm. Dynamic programming, however, provides a solution for *any* event $(x, i)$ such that $i \in \mathbb{I}_{0:N-1}$ and $x \in \mathcal{X}_i$.

Optimal control, in the classic sense of determining a control that minimizes a cost over the interval $[0, T]$, is generally time varying (at

event $(x, i)$, $i \in \mathbb{I}_{0:N}$, the optimal control is $\mu_i(x) = \kappa_{N-i}(x)$). Under fairly general conditions, $\mu_i(\cdot) \to \kappa_\infty(\cdot)$ as $N \to \infty$ where $\kappa_\infty(\cdot)$ is the stationary infinite horizon optimal control law. MPC and RHC, on the other hand, employ the time-invariant control $\kappa_N(x)$ for all $i \in \mathbb{I}_{\geq 0}$. Thus the state and control trajectories $\mathbf{x}_{\text{mpc}}(x)$ and $\mathbf{u}_{\text{mpc}}(x)$ generated by MPC for an initial event $(x, 0)$ satisfy the difference equations

$$\begin{aligned} x(0) &= x & u(0) &= \kappa_N(x) \\ x(i+1) &= f(x(i), u(i)) & u(i) &= \kappa_N(x(i)) \end{aligned}$$

and can be seen to differ in general from $\mathbf{x}^0(x)$ and $\mathbf{u}^0(x)$, which satisfy (2.13) and (2.14), and, hence, are *not* optimal for $\mathbb{P}_N(x)$.

Before leaving this section, we obtain some properties of the solution to each partial problem $\mathbb{P}_j(x)$. For this, we require a few definitions.

**Definition 2.10** (Positive and control invariant sets)**.**

(a) A set $X \subseteq \mathbb{R}^n$ is positive invariant for $x^+ = f(x)$ if $x \in X$ implies $f(x) \in X$.

(b) A set $X \subseteq \mathbb{R}^n$ is control invariant for $x^+ = f(x, u)$, $u \in \mathbb{U}$, if, for all $x \in X$, there exists a $u \in \mathbb{U}$ such that $f(x, u) \in X$.

We recall from our standing assumptions 2.2 and 2.3 that $f(\cdot)$, $\ell(\cdot)$ and $V_f(\cdot)$ are continuous, that $\mathbb{X}$ and $\mathbb{X}_f$ are closed, $\mathbb{U}$ is compact and that each of these sets contains the origin.

**Proposition 2.11** (Existence of solutions to DP recursion)**.** *Suppose Assumptions 2.2 and 2.3 hold. Then*

*(a) For all $j \geq 0$, the cost function $V_j(\cdot)$ is continuous in $\mathcal{Z}_j$, and, for each $x \in \mathcal{X}_j$, the control constraint set $\mathcal{U}_j(x)$ is compact and a solution $\mathbf{u}^0(x) \in \mathcal{U}_j(x)$ to $\mathbb{P}_j(x)$ exists.*

*(b) If $\mathcal{X}_0 := \mathbb{X}_f$ is control invariant for $x^+ = f(x, u)$, $u \in \mathbb{U}$, then, for each $j \in \mathbb{I}_{\geq 0}$, the set $\mathcal{X}_j$ is also control invariant, $\mathcal{X}_j \supseteq \mathcal{X}_{j-1}$, and $0 \in \mathcal{X}_j$. In addition, the set $\mathcal{X}_N$ is positive invariant for $x^+ = f(x, \kappa_N(x))$.*

*(c) For each $j \geq 0$, the set $\mathcal{X}_j$ is closed.*

*Proof.*

(a) This proof is almost identical to the proof of Proposition 2.4.

(b) By assumption, $\mathcal{X}_0 = \mathbb{X}_f \subseteq \mathbb{X}$ is control invariant. By (2.12)

$$\mathcal{X}_1 = \{x \in \mathbb{X} \mid \exists u \in \mathbb{U} \text{ such that } f(x, u) \in \mathcal{X}_0\}$$

Since $\mathcal{X}_0$ is control invariant for $x^+ = f(x,u), u \in \mathbb{U}$, for every $x \in \mathcal{X}_0$ there exist a $u \in \mathbb{U}$ such that $f(x,u) \in \mathcal{X}_0$ so that $x \in \mathcal{X}_1$. Hence $\mathcal{X}_1 \supseteq \mathcal{X}_0$. Since for every $x \in \mathcal{X}_1$, there exists a $u \in \mathbb{U}$ such that $f(x,u) \in \mathcal{X}_0 \subseteq \mathcal{X}_1$, it follows that $\mathcal{X}_1$ is control invariant for $x^+ = f(x,u)$, $u \in \mathbb{U}$. If for some integer $j \in \mathbb{I}_{\geq 0}$, $\mathcal{X}_{j-1}$ is control invariant for $x^+ = f(x,u)$, it follows by similar reasoning that $\mathcal{X}_j \supseteq \mathcal{X}_{j-1}$ and that $\mathcal{X}_j$ is control invariant. By induction $\mathcal{X}_j$ is control invariant and $\mathcal{X}_j \supseteq \mathcal{X}_{j-1}$ for all $j > 0$. Hence $0 \in \mathcal{X}_j$ for all $j \in \mathbb{I}_{\geq 0}$. That $\mathcal{X}_N$ is positive invariant for $x^+ = f(x, \kappa_N(x))$ follows from (2.11), which shows that $\kappa_N(\cdot)$ steers every $x \in \mathcal{X}_N$ into $\mathcal{X}_{N-1} \subseteq \mathcal{X}_N$.

(c) By Assumption 2.3, $\mathcal{X}_0 = \mathbb{X}_f$ is closed. Suppose, for some $j \in \mathbb{I}_{\geq 1}$, that $\mathcal{X}_{j-1}$ is closed. Then $\mathcal{Z}_j := \{(x,u) \in \mathbb{Z} \mid f(x,u) \in \mathcal{X}_{j-1}\}$ is closed since $f(\cdot)$ is continuous. To prove that $\mathcal{X}_j$ is closed, take any sequence $\{x_i\}$ in $\mathcal{X}_j$ that converges to, say, $\bar{x}$. For each $i$, select a $u_i \in \mathbb{U}$ such that $(x_i, u_i) \in \mathcal{Z}_j$. Then, since $\mathbb{U}$ is compact, there exists a subsequence of $\{(x_i, u_i)\}$, indexed by $I$, such that $x_i \to \bar{x}$ and $u_i \to \bar{u}$ as $i \to \infty, i \in I$. Since $\mathbb{X}$ is closed, $\mathbb{U}$ is compact, and $\mathcal{Z}_j$ is closed, it follows that $\bar{x} \in \mathbb{X}$, $\bar{u} \in \mathbb{U}$ and $(\bar{x}, \bar{u}) \in \mathcal{Z}_j$. Hence $\bar{x} \in \mathcal{X}_j := \{x \in \mathbb{X} \mid \exists u \in \mathbb{U}$ such that $(\mathrm{x}, \mathrm{u}) \in \mathcal{Z}_\mathrm{j}\}$ so that $\mathcal{X}_j$ is closed. By induction $\mathcal{X}_j$ is closed for all $j \in \mathbb{I}_{\geq 0}$. ■

The fact that $\mathcal{X}_N$ is positive invariant for $x^+ = f(x, \kappa_N(x))$ can also be established by observing that $\mathcal{X}_N$ is the set of states $x$ in $\mathbb{X}$ for which there exists a $\mathbf{u}$ that is feasible for $\mathbb{P}_N(x)$, i.e., for which there exists a control $\mathbf{u}$ satisfying the control, state and terminal constraints. It is shown in the next section that for every $x \in \mathcal{X}_N$, there exists a feasible control sequence $\tilde{\mathbf{u}}$ for $\mathbb{P}_N(x^+)$ where $x^+ = f(x, \kappa_N(x))$ is the successor state provided that $\mathbb{X}_f$ is control invariant, i.e., $\mathcal{X}_N$ is positive invariant for $x^+ = f(x, \kappa_N(x))$ if $\mathbb{X}_f$ is control invariant. An important practical consequence is that if $\mathbb{P}_N(x(0))$ can be solved for the initial state $x(0)$, then $\mathbb{P}_N(x(i))$ can be solved for any subsequent state $x(i)$ of the controlled system $x^+ = f(x, \kappa_N(x))$, a property that is sometimes called recursive feasibility. Uncertainty, in the form of additive disturbances, model error or state estimation error, may destroy this important property; techniques to restore this property when uncertainty is present are discussed in Chapter 3.

## 2.4 Stability

### 2.4.1 Introduction

To establish stability we employ Lyapunov theorems such as Theorem B.13 in Appendix B. Because we are considering the regulator problem in this chapter, we are concerned with asymptotic or exponential stability of the origin. Hence, we replace $\mathcal{A}$ in Theorem B.13 of Appendix B by $\{0\}$, the set consisting of a single point, the origin. Thus, the origin is asymptotically stable with a region of attraction $X$ for the system $x^+ = f(x)$ if there exist: a Lyapunov function $V$, a positive invariant set $X$, two $\mathcal{K}_\infty$ functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$, and a positive definite function $\alpha_3(\cdot)$ satisfying

$$V(x) \geq \alpha_1(|x|) \tag{2.15}$$

$$V(x) \leq \alpha_2(|x|) \tag{2.16}$$

$$V(f(x)) \leq V(x) - \alpha_3(|x|) \tag{2.17}$$

for all $x \in X$. Recall that $\alpha : \mathbb{R} \to \mathbb{R}_{\geq 0}$ is a $\mathcal{K}_\infty$ function if it is continuous, strictly increasing, zero at zero, and is unbounded; and $\alpha$ is a positive definite function if it is continuous and positive everywhere except at the origin. Our task in this chapter is to find a function $V(\cdot)$ with these properties for the MPC system $x^+ = f(x, \kappa_N(x))$.

A standard approach to establish stability is to employ the value function of an infinite horizon optimal control problem as a Lyapunov function. This suggests the use of $V_N^0(\cdot)$, the value function for the finite horizon optimal control problem whose solution yields the model predictive controller, as a Lyapunov function. It is simple to show, under mild assumptions on $\ell(\cdot)$, that $V_N^0(\cdot)$ has property (2.15) for all $x \in \mathcal{X}_N$. The value function $V_\infty(\cdot)$ for infinite horizon optimal control problems does satisfy, under mild conditions, $V_\infty^0(f(x, \kappa_\infty(x))) = V_\infty^0(x) - \ell(x, \kappa_\infty(x))$ thereby ensuring satisfaction of property (2.17). Since, as is often pointed out, optimality does not imply stability, this property does not usually hold when the horizon is finite. One of the main tasks of this chapter is show that if the "ingredients" $V_f(\cdot)$, $\ell(\cdot)$, and $\mathbb{X}_f$ of the finite horizon optimal control problem are chosen appropriately, then $V_N^0(f(x, \kappa_N(x))) \leq V_N^0(x) - \ell(x, \kappa_N(x))$ for all $x$ in $\mathcal{X}_N$ enabling property (2.17) to be obtained. Property (2.16), an upper bound on the value function, is more difficult to establish. We show subsequently that the choice of "ingredients" that ensures satisfaction of property (2.17) also ensures satisfaction of property (2.16) but only

for all $x$ in $\mathbb{X}_f$ rather than for all $x \in \mathcal{X}_N$. We therefore also address the problem of establishing asymptotic or exponential stability of the origin even if property (2.16) holds only for $x$ in $\mathbb{X}_f$.

We now address a point that we have glossed over. The solution to an optimization problem is not necessarily unique. Thus $\mathbf{u}^0(x)$ and $\kappa_N(x)$ may be set valued; any point in the set $\mathbf{u}^0(x)$ is a solution of $\mathbb{P}_N(x)$. Similarly $\mathbf{x}^0(x)$ is set valued. Uniqueness may be obtained by choosing that element in the set $\mathbf{u}^0(x)$ that has least norm. To avoid expressions such as "let $\mathbf{u}$ be any element of the minimizing set $\mathbf{u}^0(x)$," we shall, in the sequel, use $\mathbf{u}^0(x)$ to denote any sequence in the set of minimizing sequences and use $\kappa_N(x)$ to denote $u^0(0;x)$, the first element of this sequence.

### 2.4.2 Stabilizing Conditions: No State Constraints

To show as simply as possible that the descent property (2.17) holds if $V_f(\cdot)$ and $\mathbb{X}_f$ are chosen appropriately, we consider first the case when there are no state or terminal constraints, i.e., $\mathbb{X} = \mathbb{X}_f = \mathbb{R}^n$, so that the only constraint is the control constraint. Hence $\mathcal{U}_N(x) = \mathbb{U}^N$, which is independent of $x$. For this case, $\mathcal{X}_j = \mathbb{R}^n$ for all $j \in \{1, 2, \ldots, N\}$. Let $x$ be any state in $\mathcal{X}_N = \mathbb{R}^n$ at time 0. Then

$$V_N^0(x) = V_N(x, \mathbf{u}^0(x))$$

in which

$$\mathbf{u}^0(x) = \left\{u^0(0;x), u^0(1;x), \ldots, u^0(N-1;x)\right\}$$

is any minimizing control sequence. The resultant optimal state sequence is

$$\mathbf{x}^0(x) = \left\{x^0(0;x), x^0(1;x), \ldots, x^0(N;x)\right\}$$

where $x^0(0;x) = x$ and $x^0(1;x) = x^+$. The successor state to $x$ at time 0 is $x^+ = f(x, \kappa_N(x)) = x^0(1;x)$ at time 1 where $\kappa_N(x) = u^0(0;x)$, and

$$V_N^0(x^+) = V_N(x^+, \mathbf{u}^0(x^+))$$

in which

$$\mathbf{u}^0(x^+) = \left\{u^0(0;x^+), u^0(1;x^+), \ldots, u^0(N-1;x^+)\right\}$$

It is difficult to compare $V_N^0(x)$ and $V_N^0(x^+)$ directly, but

$$V_N^0(x^+) = V_N(x^+, \mathbf{u}^0(x^+)) \leq V_N(x^+, \tilde{\mathbf{u}})$$

where $\tilde{\mathbf{u}}$ is any feasible control sequence for $\mathbb{P}_N(x^+)$, i.e., any control sequence in $\mathbb{U}^N$. To facilitate comparison of $V_N(x^+, \tilde{\mathbf{u}})$ with $V_N^0(x) = V_N(x, \mathbf{u}^0(x))$, we choose

$$\tilde{\mathbf{u}} = \left\{u^0(1;x), \dots, u^0(N-1;x), u\right\}$$

where $u$ still has to be chosen. Comparing $\tilde{\mathbf{u}}$ with $\mathbf{u}^0(x)$ shows that $\tilde{\mathbf{x}}$, the state sequence due to control sequence $\tilde{\mathbf{u}}$, is

$$\tilde{\mathbf{x}} = \left\{x^0(1;x), x^0(2;x), \dots, x^0(N;x), f(x^0(N;x), u)\right\}$$

in which $x^0(1;x) = x^+ = f(x, \kappa_N(x))$; since there are no state or terminal constraints, the state sequence $\tilde{\mathbf{x}}$ is clearly feasible if $u \in \mathbb{U}$. Since $\mathbf{x}^0$ coincides with $\tilde{\mathbf{x}}$ and $\mathbf{u}(\cdot)$ coincides with $\tilde{\mathbf{u}}$ for $i = 1, 2, \dots, N-1$ (but not for $i = N$), a simple calculation yields

$$\begin{aligned} V_N^0(x) &= V_N(x, \mathbf{u}^0(x)) \\ &= \ell(x, \kappa_N(x)) + \sum_{j=1}^{N-1} \ell(x^0(j;x), u^0(j;x)) + V_f(x^0(N;x)) \end{aligned}$$

so that

$$\begin{aligned} V_N(x^+, \tilde{\mathbf{u}}) = V_N^0(x) - \ell(x, \kappa_N(x)) - V_f(x^0(N;x)) + \\ \ell(x^0(N;x), u) + V_f(f(x^0(N;x), u)) \end{aligned}$$

in which $x^+ = f(x, \kappa_N(x))$. Since $V_N^0(x^+) \le V_N(x^+, \tilde{\mathbf{u}})$, it follows that

$$V_N^0(f(x, \kappa_N(x))) - V_N^0(x) \le -\ell(x, \kappa_N(x)) \tag{2.18}$$

for all $x \in \mathbb{R}^n$ *provided that* for all $x \in \mathbb{R}^n$, there exists a $u \in \mathbb{U}$ such that

$$V_f(f(x,u)) - V_f(x) + \ell(x,u) \le 0 \tag{2.19}$$

A continuous positive definite function $V_f(\cdot)$ satisfying inequality (2.19) for all $x \in \mathbb{R}^n$ with a positive definite $\ell(\cdot)$ is a *global control-Lyapunov function* (CLF). If $V_f(\cdot)$ is a global CLF, the value function $V_N^0(\cdot)$ has the desired descent property (2.18). Global asymptotic stability of the origin for the system $x^+ = f(x, \kappa_N(x))$ under MPC may be established. If $V_f(\cdot)$ is a global CLF satisfying (2.19), however, there exists a control law $\kappa_f(\cdot)$ satisfying $V_f(f(x, \kappa_f(x))) \le V_f(x) - \ell(x, \kappa_f(x))$ for all $x \in \mathbb{R}^n$. Global asymptotic stability of the origin for the system $x^+ = f(x, \kappa_f(x))$ may be established. In this case MPC is *not required* to stabilize the system, though it may provide superior performance.

### 2.4.3 Stabilizing Conditions: Constrained Problems

In this section we consider the case when state and control constraints (2.2) are present. MPC is stabilizing if a global CLF is employed as the terminal cost. A global CLF is seldom available, however, either because the system is nonlinear or because constraints are present. Hence, we must set our sights lower and employ as our terminal cost function $V_f(\cdot)$, a local CLF, one that is defined only on a neighborhood $\mathbb{X}_f$ of the origin where $\mathbb{X}_f \subseteq \mathbb{X}$. A consequent requirement is that the terminal state must be constrained, explicitly or implicitly, to lie in $\mathbb{X}_f$. Our stabilizing condition now takes the form:

**Assumption 2.12** (Basic stability assumption)**.**

$$\min_{u \in \mathbb{U}} \{V_f(f(x,u)) + \ell(x,u) \mid f(x,u) \in \mathbb{X}_f\} \leq V_f(x), \ \forall x \in \mathbb{X}_f$$

This assumption implicitly requires that for each $x \in \mathbb{X}_f$, there exists a $u \in \mathbb{U}$ such that $f(x,u) \in \mathbb{X}_f$, i.e., Assumption 2.12 implies the following assumption.

**Assumption 2.13** (Implied invariance assumption)**.** The set $\mathbb{X}_f$ is control invariant for the system $x^+ = f(x,u)$.

Assumptions 2.12 and 2.13 specify properties which, if possessed by the terminal cost function and terminal constraint set, enable us to employ the value function $V_N^0(\cdot)$ for the optimal control problem $\mathbb{P}_N$ as a Lyapunov function. The important descent and monotonicity properties of $V_N^0(\cdot)$ are established in Lemmas 2.14 and 2.15.

**Lemma 2.14** (Optimal cost decrease)**.** *Suppose, as usual, that Assumptions 2.2 and 2.3 hold, and that Assumptions 2.12 (and 2.13) hold. Then*

$$V_N^0(f(x, \kappa_N(x))) \leq V_N^0(x) - \ell(x, \kappa_N(x))$$

*for all* $x \in \mathcal{X}_N$.

*Proof.* Let $x$ be any point in $\mathcal{X}_N$. Then $V_N^0(x) = V_N(x, \mathbf{u}^0(x))$ where

$$\mathbf{u}^0(x) = \{u^0(0;x), u^0(1;x), \ldots, u^0(N-1;x)\}$$

and $u^0(0;x) = \kappa_N(x)$; the control sequence is feasible for $\mathbb{P}_N(x)$ because it satisfies all control, state, and terminal constraints. The corresponding state sequence is

$$\mathbf{x}^0(x) = \{x^0(0;x), x^0(1;x), \ldots, x^0(N;x)\}$$

where $x^0(0;x) = x$, $x^0(1;x) = f(x, \kappa_N(x))$ and $x^0(N;x) \in \mathbb{X}_f$. At the successor state $x^+ = x^0(1;x)$, we choose, as before, the nonoptimal control sequence $\tilde{\mathbf{u}}$ defined by

$$\tilde{\mathbf{u}} := \{u^0(1;x), \ldots, u^0(N-1;x), u\}$$

where $u$ is still to be chosen. The resultant state sequence is

$$\tilde{\mathbf{x}} = \{x^0(1;x), \ldots, x^0(N;x), f(x^0(N;x), u)\}$$

The control sequence $\tilde{u}(\cdot)$ is feasible, but not necessarily optimal, for $\mathbb{P}_N(x^0(1;x))$ provided that $f(x^0(N;x), u) \in \mathbb{X}_f$. We obtain as before

$$V_N^0(f(x, \kappa_N(x))) \leq V_N^0(x) - \ell(x, \kappa_N(x))$$

provided now that for all $x \in \mathbb{X}_f$, there exists a $u \in \mathbb{U}$ such that

$$V_f(f(x,u)) \leq V_f(x) - \ell(x,u), \text{ and } f(x,u) \in \mathbb{X}_f$$

which is true by Assumptions 2.12 and 2.13. ■

Lemma 2.14 holds if $\mathbb{U}$ is closed but not necessarily bounded and can be used, with suitable assumptions on $\ell(\cdot)$, to establish asymptotic stability of the origin. The descent property established in Lemma 2.14 may be established also using a monotonicity property of the value function.

### 2.4.4 Monotonicity of the Value Function

If Assumptions 2.12 (and 2.13) hold, the value function sequence $\{V_j^0(\cdot)\}$ has an interesting monotonicity property, first established for the unconstrained linear quadratic regulator problem, namely, for given $x$, the value $V_j^0(x)$ decreases as the time to go $j$ increases. We prove this in Lemma 2.15.

**Lemma 2.15** (Monotonicity of the value function)**.** *Suppose, as usual, that Assumptions 2.2 and 2.3 hold, and that Assumptions 2.12 (and 2.13) hold. Then*

$$\begin{aligned} V_{j+1}^0(x) &\leq V_j^0(x) \qquad \forall x \in \mathcal{X}_j,\ \forall j \in \mathbb{I}_{0:N-1} \\ V_N^0(x) &\leq V_f(x) \qquad \forall x \in \mathbb{X}_f \end{aligned}$$

*Proof.* From the DP recursion (2.10)

$$V_1^0(x) = \min_{u\in\mathbb{U}}\{\ell(x,u) + V_0^0(f(x,u)) \mid f(x,u) \in \mathcal{X}_0\}$$

But $V_0^0(\cdot) := V_f(\cdot)$ and $\mathcal{X}_0 := \mathbb{X}_f$. Also, by Assumption 2.12,

$$\min_{u\in\mathbb{U}}\{\ell(x,u) + V_f(f(x,u))\} \le V_f(x) \qquad \forall x \in \mathbb{X}_f$$

so that

$$V_1^0(x) \le V_0^0(x) \qquad \forall x \in \mathcal{X}_0$$

Next, suppose that for some $j \ge 1$,

$$V_j^0(x) \le V_{j-1}^0(x) \qquad \forall x \in \mathcal{X}_{j-1}$$

Then, using the DP equation (2.10)

$$\begin{aligned} V_{j+1}^0(x) - V_j^0(x) &= \ell(x,\kappa_{j+1}(x)) + V_j^0(f(x,\kappa_{j+1}(x))) \\ &\quad - \ell(x,\kappa_j(x)) - V_{j-1}^0(f(x,\kappa_j(x))) \qquad \forall x \in \mathcal{X}_j \subseteq \mathcal{X}_{j+1} \end{aligned}$$

Since $\kappa_j(x)$ may *not* be optimal for $\mathbb{P}_{j+1}(x)$ for all $x \in \mathcal{X}_j \subseteq \mathcal{X}_{j+1}$, we have

$$\begin{aligned} V_{j+1}^0(x) - V_j^0(x) &\le \ell(x,\kappa_j(x)) + V_j^0(f(x,\kappa_j(x))) \\ &\quad - \ell(x,\kappa_j(x)) - V_{j-1}^0(f(x,\kappa_j(x))) \qquad \forall x \in \mathcal{X}_j \end{aligned}$$

Also, from (2.12), $x \in \mathcal{X}_j$ implies $f(x,\kappa_j(x)) \in \mathcal{X}_{j-1}$ so that, by assumption, $V_j^0(f(x,\kappa_j(x))) \le V_{j-1}^0(f(x,\kappa_j(x)))$ for all $x \in \mathcal{X}_j$. Hence

$$V_{j+1}^0(x) \le V_j^0(x) \qquad \forall x \in \mathcal{X}_j$$

By induction

$$V_{j+1}^0(x) \le V_j^0(x) \qquad \forall x \in \mathcal{X}_j,\ \forall j \in \{1,2,\ldots,N-1\}$$

It then follows that $V_N^0(x) \le V_0^0(x) := V_f(x)$ for all $x \in \mathcal{X}_0 := \mathbb{X}_f$. ■

Lemma 2.15 also holds if $\mathbb{U}$ is closed but not bounded. The monotonicity property can be used to establish the descent property of $V_N^0(\cdot)$ proved in Lemma 2.14 by noting that

$$\begin{aligned} V_N^0(x) &= \ell(x,\kappa_N(x)) + V_{N-1}^0(f(x,\kappa_N(x))) \\ &= \ell(x,\kappa_N(x)) + V_N^0(f(x,\kappa_N(x))) + \\ &\quad [V_{N-1}^0(f(x,\kappa_N(x))) - V_N^0(f(x,\kappa_N(x)))] \end{aligned}$$

so that using the monotonicity property

$$\begin{aligned} V_N^0(f(x,\kappa_N(x))) &= V_N^0(x) - \ell(x,\kappa_N(x)) + \\ &\quad [V_N^0(f(x,\kappa_N(x))) - V_{N-1}^0(f(x,\kappa_N(x)))] \\ &\le V_N^0(x) - \ell(x,\kappa_N(x)) \qquad \forall x \in \mathcal{X}_N \end{aligned}$$

which is the desired descent property.

### 2.4.5 Further Properties of the Value Function $V_N^0(\cdot)$

Lemma 2.14 shows that the value function $V_N^0(\cdot)$ has a descent property that makes it a suitable candidate for a Lyapunov function that may be used to establish stability of the origin for a wide variety of MPC systems. To proceed, we postulate two alternative conditions on the stage cost $\ell(\cdot)$ and terminal cost $V_f(\cdot)$ required to show that $V_N^0(\cdot)$ has the properties given in Appendix B, which are sufficient to establish stability of the origin. Our additional assumption is:

**Assumption 2.16** (Bounds on stage and terminal costs)**.**
(a) The stage cost $\ell(\cdot)$ and the terminal cost $V_f(\cdot)$ satisfy

$$\begin{aligned} \ell(x,u) &\ge \alpha_1(|x|) \quad \forall x \in \mathcal{X}_N,\ \forall u \in \mathbb{U} \\ V_f(x) &\le \alpha_2(|x|) \quad \forall x \in \mathbb{X}_f \end{aligned}$$

in which $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ are $\mathcal{K}_\infty$ functions, or

(b) The stage cost $\ell(\cdot)$ and the terminal cost $V_f(\cdot)$ satisfy

$$\begin{aligned} \ell(x,u) &\ge c_1|x|^a \quad \forall x \in \mathcal{X}_N,\ \forall u \in \mathbb{U} \\ V_f(x) &\le c_2|x|^a \quad \forall x \in \mathbb{X}_f \end{aligned}$$

for some $c_1 > 0$, $c_2 > 0$, and $a > 0$.

Note that Assumption 2.16(b) implies 2.16(a) and that both Assumptions 2.16(a) and 2.16(b) are satisfied with $a = 2$ if $\ell(x,u) = (1/2)(x'Qx + u'Ru)$ and $Q$ and $R$ are positive definite. With this extra assumption, $V_N^0(\cdot)$ has the properties summarized in the following result.

**Proposition 2.17** (Optimal value function properties)**.**
*(a) Suppose that Assumptions 2.2, 2.3, 2.12, 2.13, and 2.16(a) are satisfied. Then there exist $\mathcal{K}_\infty$ functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ such that $V_N^0(\cdot)$ has*

*the following properties*

$$\begin{aligned} V_N^0(x) &\geq \alpha_1(|x|) && \forall x \in \mathcal{X}_N \\ V_N^0(x) &\leq \alpha_2(|x|) && \forall x \in \mathbb{X}_f \\ V_N^0(f(x, \kappa_N(x))) &\leq V_N^0(x) - \alpha_1(|x|) && \forall x \in \mathcal{X}_N \end{aligned}$$

*(b) Suppose that Assumptions 2.2, 2.3, 2.12, 2.13, and 2.16(b) are satisfied. Then there exist positive constants $c_1$, $c_2$, and $a$ such that $V_N^0(\cdot)$ has the following properties*

$$\begin{aligned} V_N^0(x) &\geq c_1|x|^a && \forall x \in \mathcal{X}_N \\ V_N^0(x) &\leq c_2|x|^a && \forall x \in \mathbb{X}_f \\ V_N^0(f(x, \kappa_N(x))) &\leq V_N^0(x) - c_1|x|^a && \forall x \in \mathcal{X}_N \end{aligned}$$

*Proof.*

(a) The first inequality follows from Assumption 2.16(a) and the fact that $V_N^0(x) \geq \ell(x, \kappa_N(x))$. The second inequality follows from Lemma 2.15 and Assumption 2.16. Finally, the third inequality follows from Lemma 2.14 and Assumption 2.16(a).

(b) The proof of this part is similar to the previous. ■

These properties are almost identical to those required in Theorems B.11 and B.13 in Appendix B to establish asymptotic stability of the origin with a region of attraction $\mathcal{X}_N$. The second property falls short because the upper bound holds for all $x$ in $\mathbb{X}_f$ rather than for all $x$ in $\mathcal{X}_N$. Despite this, asymptotic stability of the origin with a region of attraction $\mathcal{X}_N$ can still be established when $\mathbb{X}_f$ contains the origin in its interior as we show subsequently. Alternatively, the second inequality is sometimes assumed to hold for all $x \in \mathcal{X}_N$, in which case asymptotic stability of the origin can be established using standard theorems in Appendix B; see the subsequent Assumption 2.23. Finally, as we show next, $V_N^0(x) \leq \alpha_2(|x|)$ for all $x \in \mathbb{X}_f$ implies, under some mild assumptions, that $V_N^0(x) \leq \alpha_2(|x|)$ for all $x \in \mathcal{X}_N$ so that, under these assumptions, the results of Appendix B can again be used to prove asymptotic stability of the origin. In the next result, the set $X$ may be $\mathcal{X}_N$, if $\mathcal{X}_N$ is compact, or a sublevel set of $V_N^0(\cdot)$.

**Proposition 2.18** (Extension of upper bound to compact set)**.** *Suppose that Assumptions 2.2, 2.3, 2.12, and 2.13 hold, that $\mathbb{X}_f$ contains the origin in its interior, and that $\mathbb{X}_f \subseteq X$ where $X$ is a compact set in $\mathbb{R}^n$. If*

*there exists a $\mathcal{K}_\infty$ function $\alpha(\cdot)$ such that $V_N^0(x) \leq \alpha(|x|)$ for all $x \in \mathbb{X}_f$, then there exists another $\mathcal{K}_\infty$ function $\beta(\cdot)$ such that $V_N^0(x) \leq \beta(|x|)$ for all $x \in X$.*

*Proof.* Because the origin lies in the interior of $\mathbb{X}_f$, there exists a $d > 0$ such that $\{x \mid |x| \leq d\} \subset \mathbb{X}_f$. Let $e = \max\{\alpha(|x|) \mid |x| \leq d\} > 0$; then $\alpha(|x|) \geq e$ for all $x \in X \setminus \mathbb{X}_f$. Since $X$ is compact by assumption, $\mathbb{U}$ is compact by Assumption 2.3, and $V_N(\cdot)$ continuous by Proposition 2.4, there exists an upper bound $c > e$ for $V_N(\cdot)$ on $X \times \mathbb{U}^N$ and, hence, for $V_N^0(\cdot)$ on $X$. Thus $\beta(\cdot) := (c/e)\alpha(\cdot)$ is a $\mathcal{K}_\infty$ function satisfying $\beta(|x|) \geq \alpha(|x|)$ for all $x$ in $X$ and $\beta(|x|) \geq c$ for all $x \in X \setminus \mathbb{X}_f$. Hence $\beta(\cdot)$ is a $\mathcal{K}_\infty$ function satisfying $V_N^0(x) \leq \beta(|x|)$ for all $x \in X$. ■

An immediate consequence of Propositions 2.17 and 2.18 is the following result.

**Proposition 2.19** (Lyapunov function on $\mathcal{X}_N$)**.** *Suppose Assumptions 2.2, 2.3, 2.12, 2.13, and 2.16 are satisfied, that $\mathbb{X}_f$ has an interior containing the origin, and that $\mathcal{X}_N$ is bounded. Then, for all $x \in \mathcal{X}_N$*

$$V_N^0(x) \geq \alpha_1(|x|) \tag{2.20}$$

$$V_N^0(x) \leq \alpha_2(|x|) \tag{2.21}$$

$$V_N^0(f(x, \kappa_N(x))) \leq V_N^0(x) - \alpha_1(|x|) \tag{2.22}$$

*in which $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ are $\mathcal{K}_\infty$ functions.*

*Proof.* The result follows directly from Proposition 2.18 since the assumption that the set $\mathcal{X}_N$ is bounded, coupled with the fact that it is closed, as shown in Proposition 2.11, implies that it is compact. ■

Hence, if the hypotheses of Proposition 2.19 are satisfied, Theorems B.11 and B.13 in Appendix B may be used to establish asymptotic stability of the origin in $\mathcal{X}_N$. Sufficient conditions for the boundedness of $\mathcal{X}_N$ are provided by the next result. Recall $f_{\mathbb{Z}}^{-1}(\cdot)$ is given in Definition 2.9.

**Proposition 2.20** (Boundedness of $\mathcal{X}_j$)**.** *If either $\mathbb{X}$ is bounded or $\mathbb{X}_f$ is bounded and $f_{\mathbb{Z}}^{-1}(\cdot)$ is bounded on bounded sets, then, for all $j \in \mathbb{I}_{\geq 0}$, $\mathcal{X}_j$ is bounded.*

*Proof.* That $\mathcal{X}_N$ is bounded if $\mathbb{X}$ is bounded follows immediately from the fact that, by definition, $\mathcal{X}_N \subseteq \mathbb{X}$. Assume then that $\mathbb{X}_f$ is bounded and $f_{\mathbb{Z}}^{-1}(\cdot)$ is bounded on bounded sets. Then the set $\mathcal{Z}_1 = f_{\mathbb{Z}}^{-1}(\mathbb{X}_f)$ is

bounded and, hence, so is the set $\mathcal{X}_1$. Suppose, for some $j > 0$, the set $\mathcal{Z}_{j-1}$ is bounded; then its projection onto the set $\mathcal{X}_{j-1}$ is bounded and so is the set $\mathcal{Z}_j = f_{\mathbb{Z}}^{-1}(\mathcal{X}_j)$. Thus the set $\mathcal{X}_j$ is bounded. By induction $\mathcal{X}_j$ is bounded for all $j \in \mathbb{I}_{\geq 0}$. ■

When $f(\cdot)$ is linear, i.e., $f(x,u) = Ax + Bu$, then $f_{\mathbb{Z}}^{-1}(\cdot)$ is bounded on bounded sets if $A$ is nonsingular. The matrix $A$ is always nonsingular when $A$ and $B$ are obtained by sampling a continuous time system $\dot{x} = A_c x + B_c u$ with $u$ constant between sampling instants. In this case $A = \exp(A_c \Delta)$ and $B = \int_0^\Delta \exp(A_c(\Delta - s))B ds$ so that $A$ is invertible. To show that $f_{\mathbb{Z}}^{-1}(\cdot)$ is bounded on bounded sets, let $X$ be an arbitrary bounded set in $\mathbb{R}^n$ and let $x'$ be an arbitrary point $x' \in X$. Then $f^{-1}(x') = \{(x,u) \mid Ax + Bu = x'\}$. Any $(x,u)$ in $f^{-1}(x')$ satisfies $x = A^{-1}x' - A^{-1}Bu$ so that $x$ lies in the bounded set $A^{-1}X \oplus (-A^{-1}B\mathbb{U})$ and $u$ lies in the bounded set $\mathbb{U}$. Hence both $f^{-1}(X)$ and $f_{\mathbb{Z}}^{-1}(X)$ lie in the bounded set $A^{-1}X \oplus (-A^{-1}B)$. A similar result holds for nonlinear systems. If $f(\cdot)$ is obtained by sampling a continuous time system $\dot{x} = f_c(x,u)$ with period $\Delta$ and $u$ constant between sampling instants, then $f(\cdot)$ is defined by

$$f(x,u) = x + \int_0^\Delta f_c(x(s),u)ds$$

where $x(s)$ is the solution of $\dot{x} = f_c(x,u)$ at time $s$ if $x$ is the state at time zero and $u$ is the constant input in the interval $[0,\Delta]$.

**Proposition 2.21** (Properties of discrete time system)**.** *Suppose that*

*(a) $f_c(\cdot)$ is continuous.*

*(b) There exists a positive constant $c$ such that*

$$|f_c(x',u) - f_c(x,u)| \leq c|x' - x| \quad \forall\, x,\ x' \in \mathbb{R}^n,\ u \in \mathbb{U}$$

*Then $f(\cdot)$ and $f_{\mathbb{Z}}^{-1}(\cdot)$ are bounded on bounded sets.*

The proof of Proposition 2.21 is discussed in Exercise 2.2. Proposition 2.19 shows that if the terminal constraint set $\mathbb{X}_f$ contains the origin in its interior and if $\mathcal{X}_N$ is bounded, which is often the case, then standard stability theorems, such as Theorems B.11 and B.12 in Appendix B, may be used to establish asymptotic stability of the origin. When $\mathbb{X}_f$ contains the origin in its interior but $\mathcal{X}_N$ is unbounded, asymptotic stability of the origin can still be established using the next result that is a slight generalization of Theorem B.12 in Appendix B.

**Theorem 2.22** (Asymptotic stability with unbounded region of attraction)**.** *Suppose $X \subset \mathbb{R}^n$ and $\mathbb{X}_f \subset X$ are positive invariant for the system $x^+ = f(x)$, that $\mathbb{X}_f \subset X$ is closed and contains the origin in its interior, and that there exist a function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and two $\mathcal{K}_\infty$ functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ such that*

$$V(x) \geq \alpha_1(|x|) \qquad \forall x \in X \tag{2.23}$$

$$V(x) \leq \alpha_2(|x|) \qquad \forall x \in \mathbb{X}_f \tag{2.24}$$

$$V(f(x)) - V(x) \leq -\alpha_1(|x|) \qquad \forall x \in X \tag{2.25}$$

*Then the origin is asymptotically stable with a region of attraction $X$ for the system $x^+ = f(x)$.*

*Proof.*

**Stability.** Because $\mathbb{X}_f$ contains the origin in its interior, there exists a $\delta_1 > 0$ such that $\delta_1 \mathcal{B} \subset \mathbb{X}_f$; here $\mathcal{B}$ denotes the closed unit ball in $\mathbb{R}^n$. Let $\delta \in (0, \delta_1] > 0$ be arbitrary. Let $\phi(i; x)$ denote the solution of $x^+ = f(x)$ at time $i$ if the initial state is $x$. Suppose that $|x| \leq \delta$ so that $x \in \mathbb{X}_f$. It follows from (2.24) that $V(x) \leq \alpha_2(\delta)$ and from (2.25) that $V(\phi(i; x)) \leq V(x) \leq \alpha_2(\delta)$ for all $i \in \mathbb{I}_{\geq 0}$. From (2.23), $|\phi(i; x)| \leq \alpha_1^{-1}(V(x(i))) \leq (\alpha_1^{-1} \circ \alpha_2)(\delta)$ for all $i \in \mathbb{I}_{\geq 0}$. Hence for all $\varepsilon > 0$, there exists a $\delta > 0$, $\delta := \min\{\delta_1, (\alpha_1^{-1} \circ \alpha_2)^{-1}(\varepsilon)\}$, such that $|x| \leq \delta$ implies that $|\phi(i; x)| \leq \varepsilon$ for all $i \in \mathbb{I}_{\geq 0}$. Stability of the origin is established.

**Attractivity.** The proof of attractivity is similar to the proof of attractivity in Theorem B.11 of Appendix B. ■

Hence, if we add to the hypotheses of Proposition 2.17 the assumption that $\mathbb{X}_f$ contains the origin in its interior, we can use Theorem 2.22 to establish the asymptotic stability of the origin with a region of attraction $\mathcal{X}_N$ for the system $x^+ = f(x, \kappa_N(x))$.

In situations where $\mathbb{X}_f$ does not have an interior, such as when $\mathbb{X}_f = \{0\}$, we cannot establish an upper bound for $V_N^0(\cdot)$ from Assumptions 2.12 and 2.13, and resort to the following assumption.

**Assumption 2.23** (Weak controllability)**.** There exists a $\mathcal{K}_\infty$ function $\alpha(\cdot)$ such that

$$V_N^0(x) \leq \alpha(|x|) \ \forall x \in \mathcal{X}_N$$

Assumption 2.23 is weaker than a controllability assumption though it bounds the cost of steering an initial state $x$ to $\mathbb{X}_f$. It confines attention to those initial states that can be steered to $\mathbb{X}_f$ in $N$ steps while satisfying the control and state constraints, and merely requires that the cost of doing so is not excessive.

### 2.4.6 Summary

In the sequel we apply the previous results to establish asymptotic or exponential stability of a wide range of MPC systems. To facilitate application, we summarize these results and some of their consequences in the following theorem. Since $\kappa_N(\cdot)$ may be set valued, statements of the form $\kappa_N(x)$ has property $A$ in the sequel should be interpreted as every $u$ in $\kappa_N(x)$ has property $A$.

**Theorem 2.24** (MPC stability)**.**

*(a) Suppose that Assumptions 2.2, 2.3, 2.12, 2.13, and 2.16(a) are satisfied and that $\mathcal{X}_N = \mathbb{X}_f = \mathbb{R}^n$ so that $V_f(\cdot)$ is a global CLF. Then the origin is globally asymptotically stable for $x^+ = f(x, \kappa_N(x))$. If, in addition, Assumption 2.16(b) is satisfied, then the origin is globally exponentially stable.*

*(b) Suppose that Assumptions 2.2, 2.3, 2.12, 2.13, and 2.16(a) are satisfied and that $\mathbb{X}_f$ contains the origin in its interior. Then the origin is asymptotically stable with a region of attraction $\mathcal{X}_N$ for the system $x^+ = f(x, \kappa_N(x))$. If, in addition, Assumption 2.16(b) is satisfied and $\mathcal{X}_N$ is bounded, then the origin is exponentially stable with a region of attraction $\mathcal{X}_N$ for the system $x^+ = f(x, \kappa_N(x))$; if $\mathcal{X}_N$ is unbounded, then the origin is exponentially stable with a region of attraction that is any sublevel set of $V_N^0(\cdot)$.*

*(c) Suppose that Assumptions 2.2, 2.3, 2.12, 2.13, and 2.23 are satisfied and that $\ell(\cdot)$ satisfies $\ell(x, u) \geq \alpha_1(|x|)$ for all $x \in \mathcal{X}_N$, all $u \in \mathbb{U}$, where $\alpha_1(\cdot)$ is a $\mathcal{K}_\infty$ function. Then the origin is asymptotically stable with a region of attraction $\mathcal{X}_N$ for the system $x^+ = f(x, \kappa_N(x))$. If $\ell(\cdot)$ satisfies $\ell(x, u) \geq c_1|x|^a$ for all $x \in \mathcal{X}_N$, all $u \in \mathbb{U}$, and Assumption 2.23 is satisfied with $\alpha(r) = c_2 r^a$ for some $c_1 > 0$, $c_2 > 0$ and $a > 0$, then the origin is exponentially stable with a region of attraction $\mathcal{X}_N$ for the system $x^+ = f(x, \kappa_N(x))$.*

*(d) Suppose that Assumptions 2.2, 2.3, 2.12, and 2.13 are satisfied, that $\ell(\cdot)$ satisfies $\ell(x, u) \geq c_1|x|^a + c_1|u|^a$, and that Assumption 2.23 is*

*satisfied with $\alpha(r) = c_2 r^a$ for some $c_1 > 0$, $c_2 > 0$, and $a > 0$. Then $|\kappa_N(x)| \le c|x|$ for all $x \in \mathcal{X}_N$ where $c = (c_2/c_1)^{1/a}$.*

*Proof.*

(a) Since $\mathbb{X}_f = \mathbb{R}^n$, Lemmas 2.14 and 2.15 ensure the existence of $\mathcal{K}_\infty$ functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ such that the value function $V_N^0(\cdot)$ satisfies

$$\begin{aligned} V_N^0(x) &\ge \alpha_1(|x|) \\ V_N^0(f(x, \kappa_N(x))) &\le V_N^0(x) - \alpha_1(|x|) \\ V_N^0(x) &\le \alpha_2(|x|) \end{aligned}$$

for all $x \in \mathbb{R}^n$. Asymptotic stability of the origin follows from Theorem B.11 in Appendix B. When Assumption 2.16(b) is satisfied, global exponential stability of the origin follows as in the proof of the next part with $\mathcal{X}_N = \mathbb{R}^n$.

(b) If Assumption 2.16(a) is satisfied, asymptotic stability of the origin follows from Proposition 2.17 and Theorem 2.22. If Assumption 2.16(b) is satisfied and $\mathcal{X}_N$ is bounded, it follows from Propositions 2.18 and 2.19 that there exists $c_2$ sufficiently large such that the value function satisfies

$$V_N^0(x) \ge c_1|x|^a \tag{2.26}$$

$$V_N^0(f(x, \kappa_N(x))) \le V_N^0(x) - c_1|x|^a \tag{2.27}$$

$$V_N^0(x) \le c_2|x|^a \tag{2.28}$$

for all $x \in \mathcal{X}_N$. Consider any initial state $x \in \mathcal{X}_N$, and let $x(i)$ denote the solution at time $i$ of the difference equation $x^+ = f(x, \kappa_N(x))$ with initial condition $x(0) = x$. Since, by Proposition 2.11, $\mathcal{X}_N$ is positive invariant for $x^+ = f(x, \kappa_N(x))$, the entire sequence $\{x(i)\}$ lies in $\mathcal{X}_N$ if the initial state $x$ lies in $\mathcal{X}_N$. Hence $\{x(i)\}$ satisfies

$$V_N^0(x(i+1)) \le V_N^0(x(i)) - c_1|x(i)|^a \le (1 - c_1/c_2)V_N^0(x(i))$$

for all $i \in \mathbb{I}_{\ge 0}$. It follows that

$$V_N^0(x(i)) \le \gamma^i V_N^0(x(0))$$

for all $i \in \mathbb{I}_{\ge 0}$ in which $\gamma := (1 - c_1/c_2) \in (0, 1)$. Hence

$$|x(i)|^a \le (1/c_1)V_N^0(x(i)) \le (1/c_1)\gamma^i V_N^0(x(0)) \le (c_2/c_1)\gamma^i |x(0)|^a$$

so that

$$|x(i)| \leq c\delta^i |x(0)| \qquad \forall x(0) \in \mathcal{X}_N \quad \forall i \in \mathbb{I}_{\geq 0}$$

in which $c := (c_2/c_1)^{1/a}$ and $\delta := \gamma^{1/a} \in (0,1)$. Since $x(i) \in \mathcal{X}_N$ for all $i \in \mathbb{I}_{\geq 0}$, it follows that the origin is exponentially stable with a region of attraction $\mathcal{X}_N$ for $x^+ = f(x, \kappa_N(x))$. Consider now the case when $\mathcal{X}_N$ is unbounded. It follows from (2.26) that any sublevel set of $V_N^0(\cdot)$ is bounded, and, from (2.27), is positive invariant for $x^+ = f(x, \kappa_N(x))$. The origin is exponentially stable with a region of attraction equal to any sublevel set of $V_N^0(\cdot)$, which follows by similar reasoning for the case when $\mathcal{X}_N$ is bounded by replacing $\mathcal{X}_N$ with the bounded sublevel set of $V_N^0(\cdot)$.

(c) It follows from the proof of Proposition 2.17 and Assumption 2.23 that $V_N^0(\cdot)$ satisfies (2.20)–(2.22) for all $x \in \mathcal{X}_N$. Since $\mathcal{X}_N$ is positive invariant, it follows from Theorem B.13 in Appendix B that the origin is asymptotically stable with a region of attraction $\mathcal{X}_N$ for $x^+ = f(x, \kappa_N(x))$. Suppose now that $\ell(\cdot)$ satisfies $\ell(x,u) \geq c_1|x|^a$ for all $x \in \mathcal{X}_N$, all $u \in \mathbb{U}$ and Assumption 2.23 is satisfied with $\alpha(r) = c_2 r^a$ for some $c_1 > 0$, $c_2 > 0$, and $a > 0$. It follows that $V_N^0(\cdot)$ satisfies (2.26)–(2.28) for all $x \in \mathcal{X}_N$. Exponential stability of the origin for $x^+ = f(x, \kappa_N(x))$ follows by the same reasoning employed in the proof of part (b).

(d) It follows the assumption on $\ell(\cdot)$ and Assumption 2.23 that $c_2|x|^a \geq V_N^0(x) \geq c_1|\kappa_N(x)|^a$ so that $|\kappa_N(x)|^a \leq (c_2/c_1)|x|^a$, which implies $|\kappa_N(x)| \leq (c_2/c_1)^{1/a}|x|$ for all $x \in \mathcal{X}_N$. ■

### 2.4.7 Controllability and Observability

We have not yet made any assumptions on controllability (stabilizability) or observability (detectability) of the system (2.1) being controlled, which may be puzzling since such assumptions are commonly required in optimal control to, for example, establish existence of a solution to the optimal control problem. The reasons for this omission are that such assumptions are implicitly required, at least locally, for the basic stability Assumption 2.12, and that we restrict attention to $\mathcal{X}_N$, the set of states that can be steered to $\mathbb{X}_f$ in $N$ steps satisfying all constraints.

For example, one version of MPC uses a target set $\mathbb{X}_f = \{0\}$, so that the optimal control problem requires determination of an optimal trajectory terminating at the origin; clearly some assumption on controllability to the origin such as Assumption 2.23 is required. Similarly,

if the system being controlled is linear, and the constraints polytopic or polyhedral, a common choice for $\mathbb{X}_f$ is the maximal invariant constraint admissible set for a controlled system where the controller is linear and stabilizing. The terminal constraint set $\mathbb{X}_f$ is then the set $\{x \mid x(i) \in \mathbb{X}, Kx(i) \in \mathbb{U}\}$ where $x(i)$ is the solution at time $i$ of $x^+ = (A + BK)x$, and $u = \kappa_f(x) = Kx$ is a stabilizing control law. Stabilizability of the system being controlled is then required; see Section C.3 of Appendix C for a brief exposition of invariant sets.

Detectability assumptions also are required, mainly in proofs of asymptotic or exponential stability. For example, if the stage cost satisfies $\ell(x,u) = (1/2)(|y|^2 + |u|_R^2)$ where $y = Cx$, the stability proofs commonly establish that $y(k) = Cx(k)$ tends to zero as $k$ tends to infinity. To deduce from this fact that $x(k) \to 0$ requires a detectability assumption on the system $x^+ = f(x,u)$, $y = Cx$. If $C$ is invertible, as we sometimes assume, the system is detectable (since $y(k) \to 0$ implies $x(k) \to 0$).

The requisite assumptions of stabilizability and detectability are made later in the context of discussing specific forms of MPC.

### 2.4.8 Time-Varying Systems

Most of the control problems discussed in this book are time invariant. Time-varying problems do arise in practice, however, even if the system being controlled is time invariant. One example occurs when an observer or filter is used to estimate the state of the system being controlled since bounds on the state estimation error are often time varying. In the deterministic case, for example, state estimation error decays exponentially to zero. In this section, which may be omitted in the first reading, we show how MPC may be employed for time-varying systems.

**The problem.** The time-varying nonlinear system is described by

$$x^+ = f(x,u,i)$$

where $x$ is the current state at time $i$, $u$ the current control, and $x^+$ the successor state at time $i+1$. For each integer $i$, the function $f(\cdot,i)$ is assumed to be continuous. The solution of this system at time $k$ given that the initial state is $x$ at time $i$ is denoted by $\phi(k;x,i,\mathbf{u})$; the solution now depends on both the initial time $i$ and current time $k$ rather than merely on the difference $k-i$ as in the time-invariant case. The cost

$V_N(x,i,\mathbf{u})$ also depends on the initial time $i$ and is defined by

$$V_N(x,i,\mathbf{u}) := \sum_{k=i}^{i+N-1} \ell(x(k),u(k),k) + V_f(x(i+N),i+N)$$

in which $x(k) := \phi(k;x,i,\mathbf{u})$, $\mathbf{u} = \{u(i),u(i+1),\ldots,u(i+N-1)\}$, and the stage cost $\ell(\cdot)$ and terminal cost $V_f(\cdot)$ are time varying. The state and control constraints are also time varying

$$x(k) \in \mathbb{X}(k) \qquad u(k) \in \mathbb{U}(k)$$

for all $k$. In addition, there is a time-varying terminal constraint

$$x(i+N) \in \mathbb{X}_f(i+N)$$

in which $i$ is the current time. The time-varying optimal control problem at event $(x,i)$ is $\mathbb{P}_N(x,i)$ defined by

$$\mathbb{P}_N(x,i): \quad V_N^0(x,i) = \min\{V_N(x,i,\mathbf{u}) \mid \mathbf{u} \in \mathcal{U}_N(x,i)\}$$

in which $\mathcal{U}_N(x,i)$ is the set of control sequences $\mathbf{u} = \{u(i),u(i+1),\ldots,u(i+N-1)\}$ satisfying the state, control and terminal constraints, i.e.,

$$\mathcal{U}_N(x,i) := \{\mathbf{u} \mid (x,\mathbf{u}) \in \mathbb{Z}_N(i)\}$$

in which, for each $i$, $\mathbb{Z}_N(i) \subset \mathbb{R}^n \times \mathbb{R}^{Nm}$ is defined by

$$\mathbb{Z}_N(i) := \Big\{(x,\mathbf{u}) \mid u(k) \in \mathbb{U}(k), \quad \phi(k;x,i,\mathbf{u}) \in \mathbb{X}(k), \forall k \in \mathbb{I}_{i,i+N-1}, \\ \phi(i+N;x,i,\mathbf{u}) \in \mathbb{X}_f(i+N)\Big\}$$

For each time $i$, the domain of $V_N^0(\cdot,i)$ is $\mathcal{X}_N(i)$ where

$$\begin{aligned}\mathcal{X}_N(i) &:= \{x \in \mathbb{X}(i) \mid \mathcal{U}_N(x,i) \neq \emptyset\} \\ &= \{x \in \mathbb{R}^n \mid \exists \mathbf{u} \text{ such that } (x,\mathbf{u}) \in \mathbb{Z}_N(i)\}\end{aligned}$$

which is the projection of $\mathbb{Z}_N(i)$ onto $\mathbb{R}^n$. Our standing assumptions (2.2 and 2.3) are replaced, in the time-varying case, by

**Assumption 2.25** (Continuity of system and cost; time-varying case)**.** The functions $f(\cdot)$, $\ell(\cdot)$, and $V_f(\cdot)$ are continuous; for all $i \in \mathbb{I}_{\geq 0}$, $f(0,0,i) = 0$, $\ell(0,0,i) = 0$, and $V_f(0,i) = 0$.

**Assumption 2.26** (Properties of constraint sets; time-varying case)**.** For all $i \in \mathbb{I}_{\geq 0}$, $\mathbb{X}(i)$ and $\mathbb{X}_f(i)$ are closed, $\mathbb{X}_f(i) \subset \mathbb{X}(i)$ and $\mathbb{U}(i)$ are compact; each set contains the origin.

Because of the time-varying nature of the problem, we need to extend our definitions of invariance and control invariance.

**Definition 2.27** (Time-varying control invariant sets)**.** The sequence of sets $\{\mathcal{X}(i) \mid i \in \mathbb{I}_{\geq 0}\}$ is said to be time-varying control invariant for the time-varying system $x^+ = f(x, u, i)$ if, for each $i \in \mathbb{I}_{\geq 0}$, for each $x \in \mathcal{X}(i)$, there exists a $u \in \mathbb{U}(i)$ such that $x^+ = f(x, u, i) \in \mathcal{X}(i+1)$. The sequence of sets $\{\mathcal{X}(i) \mid i \in \mathbb{I}_{\geq 0}\}$ is said to be time-varying positive invariant for the time-varying system $x^+ = f(x, i)$ if, for each $x \in \mathcal{X}(i)$, $x^+ = f(x, i) \in \mathcal{X}(i+1)$.

A sequence of sets $\{\mathcal{X}(i)\}$ is a *tube*, and time-varying positive invariance of the sequence is positive invariance of the tube. If $(x, i)$ lies in the tube, i.e., if $x \in X(i)$ for some $i \in \mathbb{I}_{\geq 0}$, then all solutions of $x^+ = f(x, i)$ starting at event $(x, i)$ remain in the tube. The following results, which are analogs of the results for time-invariant systems given previously, are stated without proof.

**Proposition 2.28** (Continuous system solution; time-varying case)**.** *Suppose Assumptions 2.25 and 2.26 are satisfied. For each initial time $i$ and final time $j$, the function $(x, \mathbf{u}) \mapsto \phi(j; x, i, \mathbf{u})$ is continuous.*

**Proposition 2.29** (Existence of solution to optimal control problem; time-varying case)**.** *Suppose Assumptions 2.25 and 2.26 are satisfied. Then for each time $i \in \mathbb{I}_{\geq 0}$*

*(a) The function $(x, \mathbf{u}) \mapsto V_N(x, i, \mathbf{u})$ is continuous in $\mathbb{Z}_N(i)$.*

*(b) For each $x \in \mathcal{X}_N(i)$, the control constraint set $\mathcal{U}_N(x, i)$ is compact.*

*(c) For each $x \in \mathcal{X}_N(i)$, a solution to $\mathbb{P}_N(x, i)$ exists.*

*(d) $\mathcal{X}_N(i)$ is closed.*

*(e) If $\{\mathbb{X}_f(i)\}$ is time-varying control invariant for $x^+ = f(x, u, i)$, then $\{\mathcal{X}_N(i)\}$ is time-varying control invariant for $x^+ = f(x, u, i)$ and time-varying positive invariant for $x^+ = f(x, \kappa_N(x, i), i)$.*

*(f)* $0 \in \mathcal{X}_N(i)$.

**Stability.** As before, the receding horizon control law $\kappa_N(\cdot)$, which is now time varying, is not necessarily optimal or stabilizing. By choosing the time-varying "ingredients" $V_f(\cdot)$ and $\mathbb{X}_f$ in the optimal control

problem appropriately, however, stability can be ensured, as we now show. We replace the stability assumptions (2.12 and 2.13) by their time-varying extension.

**Assumption 2.30** (Basic stability assumption; time-varying case)**.** For all $i \in \mathbb{I}_{\geq 0}$, $\min_{u \in \mathbb{U}_i}\{V_f(f(x,u,i),i+1)+\ell(x,u,i) \mid f(x,u,i) \in \mathbb{X}_f(i+1)\} \leq V_f(x,i),\ \forall x \in \mathbb{X}_f(i)$.

This assumption implicitly requires that the sets $\{\mathbb{X}_f(i)\}$ are *time-varying* positive invariant in the following sense.

**Assumption 2.31** (Implied invariance assumption; time-varying case)**.** For each $i \in \mathbb{I}_{\geq 0}$ and each $x \in \mathbb{X}_f(i)$, there exists a $u \in \mathbb{U}(i)$ such that $f(x,u,i) \in \mathbb{X}_f(i+1)$.

A direct consequence of Assumption 2.31 is the extension of Lemma 2.14, namely, that the time-varying value function $V_N^0(\cdot)$ has the descent property that its value at $(f(x,\kappa_N(x,i),i),i+1)$ is less than its value at $(x,i)$ by an amount $\ell(x,\kappa_N(x,i),i)$.

**Lemma 2.32** (Optimal cost decrease; time-varying case)**.** *Suppose Assumptions 2.25, 2.26, 2.30 and 2.31 hold. Then,*

$$V_N^0(f(x,\kappa_N(x,i),i),i+1) \leq V_N^0(x,i) - \ell(x,\kappa_N(x,i),i) \qquad (2.29)$$

*for all $x \in \mathcal{X}_N(i)$, all $i \in \mathbb{I}_{\geq 0}$.*

**Lemma 2.33** (MPC cost is less than terminal cost)**.** *Suppose Assumptions 2.25, 2.26, 2.30 and 2.31 hold. Then,*

$$V_N^0(x,i) \leq V_f(x,i) \qquad \forall x \in \mathbb{X}_f(i), \quad \forall i \in \mathbb{I}_{\geq 0}$$

The proofs of Lemmas 2.32 and 2.33 are left as Exercises 2.9 and 2.10. Determination of a time-varying terminal cost $V_f(\cdot)$ and time-varying terminal constraint set $\mathbb{X}_f$ is complex. Fortunately there are a few important cases where choice of time-invariant terminal cost and constraint set is possible. The first possibility is $\mathbb{X}_f = \{0\}$ and $V_f(0) = 0$; this choice satisfies Assumptions 2.12 and 2.13, as already demonstrated, as well as Assumptions 2.30 and 2.31. The second possibility arises when $f(\cdot)$ is time invariant, which is the case when output feedback rather than state feedback is employed. In this case, discussed more fully in Chapter 5, time-varying bounds on state estimation error may lead to time-varying constraints even though the underlying system is time invariant. We therefore make the following assumption.

**Assumption 2.34** (Bounds on stage and terminal costs; time-varying case)**.**

(a) The terminal cost $V_f(\cdot)$ and terminal constraint set $\mathbb{X}_f$ are time invariant.

(b) The stage cost $\ell(\cdot)$ and the terminal cost $V_f(\cdot)$ satisfy, for all $i \in \mathbb{I}_{\geq 0}$

$$\begin{aligned} \ell(x,u,i) &\geq \alpha_1(|x|) \qquad \forall x \in \mathcal{X}_N(i),\ \forall u \in \mathbb{U}(i) \\ V_f(x,i) &\leq \alpha_2(|x|) \qquad \forall x \in \mathbb{X}_f \end{aligned}$$

in which $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ are $\mathcal{K}_\infty$ functions.

Our next result is an analog of Proposition 2.17, and follows fairly simply from Lemmas 2.32 and 2.33 and our assumptions.

**Proposition 2.35** (Optimal value function properties; time-varying case)**.** *Suppose Assumptions 2.25, 2.26, 2.30, 2.31, and 2.34 are satisfied. Then there exist two $\mathcal{K}_\infty$ functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ such that, for all $i \in \mathbb{I}_{\geq 0}$*

$$\begin{aligned} V_N^0(x,i) &\geq \alpha_1(|x|) && \forall x \in \mathcal{X}_N(i) \\ V_N^0(x,i) &\leq \alpha_2(|x|) && \forall x \in \mathbb{X}_f \\ V_N^0(f(x,\kappa_N(x,i),i+1)) &\leq V_N^0(x,i) - \alpha_1(|x|) && \forall x \in \mathcal{X}_N(i) \end{aligned}$$

We can deal with the obstacle posed by the fact that the upper bound on $V_N^0(\cdot)$ holds only in $\mathbb{X}_f$ in much the same way as we did previously for the time-invariant case. For simplicity, however, we invoke instead a uniform controllability assumption.

**Assumption 2.36** (Uniform weak controllability)**.** There exists a $\mathcal{K}_\infty$ function $\alpha_1(\cdot)$ such that

$$V_N^0(x,i) \leq \alpha(|x|) \qquad \forall x \in \mathcal{X}_N(i),\ \forall i \in \mathbb{I}_{\geq 0}$$

If Assumptions 2.25, 2.26, 2.30, 2.31, 2.34, and 2.36 are satisfied, it follows from the proof of Proposition 2.35 that, for all $i \in \mathbb{I}_{\geq 0}$

$$\begin{aligned} V_N^0(x,i) &\geq \alpha_1(|x|) && \forall x \in \mathcal{X}_N(i) \quad (2.30) \\ V_N^0(x,i) &\leq \alpha_2(|x|) && \forall x \in \mathcal{X}_N(i) \quad (2.31) \\ V_N^0(f(x,\kappa_N(x,i),i+1)) &\leq V_N^0(x,i) - \alpha_1(|x|) && \forall x \in \mathcal{X}_N(i) \quad (2.32) \end{aligned}$$

Since the bounds in inequalities (2.30)–(2.32) hold independently of time $i \in \mathbb{I}_{\geq 0}$, we may employ Theorems B.11 and B.13 of Appendix B with minor modification to obtain the following stability result.

| Assumption | Title | Page |
|---|---|---|
| 2.2 | Continuity of system and cost | 97 |
| 2.3 | Properties of constraint sets | 97 |
| 2.12 | Basic stability assumption | 115 |
| 2.13 | Implied invariance assumption | 115 |
| 2.16 | Bounds on stage and terminal costs | 118 |
| 2.23 | Weak controllability | 122 |

**Table 2.1:** Stability assumptions; time-invariant case.

| Assumption | Title | Page |
|---|---|---|
| 2.25 | Continuity of system and cost | 127 |
| 2.26 | Properties of constraint sets | 128 |
| 2.30 | Basic stability assumption | 129 |
| 2.31 | Implied invariance assumption | 129 |
| 2.34 | Bounds on stage and terminal costs | 130 |
| 2.36 | Uniform weak controllability | 130 |

**Table 2.2:** Stability assumptions; time-varying case.

**Theorem 2.37** (MPC stability; time-varying case)**.** *Suppose Assumptions 2.25, 2.26, 2.30, 2.31, 2.34, and 2.36 hold. Then, for each initial time $i \in \mathbb{I}_{\geq 0}$, the origin is asymptotically stable with a region of attraction $\mathcal{X}_N(i)$ for the time-varying system $x^+ = f(x, \kappa_N(x, j), j), j \geq i$.*

## 2.5 Examples of MPC

We already have discussed the general principles underlying the design of stabilizing model predictive controllers. The conditions on $\mathbb{X}_f$, $\ell(\cdot)$, and $V_f(\cdot)$ that guarantee stability can be implemented in a variety of ways so that MPC can take many different forms. We present in this section a representative set of examples of MPC and include in these examples the most useful forms for applications. These examples also display the roles of the six main assumptions used to guarantee closed-loop asymptotic stability. These six main assumptions are summarized in Table 2.1 for the time-invariant case and Table 2.2 for the

time-varying case. Refering back to this table may prove helpful while reading this section and comparing the various forms of MPC.

One question that is often asked is whether or not the terminal constraint is necessary. Since the conditions given previously are sufficient, necessity cannot be claimed. We discuss this further later. It is evident that the constraint arises because one often has a local, rather than a global, CLF for the system being controlled. In some situations, a global CLF *is* available; in such situations, a terminal constraint is not necessary and the terminal constraint set can be taken to be $\mathbb{R}^n$.

All model predictive controllers determine the control action to be applied to the system being controlled by solving, at each state, an optimal control that is usually constrained. If the constraints in the optimal control problem include hard state constraints, then the feasible region $\mathcal{X}_N$ is a subset of $\mathbb{R}^n$. The analysis given previously shows that if the initial state $x(0)$ lies in $\mathcal{X}_N$, so do all subsequent states, a property known as *recursive feasibility*; this property holds if all the assumptions made in our analysis hold. It is always possible, however, for unanticipated events to cause the state to become infeasible. In this case, the optimal control problem, as stated, cannot be solved, and the controller fails. It is therefore desirable, if this does not conflict with design aims, to employ soft state constraints in place of hard constraints. Otherwise, any implementation of the algorithms described subsequently should be modified to include a feature that enables recovery from faults that cause infeasibility. One remedy is to replace the hard constraints by soft constraints when the current state is infeasible, thereby restoring feasibility, and to revert back to the hard constraints as soon as they are satisfied by the current state.

### 2.5.1 Unconstrained Systems

For unconstrained systems, $\mathbb{U} = \mathbb{R}^m$ and $\mathbb{X} = \mathbb{R}^n$ so that Assumption 2.3 that postulates $\mathbb{U}$ is compact does not hold.

#### 2.5.1.1 Linear Time-Invariant Systems

Here $f(x,u) = Ax + Bu$ and $\ell(x,u) = (1/2)(|x|_Q^2 + |u|_R^2)$ where $Q > 0$ and $R > 0$. If $(A,B)$ is stabilizable, there exists a stabilizing controller $u = Kx$. Let $A_K := A + BK$, $Q_f := Q + K'RK$ and let $V_f : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be defined by $V_f(x) := (1/2)x'P_fx$ where $P_f > 0$ satisfies the Lyapunov equation

$$A_K'P_fA_K + Q_f = P_f$$

Since $V_N^0(x) \geq \ell(x, \kappa_N(x)) \geq (1/2)|x|_Q^2$, it follows that there exist $c_1 > 0$ and $c_2 > 0$ such that

$$V_N^0(x) \geq c_1|x|^2 \qquad V_f(x) \leq c_2|x|^2 \quad \forall x \in \mathbb{R}^n$$

With $f(\cdot)$, $\ell(\cdot)$, and $V_f(\cdot)$ defined this way, problem $\mathbb{P}_N(x)$ is an unconstrained parametric quadratic program[4] of the form $\min_{\mathbf{u}}(1/2)x'Lx + x'M\mathbf{u} + (1/2)\mathbf{u}'N\mathbf{u}$ so that $V_N^0(\cdot)$ is a quadratic function of the parameter $x$, and $\mathbf{u}^0(\cdot)$ and $\kappa_N(\cdot)$ are linear functions of $x$. Since

$$V_f(Ax + BKx) - V_f(x) + \ell(x, Kx) = x'[A_K'P_fA_K + Q_f - P_f]x = 0$$

for all $x \in \mathbb{R}^n$,

$$V_f(Ax + BKx) = V_f(x) - \ell(x, Kx) \quad \forall x \in \mathbb{R}^n$$

so that $V_f(\cdot)$ and $\mathbb{X}_f := \mathbb{R}^n$ satisfy Assumptions 2.12 and 2.13 with $u = Kx$; $V_f(\cdot)$ is a global CLF and $\mathcal{X}_N = \mathbb{R}^n$. Hence,

$$V_N^0(f(x, \kappa_N(x))) \leq V_N^0(x) - c_1|x|^2 \quad \forall x \in \mathbb{R}^n$$

It also follows from Lemma 2.15, which does not require the assumption that $\mathbb{U}$ is compact, that

$$V_N^0(x) \leq V_f(x) \leq c_2|x|^2 \quad \forall x \in \mathbb{R}^n$$

Summarizing, we have:

> With these assumptions on $V_f(\cdot)$, $\mathbb{X}_f$, and $\ell(\cdot)$, Assumptions 2.12, 2.13, and 2.16(b) are satisfied and, as shown previously, $V_N^0(\cdot)$ satisfies (2.26)–(2.28). It follows, as shown in Theorem 2.24(a), that the origin is globally exponentially stable for $x^+ = f(x, \kappa_N(x))$.

Since $V_f(\cdot)$ is a global CLF, there exists a simple stabilizing controller, namely $u = Kx$. In this case, there is no motivation to use MPC to obtain a stabilizing controller; standard linear $H_2$ or $H_\infty$ optimal control theory may be employed to obtain satisfactory control. The situation is different in the time-varying case, which we consider next.

[4] An optimization problem is parametric if it takes the form $\min_{\mathbf{u}}\{V_N(x;\mathbf{u}) \mid \mathbf{u} \in \mathcal{U}_N(x)\}$ where $\mathbf{u}$ is the decision variable and $x$ is a parameter; the solution is a function of the parameter $x$.

#### 2.5.1.2 Linear Time-Varying Systems

Here $f(x,u,i) = A_i x + B_i u$ and $\ell(x,u,i) = (1/2)(|x|^2_{Q_i} + |u|^2_{R_i})$ where $Q_i > 0$ and $R_i > 0$ for $i \in \mathbb{I}_{\geq 0}$. Because of the time-varying nature of the problem, it is impossible to obtain a controller by solving an infinite horizon optimal control problem $\mathbb{P}_\infty(x)$. It is possible, however, to determine for each $x \in \mathcal{X}_N(i)$ and each $i \in \mathbb{I}_{\geq 0}$, the MPC action $\kappa_N(x,i)$. Hence, MPC makes it possible to solve an otherwise intractable problem.

It is difficult to determine a time-varying global CLF satisfying Assumption 2.12 that could serve as the terminal cost function $V_f(\cdot)$, so we impose the condition that $\mathbb{X}_f = \{0\}$ in which case $V_f(\cdot)$ may be chosen arbitrarily; the simplest choice is $V_f(0) = 0$. With this choice, problem $\mathbb{P}_N(x,i)$ is a time-varying unconstrained parametric quadratic program that can be easily solved online either as a parametric program or by DP. The value function is a time-varying quadratic function of the parameter $x$, and $\kappa_N(\cdot)$ a time-varying linear function of $x$. The terminal cost function and constraint set satisfy Assumptions 2.30 and 2.31. Our choice of $\ell(\cdot)$ ensures the existence of a $c_1 > 0$ such that $V^0_N(x) \geq \ell(x, \kappa_N(x)) \geq c_1|x|^2$ for all $x \in \mathbb{R}^n$. Because Assumptions 2.30 and 2.31 are satisfied, we can employ Lemma 2.32 to show that $V^0_N(\cdot)$ satisfies the descent property in (2.29). Finally, if we assume that controllability Assumption 2.36 is satisfied, we obtain an upper bound for $V^0_N(\cdot)$. Summarizing, we have:

> With these assumptions on $V_f(\cdot)$ and $\mathbb{X}_f$, Assumptions 2.30, 2.31, and 2.34 are satisfied and if, in addition, Assumption 2.36 is satisfied, then, as shown previously, $V^0_N(\cdot)$ satisfies (2.30)–(2.32). It follows from Theorem 2.37 that, for each initial time $i$, the origin is asymptotically stable with a region of attraction $\mathcal{X}_N(i)$ for the time-varying system $x^+ = A_j x + B_j \kappa_N(x,j)$, $j \geq i$.

#### 2.5.1.3 Nonlinear Systems

Generally, when the system is nonlinear, albeit unconstrained, it is difficult to obtain a global CLF. We next present two forms of MPC. In the first, which is the simplest, the target set is the origin $\mathbb{X}_f = \{0\}$. In the second, $\mathbb{X}_f$ is a positive invariant ellipsoidal set for the system with linear control based on the linearization of the nonlinear system at the

origin. The system to be controlled is

$$x^+ = f(x,u)$$

in which $f(\cdot)$ is continuous. The cost function $V_N(\cdot)$ is defined as before by

$$V_N(x,\mathbf{u}) = \sum_{i=0}^{N-1} \ell(x(i),u(i)) + V_f(x(N))$$

where, for each $i$, $x(i) := \phi(i;x,\mathbf{u})$, the solution of $x^+ = f(x,u)$ at time $i$ if the initial state is $x$ at time 0 and the control is $\mathbf{u}$. Unless $V_f(\cdot)$ is a global CLF, a terminal constraint set $\mathbb{X}_f$ is required, so the optimal control problem solved online is

$$\mathbb{P}_N(x): \quad V_N^0(x) = \min_{\mathbf{u}} \{V_N(x,\mathbf{u}) \mid \mathbf{u} \in \mathcal{U}_N(x)\}$$

in which, in the absence of state and control constraints,

$$\mathcal{U}_N(x) := \{\mathbf{u} \mid \phi(N;x,\mathbf{u}) \in \mathbb{X}_f\}$$

Problem $\mathbb{P}_N(x)$ is an unconstrained nonlinear parametric program so that global solutions are not usually possible. We ignore this difficulty here and assume in this section that the global solution for any $x$ may be computed online. We address the problem when this is not possible in Section 2.8.

**Case 1.** $\mathbb{X}_f = \{0\}$, $V_f(0) = 0$. This is the simplest case. As before, we note that Assumptions 2.12 and 2.13 hold if the origin is an equilibrium point, i.e., if $f(0,0) = 0$. If, in addition, we assume that $\ell(\cdot)$ satisfies Assumption 2.16(a), namely that there exists a $\mathcal{K}_\infty$ function $\alpha_1(\cdot)$ such that $\ell(\cdot)$ satisfies $\ell(x,u) \geq \alpha_1(|x|)$ for all $(x,u) \in \mathbb{R}^n \times \mathbb{R}^m$, then, for all $x \in \mathcal{X}_N$

$$\begin{aligned} &V_N^0(x) \geq \ell(x,\kappa_N(x)) \geq \alpha_1(|x|) \\ &V_N^0(f(x,\kappa_N(x))) - V_N^0(x) \leq -\ell(x,\kappa_N(x)) \leq -\alpha_1(|x|) \end{aligned}$$

where the latter inequality is a consequence of Lemma 2.14. If we also assume the controllability Assumption 2.23 is satisfied, then

$$V_N^0(x) \leq \alpha(|x|) \ \forall x \in \mathcal{X}_N$$

Summarizing, we have:

If these assumptions on $V_f(\cdot)$, $\mathbb{X}_f$, and $\ell(\cdot)$ hold, and Assumptions 2.2 and 2.3 are satisfied, then Assumptions 2.12, 2.13, and 2.16(a) are satisfied. If, in addition, the controllability Assumption 2.23 is satisfied, then it follows from Theorem 2.24(b) that the origin is asymptotically stable with a region of attraction $\mathcal{X}_N$ for $x^+ = f(x, \kappa_N(x))$.

**Case 2.** $V_f(x) = (1/2)|x|_P^2$, $\mathbb{X}_f = \{x \mid V_f(x) \le a\}$. In this case we obtain a terminal cost function $V_f(\cdot)$ and a terminal constraint set $\mathbb{X}_f$ by linearization of the nonlinear system $x^+ = f(x,u)$ at the origin. Hence, for the purpose of this case we assume $f(\cdot)$ and $\ell(\cdot)$ are twice continuously differentiable. Suppose then the linearized system is

$$x^+ = Ax + Bu$$

where $A := f_x(0,0)$ and $B := f_u(0,0)$. We assume that $(A,B)$ is stabilizable and we choose any controller $u = Kx$ such that the origin is globally exponentially stable for the system $x^+ = A_K x$, $A_K := A + BK$, i.e., such that $A_K$ is stable. Suppose also that the stage cost $\ell(\cdot)$ is defined by $\ell(x,u) := (1/2)(|x|_Q^2 + |u|_R^2)$ where $Q$ and $R$ are positive definite; hence $\ell(x,Kx) = (1/2)x'Q^*x$ where $Q^* := (Q + K'RK)$. Let $P$ be defined by the Lyapunov equation

$$A_K' P A_K + 2Q^* = P$$

The reason for the factor 2 will become apparent soon. Since $Q^*$ is positive definite and $A_K$ is stable, $P$ is positive definite. Let the terminal cost function $V_f(\cdot)$ be defined by

$$V_f(x) := (1/2)x'Px$$

Clearly $V_f(\cdot)$ is a global CLF for the linear system $x^+ = Ax+Bu$. Indeed, it follows from its definition that $V_f(\cdot)$ satisfies

$$V_f(A_K x) + x'Q^*x - V_f(x) = 0 \quad \forall x \in \mathbb{R}^n \tag{2.33}$$

Consider now the nonlinear system $x^+ = f(x,u)$ with linear control $u = Kx$. The controlled system satisfies

$$x^+ = f(x, Kx)$$

We wish to show that $V_f(\cdot)$ is a local CLF for $x^+ = f(x,u)$ in some neighborhood of the origin; specifically, we wish to show there exists an $a \in (0,\infty)$ such that

$$V_f(f(x,Kx)) + (1/2)x'Q^*x - V_f(x) \le 0 \quad \forall x \in W(a) \tag{2.34}$$

where, for all $a > 0$, $W(a) := \text{lev}_a V_f = \{x \mid V_f(x) \le a\}$ is a sublevel set of $V_f$. Since $P$ is positive definite, $W(a)$ is an ellipsoid with the origin as its center. Comparing inequality (2.34) with (2.33), we see that (2.34) is satisfied if

$$V_f(f(x,Kx)) - V_f(A_K x) \le (1/2)x'Q^*x \quad \forall x \in W(a) \tag{2.35}$$

Let $e(\cdot)$ be defined as follows

$$e(x) := f(x,Kx) - A_K x$$

so that

$$V_f(f(x,Kx)) - V_f(A_K x) = (A_K x)'Pe(x) + (1/2)e(x)'Pe(x) \tag{2.36}$$

By definition, $e(0) = f(0,0) - A_K 0 = 0$ and $e_x(x) = f_x(x,Kx) + f_u(x,Kx)K - A_K$. It follows that $e_x(0) = 0$. Since $f(\cdot)$ is twice continuously differentiable, for any $\delta > 0$, there exists a $c_\delta > 0$ such that $|e_{xx}(x)| \le c_\delta$ for all $x$ in $\delta\mathcal{B}$. From Proposition A.11 in Appendix A,

$$\begin{aligned} |e(x)| &= \left| e(0) + e_x(0)x + \int_0^1 (1-s)x' e_{xx}(sx)x\,ds \right| \\ &\le \int_0^1 (1-s)c_\delta |x|^2 ds \le (1/2)c_\delta |x|^2 \end{aligned}$$

for all $x$ in $\delta\mathcal{B}$. From (2.36), we see that there exists an $\varepsilon \in (0,\delta]$ such that (2.35), and, hence, (2.34), is satisfied for all $x \in \varepsilon\mathcal{B}$. Because of our choice of $\ell(\cdot)$, there exists a $c_1 > 0$ such that $V_f(x) \ge \ell(x,Kx) \ge c_1|x|^2$ for all $x \in \mathbb{R}^n$. It follows that $x \in W(a)$ implies $|x| \le \sqrt{a/c_1}$. We can choose $a$ to satisfy $\sqrt{a/c_1} = \varepsilon$. With this choice, $x \in W(a)$ implies $|x| \le \varepsilon \le \delta$, which, in turn, implies (2.34) is satisfied.

We conclude that there exists an $a > 0$ such that $V_f(\cdot)$ and $\mathbb{X}_f := W(a)$ satisfy Assumptions 2.12 and 2.13. For each $x \in \mathbb{X}_f$ there exists a $u = Kx$ such that $V_f(x,u) \le V_f(x) - \ell(x,u)$ since $\ell(x,Kx) = (1/2)x'Q^*x$. Our assumption that $\ell(x,u) = (1/2)(x'Qx + u'Ru)$ where $Q$ and $R$ are positive definite, and our definition of $V_f(\cdot)$ ensure the existence of positive constants $c_1$ and $c_2$ such that $V_N^0(x) \ge c_1|x|^2$ for all $\mathbb{R}^n$, and $V_f(x) \le c_2|x|^2$ for all $x \in \mathbb{X}_f$ thereby satisfying Assumption 2.16. The set $\mathcal{X}_N = \mathbb{R}^n$ because the optimal control problem $\mathbb{P}_N(x)$ has no state or terminal constraints. Finally, by definition, the set $\mathbb{X}_f$ contains the origin in its interior. Summarizing, we have:

> If these assumptions on $V_f(\cdot)$, $\mathbb{X}_f$, and $\ell(\cdot)$ hold, and Assumptions 2.2 and 2.3 are satisfied, then Assumptions 2.12,

2.13, and 2.16(b) are satisfied, $X_f$ contains the origin in its interior, and $\mathcal{X}_N = \mathbb{R}^n$. Hence, by Theorem 2.24(b), the origin is globally asymptotically stable for $x^+ = f(x, \kappa_N(x))$. Also, by Theorem 2.24(b), the origin is exponentially stable for $x^+ = f(x, \kappa_N(x))$ with any sublevel set of $V_N^0(\cdot)$ as a region of attraction.

### 2.5.2 Systems with Control Constraints

Usually, when constraints and/or nonlinearities are present, it is impossible to obtain a *global* CLF to serve as the terminal cost function $V_f(\cdot)$. There are, however, a few special cases where this is possible; we examine two such cases in this section.

#### 2.5.2.1 Linear Stable Systems

The system to be controlled is $x^+ = Ax + Bu$ where $A$ is stable (its eigenvalues lie strictly inside the unit circle) and the control $u$ is subject to the constraint $u \in \mathbb{U}$ where $\mathbb{U}$ is compact and contains the origin in its interior. The stage cost is $\ell(x, u) = (1/2)(x'Qx + u'Ru)$ where $Q$ and $R$ are positive definite. To establish stability of the systems under MPC (or RHC), we wish to obtain a *global* CLF to serve as the terminal cost function $V_f(\cdot)$. This is usually difficult because any linear control law $u = Kx$, say, will transgress the control constraint for $x$ sufficiently large. In other words, it is usually impossible to find a $V_f(\cdot)$ such that there exists a $u \in \mathbb{U}$ satisfying $V_f(Ax + Bu) \leq V_f(x) - \ell(x, u)$ for all $x$ in $\mathbb{R}^n$. Since $A$ is stable, however, it is possible to obtain a Lyapunov function for the autonomous system $x^+ = Ax$ that is a suitable candidate for $V_f(\cdot)$; in fact, for all $Q > 0$, there exists a $P > 0$ such that

$$A'PA + Q = P$$

Let $V_f(\cdot)$ be defined by

$$V_f(x) = (1/2)x'Px$$

With $f(\cdot)$, $\ell(\cdot)$, and $V_f(\cdot)$ defined thus, $\mathbb{P}_N(x)$ is a parametric quadratic problem if the constraint set $\mathbb{U}$ is polyhedral and global solutions may be computed online. The terminal cost function $V_f(\cdot)$ satisfies

$$V_f(Ax) + (1/2)x'Qx - V_f(x) = (1/2)x'(A'PA + Q - P)x = 0$$

for all $x \in \mathbb{X}_f := \mathbb{R}^n$. We see that for all $x \in \mathbb{X}_f$, there exists a $u$, namely $u = 0$, such that $V_f(Ax + Bu) \leq V_f(x) - \ell(x, u)$; $\ell(x, u) = (1/2)x'Qx$ when $u = 0$. Since there are no state or terminal constraints, $\mathcal{X}_N = \mathbb{R}^n$. It follows that there exist positive constants $c_1$ and $c_2$ such that

$$
\begin{aligned}
V_N^0(x) &\geq c_1|x|^2 \\
V_N^0(f(x, \kappa_N(x))) &\leq V_N^0(x) - c_1|x|^2 \\
V_N^0(x) &\leq c_2|x|^2
\end{aligned}
$$

for all $x \in \mathcal{X}_N = \mathbb{R}^n$. Summarizing, we have:

> If these assumptions on $V_f(\cdot)$, $\mathbb{X}_f$, and $\ell(\cdot)$ hold, and Assumption 2.3 is satisfied, then Assumptions 2.12, 2.13, and 2.16(b) are satisfied and $\mathcal{X}_N = \mathbb{X}_f = \mathbb{R}^n$. It follows from Theorem 2.24(a) that the origin is globally, exponentially stable for the controlled system $x^+ = Ax + B\kappa_N(x)$.

An extension of this approach for unstable $A$ is used in Chapter 6.

#### 2.5.2.2 Neutrally Stable Systems

The system to be controlled is, again, $x^+ = Ax + Bu$, but $A$ is now *neutrally* stable[5] and the control $u$ is subject to the constraint $u \in \mathbb{U}$ where $\mathbb{U}$ is compact, contains the origin in its interior, and has the form

$$
\mathbb{U} = \{u \in \mathbb{R}^m \mid u_i \in [a_i, b_i],\ i \in \mathbb{I}_{1:m}\}
$$

where $a_i < 0 < b_i$. The linear system $x^+ = Ax$ is therefore Lyapunov stable but not asymptotically stable. We assume that the pair $(A, B)$ is controllable. This problem is much more challenging than the problem considered immediately above since control has to be applied to make the system asymptotically stable, and this control can transgress the control constraints. Any linear control law $u = Kx$, no matter how small $K$, transgresses the control constraints for large enough $x$. Recent research, however, has demonstrated the existence of a *global* CLF for $x^+ = Ax + Bu$ where $A$ is neutrally stable; the Lyapunov function is, unusually, based on a *nonlinear* control law of the form $u = \text{sat}(Kx)$ where $\text{sat}(\cdot)$ is the vector saturation function defined by

$$
\text{sat}(u) := \begin{bmatrix} \text{sat}(u_1) & \text{sat}(u_2) & \cdots & \text{sat}(u_m) \end{bmatrix}'
$$

[5] A linear system is neutrally stable if some of the eigenvalues of $A$ lie on the unit circle and are simple, and the remaining eigenvalues lie within the unit circle.

in which $u_i$ is the $i$th component of the vector $u$, and

$$\text{sat}(u_i) := \begin{cases} b_i & u_i \geq b_i \\ u_i & u_i \in [a_i, b_i] \\ a_i & u_i \leq a_i \end{cases}$$

If $A$ is neutrally stable, there exists a $P > 0$ such that

$$A'PA \leq P$$

Note that this is weaker than the corresponding result in the previous section. If $\kappa$ satisfies

$$\kappa B'PB < I$$

however, then the linear control law $u = Kx$ in which

$$K := -\kappa B'PA$$

globally stabilizes the *unconstrained* system, i.e., the matrix $A_K := A + BK$ is stable. Hence, for all $Q^* > 0$, there exists a positive definite matrix $P^*$ satisfying

$$A_K' P^* A_K + Q^* = P^*$$

Let $\kappa_f(\cdot)$ denote the *nonlinear* control law defined by

$$\kappa_f(x) := \text{sat}(Kx)$$

Then, as shown in Kim, Yoon, Jadbabaie, and Persis (2004), there exists a $\lambda > 0$ such that $V_f(\cdot)$ defined by

$$V_f(x) := (1/2)x'P^*x + \lambda(x'Px)^{3/2}$$

is a *global* CLF for $x^+ = Ax + Bu$, satisfying

$$V_f(Ax + B\kappa_f(x)) - V_f(x) + (1/2)|x|^2_{Q^*} \leq 0 \tag{2.37}$$

for all $x \in \mathbb{R}^n$. Suppose now the optimal control problem defining the receding horizon controller (or model predictive controller) is

$$\mathbb{P}_N(x): \quad V_N^0(x) = \min_{\mathbf{u}} \{V_N(x, \mathbf{u}) \mid \mathbf{u} \in \mathbb{U}^N\}$$

in which the cost $V_N(\cdot)$ is defined by

$$V_N(x, \mathbf{u}) := \sum_{i=0}^{N-1} \ell(x(i), u(i)) + V_f(x(N))$$

and, for all $i$, $x(i) = \phi(i; x, \mathbf{u})$, the solution of $x^+ = Ax + Bu$ at time $i$ if the initial state at time 0 is $x$ and the control sequence is $\mathbf{u}$. The stage cost is

$$\ell(x, u) := (1/2)(|x|_Q^2 + |u|_R^2)$$

where $Q$ and $R$ are positive definite and $R$ is diagonal. We wish to ensure that $V_f(\cdot)$, defined previously, and $\mathbb{X}_f := \mathbb{R}^n$ satisfy Assumption 2.12 so that the system with MPC has satisfactory stability properties. But Assumptions 2.12 and 2.13 are satisfied for all $x \in \mathbb{R}^n$ if

$$V_f(Ax + B\kappa_f(x)) - V_f(x) + \ell(x, \kappa_f(x)) \le 0 \tag{2.38}$$

It follows from (2.37) that Assumption 2.12 is satisfied if

$$(1/2)x'Q^*x \ge \ell(x, \kappa_f(x)) = (1/2)x'Qx + (1/2)\kappa_f(x)'R\kappa_f(x) \tag{2.39}$$

We can achieve this by choosing $Q^*$ appropriately. Suppose

$$Q^* := Q + K'RK$$

Then

$$x'Q^*x = x'Qx + (Kx)'RKx$$

But

$$\kappa_f(x)'R\kappa_f(x) = (\text{sat}(Kx))'R\text{sat}(Kx)$$

Since $R$ is diagonal and positive definite, and since $(\text{sat}(a))^2 \le a^2$ if $a$ is a scalar, we have

$$\kappa_f(x)'R\kappa_f(x) \le (\text{sat}(Kx))'R\text{sat}(Kx) \le (Kx)'RKx \ \forall x \in \mathbb{R}^n$$

It follows that (2.39) and, hence, (2.38) are satisfied. Therefore, with $V_f(\cdot)$ as defined previously, $\mathbb{X}_f := \mathbb{R}^n$, and $\ell(\cdot)$ as defined previously, Assumptions 2.12, 2.13, and 2.16(b) are satisfied. Summarizing, we have:

> If these assumptions on $V_f(\cdot)$, $\mathbb{X}_f$, and $\ell(\cdot)$ hold, and Assumption 2.3 is satisfied, then Assumptions 2.12, 2.13, and 2.16(b) are satisfied, and $\mathcal{X}_N = \mathbb{X}_f = \mathbb{R}^n$. It follows from Theorem 2.24 that the origin is globally exponentially stable for the controlled system $x^+ = Ax + B\kappa_N(x)$.

Because $V_f(\cdot)$ is not quadratic, the optimal control problem $\mathbb{P}_N(x)$ is no longer a quadratic program.

### 2.5.3 Systems with Control and State Constraints

We turn now to the consideration of systems with control and state constraints. In this situation determination of a global CLF is usually difficult if not impossible. Hence we show how local CLFs may be determined together with an invariant region in which they are valid.

#### 2.5.3.1 Linear Systems

The system to be controlled is $x^+ = Ax + Bu$ where $A$ is not necessarily stable, the control $u$ is subject to the constraint $u \in \mathbb{U}$ where $\mathbb{U}$ is compact and contains the origin in its interior, and the state $x$ is subject to the constraint $x \in \mathbb{X}$ where $\mathbb{X}$ is closed and contains the origin in its interior. The stage cost is $\ell(x,u) = (1/2)(x'Qx + u'Ru)$ where $Q$ and $R$ are positive definite. Because of the constraints, it is difficult to obtain a global CLF. Hence we restrict ourselves to the more modest goal of obtaining a local CLF and proceed as follows. If $(A,B)$ is stabilizable, the solution to the infinite horizon *unconstrained* optimal control problem $\mathbb{P}_\infty^{\text{uc}}(x)$ is known; the value function for this problem is $V_\infty^{\text{uc}}(x) = (1/2)x'Px$ where $P$ is the unique (in the class of positive semidefinite matrices) solution to the discrete algebraic Riccati equation

$$P = A_K' P A_K + Q^*$$

in which $A_K := A + BK$, $Q^* := Q + K'RK$, and $u = Kx$, in which $K$ is defined by

$$K := -(B'PB + R)^{-1}B'PA'$$

is the optimal controller. The value function $V_\infty^{\text{uc}}(\cdot)$ for the infinite horizon unconstrained optimal control problem $\mathbb{P}_\infty^{\text{uc}}(x)$ satisfies

$$V_\infty^{\text{uc}}(x) = \min_u \{\ell(x,u) + V_\infty^{\text{uc}}(Ax + Bu)\} = \ell(x,Kx) + V_\infty^{\text{uc}}(A_K x)$$

It is known that $P$ is positive definite. We define the terminal cost $V_f(\cdot)$ by

$$V_f(x) := V_\infty^{\text{uc}}(x) = (1/2)x'Px$$

If $\mathbb{X}$ and $\mathbb{U}$ are polyhedral, problem $\mathbb{P}_N(x)$ is a parametric quadratic program that may be solved online using standard software. The terminal cost function $V_f(\cdot)$ satisfies

$$V_f(A_K x) + (1/2)x'Q^*x - V_f(x) \le 0 \ \forall x \in \mathbb{R}^n$$

The controller $u = Kx$ does not necessarily satisfy the control and state constraints, however. The terminal constraint set $\mathbb{X}_f$ must be chosen with this requirement in mind. We may choose $\mathbb{X}_f$ to be the maximal invariant constraint admissible set for $x^+ = A_K x$; this is the largest set $W$ with respect to inclusion[6] satisfying: (a) $W \subseteq \{x \in \mathbb{X} \mid Kx \in \mathbb{U}\}$, and (b) $x \in W$ implies $x(i) = A_K^i x \in W$ for all $i \geq 0$. Thus $\mathbb{X}_f$, defined this way, is control invariant[7] for $x^+ = Ax + Bu$, $u \in \mathbb{U}$. If the initial state $x$ of the system is in $\mathbb{X}_f$, the controller $u = Kx$ maintains the state in $\mathbb{X}_f$ and satisfies the state and control constraints for all future time ($x(i) = A_K^i x \in \mathbb{X}_f \subset \mathbb{X}$ and $u(i) = Kx(i) \in \mathbb{U}$ for all $i \geq 0$). Hence, with $V_f(\cdot)$, $\mathbb{X}_f$, and $\ell(\cdot)$ as defined previously, Assumptions 2.12, 2.13, and 2.16(b) are satisfied. Summarizing, we have:

> If these assumptions on $V_f(\cdot)$, $\mathbb{X}_f$, and $\ell(\cdot)$ hold, and Assumption 2.3 is satisfied, then Assumptions 2.12, 2.13, and 2.16(b) are satisfied, and $\mathbb{X}_f$ contains the origin in its interior. Hence, by Theorem 2.24, the origin is asymptotically stable with a region of attraction $\mathcal{X}_N$ for the controlled system $x^+ = Ax + B\kappa_N(x)$, and exponentially stable with a region of attraction any sublevel set of $V_N^0(\cdot)$.

It is, of course, not necessary to choose $K$ and $V_f(\cdot)$ as above. Any $K$ such that $A_K = A + BK$ is stable may be chosen, and $P$ may be obtained by solving the Lyapunov equation $A_K' P A_K + Q = P$. With $V_f(x) := (1/2)x'Px$ and $\mathbb{X}_f$ the maximal constraint admissible set for $x^+ = A_K x$, the origin may be shown, as above, to be asymptotically stable with a region of attraction $\mathcal{X}_N$ for $x^+ = Ax + B\kappa_N(x)$, and exponentially stable with a region of attraction any sublevel set of $V_N^0(\cdot)$. The optimal control problem is, again, a quadratic program. The terminal set $\mathbb{X}_f$ may be chosen, as above, to be the maximal invariant constraint admissible set for $x^+ = A_K x$, or it may be chosen to be a suitably small sublevel set of $V_f(\cdot)$; by suitably small, we mean small enough to ensure $\mathbb{X}_f \subseteq \mathbb{X}$ and $K\mathbb{X}_f \subseteq \mathbb{U}$. The set $\mathbb{X}_f$, if chosen this way, is ellipsoidal, a subset of the maximal constraint admissible set, and is positive invariant for $x^+ = A_K x$. The disadvantage of this choice is that $\mathbb{P}_N(x)$ is no longer a quadratic program, though it remains a convex program for which software exists.

[6] $W \in \mathcal{W}$ is the largest set in $\mathcal{W}$ with respect to inclusion if $W' \subseteq W$ for any $W' \in \mathcal{W}$.

[7] A set $X$ is positive invariant for $x^+ = f(x)$ if $x \in X$ implies $x^+ = f(x) \in X$; a set $X$ is control invariant for $x^+ = f(x,u)$, $u \in \mathbb{U}$ if $x \in X$ implies the existence of a $u \in \mathbb{U}$ such that $x^+ = f(x,u) \in X$.

The choice $V_f(\cdot) = V_\infty^{\text{uc}}(\cdot)$ results in an interesting property of the closed-loop system $x^+ = Ax + B\kappa_N(x)$. Generally, the terminal constraint set $\mathbb{X}_f$ is *not* positive invariant for the controlled system $x^+ = Ax + B\kappa_N(x)$. Thus, in solving $\mathbb{P}_N(x)$ for an initial state $x \in \mathbb{X}_f$, the "predicted" state sequence $\mathbf{x}^0(x) = \{x^0(0;x), x^0(1;x), \ldots, x^0(N;x)\}$ starts and ends in $\mathbb{X}_f$ but does not necessarily remain in $\mathbb{X}_f$. Thus $x^0(0;x) = x \in \mathbb{X}_f$ and $x^0(N;x) \in \mathbb{X}_f$, because of the terminal constraint in the optimal control problem, but, for any $i \in \mathbb{I}_{1:N-1}$, $x^0(i;x)$ may lie outside of $\mathbb{X}_f$. In particular, $x^+ = Ax + B\kappa_N(x) = x^0(1;x)$ may lie outside of $\mathbb{X}_f$; $\mathbb{X}_f$ is *not* necessarily positive invariant for the controlled system $x^+ = Ax + B\kappa_N(x)$.

Consider now the problem $\mathbb{P}_N^{\text{uc}}(x)$ defined in the same way as $\mathbb{P}_N(x)$ except that *all* constraints are omitted so that $\mathcal{U}_N(x) = \mathbb{R}^{Nm}$

$$\mathbb{P}_N^{\text{uc}}(x): \qquad V_N^{\text{uc}}(x) = \min_{\mathbf{u}} V_N(x, \mathbf{u})$$

in which $V_N(\cdot)$ is defined as previously by

$$V_N(x, \mathbf{u}) := \sum_{i=0}^{N-1} \ell(x(i), u(i)) + V_f(x(N))$$

with $V_f(\cdot)$ the value function for the infinite horizon unconstrained optimal control problem, i.e., $V_f(x) := V_\infty^{\text{uc}}(x) = (1/2)x'Px$. With these definitions, it follows that

$$V_N^{\text{uc}}(x) = V_\infty^{\text{uc}}(x) = V_f(x) = (1/2)x'Px$$
$$\kappa_N^{\text{uc}}(x) = Kx, \quad K = -(B'PB + R)^{-1}B'PA$$

for all $x \in \mathbb{R}^n$; $u = Kx$ is the optimal controller for the unconstrained infinite horizon problem. But $\mathbb{X}_f$ *is* positive invariant for $x^+ = A_K x$.

We now claim that with $V_f(\cdot)$ chosen to equal to $V_\infty^{\text{uc}}(\cdot)$, the terminal constraint set $\mathbb{X}_f$ *is* positive invariant for $x^+ = Ax + B\kappa_N(x)$. We do this by showing that $V_N^0(x) = V_N^{\text{uc}}(x) = V_\infty^{\text{uc}}(x)$ for all $x \in \mathbb{X}_f$, so that the associated control laws are the same, i.e., $\kappa_N(x) = Kx$. First, because $\mathbb{P}_N^{\text{uc}}(x)$ is identical with $\mathbb{P}_N(x)$ except for the absence of all constraints, we have

$$V_N^{\text{uc}}(x) = V_f(x) \le V_N^0(x) \quad \forall x \in \mathcal{X}_N \supseteq \mathbb{X}_f$$

Second, from Lemma 2.15,

$$V_N^0(x) \le V_f(x) \quad \forall x \in \mathbb{X}_f$$

Hence $V_N^0(x) = V_N^{\text{uc}}(x) = V_f(x)$ for all $x \in \mathbb{X}_f$. That $\kappa_N(x) = Kx$ for all $x \in \mathbb{X}_f$ follows from the uniqueness of the solutions to the problems $\mathbb{P}_N(x)$ and $\mathbb{P}_N^{\text{uc}}(x)$. Summarizing, we have:

> If $V_f(\cdot)$ is chosen to be the value function for the unconstrained infinite horizon optimal control problem, if $u = Kx$ is the associated controller, and if $\mathbb{X}_f$ is invariant for $x^+ = A_K x$, then $\mathbb{X}_f$ is also positive invariant for the controlled system $x^+ = Ax + B\kappa_N(x)$. Also $\kappa_N(x) = Kx$ for all $x \in \mathbb{X}_f$.

#### 2.5.3.2 Nonlinear Systems

The system to be controlled is

$$x^+ = f(x, u)$$

in which $f(\cdot)$ is assumed to be twice continuously differentiable. The system is subject to state and control constraints

$$x \in \mathbb{X} \qquad u \in \mathbb{U}$$

in which $\mathbb{X}$ is closed and $\mathbb{U}$ is compact; each set contains the origin in its interior. The cost function is defined by

$$V_N(x, \mathbf{u}) = \sum_{i=0}^{N-1} \ell(x(i), u(i)) + V_f(x(N))$$

in which, for each $i$, $x(i) := \phi(i; x, \mathbf{u})$, the solution of $x^+ = f(x, u)$ at time $i$ if the initial state is $x$ at time 0 and the control is $\mathbf{u}$. The stage cost $\ell(\cdot)$ is defined by

$$\ell(x, u) := (1/2)(|x|_Q^2 + |u|_R^2)$$

in which $Q$ and $R$ are positive definite. The optimal control problem $\mathbb{P}_N(x)$ is defined by

$$\mathbb{P}_N(x): \quad V_N^0(x) = \min_{\mathbf{u}} \{V_N(x, \mathbf{u}) \mid \mathbf{u} \in \mathcal{U}_N(x)\}$$

in which $\mathcal{U}_N(x)$ is defined by (2.6) and includes the terminal constraint $x(N) = \phi(N; x, \mathbf{u}) \in \mathbb{X}_f$ (in addition to the state and control constraints). Our first task is to choose the ingredients $V_f(\cdot)$ and $\mathbb{X}_f$ of

the optimal control problem to ensure asymptotic stability of the origin for the controlled system. We proceed as in Section 2.5.1.3, i.e., we linearize the system at the origin to obtain the linear model

$$x^+ = Ax + Bu$$

in which $A = f_x(0,0)$ and $B = f_u(0,0)$ and assume, as before, that $(A,B)$ is stabilizable. We choose any controller $u = Kx$ such that $A_K$ is stable. Choose $Q^* := (Q + K'RK)$ and let $P$ be defined by the Lyapunov equation

$$A_K' P A_K + 2Q^* = P$$

The terminal cost function $V_f(\cdot)$ is again chosen to be

$$V_f(x) := (1/2)x'Px$$

and $\mathbb{X}_f$ is chosen to be a sublevel set $W(a) := \text{lev}_a V_f := \{x \mid V_f(x) \le a\}$ for some suitably chosen constant $a$. As shown in Section 2.5.1.3, under the assumptions made previously, there exists an $a > 0$ such that

$$V_f(f(x,Kx)) + \ell(x,Kx) - V_f(x) \le 0 \ \forall x \in \mathbb{X}_f := W(a)$$

in which $x^+ = f(x,Kx)$ describes the nonlinear system if the linear controller $u = Kx$ is employed. To take into account the state and control constraints, we reduce $a$ if necessary to satisfy, in addition,

$$\mathbb{X}_f \subseteq \mathbb{X} \qquad K\mathbb{X}_f \subseteq \mathbb{U}$$

With $f(\cdot)$, $\ell(\cdot)$, and $V_f(\cdot)$ defined thus, $\mathbb{P}_N(x)$ is a constrained parametric nonlinear optimization problem for which global solutions cannot necessarily be obtained online; we temporarily ignore this problem. Because $\mathbb{X}_f$ is a sublevel set of $V_f(\cdot)$, it is positive invariant for $x^+ = f(x,Kx)$. It follows that $V_f(\cdot)$ and $\mathbb{X}_f$ satisfy Assumptions 2.12 and 2.13. Summarizing, we have:

> If these assumptions on $V_f(\cdot)$, $\mathbb{X}_f$, and $\ell(\cdot)$ hold, and Assumptions 2.2 and 2.3 are satisfied, then Assumptions 2.12, 2.13, and 2.16(b) are satisfied, and $X_f$ contains the origin in its interior. Hence, by Theorem 2.24(b), the origin is asymptotically stable for $x^+ = f(x,\kappa_N(x))$ in $\mathcal{X}_N$ and exponentially stable for $x^+ = f(x,\kappa_N(x))$ in any sublevel set of $V_N^0(\cdot)$.

Asymptotic stability of the origin in $\mathcal{X}_N$ may also be established when $\mathbb{X}_f := \{0\}$ if Assumption 2.23 is invoked.

## 2.6 Is a Terminal Constraint Set $\mathbb{X}_f$ Necessary?

While addition of a terminal cost $V_f(\cdot)$ does not materially affect the optimal control problem, addition of a terminal constraint $x(N) \in \mathbb{X}_f$, which is a state constraint, may have a significant effect. In particular, problems with only control constraints are usually easier to solve. So if state constraints are not present or if they are handled by penalty functions (soft constraints), it is highly desirable to avoid the addition of a terminal constraint. Moreover, it is possible to establish continuity of the value function for a range of optimal control problems *if* there are no state constraints; continuity of the value function ensures a degree of robustness (see Chapter 3). It is therefore natural to ask if the terminal constraint can be omitted without affecting stability. There are several answers to this question.

### 2.6.1 Replacing the Terminal Constraint by a Terminal Cost

A reasonably simple procedure is to replace the terminal constraint $x(N) \in \mathbb{X}_f$ by a terminal cost that is sufficiently large to ensure automatic satisfaction of the terminal constraint.

We assume, as in the examples of MPC discussed in Section 2.5, that the terminal cost function $V_f(\cdot)$, the constraint set $\mathbb{X}_f$, and the stage cost $\ell(\cdot)$ for the optimal control problem $\mathbb{P}_N(x)$ are chosen to satisfy Assumptions 2.12, 2.13 and 2.16 so that there exists a local control law $\kappa_f : \mathbb{X}_f \to \mathbb{U}$ such that $\mathbb{X}_f \subseteq \{x \in \mathbb{X} \mid \kappa_f(x) \in \mathbb{U}\}$ is positive invariant for $x^+ = f(x, \kappa_f(x))$ and $V_f(f(x, \kappa_f(x))) + \ell(x, \kappa_f(x)) \leq V_f(x)$ for all $x \in \mathbb{X}_f$. We assume that the function $V_f(\cdot)$ is defined on $\mathbb{X}$ even though it possesses the property $V_f(f(x, \kappa_f(x))) + \ell(x, \kappa_f(x)) \leq V_f(x)$ only in $\mathbb{X}_f$. In many cases, even if the system being controlled is nonlinear, $V_f(\cdot)$ is quadratic and positive definite, and $\kappa_f(\cdot)$ is linear. The set $\mathbb{X}_f$ may be chosen to be a sublevel set of $V_f(\cdot)$ so that $\mathbb{X}_f = W(a) := \{x \mid V_f(x) \leq a\}$ for some $a > 0$. We discuss in the sequel a modified form of the optimal control problem $\mathbb{P}_N(x)$ in which the terminal cost $V_f(\cdot)$ is replaced by $\beta V_f(\cdot)$ and the terminal constraint $\mathbb{X}_f$ is omitted, and show that if $\beta$ is sufficiently large the solution of the modified optimal control problem is such that the optimal terminal state nevertheless lies in $\mathbb{X}_f$ so that terminal constraint is implicitly satisfied.

For all $\beta \geq 1$, let $\mathbb{P}_N^\beta(x)$ denote the modified optimal control problem defined by

$$\hat{V}_N^\beta(x) = \min_{\mathbf{u}} \{V_N^\beta(x, \mathbf{u}) \mid \mathbf{u} \in \hat{\mathcal{U}}_N(x)\}$$

in which the cost function to be minimized is now

$$V_N^\beta(x,\mathbf{u}) := \sum_{i=0}^{N-1} \ell(x(i),u(i)) + \beta V_f(x(N))$$

in which, for all $i$, $x(i) = \phi(i;x,\mathbf{u})$, the solution at time $i$ of $x^+ = f(x,u)$ when the initial state is $x$ and the control sequence is $\mathbf{u}$. The control constraint set $\hat{\mathcal{U}}_N(x)$ ensures satisfaction of the state and control constraints, but not the terminal constraint, and is defined by

$$\hat{\mathcal{U}}_N(x) := \{\mathbf{u} \mid u(i) \in \mathbb{U},\ x(i) \in \mathbb{X},\ i \in \mathbb{I}_{0,N-1}, x(N) \in \mathbb{X}\}$$

The cost function $V_N^\beta(\cdot)$ with $\beta = 1$ is identical to the cost function $V_N(\cdot)$ employed in the standard problem $\mathbb{P}_N$ considered previously. Let $\hat{\mathcal{X}}_N := \{x \in \mathbb{X} \mid \hat{\mathcal{U}}_N(x) \neq \emptyset\}$ denote the domain of $\hat{V}^\beta(\cdot)$; let $\mathbf{u}^\beta(x)$ denote the solution of $\mathbb{P}_N^\beta(x)$; and let $\mathbf{x}^\beta(x)$ denote the associated optimal state trajectory. Thus

$$\mathbf{u}^\beta(x) = \{u^\beta(0;x), u^\beta(1;x), \ldots, u^\beta(N-1;x)\}$$
$$\mathbf{x}^\beta(x) = \{x^\beta(0;x), x^\beta(1;x), \ldots, x^\beta(N;x)\}$$

where $x^\beta(i;x) := \phi(i;x,\mathbf{u}^\beta(x))$ for all $i$. The implicit MPC control law is $\kappa_N^\beta(\cdot)$ where $\kappa_N^\beta(x) := u^\beta(0;x)$. Neither $\hat{\mathcal{U}}_N(x)$ nor $\hat{\mathcal{X}}_N$ depend on the parameter $\beta$. It can be shown (Exercise 2.11) that *the pair* $(\beta V_f(\cdot), \mathbb{X}_f)$ *satisfies Assumptions 2.12 and 2.13* if $\beta \geq 1$, since these assumptions are satisfied by the pair $(V_f(\cdot), \mathbb{X}_f)$. The absence of the terminal constraint $x(N) \in \mathbb{X}_f$ in problem $\mathbb{P}_N^\beta(x)$, which is otherwise the same as the normal optimal control problem $\mathbb{P}_N(x)$ when $\beta = 1$, ensures that $\hat{V}_N^1(x) \leq V_N^0(x)$ for all $x \in \mathcal{X}_N$ and that $\mathcal{X}_N \subseteq \hat{\mathcal{X}}_N$ where $V_N^0(\cdot)$ is the value function for $\mathbb{P}_N(x)$ and $\mathcal{X}_N$ is the domain of $V_N^0(\cdot)$.

The next task is to show the existence of a $\beta \geq 1$ such that $x^\beta(N;x) = \phi(N;x,\mathbf{u}^\beta) \in \mathbb{X}_f$ for all $x$ in some compact set, also to be determined. To proceed, let the terminal equality constrained optimal problem $\mathbb{P}_N^c(x)$ be defined by

$$V_N^c(x) = \min_{\mathbf{u}} \{J_N(x,\mathbf{u}) \mid \mathbf{u} \in \mathcal{U}_N^c(x)\}$$

in which $J_N(\cdot)$ and $\mathcal{U}_N^c(\cdot)$ are defined by

$$J_N(x,\mathbf{u}) := \sum_{i=0}^{N-1} \ell(x(i),u(i))$$
$$\mathcal{U}_N^c(x) := \hat{\mathcal{U}}_N(x) \cap \{\mathbf{u} \mid \phi(N;x,\mathbf{u}) = 0\}$$

In the definition of $J_N(\cdot)$, $x(i) := \phi(i; x, \mathbf{u})$. Let $\mathbf{u}^c$ denote the solution of $\mathbb{P}_N^c(x)$ and let $\mathcal{X}_N^c := \{x \in \mathbb{X} \mid \mathcal{U}_N^c(x) \neq \emptyset\}$ denote the domain of $V_N^c(\cdot)$. We assume that $\mathcal{X}_N^c$ is compact and contains the origin in its interior. Clearly $\mathcal{U}_N^c(x) \subseteq \hat{\mathcal{U}}_N(x)$ and $\mathcal{X}_N^c \subseteq \mathcal{X}_N$. We also assume that there exists a $\mathcal{K}_\infty$ function $\alpha^c(\cdot)$ such that

$$V_N^c(x) \leq \alpha^c(|x|)$$

for all $x \in \mathcal{X}_N^c$; this is essentially a controllability assumption. The value function for the modified problem $\mathbb{P}_N^\beta(x)$ satisfies

$$\begin{aligned}\hat{V}_N^\beta(x) &= J_N(x, \mathbf{u}_N^\beta(x)) + \beta V_f(x_N^\beta(N; x)) \\ &\leq J_N(x, \mathbf{u}_N^c(x)) = V_N^c(x) \leq \alpha^c(|x|)\end{aligned}$$

for all $x \in \mathcal{X}_N^c$ where the first inequality follows from the fact that $\beta V_f(x_N^c(N; x)) = 0$ and $\mathbf{u}_N^c$ is not optimal for $\mathbb{P}_N^\beta(x)$; here $x_N^c(N; x) := \phi(N; x, \mathbf{u}^c(x))$. Hence

$$\beta V_f(x_N^\beta(N; x)) \leq \alpha^c(|x|)$$

for all $x \in \mathcal{X}_N^c$. Since $\mathcal{X}_N^c$ is compact, there exists a finite $\beta$ such that $V_f(x_N^\beta(N; x)) \leq a$ for all $x \in \mathcal{X}_N^c$. Hence, there exists a finite $\beta > 1$ such that $x_N^\beta(N; x) \in \mathbb{X}_f$ for all $x \in \mathcal{X}_N^c$.

Suppose then that $\beta$ is sufficiently large to ensure $x_N^\beta(N; x) \in \mathbb{X}_f$ for all $x \in \mathcal{X}_N^c$. Then the origin is asymptotically or exponentially stable for $x^+ = f(x, \kappa_N^\beta(x))$ with a region of attraction $\mathcal{X}_N^c$.

### 2.6.2 Omitting the Terminal Constraint

A related procedure is merely to omit the terminal constraint and to require that the initial state lies in a subset of $\mathcal{X}_N$ that is sufficiently small or that $N$ is sufficiently large to ensure that the origin is asymptotically stable for the resultant controller. In either approach, the terminal cost may be modified. Here we examine the first alternative and assume, in the sequel, that $V_f(\cdot)$, $\mathbb{X}_f$ and $\ell(\cdot)$ satisfy Assumptions 2.12, 2.13, and 2.16, and that $\mathbb{X}_f := \{x \mid V_f(x) \leq a\}$ for some $a > 0$. Problem $\mathbb{P}_N^\beta(x)$ and the associated MPC control law $\kappa_N^\beta(\cdot)$ are defined in Section 2.6.1. Limon, Alamo, Salas, and Camacho (2006) show that the origin is asymptotically stable for $x^+ = f(x, \kappa_N^\beta(x))$ and each $\beta \geq 1$, with a region of attraction that depends on the parameter $\beta$ by establishing the following results.

**Lemma 2.38** (Entering the terminal region)**.** *Suppose* $\mathbf{u}^\beta(x)$ *is optimal for the terminally unconstrained problem* $\mathbb{P}_N^\beta(x)$, $\beta \geq 1$, *and that* $\mathbf{x}^\beta(x)$ *is the associated optimal state trajectory. If* $x^\beta(N;x) \notin \mathbb{X}_f$, *then* $x^\beta(i;x) \notin \mathbb{X}_f$ *for all* $i \in \mathbb{I}_{0:N-1}$.

*Proof.* Since, as shown in Exercise 2.11, $\beta V_f(f(x,\kappa_f(x))) \leq \beta V_f(x) - \ell(x,\kappa_f(x))$ and $f(x,\kappa_f(x)) \in \mathbb{X}_f$ for all $x \in \mathbb{X}_f$, all $\beta \geq 1$, it follows that for all $x \in \mathbb{X}_f$ and all $i \in \mathbb{I}_{0:N-1}$

$$\beta V_f(x) \geq \sum_{j=i}^{N-1} \ell(x^f(j;x,i), u^f(j;x,i)) + \beta V_f(x^f(N;x,i)) \geq \hat{V}_{N-i}^\beta(x)$$

in which $x^f(j;x,i)$ is the solution of $x^+ = f(x,\kappa_f(x))$ at time $j$ if the initial state is $x$ at time $i$, $u^f(j;x,i) = \kappa_f(x^f(j;x,i))$, and $\kappa_f(\cdot)$ is the local control law that satisfies the stability assumptions. The second inequality follows from the fact that the control sequence $\{u^f(j;x,i) \mid i \in \mathbb{I}_{i:N-1}\}$ is feasible for $\mathbb{P}_N^\beta(x)$ if $x \in \mathbb{X}_f$. Suppose contrary to what is to be proved, that there exists a $i \in \mathbb{I}_{0:N-1}$ such that $x^\beta(i;x) \in \mathbb{X}_f$. By the principle of optimality, the control sequence $\{u^\beta(i;x), u^\beta(i+1;x), \ldots, u^\beta(N-1;x)\}$ is optimal for $\mathbb{P}_{N-i}^\beta(x^\beta(i;x))$. Hence

$$\beta V_f(x^\beta(i;x)) \geq \hat{V}_{N-i}^\beta(x^\beta(i;x)) \geq \beta V_f(x^\beta(N;x)) > \beta a$$

since $x^\beta(N;x) \notin \mathbb{X}_f$ contradicting the fact that $x^\beta(i;x) \in \mathbb{X}_f$. This proves the lemma. ∎

For all $\beta \geq 1$, let the set $\Gamma_N^\beta$ be defined by

$$\Gamma_N^\beta := \{x \mid \hat{V}_N^\beta(x) \leq Nd + \beta a\}$$

We assume in the sequel that there exists a $d > 0$ such $\ell(x,u) \geq d$ for all $x \in \mathbb{X} \setminus \mathbb{X}_f$ and all $u \in \mathbb{U}$. The following result is due to Limon et al. (2006).

**Theorem 2.39** (MPC stability; no terminal constraint)**.** *The origin is asymptotically or exponentially stable for the closed-loop system* $x^+ = f(x,\kappa_N^\beta(x))$ *with a region of attraction* $\Gamma_N^\beta$. *The set* $\Gamma_N^\beta$ *is positive invariant for* $x^+ = f(x,\kappa_N^\beta(x))$.

*Proof.* From the Lemma, $x^\beta(N;x) \notin \mathbb{X}_f$ implies $x^\beta(i;x) \notin \mathbb{X}_f$ for all $i \in \mathbb{I}_{0:N}$. This, in turn, implies

$$\hat{V}_N^\beta(x) > Nd + \beta a$$

so that $x \notin \Gamma_N^\beta$. Hence $x \in \Gamma_N^\beta$ implies $x^\beta(N;x) \in \mathbb{X}_f$. It then follows, since $\beta V_f(\cdot)$ and $\mathbb{X}_f$ satisfy Assumptions 2.12 and 2.13, that the origin is asymptotically or exponentially stable for $x^+ = f(x, \kappa_N^\beta(x))$ with a region of attraction $\Gamma_N^\beta$. It also follows that $x \in \Gamma_N^\beta(x)$ implies

$$\hat{V}_N^\beta(x^\beta(1;x)) \le \hat{V}_N^\beta(x) - \ell(x, \kappa_N^\beta(x)) \le \hat{V}_N^\beta(x) \le Nd + \beta a$$

so that $x^\beta(1;x) = f(x, \kappa_N^\beta(x)) \in \Gamma_N^\beta$. Hence $\Gamma_N^\beta$ is positive invariant for $x^+ = f(x, \kappa_N^\beta(x))$. ■

Limon et al. (2006) then proceed to show that $\Gamma_N^\beta$ increases with $\beta$ or, more precisely, that $\beta_1 \le \beta_2$ implies that $\Gamma_N^{\beta_1} \subseteq \Gamma_N^{\beta_2}$. They also show that for any $x$ steerable to the interior of $\mathbb{X}_f$ by a feasible control, there exists a $\beta$ such that $x \in \Gamma_N^\beta$.

An attractive alternative is described by Hu and Linnemann (2002) who merely require that the state and control constraint sets, $\mathbb{X}$ and $\mathbb{U}$ respectively, are closed. Their approach uses, as usual, a terminal cost function $V_f : \mathbb{X}_f \to \mathbb{R}$, a terminal constraint set $\mathbb{X}_f$, and a stage cost $\ell(\cdot)$ that satisfy Assumptions 2.12, 2.13, and 2.16. Let $\mathbb{X}_f$ be a sublevel set of $V_f(\cdot)$ defined by

$$\mathbb{X}_f := \{x \in \mathbb{X} \mid V_f(x) \le a\}$$

for some $a > 0$. Then the extended function $V_f^e : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$V_f^e(x) := \begin{cases} V_f(x) & x \in \mathbb{X}_f \\ a & x \notin \mathbb{X}_f \end{cases}$$

The function $V_f^e(\cdot)$ is continuous but not continuously differentiable; we show later how the definition may be modified to ensure continuous differentiability, a desirable property for optimization algorithms. The optimization problem $\mathbb{P}_N^e(x)$ solved online is defined by

$$\hat{V}_N^e(x) := \min_{\mathbf{u}} \{V_N^e(x, \mathbf{u}) \mid \mathbf{u} \in \hat{\mathcal{U}}_N(x)\}$$

in which, with $x(N) := \phi(N; x, \mathbf{u})$,

$$V_N^e(x, \mathbf{u}) := J_N(x, \mathbf{u}) + V_f^e(x(N))$$

and $J_N(\cdot)$ and $\hat{\mathcal{U}}_N(x)$ are defined in Section 2.6.1. Let $\mathbf{u}^e(x)$ denote the solution of $\mathbb{P}_N^e(x)$ and $\mathbf{x}^e(x)$ the associated state trajectory where

$$\mathbf{u}^e(x) = \{u^e(0;x), u^e(1;x), \ldots, u^e(N-1;x)\}$$
$$\mathbf{x}^e(x) = \{x^e(0;x), x^e(1;x), \ldots, x^e(N;x)\}$$

The implicit MPC control law is $\kappa_N^e(\cdot)$ defined by

$$\kappa_N^e(x) := u^e(0;x)$$

We now define a restricted set $\mathcal{X}_N^e$ of initial states by

$$\mathcal{X}_N^e := \{x \mid x^e(N;x) \in \mathbb{X}_f\}$$

Hence, the terminal state of any optimal state trajectory with initial state $x \in \mathcal{X}_N^e$ lies in $\mathbb{X}_f$. It follows, by the usual arguments, that for all $x \in \mathcal{X}_N^e$

$$\hat{V}_N^e(x^+) \leq \hat{V}_N^e(x) - \ell(x, \kappa_N^e(x))$$

where $x^+ := f(x, \kappa_N^e(x)) = x^e(1;x)$. If $\mathcal{X}_N^e$ is positive invariant for $x^+ = f(x, \kappa_N^e(x))$, the origin is asymptotically stable for the system $x^+ := f(x, \kappa_N^e(x))$ with a region of attraction $\mathcal{X}_N^e$. Note, however, that $x \in \mathcal{X}_N^e$ does not necessarily imply that $x^+ = f(x, \kappa_N^e(x)) \in \mathcal{X}_N^e$. Hu and Linnemann (2002) show that $x \in \mathcal{X}_N^e$ implies

$$V_f^e(x^e(N;x^+)) \leq V_f^e(x^e(N-1;x^+)) - \ell(x^e(N-1;x^+), u^e(N-1;x^+))$$

The proof of this inequality is Exercise 2.12. If $x^e(N-1;x^+) = 0$, then $x^e(N;x^+) = 0 \in \mathbb{X}_f$ so that $x^+ \in \mathcal{X}_N^e$. On the other hand, if $x^e(N-1;x^+) \neq 0$, then, from the last inequality, $V_f^e(x^e(N;x^+)) < V_f^e(x^e(N-1;x^+))$. It follows from the definition of $V_f^e(\cdot)$ that $x^e(N;x^+) \in \mathbb{X}_f$, which implies that $x^+ \in \mathcal{X}_N^e$. Hence $\mathcal{X}_N^e$ is positive invariant for $x^+ = f(x, \kappa_N^e(x))$. It follows that the origin is asymptotically stable for $x^+ = f(x, \kappa_N^e(x))$ with a region of attraction $\mathcal{X}_N^e$.

For implementation, it is desirable that $V_f^e(\cdot)$ be continuously differentiable; standard optimization algorithms usually require this property. The essential property that $V_f^e(\cdot)$ should have to ensure asymptotic stability of the origin is that, for any $x \in \mathcal{X}_N^e$, $V_f^e(y) - V_f^e(x) \leq -\ell(x,u)$ for all $u \in \mathbb{U}$ implies that $y \in \mathbb{X}_f$. Suppose, then, that we choose $V_f^e(\cdot)$ to be a continuously differentiable $\mathcal{K}$ function that is equal to $V_f(\cdot)$ in $\mathbb{X}_f$ and is bounded by $a + d/2$ outside $\mathbb{X}_f$ where $d$ is such that $\ell(x,u) \geq d$ for all $x \notin \mathbb{X}_f$, all $u \in \mathbb{U}$. We consider two cases.

(a) Suppose $x \in \mathcal{X}_N^e \setminus \mathbb{X}_f$ and $V_f^e(y) - V_f^e(x) \leq -\ell(x,u)$. Then $V_f^e(y) - V_f^e(x) \leq -\ell(x,u) \leq -d$ for any $u \in \mathbb{U}$. Suppose, contrary to what we wish to prove, that $y \notin \mathbb{X}_f$. The definition of $V_f^e(\cdot)$ implies that $|V_f^e(y) - V_f^e(x)| \leq d/2$, a contradiction. Hence $y \in \mathbb{X}_f$.

(b) Suppose $x \in \mathbb{X}_f$ and $V_f^e(y) - V_f^e(x) \leq -\ell(x,u)$. Then $V_f^e(y) \leq V_f(x) \leq a$ which implies that $y \in \mathbb{X}_f$.

Hence the continuously differentiable version of $V_f^e(\cdot)$ has the essential property stated above so that $\mathcal{X}_N^e$ is positive invariant for $x^+ = f(x, \kappa_N^e(x))$ and the origin is asymptotically stable for $x^+ = f(x, \kappa_N^e(x))$ with a region of attraction $\mathcal{X}_N^e$.

If $V_f(x)$ is equal to the optimal infinite horizon cost for all $x \in \mathbb{X}_f$, then $V_N^e(x)$ is also equal to the optimal infinite horizon cost for all $x \in \mathcal{X}_N^e$.

## 2.7 Stage Cost $\ell(\cdot)$ not Positive Definite

In the analysis above we assume that the function $(x,u) \mapsto \ell(x,u)$ is positive definite; more precisely, we assume that there exists a $\mathcal{K}_\infty$ function $\alpha_1(\cdot)$ such that $\ell(x,u) \geq \alpha_1(|x|)$ for all $(x,u)$. Often we assume that $\ell(\cdot)$ is quadratic, satisfying $\ell(x,u) = (1/2)(x'Qx + u'Ru)$ where $Q$ and $R$ are positive definite. In this section we consider the case where the stage cost is $\ell(y,u)$ where $y = h(x)$ and the function $h(\cdot)$ is not necessarily invertible. An example is the quadratic stage cost $\ell(y,u) = (1/2)(|y|^2 + u'Ru)$ where $y = Cx$ and $C$ is not invertible; hence the stage cost is $(1/2)(x'Qx + u'Ru)$ where $Q = C'C$ is merely positive semidefinite. Since now $\ell(\cdot)$ does not satisfy $\ell(x,u) \geq \alpha_1(|x|)$ for all $(x,u)$ and some $\mathcal{K}_\infty$ function $\alpha_1(\cdot)$, we have to make an additional assumption in order to establish asymptotic stability of the origin for the closed-loop system. An appropriate assumption is detectability, or input/output-to-state-stability (IOSS) that ensures the state goes to zero as the output and input go to zero. We recall Definition B.42, restated here.

**Definition 2.40** (Input/output-to-state stable (IOSS))**.** The system $x^+ = f(x,u)$, $y = h(x)$ is IOSS if there exist functions $\beta(\cdot) \in \mathcal{KL}$ and $\gamma_1(\cdot)$, $\gamma_2(\cdot) \in \mathcal{K}$ such that for every initial state $x_0 \in \mathbb{R}^n$, every control sequence $\mathbf{u}$, and all $i \geq 0$.

$$|x(i)| \leq \max\{\beta(|x|, i), \gamma_1(\|\mathbf{u}\|_{0:i-1}), \gamma_2(\|\mathbf{y}\|_{0:i})\}$$

where $x(i) := \phi(i; x, \mathbf{u})$, the solution of $x^+ = f(x,u)$ at time $i$ if the initial state is $x$ and the input sequence is $\mathbf{u}$; $y(i) := h(x(i))$, $\|\mathbf{u}\|_{0:i-1}$ is the max norm of the sequence $\{u(0), u(1), \ldots, u(i-1)\}$ and $\|\mathbf{y}\|_{0:i}$ is the max norm of the sequence $\{y(0), y(1), \ldots, y(i)\}$.

We assume, as usual, that Assumptions 2.2, 2.3, 2.12, and 2.13 are satisfied but in place of Assumption 2.16 we assume that there exists $\mathcal{K}_\infty$ functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ such that

$$\ell(y,u) \geq \alpha_1(|y|) + \alpha_1(|u|) \qquad V_f(x) \leq \alpha_2(|x|)$$

for all $(y,u)$ and all $x$. We also assume that the system $x^+ = f(x,u)$, $y = h(x)$ is IOSS and that $\mathbb{X}_f$ has an interior. Under these assumptions, the value function $V_N^0(\cdot)$ has the following properties

$$\begin{aligned} &V_N^0(x) \geq \alpha_1|h(x)| && \forall x \in \mathcal{X}_N \\ &V_N^0(f(x,\kappa_N(x))) \leq V_N^0(x) - \alpha_1(|h(x)|) && \forall x \in \mathcal{X}_N \\ &V_N^0(x) \leq \alpha_2(|x|) && \forall x \in \mathbb{X}_f \end{aligned}$$

That $V_N^0(f(x,\kappa_N(x))) \leq V_N^0(x) - \ell(h(x),\kappa_N(x))$ follows from the basic stability assumption. The fact that $h(x)$ appears in the first and second inequalities instead of $x$ complicates analysis and makes it necessary to assume the IOSS property. We require the following result:

**Proposition 2.41** (Convergence of state under IOSS)**.** *Assume that the system $x^+ = f(x,u)$, $y = h(x)$ is IOSS and that $u(i) \to 0$ and $y(i) \to 0$ as $i \to \infty$. Then $x(i) = \phi(i;x,\mathbf{u}) \to 0$ as $i \to \infty$ for any initial state $x$.*

This proof of this result is discussed in Exercise 2.16.

Given the IOSS property, one can establish that the origin is attractive for closed-loop system with a region of attraction $\mathcal{X}_N$. For all $x \in \mathcal{X}_N$, all $i \in \mathbb{I}_{\geq 0}$, let $x(i;x) := \phi(i;x,\kappa_N(\cdot))$, the solution at time $i$ of $x^+ = f(x,\kappa_N(x))$ if the initial state is $x$, $y(i;x) := h(x(i;x))$ and $u(i;x) := \kappa_N(x(i;x))$. It follows from the properties of the value function that, for any initial state $x \in \mathcal{X}_N$, the sequence $\{V_N^0(x(i;x))\}$ is nonincreasing and bounded below by zero, so that $V_N^0(x(i;x)) \to c \geq 0$ as $i \to \infty$. Since $V_N^0(x(i+1)) \leq V_N^0(x(i)) - \ell(x(i;x),y(i;x))$, it follows that $\ell(y(i;x),u(i;x)) \to 0$ and, hence, that $y(i;x) \to 0$ and $u(i;x) \to 0$ as $i \to \infty$. From Proposition 2.41, $x(i;x) \to 0$ as $i \to \infty$ for any initial state $x \in \mathcal{X}_N$.

The stability property also is not difficult to establish. Suppose the initial state $x$ satisfies $|x| \leq \delta$ where $\delta$ is small enough to ensure that $\delta\mathcal{B} \subset \mathbb{X}_f$. Then $V_N^0(x) \leq \alpha_2(|x|)$ and, since $\{V_N^0(x(i;x))\}$ is nonincreasing, $V_N^0(x(i;x)) \leq \alpha_2(|x|)$ for all $i \in \mathbb{I}_{\geq 0}$. Since $\alpha_2(|x|) \geq V_N^0(x(i;x)) \geq \ell(y(i;x),u(i;x)) \geq \alpha_1(|y(i;x)|) + \alpha_1(|u(i;x)|)$, it follows that $|y(i;x)| \leq \alpha_3(|x|)$ and $|u(i;x)| \leq \alpha_3(|x|)$ for all $x \in \mathcal{X}_N$,

all $i \in \mathbb{I}_{\geq 0}$ where $\alpha_3(\cdot)$ is a $\mathcal{K}$ function defined by $\alpha_3 := \alpha_1^{-1} \circ \alpha_2$, i.e., $\alpha_3(r) = \alpha_1^{-1}(\alpha_2(r))$ for all $r \geq 0$. Hence $x(i;x)$ satisfies

$$|x(i;x)| \leq \max\{\beta(\delta, i), \alpha_3(\delta)\} \leq \max\{\beta(\delta, 1), \alpha_3(\delta)\}$$

for all $x \in \mathcal{X}_N$, all $i \in \mathbb{I}_{\geq 0}$. Thus, for all $\varepsilon > 0$, there exists a $\delta > 0$, such that $|x| \leq \delta$ implies $|x(i;x)| \leq \varepsilon$ for all $i \in \mathbb{I}_{\geq 0}$. We have established stability of the origin for $x^+ = f(x, \kappa_N(x))$. Hence the origin is asymptotically stable for the closed-loop system $x^+ = f(x, \kappa_N(x))$ with a region of attraction $\mathcal{X}_N$.

In earlier MPC literature, observability rather than detectability was often employed as the extra assumption required to establish asymptotic stability. Exercise 2.15 discusses this approach.

## 2.8 Suboptimal MPC

There is a significant practical problem that we have not addressed, namely that if the optimal control problem $\mathbb{P}_N(x)$ solved online is not convex, which is usually the case when the system is nonlinear, the global minimum of $V_N(x, \mathbf{u})$ in $\mathcal{U}_N(x)$ cannot usually be determined. Since we assume, in the stability theory given previously, that the global minimum *is* achieved, we have to consider the impact of this unpalatable fact. It is possible, as we show in this section, to achieve stability *without* requiring globally optimal solutions of $\mathbb{P}_N(x)$. Roughly speaking, all that is required is at state $x$, a feasible solution $\mathbf{u} \in \mathcal{U}_N(x)$ is found giving a cost $V_N(x, \mathbf{u})$ lower than the cost $V_N(w, \mathbf{v})$ at the previous state $w$ due to the previous control sequence $\mathbf{v} \in \mathcal{U}_N(w)$.

Consider then the usual optimal control problem with the terminal cost $V_f(\cdot)$ and terminal constraint set $\mathbb{X}_f$ satisfying Assumptions 2.12 and 2.13; the state constraint set $\mathbb{X}$ is assumed to be closed and the control constraint set $\mathbb{U}$ to be compact. In addition, we assume that $V_f(\cdot)$ satisfies $V_f(x) \geq \alpha_f(|x|)$ and $V_f(x) \leq \gamma_f(|x|)$ for all $x \in \mathbb{X}_f$ where $\alpha_f(\cdot)$ and $\gamma_f(\cdot)$ are $\mathcal{K}_\infty$ functions. These conditions are satisfied, for example, if $V_f(\cdot)$ is a positive definite quadratic function and $\mathbb{X}_f$ is a sublevel set of $V_f(\cdot)$. The set $\mathbb{X}_f$ is assumed to be a sublevel set of $V_f(\cdot)$, i.e., $\mathbb{X}_f = \{x \mid V_f(x) \leq r\}$ for some $r > 0$. We also make the standard Assumption 2.16(a) that $\ell(x, u) \geq \alpha_1(|x|)$ for all $(x, u) \in \mathbb{X} \times \mathbb{U}$ where $\alpha_1(\cdot)$ is a $\mathcal{K}_\infty$ function, which is satisfied if $\ell(\cdot)$ is a positive definite quadratic function. Let $\mathcal{X}_N$ denote, as before, the set of $x$ for which a control sequence $\mathbf{u}$ exists that satisfies the state,

control and terminal constraints, i.e., $\mathcal{X}_N := \{x \in \mathbb{X} \mid \mathcal{U}_N(x) \neq \emptyset\}$ where $\mathcal{U}_N(x)$ is defined by (2.6).

The basic idea behind the suboptimal model predictive controller is simple. Suppose that the current state is $x$ and that $\mathbf{u} = \{u(0), u(1), \ldots, u(N-1)\} \in \mathcal{U}_N(x)$ is a feasible control sequence for $\mathbb{P}_N(x)$. The first element $u(0)$ of $\mathbf{u}$ is applied to the system $x^+ = f(x,u)$. In the absence of uncertainty, the next state is equal to the predicted state $x^+ = f(x,u(0))$. Consider the control sequence $\mathbf{u}^+$ defined by

$$\mathbf{u}^+ = \{u(1), u(2), \ldots, u(N-1), \kappa_f(x(N))\} \tag{2.40}$$

in which $x(N) = \phi(N; x, \mathbf{u})$ and $\kappa_f(\cdot)$ is a local control law with the property that $u = \kappa_f(x)$ satisfies Assumption 2.12 for all $x \in \mathbb{X}_f$. The existence of such a $\kappa_f(\cdot)$, which is usually of the form $\kappa_f(x) = Kx$, is implied by Assumption 2.12. Then, as shown in Section 2.4.3, the control sequence $\mathbf{u}^+ \in \mathcal{U}_N(x)$ satisfies

$$V_N(x^+, \mathbf{u}^+) + \ell(x, u(0)) \leq V_N(x, \mathbf{u}) \tag{2.41}$$

and, hence

$$V_N(x^+, \mathbf{u}^+) \leq V_N(x, \mathbf{u}) - \alpha_1(|x|) \tag{2.42}$$

No optimization is required to get the cost reduction $\ell(x,u(0))$ given by (2.41); in practice the control sequence $\mathbf{u}^+$ can be improved by several iterations of an optimization algorithm. Inequality (2.42) is reminiscent of the inequality $V_N^0(x^+) \leq V_N^0(x) - \ell(x, \kappa_N(x))$ that provides the basis for establishing asymptotic stability of the origin for the controlled systems previously analyzed and suggests that the simple algorithm described previously, which places very low demands on the online optimization algorithm, may also ensure asymptotic stability of the origin. This is almost true. The obstacle to applying standard Lyapunov theory is that there is no obvious Lyapunov function because, at each state $x^+$, there exist many control sequences $\mathbf{u}^+$ satisfying $V_N(x^+, \mathbf{u}^+) \leq V_N(x, \mathbf{u}) - \alpha_1(|x|)$. The function $(x, \mathbf{u}) \mapsto V_N(x, \mathbf{u})$ is *not* a function of $x$ only and may have many different values for each $x$; therefore it cannot play the role of the function $V_N^0(x)$ used previously. Moreover, the controller can generate, for a given initial state, many different trajectories, all of which have to be considered.

Global attractivity of the origin in $\mathcal{X}_N$, however, may be established. For all $x(0) \in \mathcal{X}_N$, let $\{(x(0), \mathbf{u}(0)), (x(1), \mathbf{u}(1)), \ldots\}$ denote *any* infinite sequence generated by the controlled system and satisfying, therefore, $V_N(x(i+1), \mathbf{u}(i+1)) \leq V_N(x(i), \mathbf{u}(i)) - \alpha_1(|x(i)|)$ for all $i$. Then

$\{V_N(x(i), \mathbf{u}(i)) \mid i \in \mathbb{I}_{\geq 0}\}$ is a nonincreasing sequence bounded below by zero. Hence $V_N(x(i), \mathbf{u}(i)) \to V_N^* \geq 0$ so that $V_N(x(i+1), \mathbf{u}(i+1)) - V_N(x(i), \mathbf{u}(i)) \to 0$ as $i \to \infty$. We deduce, from (2.42), that $\alpha_1(|x(i)|) \to 0$ so that $x(i) \to 0$ as $i \to \infty$.

Establishing stability of the origin is more difficult for reasons given previously and requires a minor modification of the controller when the state $x$ is close to the origin. The modification we make to the controller is to require that $\mathbf{u}$ satisfies the following requirement when $x$ lies in $\mathbb{X}_f$

$$V_N(x, \mathbf{u}) \leq V_f(x) \qquad f(x, u(0)) \in \mathbb{X}_f \tag{2.43}$$

where $u(0)$ is the first element in $\mathbf{u}$. Stability of the origin can be established using (2.42), (2.43) and the properties of $V_f(\cdot)$ as shown subsequently. Inequality (2.43) is achieved quite simply by using the control law $u = \kappa_f(x)$ to generate the control $u$ when $x \in \mathbb{X}_f$. Let $\mathbf{x}(x; \kappa_f)$ and $\mathbf{u}(x; \kappa_f)$ denote the state and control sequences generated in this way when the initial state is $x$; these sequences satisfy

$$x^+ = f(x, \kappa_f(x)) \qquad u = \kappa_f(x)$$

with initial condition $x(0) = x$, so that $x(0; x, \kappa_f) = x$, $x(1; x, \kappa_f) = f(x, \kappa_f(x))$, $x(2; x, \kappa_f) = f(x(1; x, \kappa_f)), \kappa_f(x(1; x, \kappa_f)))$, etc. Since Assumption 2.12 is satisfied,

$$V_f(x) \geq \ell(x, \kappa_f(x)) + V_f(f(x, \kappa_f(x)))$$

which, when used iteratively, implies

$$V_f(x) \geq \sum_{i=0}^{N-1} \ell(x(i; x, \kappa_f), \kappa_f(x(i; x, \kappa_f))) + V_f(x(N; x, \kappa_f))$$

Hence, for all $x \in \mathbb{X}_f$

$$\begin{aligned} V_N(x, \mathbf{u}(x; \kappa_f)) &= \sum_{i=0}^{N-1} \ell(x(i; x, \kappa_f), \kappa_f(x(i; x, \kappa_f))) + V_f(x(N; x, \kappa_f)) \\ &\leq V_f(x) \end{aligned}$$

as required. Also, it follows from Assumption 2.12 and the definition of $\kappa_f(\cdot)$ that $x^+ = f(x, u(0)) \in \mathbb{X}_f$ if $x \in \mathbb{X}_f$. Thus the two conditions in (2.43) are satisfied by $\mathbf{u}(x; \kappa_f)$. If desired, $\mathbf{u}(x; \kappa_f)$ may be used for the current control sequence $\mathbf{u}$ or as a "warm start" for an optimization algorithm yielding an improved control sequence. In any case, if (2.43) is satisfied, stability of the origin may be established as follows.

Let $\delta > 0$ be arbitrary but small enough to ensure $\delta\mathcal{B} \subset \mathbb{X}_f$. Suppose the initial state $x(0)$ satisfies $|x(0)| \leq \delta$ so that $x(0) \in \mathbb{X}_f$. As before, let $\{(x(i), \mathbf{u}(i))\}$ denote *any* state-control sequence with initial state $x(0) \in \delta\mathcal{B}$ generated by the suboptimal controller. From (2.42) and (2.43) we deduce that $V_N(x(i), \mathbf{u}(i)) \leq V_N(x(0), \mathbf{u}(0)) \leq V_f(x(0)) \leq \gamma_f(|x(0)|) \leq \gamma_f(\delta)$ for all $i \in \mathbb{I}_{\geq 0}$, all $x(0) \in \delta\mathcal{B}$. It follows from our assumption on $\ell(\cdot)$ that $V_N(x, \mathbf{u}) \geq \alpha_1(|x|)$ for all $x \in \mathcal{X}_N$, all $\mathbf{u} \in \mathbb{U}^N$. Hence

$$\alpha_1(|x(i)|) \leq V_N(x(i), \mathbf{u}(i)) \leq \gamma_f(\delta)$$

so that $|x(i)| \leq (\alpha_1^{-1} \circ \gamma_f)(\delta)$ for all $i \in \mathbb{I}_{\geq 0}$. Hence, for all $\varepsilon > 0$, there exists a $\delta = (\alpha_1^{-1} \circ \gamma_f)^{-1}(\varepsilon) = (\gamma_f^{-1} \circ \alpha_1)(\varepsilon) > 0$ such that $|x(0)| \leq \delta$ implies $|x(i)| \leq \varepsilon$ for all $i \in \mathbb{I}_{\geq 0}$. The origin is, therefore, stable and, hence, asymptotically stable with a region of attraction $\mathcal{X}_N$ for the controlled system.

**Suboptimal control algorithm.**

**Data:** Integer $N_{\text{iter}}$.

**Input:** Current state $x$, previous state sequence $\mathbf{w} = \{w(0), w(1), \ldots, w(N)\}$, previous control sequence $\mathbf{v} = \{v(0), v(1), \ldots, v(N-1)\}$.

**Step 1:** If $x \notin \mathbb{X}_f$, use $\{v(1), v(2), \ldots, v(N-1), \kappa_f(w(N))\}$ as a warm start for an optimization algorithm. Perform $N_{\text{iter}}$ iterations of the algorithm to obtain an improved control sequence $\mathbf{u} \in \mathcal{U}_N(x)$. Apply control $u = u(0)$ to the system being controlled.

**Step 2:** If $x \in \mathbb{X}_f$, set $u = \kappa_f(x)$ and apply $u$ to the system being controlled; or perform $N_{\text{iter}}$ steps of an optimization algorithm using $\mathbf{u}(x, \kappa_f)$, defined previously, as a warm start to obtain an improved control sequence $\mathbf{u} \in \mathcal{U}_N(x)$ satisfying (2.43) and associated state sequence $\mathbf{w}$.

A nominally stabilizing controller with very low online computational demands may be obtained by merely using the warm starts defined in the algorithm. Improved performance is obtained by using $N_{\text{iter}}$ iterations of an optimization algorithm to improve the warm start. It is more important to employ optimization in Step 1 when $x \notin \mathbb{X}_f$ since the warm start when $x \in \mathbb{X}_f$ has good performance if $\kappa_f(\cdot)$ is designed properly.

## 2.9 Tracking

In preceding sections we were concerned with regulation to the origin and the determination of conditions that ensure stability of the origin for the closed-loop system. In this section we consider the problem of tracking a constant reference signal, i.e., regulation to a set point. Assume that the system to be controlled satisfies

$$x^+ = f(x,u) \quad y = h(x)$$

and is subject to the constraints

$$x \in \mathbb{X} \qquad u \in \mathbb{U}$$

in which $\mathbb{X}$ is closed and $\mathbb{U}$ is compact. If the constant reference signal is $r$, then we wish to steer the initial state $x$ to a state $\bar{x}$ satisfying $h(\bar{x}) = r$ so that $y = r$.

### 2.9.1 No Uncertainty

We assume initially that there is no model error and no disturbance. The target state and associated steady-state control are obtained by minimizing $|\bar{u}|^2$ with respect to $(x,u)$ subject to the equality constraints

$$\begin{aligned} x &= f(x,u) \\ r &= h(x) \end{aligned}$$

and the inequality constraints $x \in \mathbb{X}$ and $u \in \mathbb{U}$. We assume that a solution exists and denote the solution by $(\bar{x}(r), \bar{u}(r))$; this notation indicates the dependence of the target state and its associated control on the reference variable $r$. We require the dimension of $r$ to be less than or equal to $m$, the dimension of $u$.

MPC may then be achieved by solving online the optimal control problem $\mathbb{P}_N(x,r)$ defined by

$$V_N^0(x,r) = \min_{\mathbf{u}} \{V_N(x,r,\mathbf{u}) \mid \mathbf{u} \in \mathcal{U}_N(x,r)\}$$

in which the cost function $V_N(\cdot)$ and the constraint set are defined by

$$\begin{aligned} V_N(x,r,\mathbf{u}) &:= \sum_{i=0}^{N-1} \ell(x(i) - \bar{x}(r), u(i) - \bar{u}(r)) + V_f(x,r) \\ \mathcal{U}_N(x,r) &:= \{\mathbf{u} \mid x(i) \in \mathbb{X},\ u(i) \in \mathbb{U}, \forall i \in \mathbb{I}_{0:N-1}; x(N) \in \mathbb{X}_f(r)\} \end{aligned}$$

In these definitions, $x(i) = \phi(i; x, \mathbf{u})$, the solution at time $i$ of $x^+ = f(x,u)$ if the initial state is $x$ and the control sequence is $\mathbf{u}$. Let $\mathbf{u}^0(x,r)$ denote the solution of $\mathbb{P}_N(x,r)$. The MPC control law is $\kappa_N(x,r)$, the first control in the sequence $\mathbf{u}^0(x,r)$. The terminal cost function $V_f(\cdot,r)$ and constraint set $\mathbb{X}_f(r)$ must be chosen to satisfy suitably modified stabilizing conditions. Since both depend on $r$, the simplest option is to choose a terminal equality constraint so that

$$V_f(\bar{x}(r),r) = 0 \qquad \mathbb{X}_f(r) = \{\bar{x}(r)\} \subset \mathbb{X}$$

If the system is linear, i.e., if $x^+ = Ax + Bu$, an alternative choice is

$$V_f(x,r) = V'_f(x - \bar{x}(r)) \qquad \mathbb{X}_f(r) = \{\bar{x}(r)\} \oplus \mathbb{X}'_f \subset \mathbb{X}$$

in which $V'_f(\cdot)$ and $\mathbb{X}'_f$ are, respectively, the terminal cost function and terminal constraint set derived in Section 2.5.3.1; the reference $r$ must satisfy the constraint that $\mathbb{X}_f(r) \subset \mathbb{X}$. With this choice, $V_f(\bar{x}(r),r) = 0$ and $\mathbb{X}_f(r)$ is control invariant for $x^+ = Ax + Bu$. It is easily shown, with either choice and appropriate assumptions, that the point $\bar{x}(r)$ is asymptotically, or exponentially, stable for the controlled system with a region of attraction $\mathcal{X}_N(r) := \{x \mid \mathcal{U}_N(x,r) \neq \emptyset\}$.

### 2.9.2 Offset-Free MPC

If uncertainty is present, in the form of model error or an unknown constant disturbance, the tracking error $y - r$ may converge to a nonzero constant vector, called the *offset*, rather than to the origin. It is possible to ensure zero offset by augmenting the system with a model of the disturbance.

We therefore assume that the system to be controlled satisfies

$$\begin{aligned} x^+ &= f(x,u) \\ d^+ &= d \\ y &= h(x) + d + v \end{aligned}$$

in which $v$ is measurement noise. If we assume, as we do everywhere in this chapter, that the state $x$ is known, then a simple filter may be used to obtain an estimate $\hat{d}$ of the unknown, but constant disturbance $d$. The filter is described by

$$\hat{d}^+ = \hat{d} + L(y - h(x) - \hat{d})$$

in which $y - h(x)$ may be regarded as a noisy measurement of $d$, since $y - h(x) = d + v$. The difference equation for the estimation error $\tilde{d} := d - \hat{d}$ is

$$\tilde{d}^+ = A_L \tilde{d} - Lv$$

in which $L$ is chosen to ensure that $A_L := I - L$ is stable. If there is zero measurement noise, $\tilde{d}(i) \to 0$ exponentially as $i \to \infty$. Since $y - h(x) = d + v$, the difference equation for $\hat{d}$ may be written as

$$\hat{d}^+ = \hat{d} + L(\tilde{d} + v)$$

Since $d$ is unknown, we have to use $\hat{d}$ for control. Hence, for the purpose of control we employ the difference equations

$$\begin{aligned} x^+ &= f(x,u) \\ \hat{d}^+ &= \hat{d} + L(\tilde{d} + v) \end{aligned}$$

If $\hat{d}$ is the current estimate of $d$, our best estimate of $d$ at any time in the future is also $\hat{d}$. Given the current state $(x, \hat{d})$ of the composite system and the current reference $r$, we determine the target state and associated control by minimizing $|u|^2$ with respect to $(x,u)$ subject to the equality constraints

$$\begin{aligned} x &= f(x,u) \\ r &= h(x) + \hat{d} \end{aligned}$$

and the inequality constraints $x \in \mathbb{X}$ and $u \in \mathbb{U}$. We assume that a solution to this problem exists and denote the solution by $(\bar{x}(r,\hat{d}), \bar{u}(r,\hat{d}))$.

MPC may then be achieved by solving online the optimal control problem $\mathbb{P}_N(x,r,\hat{d})$ defined by

$$V_N^0(x,r,\hat{d}) = \min_{\mathbf{u}} \{V_N(x,r,\mathbf{u}) \mid \mathbf{u} \in \mathcal{U}_N(x,r,\hat{d})\}$$

in which the cost function $V_N(\cdot)$ and the constraint set are defined by

$$V_N(x,r,\hat{d},\mathbf{u}) := \sum_{i=0}^{N-1} \ell(x(i) - \bar{x}(r,\hat{d}), u(i) - \bar{u}(r,\hat{d})) + V_f(x,r,\hat{d})$$

$$\mathcal{U}_N(x,r,\hat{d}) := \{\mathbf{u} \mid x(i) \in \mathbb{X},\ u(i) \in \mathbb{U}, \forall i \in \mathbb{I}_{0:N-1}; x(N) \in \mathbb{X}_f(r,\hat{d})\}$$

In these definitions, $x(i) = \phi(i;x,\mathbf{u})$, the solution at time $i$ of $x^+ = f(x,u)$ if the initial state is $x$ and the control sequence is $\mathbf{u}$. Let

$\mathbf{u}^0(x, r, \hat{d})$ denote the solution of $\mathbb{P}_N(x, r, \hat{d})$. The MPC control law is $\kappa_N(x, r, \hat{d})$, the first control in the sequence $\mathbf{u}^0(x, r, \hat{d})$. The terminal cost function $V_f(\cdot, r, \hat{d})$ and constraint set $\mathbb{X}_f(r, \hat{d})$ must be chosen to satisfy suitably modified stabilizing conditions. Since both depend on $(r, \hat{d})$, the simplest option is to choose a terminal equality constraint so that

$$V_f(\bar{x}(r, \hat{d}), r, \hat{d}) = 0 \qquad \mathbb{X}_f(r, \hat{d}) = \{\bar{x}(r, \hat{d})\} \subset \mathbb{X}$$

This constraint is equivalent to requiring that the terminal state is equal to $\bar{x}(r, \hat{d})$ in the optimal control problem $\mathbb{P}_N(x, r, \hat{d})$.

If $\hat{d}$ is constant, standard MPC theory shows, under suitable assumptions, that the constant target state $\bar{x}(r, \hat{d})$ is asymptotically stable for $x^+ = f(x, \kappa_N(x, r, \hat{d}))$ with a region of attraction $\mathcal{X}_N(r, \hat{d}) := \{x \mid \mathcal{U}_N(x, r, \hat{d}) \neq \emptyset\}$. In particular, the state $x(i)$ of the controlled system at time $i$ converges to $\bar{x}(r, \hat{d})$ as $i \to \infty$. We now assume that the disturbance $v(i) \to 0$ and, consequently, that $\hat{d}(i) \to d_s$, $x(i) \to x_s := \bar{x}(r, d_s)$, and $u(i) \to u_s := \bar{u}(r, d_s)$ as $i \to \infty$. Hence $y(i) = h(x(i)) + \hat{d}(i) + v(i) \to h(x_s) + d_s$ as $i \to \infty$. It follows from the difference equations for $x$ and $\hat{d}$ that

$$x_s = f(x_s, u_s) \qquad L(y_s - h(x_s) - d_s) = 0$$

If $L$ is invertible ($y$ and $d$ have the same dimension), it follows that

$$x_s = f(x_s, u_s) \qquad y_s = h(x_s) + d_s$$

But, since $x_s := \bar{x}(r, d_s)$ and $u_s := \bar{u}(r, d_s)$, it follows, by definition, that

$$h(x_s) + d_s = r$$

Hence $y(i) \to y_s = r$ as $i \to \infty$; the offset is asymptotically zero.

If we do not assume that $\hat{d}$ converges to a constant value, however, uncertainty in the evolution of $\hat{d}$ may cause the value function $V_N^0(x, r, \hat{d})$ to increase sufficiently often to destroy stability. Robust output MPC, discussed in Chapter 5, may have to be employed to ensure stability of a set rather than a point.

### 2.9.3 Unreachable Setpoints

In process control, steady-state optimization is often employed to determine an optimal setpoint, and MPC to steer the state of the system to

this setpoint. Because of nonzero process disturbances and discrepancies between the models employed for steady-state optimization and for control, the optimal setpoint may be unreachable. Often an unreachable setpoint is then replaced by a reachable steady-state target that is closest to it. This standard procedure is *suboptimal*, however, and does not minimize tracking error. We show in this section that by defining performance relative to the unreachable setpoint rather than to the closest reachable target, it is possible to achieve improved performance. Standard MPC theory can no longer be used to analyze stability, however, because the value function for the new problem does not necessarily decrease along trajectories of the controlled system. With an infinite horizon, the cost function for the optimal control problem that yields MPC is unbounded.

Suppose the system to be controlled is described by

$$x^+ = Ax + Bu$$

with control constraint $u \in \mathbb{U}$, in which $\mathbb{U}$ is convex and compact. The setpoint pair $(x_{\text{sp}}, u_{\text{sp}})$ is not necessarily reachable. The cost function $V(\cdot)$ for the optimal control problem is

$$V_N(x, \mathbf{u}) := \sum_{i=0}^{N-1} \ell(x(i), u(i))$$

in which $x(i) := \phi(i; x, \mathbf{u})$, the solution of the dynamic system at time $i$ if the initial state at time 0 is $x$ and the control sequence is $\mathbf{u} := \{u(0), u(1), \ldots, u(N-1)\}$. The stage cost $\ell(\cdot)$ is defined to be a quadratic function of the distance from the setpoint

$$\ell(x, u) := (1/2)(|x - x_{\text{sp}}|_Q^2 + |u - u_{\text{sp}}|_R^2)$$

in which $Q$ and $R$ are positive definite. For simplicity of exposition, a terminal constraint $x(N) = x_s$, in which $x_s$ is defined subsequently, is included in the optimal control problem $\mathbb{P}_N(x)$ whose solution yields the model predictive controller; problem $\mathbb{P}_N(x)$ is therefore defined by

$$V_N^0(x) = \min_{\mathbf{u}} \{V_N(x, \mathbf{u}) \mid \mathbf{u} \in \mathcal{U}_N(x)\}$$

in which the control sequence constraint set $\mathcal{U}_N(x)$ is defined by

$$\mathcal{U}_N(x) := \{\mathbf{u} \mid u(i) \in \mathbb{U}, i = 0, 1, \ldots, N-1,\ \phi(N; x, \mathbf{u}) = x_s\}$$

The domain of $V_N^0(\cdot)$, i.e., the set of feasible initial states for $\mathbb{P}_N(x)$, is $\mathcal{X}_N$ defined by

$$\mathcal{X}_N := \{x \mid \mathcal{U}_N(x) \neq \emptyset\}$$

For all $x \in \mathcal{X}_N$, the constraint set $\mathcal{U}_N(x)$ is compact. The set $\mathcal{X}_N$ is thus the set of states that can be steered to $x_s$ in $N$ steps by a control sequence $\mathbf{u}$ that satisfies the control constraint. It follows from its definition that $\mathcal{X}_N$ is closed and is compact if $A$ is invertible. Because $V_N(\cdot)$ is continuous, and $\mathcal{U}_N(x)$ is compact for each $x \in \mathcal{X}_N$, it follows that for each $x \in \mathcal{X}_N$, $\mathbf{u} \mapsto V_N(x, \mathbf{u})$ achieves its minimum, $V_N^0(x)$, in $\mathcal{U}_N(x)$. Let $\mathbf{u}^0(x) = \{u^0(0;x), u^0(1;x), \ldots, u^0(N-1;x)\}$ denote the solution of $\mathbb{P}_N(x)$. Following usual practice, the model predictive control at state $x$ is $\kappa_N(x) := u^0(0;x)$, the first element of the optimal control sequence $\mathbf{u}^0(x)$.

The optimal steady state $(x_s, u_s)$ is defined to be the solution of the optimization problem $\mathbb{P}_\mathrm{s}$

$$(x_s, u_s) := \arg\min_{x,u} \{\ell(x,u) \mid x = Ax + Bu,\ u \in \mathbb{U}\}$$

This problem has a solution if $0 \in \mathbb{U}$ since then $(0,0)$ satisfies the constraints. Since $Q, R > 0$, the minimizer $(x_s, u_s)$ is unique. Clearly $\ell(x_s, u_s) > 0$ unless the setpoint $(x_\mathrm{sp}, u_\mathrm{sp})$ is feasible for $\mathbb{P}_\mathrm{s}$; it is this fact that requires a nonstandard method for establishing asymptotic stability of $(x_s, u_s)$. The following theorem is proved in (Rawlings, Bonné, Jørgensen, Venkat, and Jørgensen, 2008)

**Theorem 2.42** (MPC stability with unreachable setpoint)**.** *The optimal steady state $x_s$ is asymptotically stable with a region of attraction $\mathcal{X}_N$ for the closed-loop system $x^+ = Ax + B\kappa_N(x)$ using setpoint MPC.*

This paper also discusses relaxing the terminal constraint and using instead a terminal penalty based on a terminal controller.

**Example 2.43: Unreachable setpoint MPC**

An example is presented to illustrate the advantages of the proposed setpoint tracking MPC (sp-MPC) compared to traditional target tracking MPC (targ-MPC). The regulator cost function for the proposed sp-MPC, is

$$V_N^\mathrm{sp}(x, \mathbf{u}) = \frac{1}{2} \sum_{j=0}^{N-1} |x(j) - x_\mathrm{sp}|_Q^2 + |u(j) - u_\mathrm{sp}|_R^2 + |u(j+1) - u(j)|_S^2$$

| Performance measure | targ-MPC (a=targ) | sp-MPC (a=sp) | $\Delta$ index (%) |
|---|---|---|---|
| $V_u^a$ | 0.016 | $2.2 \times 10^{-6}$ | 99.98 |
| $V_y^a$ | 3.65 | 1.71 | 53 |
| $V^a$ | 3.67 | 1.71 | 54 |

**Table 2.3:** Comparison of controller performance for Example 2.43.

in which $Q > 0$, $R, S \geq 0$, at least one of $R, S > 0$, and $x(j) = \phi(j; x, \mathbf{u})$. This system can be put in the standard form defined for terminal constraint MPC by using the augmented state $\widetilde{x}(k) := (x(k), u(k-1))$ discussed in Section 1.2.5. The regulator cost function in traditional targ-MPC is

$$V_N^{\text{targ}}(x, \mathbf{u}) = \frac{1}{2} \sum_{j=0}^{N-1} |x(j) - x_s|_Q^2 + |u(j) - u_s|_R^2 + |u(j+1) - u(j)|_S^2$$

The controller performance is assessed using the following three closed-loop control performance measures

$$V_u^a = \frac{1}{2k\Delta} \sum_{j=0}^{k-1} |u(j) - u_{\text{sp}}|_R^2 + |u(j+1) - u(j)|_S^2$$

$$V_y^a = \frac{1}{2k\Delta} \sum_{j=0}^{k-1} |x(j) - x_{\text{sp}}|_Q^2$$

$$V^a = V_u^a + V_y^a \qquad a = (\text{sp}, \text{targ})$$

in which $\Delta$ is the process sample time, and $x(j)$ and $u(j)$ are the state and control at time $j$ of the controlled system using either target (a=targ) or setpoint (a=sp) tracking MPC for a specified initial state. For each of the indices defined previously, we define the percentage improvement of sp-MPC compared with targ-MPC by

$$\Delta \text{ index } (\%) = \frac{V^{\text{targ}} - V^{\text{sp}}}{V^{\text{targ}}} \times 100$$

Consider the single-input, single-output system with transfer function

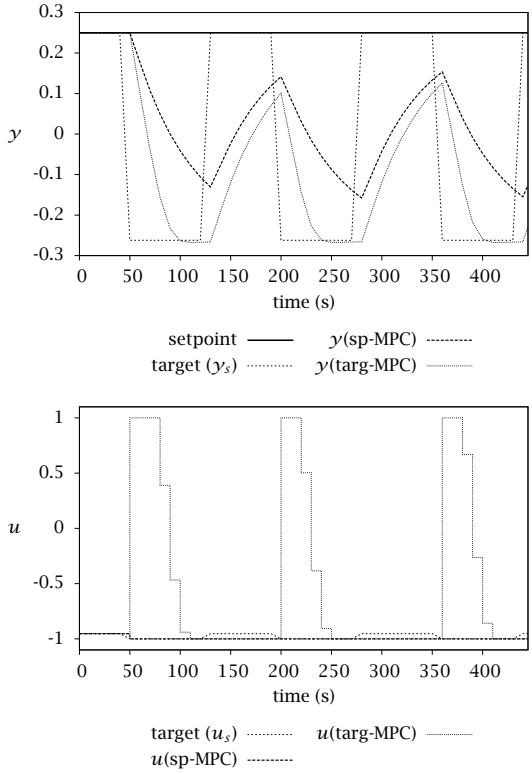



**Figure 2.6:** Closed-loop performance of sp-MPC and targ-MPC.

and state space realization

$$G(s) = \frac{-0.2623}{60s^2 + 59.2s + 1} \qquad A = \begin{bmatrix} 0.857252 & 0.884179 \\ -0.014736 & -0.015139 \end{bmatrix}$$
$$B = \begin{bmatrix} 8.56490 \\ 0.88418 \end{bmatrix} \qquad C = \begin{bmatrix} -0.0043717 & 0 \end{bmatrix}$$

sampled with $\Delta = 10$ s. The input $u$ is constrained $|u| \leq 1$. The desired output setpoint is $y_{\text{sp}} = 0.25$, which corresponds to a steady-state input value of $-0.953$. The regulator parameters are $Q_y = 10, R = 0, S = 1, Q = C'Q_yC + 0.01I_2$. A horizon length of $N = 80$ is used. In time intervals 50–130, 200–270, and 360–430, a state disturbance $d_x = [17.1, 1.77]'$ causes the input to saturate at its lower limit. The output setpoint is unreachable under the influence of this state dis-

turbance. The closed-loop performance of sp-MPC and targ-MPC under the described disturbance scenario are shown in Figure 2.6. The closed-loop performance of the two control formulations are compared in Table 2.3.

In the targ-MPC framework, the controller tries to reject the state disturbance and minimize the deviation from the new steady-state target. This requires a large, undesirable control action that forces the input to move between the upper and lower constraints. The sp-MPC framework, on the other hand, attempts to minimize the deviation from setpoint and subsequently the input just rides the lower input constraint.

The greater cost of control action in targ-MPC is shown by the cost index $V_u$ in Table 2.3. The cost of control action in targ-MPC exceeds that of sp-MPC by nearly 100%. The control in targ-MPC causes the output of the system to move away from the (unreachable) setpoint faster than the corresponding output of sp-MPC. Since the control objective is to be close to the setpoint, this undesirable behavior is eliminated by sp-MPC. □

## 2.10 Concluding Comments

MPC is an implementation, for practical reasons, of receding horizon control (RHC), in which offline determination of the RHC law $\kappa_N(\cdot)$ is replaced by online determination of its value $\kappa_N(x)$, the control action, at each state $x$ encountered during its operation. Because the optimal control problem that defines the control is a finite horizon problem, neither stability nor optimality of the cost function is achieved by a receding horizon or model predictive controller. This chapter shows how stability may be achieved by adding a terminal cost function and a terminal constraint to the optimal control problem. Adding a terminal cost function adds little or no complexity to the optimal control problem that has to be solved online and usually improves performance. Indeed, the infinite horizon value function $V_\infty^0(\cdot)$ for the constrained problem would be an ideal choice for the terminal penalty because the value function $V_N^0(\cdot)$ for the online optimal control problem would then be equal to $V_\infty^0(\cdot)$ and the controller would inherit the performance advantages of the infinite horizon controller. In addition, the actual trajectories of the controlled system would be precisely equal, in the absence of uncertainty, to those predicted by the online optimizer. Of course, if we knew $V_\infty^0(\cdot)$, the optimal infinite horizon controller $\kappa_\infty(\cdot)$

could be determined and there would be no reason to employ MPC. The infinite horizon cost $V_\infty^0(\cdot)$ is known globally only for special cases, however, such as the linear quadratic unconstrained problem. For more general problems in which constraints and/or nonlinearity are present, its value, or approximate value, in a neighborhood of the setpoint can usually be obtained and the use of this local CLF should, in general, enhance performance. Adding a terminal cost appears to be generally advantageous.

The reason for the terminal constraint is precisely the fact that the terminal penalty is usually merely a local CLF requiring the terminal state to lie in the region where the CLF is valid. Unlike the addition of a terminal penalty, however, addition of a terminal constraint may increase complexity considerably. Because efficient programs exist for solving QPs, in which the cost function to be minimized is quadratic and the constraints polyhedral, there is an argument for using polyhedral constraints. Indeed, a potential terminal constraint set for the constrained linear quadratic optimal control problem is the maximal constraint admissible set, which is polyhedral. This set is complex, however, i.e., defined by many linear inequalities, and would appear to be unsuitable for the complex control problems routinely encountered in industry.

A terminal constraint set that is considerably simpler is a suitable sublevel set of the terminal penalty, which is often a simple positive definite quadratic function resulting in a convex terminal constraint set. A disadvantage is that the terminal constraint set is now ellipsoidal rather than polytopic and conventional quadratic programs cannot be employed for the linear quadratic constrained optimal control problem. This does not appear to be a serious disadvantage, however, because the optimal control problem remains convex, so interior point methods may be readily employed.

In the nonlinear case, adding an ellipsoidal terminal constraint set does not appreciably affect the complexity of the optimal control problem. In any case, it is possible to replace the ellipsoidal terminal constraint set by a suitable modification of the terminal penalty as shown in Section 2.6.1. A more serious problem, when the system is nonlinear, is that the optimal control problem is then usually nonconvex so that global solutions, on which many theoretical results are predicated, are usually too difficult to obtain. A method for dealing with this difficulty, which also has the advantage of reducing online complexity, is suboptimal MPC described in Section 2.8.

This chapter also presents some results that contribute to an understanding of the subject but do not provide practical tools. For example, it is useful to know that the domain of attraction for many of the controllers described here is $\mathcal{X}_N$, the set of initial states controllable to the terminal constraint set, but this set cannot usually be computed. The set is, in principle, computable using the DP equations presented in this chapter, and may be computed if the system is linear and the constraints, including the terminal constraint, are polyhedral, provided that the state dimension and the horizon length are suitably small, considerably smaller than in problems routinely encountered in industry. In the nonlinear case, this set cannot usually be computed. Computation difficulties are not resolved if $\mathcal{X}_N$ is replaced by a suitable sublevel set of the value function $V_N^0(\cdot)$. Hence, in practice, both for linear and nonlinear MPC, this set has to be estimated by simulation.

## 2.11 Notes

MPC has an unusually rich history, making it impossible to summarize here the many contributions that have been made. Here we restrict attention to a subset of this literature that is closely related to the approach adopted in this book. A fuller picture is presented in the review paper (Mayne, Rawlings, Rao, and Scokaert, 2000).

The success of conventional MPC derives from the fact that for deterministic problems (no uncertainty), feedback is not required so the solution to the open-loop optimal control problem solved online for a particular initial state is the same as that obtained by solving the feedback problem using DP, for example. Lee and Markus (1967) pointed out the possibility of MPC in their book on optimal control

> One technique for obtaining a feedback controller synthesis is to measure the current control process state and then compute very rapidly the open-loop control function. The first portion of this function is then used during a short time interval after which a a new measurement of the process state is made and a new open-loop control function is computed for this new measurement. The procedure is then repeated.

Even earlier, Propoi (1963) proposed a form of MPC utilizing linear programming, for the control of linear systems with hard constraints on the control. A big surge in interest in MPC occurred when Richalet,

Rault, Testud, and Papon (1978b) advocated its use for process control. A whole series of papers, such as Richalet, Rault, Testud, and Papon (1978a), Cutler and Ramaker (1980), Prett and Gillette (1980), García and Morshedi (1986), and Marquis and Broustail (1988) helped cement its popularity in the process control industries, and MPC soon became the most useful method in modern control technology for control problems with hard constraints with thousands of applications to its credit.

The basic question of stability, an important issue since optimizing a finite horizon cost does not necessarily yield a stabilizing control, was not resolved in this early literature. Early academic research in MPC, reviewed in García, Prett, and Morari (1989), did not employ Lyapunov theory and therefore restricted attention to control of unconstrained linear systems, studying the effect of control and cost horizons on stability. Similar studies appeared in the literature on generalized predictive control (GPC) (Ydstie, 1984; Peterka, 1984; De Keyser and Van Cauwenberghe, 1985; Clarke, Mohtadi, and Tuffs, 1987) that arose to address deficiencies in minimum variance control. Interestingly enough, earlier research on RHC (Kleinman, 1970; Thomas, 1975; Kwon and Pearson, 1977) had shown indirectly that the imposition of a terminal equality constraint in the finite horizon optimal control problem ensured closed-loop stability for linear unconstrained systems. That a terminal equality constraint had an equally beneficial effect for constrained nonlinear discrete time systems was shown by Keerthi and Gilbert (1988) and for constrained nonlinear continuous time systems by Chen and Shaw (1982) and Mayne and Michalska (1990). In each of these papers, Lyapunov stability theory was employed in contrast to the then current literature on MPC and GPC.

The next advance showed that incorporation of a suitable terminal cost and terminal constraint in the finite horizon optimal control problem ensured closed-loop stability; the terminal constraint set is required to be control invariant, and the terminal cost function is required to be a local CLF. Perhaps the earliest proposal in this direction is the brief paper by Sznaier and Damborg (1987) for linear systems with polytopic constraints; in this prescient paper the terminal cost is chosen to be the value function for the *unconstrained* infinite horizon optimal control problem, and the terminal constraint set is the maximal constraint admissible set (Gilbert and Tan, 1991) for the optimal controlled system.[8] A suitable terminal cost and terminal constraint

[8] If the optimal infinite horizon controlled system is described by $x^+ = A_K x$ and if

set for constrained nonlinear continuous time systems was proposed in Michalska and Mayne (1993) but in the context of dual mode MPC. In a paper that has had considerable impact, Chen and Allgöwer (1998) showed that similar "ingredients" may be employed to stabilize constrained nonlinear continuous time systems when conventional MPC is employed. Related results were obtained by Parisini and Zoppoli (1995), and De Nicolao, Magni, and Scattolini (1996).

Stability proofs for the form of MPC proposed, but not analyzed, in Sznaier and Damborg (1987) were finally provided by Chmielewski and Manousiouthakis (1996) and Scokaert and Rawlings (1998). These papers also showed that optimal control for the *infinite* horizon constrained optimal control problem with a specified initial state is achieved if the horizon is chosen sufficiently long. A terminal constraint is not required if a global, rather than a local, CLF is available for use as a terminal cost function. Thus, for the case when the system being controlled is linear and stable, and subject to a convex control constraint, Rawlings and Muske (1993) showed, in a paper that raised considerable interest, that closed-loop stability may be obtained if the terminal constraint is omitted and the infinite horizon cost using zero control is employed as the terminal cost. The resultant terminal cost is a global CLF.

The basic principles ensuring closed-loop stability in these and many other papers including De Nicolao, Magni, and Scattolini (1998), and Mayne (2000) were distilled and formulated as "stability axioms" in the review paper Mayne et al. (2000); they appear as Assumptions 2.12, 2.13 and 2.16 in this chapter. These assumptions provide sufficient conditions for closed-loop stability for a given horizon. There is an alternative literature that shows that closed-loop stability may often be achieved if the horizon is chosen to be sufficiently long. Contributions in this direction include Primbs and Nevistić (2000), Jadbabaie, Yu, and Hauser (2001), as well as Parisini and Zoppoli (1995), Chmielewski and Manousiouthakis (1996), and Scokaert and Rawlings (1998) already mentioned. An advantage of this approach is that it avoids addition of a terminal constraint, although this may be avoided by alternative means as shown in Section 2.6.

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the constraints are $u \in \mathbb{U}$ and $x \in \mathbb{X}$, then the maximal constraint admissible set is $\{x \mid A_K^i x \in \mathbb{X},\ KA_K^i x \in \mathbb{U}\ \forall i \in \mathbb{I}_{\geq 0}\}$.

## 2.12 Exercises

### Exercise 2.1: Discontinuous MPC

Compute, for Example 2.8, $\mathcal{U}_3(x)$, $V_3^0(x)$ and $\kappa_3(x)$ at a few points on the unit circle.

### Exercise 2.2: Boundedness of discrete time model

Complete the proof of Proposition 2.21 by showing that $f(\cdot)$ and $f_{\mathbb{Z}}^{-1}(\cdot)$ are bounded on bounded sets.

### Exercise 2.3: Destabilization with state constraints

Consider a state feedback regulation problem with the origin as the setpoint (Muske and Rawlings, 1993). Let the system be

$$A = \begin{bmatrix} 4/3 & -2/3 \\ 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad C = [-2/3 \;\; 1]$$

and the controller objective function tuning matrices be

$$Q = I \qquad R = I \qquad N = 5$$

(a) Plot the unconstrained regulator performance starting from initial condition $x(0) = \begin{bmatrix} 3 & 3 \end{bmatrix}'$.

(b) Add the output constraint $y(k) \leq 0.5$. Plot the response of the constrained regulator (both input and output). Is this regulator stabilizing? Can you modify the tuning parameters $Q, R$ to affect stability as in Section 1.3.4?

(c) Change the output constraint to $y(k) \leq 1 + \epsilon, \epsilon > 0$. Plot the closed-loop response for a variety of $\epsilon$. Are any of these regulators destabilizing?

(d) Set the output constraint back to $y(k) \leq 0.5$ and add the terminal constraint $x(N) = 0$. What is the solution to the regulator problem in this case? Increase the horizon $N$. Does this problem eventually go away?

### Exercise 2.4: Computing the projection of $\mathbb{Z}$ onto $\mathcal{X}_N$

Given a polytope

$$\mathbb{Z} := \{(x, u) \in \mathbb{R}^n \times \mathbb{R}^m \mid Gx + Hu \leq \psi\}$$

write an Octave or MATLAB program to determine $\mathcal{X}$, the projection of $\mathbb{Z}$ onto $\mathbb{R}^n$

$$\mathcal{X} = \{x \in \mathbb{R}^n \mid \exists u \in \mathbb{R}^m \text{ such that } (x, u) \in \mathbb{Z}\}$$

Use algorithms 3.1 and 3.2 in Keerthi and Gilbert (1987).

To check your program, consider a system

$$x^+ = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

subject to the constraints $\mathbb{X} = \{x \mid x_1 \leq 2\}$ and $\mathbb{U} = \{u \mid -1 \leq u \leq 1\}$. Consider the MPC problem with $N = 2$, $\mathbf{u} = (u(0), u(1))$, and the set $\mathbb{Z}$ given by

$$\mathbb{Z} = \{(x, \mathbf{u}) \mid x, \phi(1; x, \mathbf{u}), \phi(2; x, \mathbf{u}) \in \mathbb{X} \text{ and } u(0), u(1) \in \mathbb{U}\}$$

Verify that the set

$$\mathcal{X}_2 := \{x \in \mathbb{R}^2 \mid \exists \mathbf{u} \in \mathbb{R}^2 \text{ such that } (x, \mathbf{u}) \in \mathbb{Z}\}$$

is given by

$$\mathcal{X}_2 = \{x \in \mathbb{R}^2 \mid Px \le p\} \qquad P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \qquad p = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$$

### Exercise 2.5: Computing the maximal output admissible set

Write an Octave or MATLAB program to determine the maximal constraint admissible set for the system $x^+ = Fx, y = Hx$ subject to the hard constraint $y \in Y$ in which $Y = \{y \mid Ey \le e\}$. Use algorithm 3.2 in Gilbert and Tan (1991).

To check your program, verify for the system

$$F = \begin{bmatrix} 0.9 & 1 \\ 0 & 0.09 \end{bmatrix} \qquad H = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

subject to the constraint $Y = \{y \mid -1 \le y \le 1\}$, and that the maximal output admissible set is given by

$$O_\infty = \{x \in \mathbb{R}^2 \mid Ax \le b\} \qquad A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 0.9 & 1.01 \\ -0.9 & -1.01 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Show that $t^*$, the smallest integer $t$ such that $O_t = O_\infty$ satisfies $t^* = 1$.

What happens to $t^*$ as $F_{22}$ increases and approaches 1. What do you conclude for the case $F_{22} \ge 1$?

### Exercise 2.6: Terminal constraint and region of attraction

Consider the system

$$x^+ = Ax + Bu$$

subject to the constraints

$$x \in \mathbb{X} \qquad u \in \mathbb{U}$$

in which

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbb{X} = \{x \in \mathbb{R}^2 \mid x_1 \le 5\} \qquad \mathbb{U} = \{u \in \mathbb{R}^2 \mid -\mathbf{1} \le u \le \mathbf{1}\}$$

and $\mathbf{1} \in \mathbb{R}^2$ is a vector of ones. The MPC cost function is

$$V_N(x, \mathbf{u}) = \sum_{i=0}^{N-1} \ell(x(i), u(i)) + V_f(x(N))$$

in which

$$\ell(x, u) = (1/2)\left(|x|_Q^2 + |u|^2\right) \qquad Q = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}$$

and $V_f(\cdot)$ is the terminal penalty on the final state.

(a) Implement unconstrained MPC with no terminal cost ($V_f(\cdot) = 0$) for a few values of $\alpha$. Choose a value of $\alpha$ for which the resultant closed loop is unstable. Try $N = 3$.

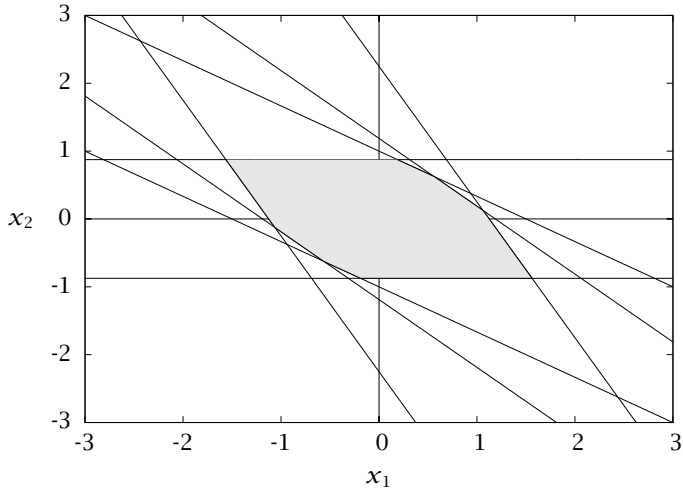


**Figure 2.7:** Region of attraction (shaded region) for constrained MPC controller of Exercise 2.6.

(b) Implement constrained MPC with no terminal cost or terminal constraint for the value of $\alpha$ obtained in the previous part. Is the resultant closed loop stable or unstable?

(c) Implement constrained MPC with terminal equality constraint $x(N) = 0$ for the same value of $\alpha$. Find the region of attraction for the constrained MPC controller using the projection algorithm from Exercise 2.4. The result should resemble Figure 2.7.

### Exercise 2.7: Infinite horizon cost to go as terminal penalty

Consider the system

$$x^+ = Ax + Bu$$

subject to the constraints

$$x \in \mathbb{X} \qquad u \in \mathbb{U}$$

in which

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\mathbb{X} = \{x \in \mathbb{R}^2 \mid -5 \leq x_1 \leq 5\} \qquad \mathbb{U} = \{u \in \mathbb{R}^2 \mid -\mathbf{1} \leq u \leq \mathbf{1}\}$$

The cost is

$$V_N(x, \mathbf{u}) := \sum_{i=0}^{N-1} \ell(x(i), u(i)) + V_f(x(N))$$

in which

$$\ell(x, u) = (1/2)(|x|_Q^2 + |u|^2) \qquad Q = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}$$

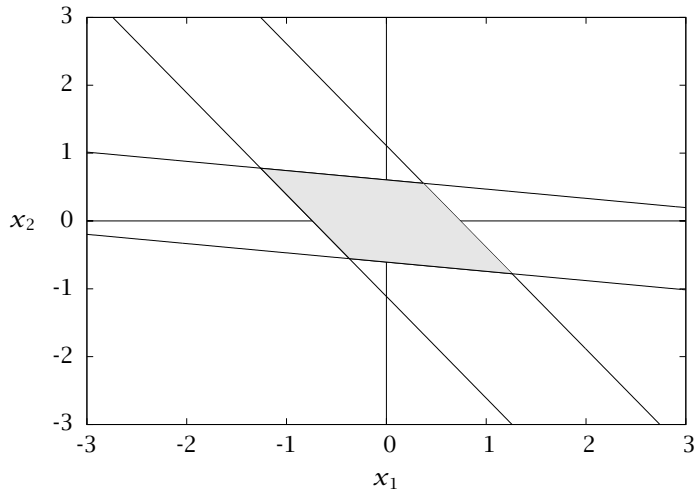


**Figure 2.8:** The region $\mathbb{X}_f$, in which the unconstrained LQR control law is feasible for Exercise 2.7.

and $V_f(\cdot)$ is the terminal penalty on the final state and $\mathbf{1} \in \mathbb{R}^2$ is a vector of all ones. Use $\alpha = 10^{-5}$ and $N = 3$ and terminal cost $V_f(x) = (1/2)x'\Pi x$ where $\Pi$ is the solution to the steady-state Riccati equation.

(a) Compute the infinite horizon optimal cost and control law for the unconstrained system.

(b) Find the region $\mathbb{X}_f$, the maximal constraint admissible set using the algorithm in Exercise 2.5 for the system $x^+ = (A+BK)x$ with constraints $x \in \mathbb{X}$ and $Kx \in \mathbb{U}$. You should obtain the region shown in Figure 2.8.

(c) Add a terminal constraint $x(N) \in \mathbb{X}_f$ and implement constrained MPC. Find $\mathcal{X}_N$, the region of attraction for the MPC problem with $V_f(\cdot)$ as the terminal cost and $x(N) \in \mathbb{X}_f$ as the terminal constraint. Contrast it with the region of attraction for the MPC problem in Exercise 2.6 with a terminal constraint $x(N) = 0$.

(d) Estimate $\bar{X}_N$, the set of initial states for which the MPC control sequence for horizon $N$ is equal to the MPC control sequence for an infinite horizon.
Hint: $x \in \bar{X}_N$ if and only if $x^0(N;x) \in \text{int}(\mathbb{X}_f)$. Why?

### Exercise 2.8: Terminal penalty with and without terminal constraint

Consider the system

$$x^+ = Ax + Bu$$

subject to the constraints

$$x \in \mathbb{X} \qquad u \in \mathbb{U}$$

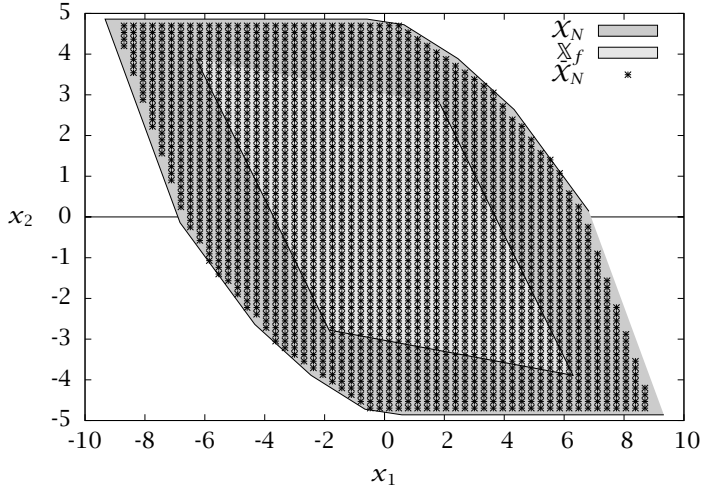


**Figure 2.9:** The region of attraction for terminal constraint $x(N) \in \mathbb{X}_f$ and terminal penalty $V_f(x) = (1/2)x'\Pi x$ and the estimate of $\bar{\mathcal{X}}_N$ for Exercise 2.8.

in which

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\mathbb{X} = \{x \in \mathbb{R}^2 \mid -15 \le x_1 \le 15\} \qquad \mathbb{U} = \{u \in \mathbb{R}^2 \mid -5 \cdot \mathbf{1} \le u \le 5 \cdot \mathbf{1}\}$$

The cost is

$$V_N(x, \mathbf{u}) = \sum_{i=0}^{N-1} \ell(x(i), u(i)) + V_f(x(N))$$

in which

$$\ell(x, u) = (1/2)(|x|_Q^2 + |u|)^2 \qquad Q = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}$$

$V_f(\cdot)$ is the terminal penalty on the final state, and $\mathbf{1} \in \mathbb{R}^2$ is a vector of ones.

Use $\alpha = 10^{-5}$ and $N = 3$ and terminal cost $V_f(x) = (1/2)x'\Pi x$ where $V_f(\cdot)$ is the infinite horizon optimal cost for the unconstrained problem.

(a) Add a terminal constraint $x(N) \in \mathbb{X}_f$, in which $\mathbb{X}_f$ is the maximal constraint admissible set for the system $x^+ = (A + BK)x$ and $K$ is the optimal controller gain for the unconstrained problem. Using the code developed in Exercise 2.7, estimate $\mathcal{X}_N$, the region of attraction for the MPC problem with this terminal constraint and terminal cost. Also estimate $\bar{\mathcal{X}}_N$, the region for which the MPC control sequence for horizon $N$ is equal to the the MPC control sequence for infinite horizon. Your results should resemble Figure 2.9

(b) Remove the terminal constraint and *estimate* the domain of attraction $\hat{\mathcal{X}}_N$ (by simulation). Compare this $\hat{\mathcal{X}}_N$ with $\mathcal{X}_N$ and $\bar{\mathcal{X}}_N$ obtained previously.

(c) Change the terminal cost to $V_f(x) = (3/2)x'\Pi x$ and repeat the previous part.

### Exercise 2.9: Decreasing property for the time-varying case

Prove Lemma 2.32.

### Exercise 2.10: Terminal cost bound for the time-varying case

Prove Lemma 2.33.

### Exercise 2.11: Modification of terminal cost

Refer to Section 2.6.1. Show that the pair $(\beta V_f(\cdot), \mathbb{X}_f)$ satisfies Assumptions 2.12 and 2.13 if $(V_f(\cdot), \mathbb{X}_f)$ satisfies these assumptions, $\beta \geq 1$, and $\ell(\cdot)$ satisfies Assumption 2.16.

### Exercise 2.12: Terminal inequality

Refer to Section 2.6.2 where the terms $V_N^e(\cdot)$, $\hat{V}_N^e(\cdot)$, $V_f^e(\cdot)$ and $\mathcal{X}_N^e$ are defined. Prove that the control sequence

$$\{u^e(0;x), u^e(0;x^+), u^e(1;x^+), \ldots, u^e(N-2;x^+)\}$$

is feasible for problem $\mathbb{P}_N^e(x)$. Use this result and the fact that $x^e(N;x)$ lies in $\mathbb{X}_f$ to establish that

$$V_f^e(x^e(N;x^+)) \leq V_f^e(x^e(N-1;x^+)) - \ell(x^e(N-1;x^+), u^e(N-1;x^+))$$

in which $x^+ := f(x, u^e(0;x))$.

### Exercise 2.13: A Lyapunov theorem for asymptotic stability

Prove the asymptotic stability result for Lyapunov functions.

**Theorem 2.44** (Lyapunov theorem for asymptotic stability)**.** *Given the dynamic system*

$$x^+ = f(x) \qquad 0 = f(0)$$

*The origin is asymptotically stable if there exist $\mathcal{K}$-functions $\alpha$, $\beta$, $\gamma$, and $r > 0$ such that Lyapunov function $V$ satisfies for $x \in r\mathcal{B}$*

$$\alpha(|x|) \leq V(x) \leq \beta(|x|)$$
$$V(f(x)) - V(x) \leq -\gamma(|x|)$$

### Exercise 2.14: An MPC stability result

Given the following nonlinear model and objective function

$$x^+ = f(x,u), \qquad 0 = f(0,0)$$
$$x(0) = x$$
$$V_N(x,\mathbf{u}) = \sum_{k=0}^{N-1} \ell(x(k), u(k))$$

Consider the terminal constraint MPC regulator

$$\min_{\mathbf{u}} V_N(x,\mathbf{u})$$

subject to

$$x^+ = f(x,u) \qquad x(0) = x \qquad x(N) = 0$$

and denote the first move in the optimal control sequence as $u^0(x)$. Given the closed-loop system

$$x^+ = f(x, u^0(x))$$

(a) Prove that the origin is asymptotically stable for the closed-loop system. State the cost function assumption and controllability assumption required so that the control problem is feasible for some set of defined initial conditions.

(b) What assumptions about the cost function $\ell(x,u)$ are required to strengthen the controller so that the origin is exponentially stable for the closed-loop system? How does the controllability assumption change for this case?

### Exercise 2.15: Stability using observability instead of IOSS

Assume that the system $x^+ = f(x,u)$, $y = h(x)$ is $\ell$-observable, i.e., there exists a $\alpha \in \mathcal{K}$ and an integer $N_o \geq 1$ such that

$$\sum_{j=0}^{N_o-1} \ell(y(i), u(i)) \geq \alpha(|x|)$$

for all $x$ and all $\mathbf{u}$; here $x(i) := \phi(i; x, \mathbf{u})$ and $y(i) := h(x(i))$. Prove the result given in Section 2.7 that the origin is asymptotically stable for the closed-loop system $x^+ = f(x, \kappa_N(x))$ using the assumption that $x^+ = f(x,u)$, $y = h(x)$ is $\ell$-observable rather than IOSS. Assume that $N \geq N_o$.

### Exercise 2.16: Input/output-to-state stability (IOSS) and convergence

Prove Proposition 2.41. Hint: consider the solution at time $k+l$ using the state at time $k$ as the initial state.

### Exercise 2.17: Equality for quadratic functions

Prove the following result which is useful for analyzing the unreachable setpoint problem.

**Lemma 2.45** (An equality for quadratic functions)**.** *Let $\mathbb{X}$ be a nonempty compact subset of $\mathbb{R}^n$, and let $\ell(\cdot)$ be a strictly convex quadratic function on $\mathbb{X}$ defined by $\ell(x) := (1/2)x'Qx + q'x + c, \quad Q > 0$. Consider a sequence $\{x(i) \mid i \in \mathbb{I}_{1:P}\}$ with mean $\bar{x}_P := (1/P)\sum_{i=1}^P x(i)$. Then the following holds*

$$\sum_{i=1}^P \ell(x(i)) = (1/2)\sum_{i=1}^P |x(i) - \bar{x}_P|_Q^2 + P\ell(\bar{x}_P)$$

It follows from this lemma that $\ell(\bar{x}_P) \leq (1/P)\sum_{i=1}^P \ell(x(i))$, which is Jensen's inequality for the special case of a quadratic function.

### Exercise 2.18: Unreachable setpoint MPC and evolution in a compact set

Prove the following lemma, which is useful for analyzing the stability of MPC with an unreachable setpoint.

**Lemma 2.46** (Evolution in a compact set)**.** *Suppose $x(0) = x$ lies in the set $\mathcal{X}_N$. Then the state trajectory $\{x(i)\}$ where, for each $i$, $x(i) = \phi_f(i;x)$ of the controlled system $x^+ = f(x)$ evolves in a compact set.*

### Exercise 2.19: MPC and multivariable, constrained systems

Consider a two-input, two-output process with the following transfer function

$$G(s) = \begin{bmatrix} \dfrac{2}{10s+1} & \dfrac{2}{s+1} \\ \dfrac{1}{s+1} & -\dfrac{4}{s+1} \end{bmatrix}$$

(a) Consider a unit setpoint change in the first output. Choose a reasonable sample time, $\Delta$. Simulate the behavior of an offset-free discrete time MPC controller with $Q = I, S = I$ and large $N$.

(b) Add the constraint $-1 \le u(k) \le 1$ and simulate the response.

(c) Add the constraint $-0.1 \le \Delta u/\Delta \le 0.1$ and simulate the response.

(d) Add significant noise to both output measurements (make the standard deviation in each output about 0.1). Retune the MPC controller to obtain good performance. Describe which controller parameters you changed and why.

### Exercise 2.20: LQR versus LAR

We are now all experts on the linear quadratic regulator (LQR), which employs a linear model and quadratic performance measure. Let's consider the case of a linear model but absolute value performance measure, which we call the linear absolute regulator (LAR)[9]

$$\min_{\mathbf{u}} \sum_{k=0}^{N-1} \left(q\,|x(k)| + r\,|u(k)|\right) + q(N)\,|x(N)|$$

For simplicity consider the following one-step controller, in which $u$ and $x$ are *scalars*

$$\min_{u(0)} V(x(0), u(0)) = |x(1)| + |u(0)|$$

subject to

$$x(1) = Ax(0) + Bu(0)$$

Draw a sketch of $x(1)$ versus $u(0)$ (recall $x(0)$ is a known parameter) and show the $x$-axis and $y$-axis intercepts on your plot. Now draw a sketch of $V(x(0), u(0))$ versus $u(0)$ in order to see what kind of optimization problem you are solving. You may want to plot both terms in the objective function individually and then add them together to make your $V$ plot. Label on your plot the places where the cost function $V$ suffers discontinuities in slope. Where is the solution in your sketch? Does it exist for all $A, B, x(0)$? Is it unique for all $A, B, x(0)$?

The motivation for this problem is to change the quadratic program (QP) of the LQR to a linear program (LP) in the LAR, because the computational burden for LPs is often smaller than QPs. The absolute value terms can be converted into linear terms with a standard trick involving slack variables.

[9] Laplace would love us for making this choice, but Gauss would not be happy.

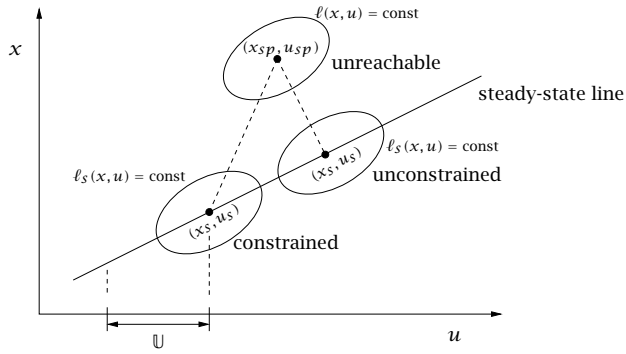


**Figure 2.10:** Inconsistent setpoint $(x_{\rm sp}, u_{\rm sp})$, unreachable stage cost $\ell(x,u)$, and optimal steady states $(x_s, u_s)$, and stage costs $\ell_s(x,u)$ for constrained and unconstrained systems.

### Exercise 2.21: Unreachable setpoints in constrained versus unconstrained linear systems

Consider the linear system with input constraint

$$x^+ = Ax + Bu \qquad u \in \mathbb{U}$$

We examine here both unconstrained systems in which $\mathbb{U} = \mathbb{R}^m$ and constrained systems in which $\mathbb{U} \subset \mathbb{R}^m$ is a convex polyhedron. Consider the stage cost defined in terms of setpoints for state and input $x_{\rm sp}$, $u_{\rm sp}$

$$\ell(x,u) = (1/2)(|x - x_{\rm sp}|_Q^2 + |u - u_{\rm sp}|_R^2)$$

in which we assume for simplicity that $Q, R > 0$. For the setpoint to be unreachable in an unconstrained problem, the setpoint must be *inconsistent*, i.e., not a steady state of the system, or

$$x_{\rm sp} \neq Ax_{\rm sp} + Bu_{\rm sp}$$

Consider also using the stage cost centered at the optimal steady state $(x_s, u_s)$

$$\ell_s(x,u) = (1/2)(\,|x - x_s|_Q^2 + |u - u_s|_R^2\,)$$

The optimal steady state satisfies

$$(x_s, u_s) = \arg\min_{x,u} \ell(x,u)$$

subject to

$$\begin{bmatrix} I - A & -B \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = 0 \qquad u \in \mathbb{U}$$

Figure 2.10 depicts an inconsistent setpoint, and the optimal steady state for unconstrained and constrained systems.

(a) For unconstrained systems, show that optimizing the cost function with terminal constraint

$$V(x, \mathbf{u}) := \sum_{k=0}^{N-1} \ell(x(k), u(k))$$

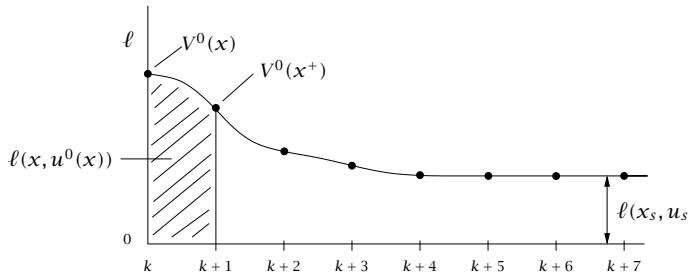


**Figure 2.11:** Stage cost versus time for the case of unreachable setpoint. The cost $V^0(x(k))$ is the area under the curve to the right of time $k$.

subject to

$$x^+ = Ax + Bu \qquad x(0) = x \qquad x(N) = x_s$$

gives the same solution as optimizing the cost function

$$V_s(x, \mathbf{u}) := \sum_{k=0}^{N-1} \ell_s(x(k), u(k))$$

subject to the same model constraint, initial condition, and terminal constraint.

Therefore, there is no reason to consider the unreachable setpoint problem further for an *unconstrained* linear system. Shifting the stage cost from $\ell(x, u)$ to $\ell_s(x, u)$ provides identical control behavior and is simpler to analyze.

> Hint. First define a third stage cost $l(x, u) = \ell(x, u) - \lambda'((I - A)x - Bu)$, and show, for any $\lambda$, optimizing with $l(x, u)$ as stage cost is the same as optimizing using $\ell(x, u)$ as stage cost. Then set $\lambda = \lambda_s$, the optimal Lagrange multiplier of the *steady-state* optimization problem.

(b) For *constrained* systems, provide a simple example that shows optimizing the cost function $V(x, \mathbf{u})$ subject to

$$x^+ = Ax + Bu \qquad x(0) = x \qquad x(N) = x_s \qquad u(k) \in \mathbb{U} \quad \text{for all } k \in \mathbb{I}_{0:N-1}$$

does *not* give the same solution as optimizing the cost function $V_s(x, \mathbf{u})$ subject to the same constraints. For *constrained* linear systems, these problems are different and optimizing the unreachable stage cost provides a new design opportunity.

### Exercise 2.22: Filing for patent

An excited graduate student shows up at your office. He begins, "Look, I have discovered a great money-making scheme using MPC." You ask him to tell you about it. "Well," he says, "you told us in class that the optimal steady state is asymptotically stable even if you use the stage cost measuring distance from the unreachable setpoint, right?" You

reply, "Yes, that's what I said." He continues, "OK, well look at this little sketch I drew," and he shows you a picture like Figure 2.11. "So imagine I use the infinite horizon cost function so the open-loop and closed-loop trajectories are identical. If the best steady state is asymptotically stable, then the stage cost asymptotically approaches $\ell(x_s, u_s)$, right?" You reply, "I guess that looks right." He then says, "OK, well if I look at the optimal cost using state $x$ at time $k$ and state $x^+$ at time $k+1$, by the principle of optimality I get the usual cost decrease"

$$V^0(x^+) \leq V^0(x) - \ell(x, u^0(x)) \tag{2.44}$$

You interrupt, "Wait, these $V^0(\cdot)$ costs are not bounded in this case!" Unfazed, the student replies, "Yeah, I realize that, but this sketch is basically correct regardless. Say we just make the horizon *really long*; then the costs are all finite and this equation becomes closer and closer to being true as we make the horizon longer and longer." You start to feel a little queasy at this point. The student continues, "OK, so if this inequality basically holds, $V^0(x(k))$ is decreasing with $k$ along the closed-loop trajectory, it is bounded below for all $k$, it converges, and, therefore, $\ell(x(k), u^0(x(k)))$ goes to zero as $k$ goes to $\infty$." You definitely don't like where this is heading, and the student finishes with, "But $\ell(x, u) = 0$ implies $x = x_{\text{sp}}$ and $u = u_{\text{sp}}$, and the setpoint is *supposed* to be unreachable. But I have proven that infinite horizon MPC can reach an *unreachable* setpoint. We should patent this!"

How do you respond to this student? Here are some issues to consider.

(a) Does the principle of optimality break down in the unreachable setpoint case?

(b) Are the open-loop and closed-loop trajectories identical in the limit of an infinite horizon controller with an unreachable setpoint?

(c) Does inequality (2.44) hold as $N \to \infty$? If so, how can you put it on solid footing? If not, why not, and with what do you replace it?

(d) Do you file for patent?

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# 3
# Robust Model Predictive Control

## 3.1 Introduction

### 3.1.1 Types of Uncertainty

Robust control concerns control of systems that are uncertain in some sense so that predicted behavior based on the *nominal* system is not identical to actual behavior. Uncertainty may arise in different ways. The system may have an additive disturbance that is unknown, the state of the system may not be perfectly known, or the model of the system that is used to determine control may be inaccurate.

A system with additive disturbance satisfies the following difference equation

$$x^+ = f(x,u) + w$$

The disturbance $w$ in constrained optimal control problems is usually assumed to be bounded since it is impossible to ensure that a system with unbounded disturbances satisfies the usual state and control constraints. More precisely, we usually assume that $w$ satisfies the constraint $w \in \mathbb{W}$ where $\mathbb{W}$ is a compact subset of $\mathbb{R}^n$ containing the origin.

The situation in which the state is not perfectly measured may be treated in several ways. In the stochastic optimal control literature, where the measured output is $y = Cx + v$ and the disturbance $w$ and measurement noise $v$ are usually assumed to be Gaussian white noise processes, the state or *hyperstate* of the optimal control problem is the conditional density of the state $x$ at time $k$ given prior measurements $\{y(0), y(1), \ldots, y(k-1)\}$. Because this density is usually difficult to compute and use, except in the linear case when it is provided by the Kalman filter, a suboptimal procedure is often adopted. In this suboptimal approach, the state $x$ is replaced by its estimate $\hat{x}$ in a control law

determined under the assumption that the state is accessible. This procedure is usually referred to as *certainty equivalence*, a term that was originally employed for the linear quadratic Gaussian (LQG) or similar cases when this procedure did not result in loss of optimality. When $f(\cdot)$ is linear, the evolution of the state estimate $\hat{x}$ may be expressed by a difference equation

$$\hat{x}^+ = f(\hat{x}, u) + \xi$$

in which $\xi$ is the *innovation process*. In controlling $\hat{x}$, we should ensure that the actual state $x$, which, if the innovation process is bounded, lies in a bounded, possibly time-varying neighborhood of $\hat{x}$, satisfies the constraints of the optimal control problem.

Finally, a system that has parametric uncertainty may be modeled as

$$x^+ = f(x, u, \theta)$$

in which $\theta$ represents parameters of the system that are known only to the extent that they belong to a compact set $\Theta$. A much studied example is

$$x^+ = Ax + Bu$$

in which $\theta := (A, B)$ may take any value in $\Theta := \text{co}\{(A_i, B_i) \mid i \in \mathcal{I}\}$ where $\mathcal{I} = \{1, 2, \ldots, I\}$, say, is an index set.

It is possible, of course, for all these types of uncertainty to occur in a single application. In this chapter we focus on the first and third types of uncertainty, namely, additive disturbance and parameter uncertainty. Output MPC, where the controller employs an estimate of the state, rather than the state itself, is treated in Chapter 5.

### 3.1.2 Feedback Versus Open-Loop Control

It is well known that feedback is required only when uncertainty is present; in the absence of uncertainty, feedback control and open-loop control are equivalent. Indeed, when uncertainty is not present, as for the systems studied in Chapter 2, the optimal control for a given initial state may be computed using either dynamic programming (DP) that provides an optimal control policy or sequence of feedback control laws, or an open-loop optimal control that merely provides a sequence of control actions. A simple example illustrates this fact. Consider the deterministic linear dynamic system defined by

$$x^+ = x + u$$

The optimal control problem, with horizon $N = 3$, is

$$\mathbb{P}_3(x): \qquad V_3^0(x) = \min_{\mathbf{u}_3} V_3(x, \mathbf{u})$$

in which $\mathbf{u} = \{u(0), u(1), u(2)\}$

$$V_3(x, \mathbf{u}) := (1/2) \sum_{i=0}^{2} [(x(i)^2 + u(i)^2)] + (1/2)x(3)^2$$

where, for each $i$, $x(i) = \phi(i; x, \mathbf{u}) = x + u(0) + u(1) + \ldots + u(i-1)$, the solution of the difference equation $x^+ = x + u$ at time $i$ if the initial state is $x(0) = x$ and the control (input) sequence is $\mathbf{u} = \{u(0), u(1), u(2)\}$; in matrix operations $\mathbf{u}$ is taken to be the column vector $[u(0), u(1), u(2)]'$. Thus

$$\begin{aligned} V_3(x, \mathbf{u}) &= (1/2)[x^2 + (x + u(0))^2 + (x + u(0) + u(1))^2 + \\ &\qquad (x + u(0) + u(1) + u(2))^2 + u(0)^2 + u(1)^2 + u(2)^2] \\ &= (3/2)x^2 + x \begin{bmatrix} 3 & 2 & 1 \end{bmatrix} \mathbf{u} + (1/2)\mathbf{u}' P_3 \mathbf{u} \end{aligned}$$

in which

$$P_3 = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

The vector form of the optimal *open-loop* control sequence for an initial state of $x$ is, therefore,

$$\mathbf{u}^0(x) = -P_3^{-1} \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}' x = -\begin{bmatrix} 0.615 & 0.231 & 0.077 \end{bmatrix}' x$$

The optimal control and state sequences are, therefore,

$$\begin{aligned} \mathbf{u}^0(x) &= \{-0.615x, -0.231x, -0.077x\} \\ \mathbf{x}^0(x) &= \{x, 0.385x, 0.154x, 0.077x\} \end{aligned}$$

To compute the optimal *feedback* control, we use the DP recursions

$$\begin{aligned} V_i^0(x) &= \min_{u \in \mathbb{R}} \{x^2/2 + u^2/2 + V_{i-1}^0(x + u)\} \\ \kappa_i^0(x) &= \arg\min_{u \in \mathbb{R}} \{x^2/2 + u^2/2 + V_{i-1}^0(x + u)\} \end{aligned}$$

with boundary condition

$$V_0^0(x) = (1/2)x^2$$

This procedure gives the value function $V_i^0(\cdot)$ and the optimal control law $\kappa_i^0(\cdot)$ at each $i$ where the subscript $i$ denotes time to go. Solving the DP recursion, for all $x \in \mathbb{R}$, all $i \in \{1,2,3\}$, yields

$$\begin{aligned} V_1^0(x) &= (3/4)x^2 & \kappa_1^0(x) &= -(1/2)x \\ V_2^0(x) &= (4/5)x^2 & \kappa_2^0(x) &= -(3/5)x \\ V_3^0(x) &= (21/26)x^2 & \kappa_3^0(x) &= -(8/13)x \end{aligned}$$

Starting at state $x$ at time 0, and applying the optimal control laws iteratively to the *deterministic* system $x^+ = x + u$ (recalling that at time $i$ the optimal control law is $\kappa_{3-i}^0(\cdot)$ since, at time $i$, $3-i$ is the time to go) yields

$$\begin{aligned} x^0(0) &= x & u^0(0) &= -(8/13)x \\ x^0(1) &= (5/13)x & u^0(1) &= -(3/13)x \\ x^0(2) &= (2/13)x & u^0(2) &= -(1/13)x \\ x^0(3;x) &= (1/13)x \end{aligned}$$

so that the optimal control and state sequences are, respectively,

$$\begin{aligned} \mathbf{u}^0(x) &= \{-(8/13)x, -(3/13)x, -(1/13)x\} \\ \mathbf{x}^0(x) &= \{x, (5/13)x, (2/13)x, (1/13)x\} \end{aligned}$$

which are identical with the optimal open-loop values computed above.

Consider next an uncertain version of the dynamic system in which uncertainty takes the simple form of an additive disturbance $w$; the system is defined by

$$x^+ = x + u + w$$

in which the only knowledge of $w$ is that it lies in the compact set $\mathbb{W} := [-1,1]$. Let $\phi(i;x,\mathbf{u},\mathbf{w})$ denote the solution of this system at time $i$ if the initial state is $x$ at time 0, and the input and disturbance sequences are, respectively, $\mathbf{u}$ and $\mathbf{w} := \{w(0), w(1), w(2)\}$. The cost now depends on the disturbance sequence — but it also depends, in contrast to the deterministic problem discussed above, on whether the control is open-loop or feedback. To discuss the latter case, we define a feedback policy $\boldsymbol{\mu}$ to be a sequence of control laws

$$\boldsymbol{\mu} := \{\mu_0(\cdot), \mu_1(\cdot), \mu_2(\cdot)\}$$

in which $\mu_i : \mathbb{R} \to \mathbb{R}$, $i = 0,1,2$; under policy $\boldsymbol{\mu}$, if the state at time $i$ is $x$, the control is $\mu_i(x)$. Let $\mathcal{M}$ denote the class of *admissible* policies, for

example those policies for which each control law $\mu_i(\cdot)$ is continuous. Then, $\phi(i; x, \boldsymbol{\mu}, \mathbf{w})$ denotes the solution at time $i \in \{0, 1, 2, 3\}$ of the following difference equation

$$x(i+1) = x(i) + \mu_i(x(i)) + w(i) \qquad x(0) = x$$

An open-loop control sequence $\mathbf{u} = \{u(0), u(1), u(2)\}$ is then merely a degenerate policy $\boldsymbol{\mu} = \{\mu_0(\cdot), \mu_1(\cdot), \mu_2(\cdot)\}$ where each control law $\mu_i(\cdot)$ satisfies

$$\mu_i(x) = u(i)$$

for all $x \in \mathbb{R}$ and all $i \in \{0, 1, 2\}$. The cost $V_3(\cdot)$ may now be defined

$$V_3(x, \boldsymbol{\mu}, \mathbf{w}) := (1/2) \sum_{i=0}^{2} [(x(i)^2 + u(i)^2)] + (1/2)x(3)^2$$

where, now, $x(i) = \phi(i; x, \boldsymbol{\mu}, \mathbf{w})$ and $u(i) = \mu_i(x(i))$. Since the disturbance is unpredictable, the value of $\mathbf{w}$ is not known at time 0, so the optimal control problem must "eliminate" it in some meaningful way so that the solution $\boldsymbol{\mu}^0(x)$ does not depend on $\mathbf{w}$. To eliminate $\mathbf{w}$, the optimal control problem $\mathbb{P}_3^*(x)$ is defined by

$$\mathbb{P}_3^*(x): \qquad V_3^0(x) := \inf_{\boldsymbol{\mu} \in \mathcal{M}} J_3(x, \boldsymbol{\mu})$$

in which the cost $J_3(\cdot)$ is defined in such a way that it does not depend on $\mathbf{w}$; inf is used rather than min in this definition since the minimum may not exist. The most popular choice for $J_3(\cdot)$ in the MPC literature is

$$J_3(x, \boldsymbol{\mu}) := \max_{\mathbf{w} \in \mathcal{W}} V_3(x, \boldsymbol{\mu}, \mathbf{w})$$

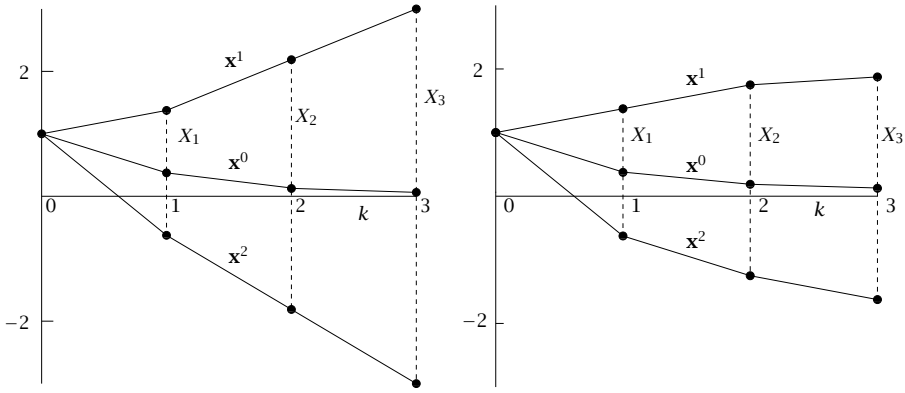
in which the disturbance $\mathbf{w}$ is assumed to lie in $\mathcal{W}$ a bounded class of admissible disturbance sequences. Alternatively, if the disturbance sequence is random, the cost $J_3(\cdot)$ may be chosen to be

$$J_3(x, \boldsymbol{\mu}) := \mathcal{E} V_3(x, \boldsymbol{\mu}, \mathbf{w})$$

in which $\mathcal{E}$ denotes "expectation" or average, over random disturbance sequences. For our purpose here, we adopt the simple cost

$$J_3(x, \boldsymbol{\mu}) := V_3(x, \boldsymbol{\mu}, \mathbf{0})$$

in which $\mathbf{0} := \{0, 0, 0\}$ is the zero disturbance sequence. In this case, $J_3(x, \boldsymbol{\mu})$ is the nominal cost, i.e., the cost associated with the nominal




(a) Open-loop trajectories.

(b) Feedback trajectories.

**Figure 3.1:** Open-loop and feedback trajectories.

system $x^+ = x + u$ in which the disturbance is neglected. With this cost function, the solution to $\mathbb{P}_3^*(x)$ is the DP solution, obtained previously, to the deterministic *nominal* optimal control problem.

We now compare two solutions to $\mathbb{P}_3(x)$: the open-loop solution in which $\mathcal{M}$ is restricted to be the set of control sequences, and the feedback solution in which $\mathcal{M}$ is the class of admissible policies. The solution to the first problem is the solution to the deterministic problem discussed previously; the optimal control sequence is

$$\mathbf{u}^0(x) = \{-(8/13)x, -(3/13)x, -(1/13)x\}$$

in which $x$ is the initial state at time 0. The solution to the second problem is the sequence of control laws determined previously, also for the deterministic problem, using *dynamic programming*; the optimal policy is $\boldsymbol{\mu}^0 = \{\mu_0^0(\cdot), \mu_1^0(\cdot), \mu_2(\cdot)\}$ where the control laws (functions) $\mu_i(\cdot)$, $i = 0, 1, 2$, are defined by

$$\begin{aligned}
\mu_0^0(x) &:= \kappa_3^0(x) = -(8/13)x \quad \forall x \in \mathbb{R} \\
\mu_1^0(x) &:= \kappa_2^0(x) = -(3/5)x \quad \forall x \in \mathbb{R} \\
\mu_2^0(x) &:= \kappa_1^0(x) = -(1/2)x \quad \forall x \in \mathbb{R}
\end{aligned}$$

The two solutions, $\mathbf{u}^0(\cdot)$ and $\boldsymbol{\mu}^0$, when applied to the uncertain system $x^+ = x + u + w$ do *not* yield the same trajectories for all disturbance sequences. This is illustrated in Figure 3.1 for the three disturbance sequences, $\mathbf{w}^0 := \{0, 0, 0\}$, $\mathbf{w}^1 := \{1, 1, 1\}$, and $\mathbf{w}^2 := \{-1, -1, -1\}$;

and initial state $x = 1$ for which the corresponding state trajectories, denoted $\mathbf{x}^0$, $\mathbf{x}^1$, and $\mathbf{x}^2$, are

**Open-loop solution.**

$$\mathbf{x}^0 = \{1, (5/13), (2/13), (1/13)\}$$
$$\mathbf{x}^1 = \{1, (18/13), (28/13), (40/13)\}$$
$$\mathbf{x}^2 = \{1, -(8/13), -(24/13), -(38/13)\}$$

**Feedback solution.**

$$\mathbf{x}^0 = \{1, (5/13), (2/13), (1/13)\}$$
$$\mathbf{x}^1 = \{1, (18/13), (101/65), (231/130)\}$$
$$\mathbf{x}^2 = \{1, -(8/13), -(81/65), -(211/130)\}$$

Even for the short horizon of 3, the superiority of the feedback solution can be seen although the feedback was designed for the deterministic (nominal) system and therefore did not take the disturbance into account. For the open-loop solution $|x^2(3) - x^1(3)| = 6$, whereas for the feedback case $|x^2(3) - x^1(3)| = 3.4$; the open-loop solution does not restrain the *spread* of the trajectories resulting from the disturbance $\mathbf{w}$. If the horizon length is $N$, for the open-loop solution, $|x^2(N) - x^1(N)| = 2N$, whereas for the feedback case $|x^2(N) - x^1(N)| \to 3.24$ as $N \to \infty$. The obvious and well-known conclusion is that feedback control is superior to open-loop control when uncertainty is present. Feedback control requires determination of a control *policy*, however, which is a difficult task if nonlinearity and/or constraints are features of the optimal control problem.

### 3.1.3 Robust MPC

An important feature of conventional, or deterministic, MPC discussed in Chapter 2 is that the solution of the open-loop optimal control problem solved online is identical to that obtained by DP for the given initial state. When uncertainty is present and the state is known or observations of the state are available, feedback control is superior to open-loop control. The optimal control problem solved online must, therefore, permit feedback in order for its solution to coincide with the DP solution. The online optimal control problem with horizon $N$ is $\mathbb{P}_N^*(x)$ in which the decision variable $\boldsymbol{\mu}$ is a sequence of control *laws* rather than $\mathbb{P}_N(x)$ in which the decision variable $\mathbf{u}$ is a sequence of control

*actions*. MPC in which the decision variable is a policy has been termed *feedback* MPC to distinguish it from conventional MPC. Both forms of MPC naturally provide feedback control since the control that is implemented depends on the current state $x$ in both cases. But the control that is applied depends on whether the optimal control problem solved is open loop, in which case the decision variable is a control sequence, or feedback, in which case the decision variable is a feedback policy.

In feedback MPC the solution to the optimal control problem $\mathbb{P}_N^*(x)$ is the policy $\boldsymbol{\mu}^0(x) = \{\mu_0^0(\cdot;x), \mu_1^0(\cdot;x), \ldots, \mu_{N-1}^0(\cdot;x)\}$. The constituent control laws are restrictions of those determined by DP and therefore depend on the initial state $x$ as implied by the notation. Thus only the value $u^0(x) = \mu_0(x;x)$ of the control law $\mu_0(\cdot;x)$ at the initial state $x$ need be determined while successive laws need only be determined over a limited range. In the example illustrated in Figure 3.1, $\mu_0(\cdot;x)$ need only be determined at the point $x = 1$, $\mu_1(\cdot;x)$ need only be determined over the interval $[-8/13, 18/13]$, and $\mu_2(\cdot;x)$ over the interval $[-81/65, 101/65]$, whereas in the DP solution these control laws are defined over the infinite interval $(-\infty, \infty)$.

While feedback MPC is superior in the presence of uncertainty, the associated optimal control problem is vastly more complex than the optimal control problem employed in deterministic MPC. The decision variable $\boldsymbol{\mu}$, being a sequence of control laws, is infinite dimensional; each law or function requires, in general, an infinite dimensional grid to specify it. The complexity is comparable to solving the DP equation, so that MPC, which in the deterministic case replaces DP with a solvable open-loop optimization problem, is not easily solved when uncertainty is present. Hence much research effort has been devoted to forms of feedback MPC that sacrifice optimality for simplicity. As in the early days of adaptive control, many different proposals have been made. These proposals for robust MPC are all simpler to implement than the optimal solution provided by DP.

At the current stage of research it is perhaps premature to select a particular approach; we have, nevertheless, selected one approach, *tube-based* MPC that we describe here and in Chapter 5. There is a good reason for our choice. It is well known that standard mathematical optimization algorithms may be used to obtain an optimal open-loop control sequence for an optimal control problem. What is less well known is that there exist algorithms, the second variation algorithms, which provide not only an optimal control sequence but also a *local* time-varying feedback law of the form $u(k) = v(k) + K(k)(x(k) - z(k))$

where $\{v(k)\}$ is the optimal open-loop control sequence and $\{z(k)\}$ the corresponding optimal open-loop state sequence. This policy provides feedback control for states $x(k)$ close to the nominal states $z(k)$. The second variation algorithms are too complex for routine use in MPC because they require computation of the second derivatives with respect to $(x, u)$ of $f(\cdot)$ and $\ell(\cdot)$. When the system is linear, the cost quadratic, and the disturbance additive, however, the optimal control law for the unconstrained infinite horizon case is $u = Kx$. This result may be expressed as a time-varying control law $u(k) = v(k) + K(x(k) - z(k))$ where the state and control sequences $\{z(k)\}$ and $\{v(k)\}$ satisfy the nominal difference equations $z^+ = Az + Bv$, $v = Kz$, i.e., the sequences $\{z(k)\}$ and $\{v(k)\}$ are optimal open-loop solutions for zero disturbance and some initial state. The time-varying control law $u(k) = v(k) + K(x(k) - z(k))$ is clearly optimal in the unconstrained case; it remains optimal for the constrained case in the neighborhood of the nominal trajectory $\{z(k)\}$ if $\{z(k)\}$ and $\{v(k)\}$ lie in the interior of their respective constraint sets.

These comments suggest that a time-varying policy of the form $u(x, k) = v(k) + K(x - z(k))$ might be adequate, at least when $f(\cdot)$ is linear. The nominal control and state sequences, $\{v(k)\}$ and $\{z(k)\}$, respectively, can be determined by solving a standard open-loop optimal control problem of the form usually employed in MPC, and the feedback matrix $K$ can be determined offline. We show that this form of robust MPC has the same order of online complexity as that conventionally used for deterministic systems. It requires a modified form of the online optimal control problem in which the constraints are simply *tightened* to allow for disturbances, thereby constraining the trajectories of the uncertain system to lie in a tube centered on the nominal trajectories. Offline computations are required to determine the modified constraints and the feedback matrix $K$. We also present, in the last section of this chapter, a modification of this tube-based procedure for nonlinear systems for which a *nonlinear* local feedback policy is required.

A word of caution is necessary. Just as nominal model predictive controllers presented in Chapter 2 may fail in the presence of uncertainty, the controllers presented in this chapter may fail if the actual uncertainty does not satisfy our assumptions, such as when a disturbance that we assume to be bounded exceeds the assumed bounds; the controlled systems are robust only to the specified uncertainties. As always, online fault diagnosis and safe recovery procedures are required

to protect the system from unanticipated events.

### 3.1.4 Tubes

The approach that we adopt is motivated by the following observation. Both open-loop and feedback control generate, in the presence of uncertainty, a *bundle* or *tube* of trajectories, each trajectory in the bundle or tube corresponding to a particular realization of the uncertainty. In Figure 3.1(a), the tube corresponding to $\mathbf{u} = \mathbf{u}^0(x)$ and initial state $x = 1$, is $\{X_0, X_1, X_2, X_3\}$ where $X_0 = \{1\}$; for each $i$, $X_i = \{\phi(i; x, \mathbf{u}, \mathbf{w}) \mid \mathbf{w} \in \mathcal{W}\}$, the set of states at time $i$ generated by all possible realizations of the disturbance sequence. State constraints must be satisfied by every trajectory in the tube. Control of uncertain systems is best viewed as control of tubes rather than trajectories; the designer chooses, for each initial state, a tube in which all realizations of the state trajectory are controlled to lie. By suitable choice of the tube, satisfaction of state and control constraints may be guaranteed for *every* realization of the disturbance sequence.

Determination of an *exact* tube $\{X_0, X_1, \ldots\}$ corresponding to a given initial state $x$ and policy $\boldsymbol{\mu}$ is difficult even for linear systems, however, and virtually impossible for nonlinear systems. Hence, in the sequel, we show how simple tubes that bound all realizations of the state trajectory may be constructed. For example, for linear systems with convex constraints, a tube $\{X_0, X_1, \ldots, \}$, where for each $i$, $X_i = \{z(i)\} \oplus Z$, $z(i)$ is the state at time $i$ of a deterministic system, $X_i$ is a polytope, and $Z$ is a positive invariant set, may be designed to bound all realizations of the state trajectory. The exact tube lies inside this simple approximation. Using this construction permits robust model predictive controllers to be designed with not much more computation than that required for deterministic systems.

### 3.1.5 Difference Inclusion Description of Uncertain Systems

Here we introduce some notation that will be useful in the sequel. A deterministic discrete time system is usually described by a difference equation

$$x^+ = f(x, u) \tag{3.1}$$

We use $\phi(k; x, i, \mathbf{u})$ to denote the solution of (3.1) at time $k$ when the initial state at time $i$ is $x$ and the control sequence is $\mathbf{u} = \{u(0), u(1), \ldots\}$; if the initial time $i = 0$, we write $\phi(k; x, \mathbf{u})$ in place of $\phi(k; (x, 0), \mathbf{u})$.

Similarly, an uncertain system may be described by the difference equation

$$x^+ = f(x, u, w) \tag{3.2}$$

in which the variable $w$ that represents the uncertainty takes values in a specified set $\mathbb{W}$. We use $\phi(k; x, i, \mathbf{u}, \mathbf{w})$ to denote the solution of (3.2) when the initial state at time $i$ is $x$ and the control and disturbance sequences are, respectively, $\mathbf{u} = \{u(0), u(1), \ldots\}$ and $\mathbf{w} = \{w(0), w(1), \ldots\}$. The uncertain system may alternatively be described by a *difference inclusion* of the form

$$x^+ \in F(x, u)$$

in which $F(\cdot)$ is a set-valued map. We use the notation $F : \mathbb{R}^n \times \mathbb{R}^m \rightsquigarrow \mathbb{R}^n$ or[1] $F : \mathbb{R}^n \times \mathbb{R}^m \to 2^{\mathbb{R}^n}$ to denote a function that maps points in $\mathbb{R}^n \times \mathbb{R}^m$ into subsets of $\mathbb{R}^n$. If the uncertain system is described by (3.2), then

$$F(x, u) = f(x, u, \mathbb{W}) := \{f(x, u, w) \mid w \in \mathbb{W}\}$$

If $x$ is the current state, and $u$ the current control, the successor state $x^+$ lies anywhere in the set $F(x, u)$. If a control policy $\boldsymbol{\mu} := \{\mu_0(\cdot), \mu_1(\cdot), \ldots\}$ is employed, the state evolves according to

$$x^+ \in F(x, \mu_k(x)) \tag{3.3}$$

in which $x$ is the current state, $k$ the current time, and $x^+$ the successor state at time $k+1$. The system described by (3.3) does not have a single solution for a given initial state; it has a solution for each possible realization $\mathbf{w}$ of the disturbance sequence. We use $S(x, i)$ to denote the set of solutions of (3.3) if the initial state is $x$ at time $i$. If $\phi(\cdot) \in S(x, i)$ then

$$\phi(t) = \phi(t; x, i, \boldsymbol{\mu}, \mathbf{w})$$

for some admissible disturbance sequence $\mathbf{w}$ where $\phi(t; x, i, \boldsymbol{\mu}, \mathbf{w})$ denotes the solution at time $t$ of

$$x^+ = f(x, \mu_k(x), w)$$

when the initial state is $x$ at time $i$ and the disturbance sequence is $\mathbf{w}$. The policy $\boldsymbol{\mu}$ is defined, as before, to be the sequence $\{\mu_0(\cdot), \mu_1(\cdot), \ldots,$

[1] For any set $X$, $2^X$ denotes the set of subsets of $X$.

$\mu_{N-1}(\cdot)\}$ of control laws. The tube $\mathbf{X} = \{X_0, X_1, \ldots\}$, discussed in Section 3.4, generated when policy $\boldsymbol{\mu}$ is employed, satisfies

$$X_{k+1} = \mathbf{F}(X_k, \mu_k(\cdot)) := \{f(x, \mu_k(x), w) \mid x \in X_k, w \in \mathbb{W}\}$$

in which $\mathbf{F}$ maps sets into sets.

## 3.2 Nominal (*Inherent*) Robustness

### 3.2.1 Introduction

Because feedback MPC is complex, it is natural to inquire if nominal MPC, i.e., MPC based on the nominal system ignoring uncertainty, is sufficiently robust to uncertainty. Before proceeding with a detailed analysis, a few comments may be helpful.

MPC uses, as a Lyapunov function, the value function of a parametric optimal control problem. Often the value function is continuous, but this is not necessarily the case, especially if state and/or terminal constraints are present. It is also possible for the value function to be continuous but the associated control law to be discontinuous; this can happen, for example, if the minimizing control is not unique.

It is important to realize that a control law may be stabilizing but not robustly stabilizing; arbitrary perturbations, no matter how small, can destabilize the system. Teel (2004) illustrates this point with the following discontinuous autonomous system ($n = 2$, $x = (x_1, x_2)$)

$$x^+ = f(x) \qquad f(x) = \begin{cases} (0, |x|) & x_1 \neq 0 \\ (0, 0) & \text{otherwise} \end{cases}$$

If the initial state is $x = (1, 1)$, then $\phi(1; x) = (0, \sqrt{2})$ and $\phi(2; x) = (0, 0)$, with similar behavior for other initial states. In fact, all solutions satisfy

$$\phi(k; x) \leq \beta(|x|, k)$$

in which $\beta$, defined by

$$\beta(|x|, k) := |x| \max\{2 - k, 0\}$$

is a $\mathcal{KL}$ function, so that the origin is *globally asymptotically stable*. Consider now a perturbed system satisfying

$$x^+ = \begin{bmatrix} \delta \\ |x| + \delta \end{bmatrix}$$

in which $\delta > 0$ is a constant perturbation that causes $x_1$ to remain strictly positive. If the initial state is $x = \varepsilon(1,1)$, then $x_1(k) = \delta$ for $k \geq 1$, and $x_2(k) > \varepsilon\sqrt{2} + k\delta \to \infty$ as $k \to \infty$, no matter how small $\delta$ and $\varepsilon$ are. Hence the origin is unstable in the presence of an arbitrarily small perturbation; global asymptotic stability is not a robust property of this system.

This example may appear contrived but, as Teel (2004) points out, it can arise in receding horizon optimal control of a *continuous system*. Consider the following system

$$x^+ = \begin{bmatrix} x_1(1-u) \\ |x|\, u \end{bmatrix}$$

in which the control $u$ is constrained to lie in the set $\mathbb{U} = [-1,1]$. Suppose we choose a horizon length $N = 2$ and choose $\mathbb{X}_f$ to be the origin. If $x_1 \neq 0$, the only feasible control sequence steering $x$ to 0 in two steps is $\mathbf{u} = \{1,0\}$; the resulting state sequence is $\{x, (0,|x|), (0,0)\}$. Since there is only one feasible control sequence, it is also optimal, and $\kappa_2(x) = 1$ for all $x$ such that $x_1 \neq 0$. If $x_1 = 0$, then the only optimal control sequence is $\mathbf{u} = \{0,0\}$ and $\kappa_2(x) = 0$. The resultant closed-loop system satisfies

$$x^+ = f(x) := \begin{bmatrix} x_1(1-\kappa_2(x)) \\ |x|\, \kappa_2(x) \end{bmatrix}$$

in which $\kappa_2(x) = 1$ if $x_1 \neq 0$, and $\kappa_2(x) = 0$ otherwise. Thus

$$f(x) = \begin{cases} (0,|x|) & x_1 \neq 0 \\ (0,0) & \text{otherwise} \end{cases} \tag{3.4}$$

The system $x^+ = f(x)$ is the discontinuous system analyzed previously. Thus, receding horizon optimal control of a continuous system has resulted in a discontinuous system that is globally asymptotically stable but has no robustness.

### 3.2.2 Difference Inclusion Description of Discontinuous Systems

Consider a discontinuous system

$$x^+ = f(x)$$

in which $f(\cdot)$ is not continuous. An example of such a system occurred in the previous subsection where $f(\cdot)$ satisfies (3.4). Solutions of this

system are very sensitive to the value of $x_1$. An infinitesimal change in $x_1$ at time 0, say, from 0 can cause a substantial change in the subsequent trajectory resulting, in this example, in a loss of robustness. To design a robust system, one must take into account, in the design process, the system's extreme sensitivity to variations in state. This can be done by *regularizing* the system (Teel, 2004). If $f(\cdot)$ is locally bounded,[2] the *regularization* $x^+ = f(x)$ is defined to be

$$x^+ \in F(x) := \bigcap_{\delta>0} \overline{f(\{x\} \oplus \delta\mathcal{B})}$$

in which $\mathcal{B}$ is the closed unit ball so that $\{x\} \oplus \delta\bar{\mathcal{B}} = \{z \mid |z - x| \le \delta\}$ and $\overline{A}$ denotes the closure of set $A$. At points where $f(\cdot)$ is continuous, $F(x) = \{f(x)\}$, i.e., $F(x)$ is the single point $f(x)$. If $f(\cdot)$ is piecewise continuous, e.g., if $f(x) = x$ if $x < 1$ and $f(x) = 2x$ if $x \ge 1$, then $F(x) = \{\lim_{x_i \to x} f(x_i)\}$, the set of all limits of $f(x_i)$ as $x_i \to x$. For our example immediately above, $F(x) = \{x\}$ if $x < 1$ and $F(x) = \{2x\}$ if $x > 1$. When $x = 1$, the limit of $f(x_i)$ as $x_i \to 1$ from below is 1 and the limit of $f(x_i)$ as $x \to 1$ from above is 2, so that $F(1) = \{1, 2\}$. The regularization of $x^+ = f(x)$ where $f(\cdot)$ is defined in (3.4) is $x^+ \in F(x)$ where $F(\cdot)$ is defined by

$$F(x) = \left\{ \begin{bmatrix} 0 \\ |x| \end{bmatrix} \right\} \qquad x_1 \neq 0 \tag{3.5}$$

$$F(x) = \left\{ \begin{bmatrix} 0 \\ |x| \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} \quad x_1 = 0 \tag{3.6}$$

If the initial state is $x = (1, 1)$, as before, then the difference inclusion generates the following tube

$$X_0 = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}, \qquad X_1 = \left\{ \begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix} \right\}, \qquad X_2 = \left\{ \begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}, \quad \dots$$

with $X_k = X_2$ for all $k \ge 2$. The set $X_k$ of possible states clearly does not converge to the origin even though the trajectory generated by the original system does. The regularization reveals that small perturbations can destabilize the system.

### 3.2.3 When Is Nominal MPC Robust?

The discussion in Section 2.1 shows that nominal MPC is not necessarily robust. It is therefore natural to ask under what conditions nominal

[2] A function $f : \mathbb{R}^p \to \mathbb{R}^n$ is locally bounded if, for every $x \in \mathbb{R}^p$, there exists a neighborhood $\mathcal{N}$ of $x$ and a $c > 0$ such that $|f(z)| \le c$ for all $z \in \mathcal{N}$.

MPC is robust. To answer this, we have to define robustness precisely. In Appendix B, we define robust stability, and robust asymptotic stability, of a set. We employ this concept later in this chapter in the design of robust model predictive controllers that for a given initial state in the region of attraction, steer *every* realization of the state trajectory to this set. Here, however, we address a slightly different question: when is nominal MPC that steers every trajectory in the region of attraction to the origin robust? Obviously, the disturbance will preclude the controller from steering the state of the perturbed system to the origin; the best that can be hoped for is that the controller will steer the state to some small neighborhood of the origin. Let the nominal (controlled) system be described by $x^+ = f(x)$ where $f(\cdot)$ is not necessarily continuous, and let the perturbed system be described by $x^+ = f(x+e)+w$. Also let $S_\delta(x)$ denote the set of solutions for the perturbed system with initial state $x$ and perturbation sequences $\mathbf{e} := \{e(0), e(1), e(2), \ldots\}$ and $\mathbf{w} := \{w(0), w(1), w(2), \ldots\}$ satisfying $\max\{\|\mathbf{e}\|, \|\mathbf{w}\|\} \leq \delta$ where, for any sequence $\boldsymbol{\nu}$, $\|\boldsymbol{\nu}\|$ denotes the sup norm, $\sup_{k\geq 0} |\nu(k)|$. The definition of robustness that we employ is (Teel, 2004):

**Definition 3.1** (Robust global asymptotic stability (GAS))**.** Let $\mathcal{A}$ be compact, and let $d(x,\mathcal{A}) := \min_a\{|a-x| \mid a \in \mathcal{A}\}$, and $|x|_{\mathcal{A}} := d(x,\mathcal{A})$. The set $\mathcal{A}$ is robust GAS for $x^+ = f(x)$ if there exists a class $\mathcal{KL}$-function $\beta(\cdot)$ such that for each $\varepsilon > 0$ and each compact set $C$, there exists a $\delta > 0$ such that for each $x \in C$ and each $\phi \in S_\delta(x)$, there holds $|\phi(k;x)|_{\mathcal{A}} \leq \beta(|x|_{\mathcal{A}}, k) + \varepsilon$ for all $k \in \mathbb{I}_{\geq 0}$.

Taking the set $\mathcal{A}$ to be the origin ($\mathcal{A} = \{0\}$) so that $|x|_{\mathcal{A}} = |x|$, we see that if the origin is robustly asymptotically stable for $x^+ = f(x)$, then, for each $\varepsilon > 0$, there exists a $\delta > 0$ such that every trajectory of the perturbed system $x^+ = f(x+e)+w$ with $\max\{\|\mathbf{e}\|, \|\mathbf{w}\|\} \leq \delta$ converges to $\varepsilon\mathcal{B}$ ($\mathcal{B}$ is the closed unit ball); this is the attractivity property. Also, if the initial state $x$ satisfies $|x| \leq \beta^{-1}(\varepsilon, 0)$, then $|\phi(k;x)| \leq \beta(\beta^{-1}(\varepsilon,0),0) + \varepsilon = 2\varepsilon$ for all $k \in \mathbb{I}_{\geq 0}$ and for all $\phi \in S_\delta$, which is the Lyapunov stability property. Here the function $\beta^{-1}(\cdot,0)$ is the inverse of the function $\alpha \mapsto \beta(\alpha,0)$.

We return to the question: under what conditions is asymptotic stability robust? This is answered by the following important result (Teel, 2004; Kellet and Teel, 2004):

**Theorem 3.2** (Lyapunov function and robust GAS)**.** *Suppose $\mathcal{A}$ is com-*

*pact and that $f(\cdot)$ is locally bounded.*[3] *The set $\mathcal{A}$ is robustly globally asymptotically stable for the system $x^+ = f(x)$ if and only if the system admits a continuous global Lyapunov function for $\mathcal{A}$.*

It is shown in Appendix B that for the system $x^+ = f(x)$, $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a global Lyapunov function for set $\mathcal{A}$ if there exist $\mathcal{K}_\infty$ functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$, and a continuous positive definite function $\rho(\cdot)$, such that for all $x \in \mathbb{R}^n$

$$\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}})$$
$$V(f(x)) \leq V(x) - \rho(|x|_{\mathcal{A}})$$

in which $|x|_{\mathcal{A}} := d(x, \mathcal{A})$, the distance of $x$ from the set $\mathcal{A}$. In MPC, the value function of the finite horizon optimal control problem that is solved online is used as a Lyapunov function. In certain cases, such as linear systems with polyhedral constraints, the value function is known to be continuous; see Proposition 7.13. Theorem 3.2, suitably modified because the region of attraction is not global, shows that asymptotic stability is robust, i.e., that asymptotic stability is not destroyed by *small* perturbations.

This result, though important, is limited in its use for applications in that it merely states the existence of a $\delta > 0$ that specifies the permitted magnitude of the perturbations; in practice its value would be required. In the next section we show how the performance of an uncertain system with disturbances of a specified magnitude may be estimated.

Theorem 3.2 characterizes robust stability of the set $\mathcal{A}$ for the system $x^+ = f(x)$ in the sense that it shows robust stability is equivalent to the existence of a continuous global Lyapunov function for the system. It is also possible to characterize robustness of $x^+ = f(x)$ by global asymptotic stability of its regularization $x^+ \in F(x)$. It is shown in Appendix B that for the system $x^+ \in F(x)$, the set $\mathcal{A}$ is *globally asymptotically stable* if there exists a $\mathcal{KL}$-function $\beta(\cdot)$ such that for each $x \in \mathbb{R}^n$ and each $\phi(\cdot) \in S(x)$, i.e., for each solution of $x^+ \in F(x)$ with initial state $x$, $\phi(k) \leq \beta(|x|, k)$ for all $k \in \mathbb{I}_{\geq 0}$. The following alternative characterization of robust stability of $\mathcal{A}$ for the system $x^+ = f(x)$ appears in (Teel, 2004).

**Theorem 3.3** (Robust GAS and regularization)**.** *Suppose $\mathcal{A}$ is compact and that $f(\cdot)$ is locally bounded. The set $\mathcal{A}$ is robust GAS for the system*

[3] A function $f : \mathcal{X} \to \mathcal{Y}$ is locally bounded if, for every $x \in \mathcal{X}$, there exists a neighborhood $\mathcal{N}$ of $x$ such that the set $f(\mathcal{N})$ in $\mathcal{Y}$ is bounded.

*$x^+ = f(x)$ if and only if the set $\mathcal{A}$ is globally asymptotically stable for $x^+ \in F(x)$, the regularization of $x^+ = f(x)$.*

We saw previously that for $f(\cdot)$ and $F(\cdot)$ defined respectively in (3.4) and (3.6), the origin is not globally asymptotically stable for the regularization $x^+ \in F(x)$ of $x^+ = f(x)$ since not every solution of $x^+ \in F(x)$ converges to the origin. Hence the origin is not robust GAS for the system $x^+ = f(x)$.

### 3.2.4 Input-to-State Stability

When an uncertain system is nominally asymptotically stable, it is sometimes possible to establish input-to-state stability (ISS) as shown in Section B.6 in Appendix B. We consider the uncertain system described by

$$x^+ = f(x, u, w) \tag{3.7}$$

in which $w$ is a bounded additive disturbance. The constraints that are required to be satisfied are

$$x(i) \in \mathbb{X} \qquad u(i) \in \mathbb{U}$$

for all $i \in \mathbb{I}_{\geq 0} := \{0, 1, 2, \ldots\}$, the set of nonnegative integers. The disturbance $w$ may take any value in the set $\mathbb{W}$. As before, $\mathbf{u}$ denotes the control sequence $\{u(0), u(1), \ldots\}$ and $\mathbf{w}$ the disturbance sequence $\{w(0), w(1), \ldots\}$; $\phi(i; x, \mathbf{u}, \mathbf{w})$ denotes the solution of (3.7) at time $i$ if the initial state is $x$, and the control and disturbance sequences are, respectively, $\mathbf{u}$ and $\mathbf{w}$.

The *nominal* system is described by

$$x^+ = \bar{f}(x, u) := f(x, u, 0) \tag{3.8}$$

and $\bar{\phi}(i; x, \mathbf{u})$ denotes the solution of the nominal system (3.8) at time $i$ if the initial state is $x$ and the control sequence is $\mathbf{u}$. The *nominal* control problem, defined subsequently, includes, for reasons discussed in Chapter 2, a terminal constraint

$$x(N) \in \mathbb{X}_f$$

The *nominal* optimal control problem is

$$\begin{aligned} \mathbb{P}_N(x): \qquad & V_N^0(x) = \min_{\mathbf{u}} \{V_N(x, \mathbf{u}) \mid \mathbf{u} \in \mathcal{U}_N(x)\} \\ & \mathbf{u}^0(x) = \arg\min_{\mathbf{u}} \{V_N(x, \mathbf{u}) \mid \mathbf{u} \in \mathcal{U}_N(x)\} \end{aligned}$$

in which $\mathbf{u}^0(x) = \{u_0^0(x), u_1^0(x), \ldots, u_{N-1}^0(x)\}$ and the nominal cost $V_N(\cdot)$ is defined by

$$V_N(x, \mathbf{u}) := \sum_{i=0}^{N-1} \ell(x(i), u(i)) + V_f(x(N)) \tag{3.9}$$

In (3.9) and (3.10), $x(i) := \bar{\phi}(i; x, \mathbf{u})$ for all $i \in \mathbb{I}_{0:N-1} = \{0, 1, 2, \ldots, N-1\}$; the set of *admissible* control sequences $\mathcal{U}_N(x)$ is defined by

$$\mathcal{U}_N(x) := \{\mathbf{u} \mid u(i) \in \mathbb{U}, \quad x(i) \in \mathbb{X} \ \forall i \in \mathbb{I}_{0:N-1}, \quad x(N) \in \mathbb{X}_f\} \tag{3.10}$$

which is the set of control sequences such that the nominal system satisfies the control, state, and terminal constraints when the initial state at time 0 is $x$. Thus, $\mathcal{U}_N(x)$ is the set of feasible controls for the nominal optimal control problem $\mathbb{P}_N(x)$. The set $\mathcal{X}_N \subset \mathbb{R}^n$, defined by

$$\mathcal{X}_N := \{x \in \mathbb{X} \mid \mathcal{U}_N(x) \neq \emptyset\}$$

is the domain of the value function $V_N^0(\cdot)$, i.e., the set of $x \in \mathbb{X}$ for which $\mathbb{P}_N(x)$ has a solution; $\mathcal{X}_N$ is also the domain of the minimizer $\mathbf{u}^0(x)$. The value of the nominal model predictive control at state $x$ is $u^0(0; x)$, the first control in the sequence $\mathbf{u}^0(x)$. Hence the *implicit* nominal MPC control law is $\kappa_N : \mathcal{X}_N \to \mathbb{U}$ defined by

$$\kappa_N(x) = u^0(0; x)$$

We assume, as before, that $\ell(\cdot)$ and $V_f(\cdot)$ are defined by

$$\ell(x, u) := (1/2)(x'Qx + u'Ru) \qquad V_f(x) := (1/2)x'P_f x$$

in which $Q$, $R$, and $P_f$ are all positive definite. We also assume that $V_f(\cdot)$ and $\mathbb{X}_f$ satisfy the standard assumption that, for $x \in \mathbb{X}_f$, there exists a $u \in \mathbb{U}$ such that $V_f(\bar{f}(x, u)) \leq V_f(x) - \ell(x, u)$ and that $\mathcal{X}_N$ is compact. Under these assumptions, as shown in Chapter 2, there exist positive constants $c_1$ and $c_2$, $c_2 > c_1$, satisfying

$$c_1|x|^2 \leq V_N^0(x) \leq c_2|x|^2 \tag{3.11}$$

$$V_N^0(\bar{f}(x, \kappa_N(x))) \leq V_N^0(x) - c_1|x|^2 \tag{3.12}$$

for all $x \in \mathcal{X}_N$. We also assume:

**Assumption 3.4** (Lipschitz continuity of value function)**.** The value function $V_N^0(\cdot)$ is Lipschitz continuous on bounded sets.

Assumption 3.4 is satisfied, as shown in Proposition 7.13, if $f(\cdot)$ is affine, $\ell(\cdot)$ and $V_f(\cdot)$ are quadratic and positive definite, $\mathbb{X}$ is polyhedral, and $\mathbb{X}_f$ and $\mathbb{U}$ are polytopic. Assumption 3.4 is also satisfied, as shown in Theorem C.29, if $V_N(\cdot)$ is Lipschitz continuous on bounded sets, $\mathbb{U}$ is compact, and there are no state constraints, i.e., if $\mathbb{X} = \mathbb{X}_f = \mathbb{R}^n$. It follows from (3.11) and (3.12) that for the nominal system under MPC, the origin is exponentially stable, with a region of attraction $\mathcal{X}_N$; the nominal system under MPC satisfies

$$x^+ = \bar{f}(x, \kappa_N(x)) \tag{3.13}$$

It also follows that there exists a $\gamma \in (0,1)$ such that

$$V_N^0(\bar{f}(x, \kappa_N(x))) \leq \gamma V_N^0(x)$$

for all $x \in \mathcal{X}_N$ so that $V_N^0(x(i))$ decays exponentially to zero as $i \to \infty$, where $x(i)$ is the state of the controlled system at time $i$ when there is no disturbance. In fact, $V_N^0(x(i)) \leq \gamma^i V_N^0(x(0))$ for all $i \in \mathbb{I}_{\geq 0}$.

We now examine the consequences of applying the nominal model predictive controller $\kappa_N(\cdot)$ to the uncertain system (3.7). The controlled uncertain system satisfies the difference equation

$$x^+ = f(x, \kappa_N(x), w) \tag{3.14}$$

in which $w$ can take any value in $\mathbb{W}$. It is obvious that the state $x(i)$ of the controlled system (3.14) cannot tend to the origin as $i \to \infty$; the best that can be hoped for is that $x(i)$ tends to and remains in some neighborhood of the origin. We shall establish this, if the disturbance $w$ is sufficiently small, using the value function $V_N^0(\cdot)$ of the nominal optimal control problem as an input-to-state stable (ISS) Lyapunov function for the controlled uncertain system (3.14). As before, $V_N^0(\cdot)$ satisfies (3.11) and (3.12). Let

$$R_c := \text{lev}_c V_N^0 = \{x \mid V_N^0(x) \leq c\}$$

be the largest sublevel set of $V_N^0(\cdot)$ contained in $\mathcal{X}_N$; the set $R_c$ is compact. Hence there exists a finite Lipschitz constant $d$ for $V_N^0(\cdot)$ in $R_c \times \mathbb{W}$. Since $R_c \subset \mathcal{X}_N$, the state constraint $x \in \mathbb{X}$ is satisfied everywhere in $R_c$. Because the uncertain system satisfies (3.14) rather than (3.13), the value function evolves along trajectories of the uncertain system according to

$$V_N^0(f(x, \kappa_N(x), w)) - V_N^0(x) \leq V_N^0(\bar{f}(x, \kappa_N(x))) - V_N^0(x) + d|w|$$

for all $w \in \mathbb{W}$, i.e., according to

$$V_N^0(f(x, \kappa_N(x), w)) \leq \gamma V_N^0(x) + d|w| \tag{3.15}$$

where $\gamma \in (0,1)$. In contrast to the nominal case, the value function does not necessarily decrease along trajectories of the uncertain system; indeed, at the origin ($x = 0$), the value function increases unless $w = 0$. The origin is *not* asymptotically stable for the uncertain system. If $\mathbb{W}$ is sufficiently small, however, a sublevel set $R_b = \{x \mid V_N^0(x) \leq b\} \subset R_c$ of $V_N^0(\cdot)$ satisfying $b < c$ is robust positive invariant for $x^+ = f(x, \kappa_N(x), w)$, $w \in \mathbb{W}$, which we show next. We assume, therefore,

**Assumption 3.5** (Restricted disturbances)**.** Let $e := \max_w \{|w| \mid w \in \mathbb{W}\}$; $e \leq (\rho - \gamma)b/d$ for some $\rho \in (\gamma, 1)$.

The first consequence of this assumption is that $R_b$ is robust positive invariant for $x^+ = f(x, \kappa_N(x), w)$, $w \in \mathbb{W}$. Suppose $x \in R_b$ so that $V_N^0(x) \leq b$. Then

$$V_N^0(f(x, \kappa_N(x), w)) \leq \gamma V_N^0(x) + d|w| \leq \gamma b + (\rho - \gamma)b \leq \rho b$$

so that $x^+ \in R_b$ for all $w \in \mathbb{W}$. A second consequence is that $R_c$ is robust positive invariant and that any $x \in R_c \setminus R_b$ is steered by the controller into $R_b$ in finite time since Assumption 3.5 implies $V_N^0(x^+) \leq \rho V_N^0(x)$ for all $x^+ = f(x, \kappa_N(x), w)$, all $x \in R_c \setminus R_b$, all $w \in \mathbb{W}$. Any trajectory with an initial state $x$ in $R_c$ remains in $R_c$ and enters, in finite time, the set $R_b$ where it then remains.

It also follows from (3.11), (3.12) and (3.15) that $V_N^0(\cdot)$ and $R_c$ satisfy Definition B.37 so that $V_N^0(\cdot)$ is an ISS-Lyapunov function in $R_c$ for the uncertain system $x^+ = f(x, \kappa_N(x), w)$, $w \in \mathbb{W}$. By Lemma B.38, the system $x^+ = f(x, \kappa_N(x), w)$, $w \in \mathbb{W}$ is ISS in $R_c$ satisfying, therefore, for some $\beta(\cdot) \in \mathcal{KL}_\infty$, some $\sigma(\cdot) \in \mathcal{K}$,

$$|\phi(i; x, \mathbf{w}_i)| \leq \beta(|x|, i) + \sigma(\|\mathbf{w}_i\|) \leq \beta(|x|, i) + \sigma(e)$$

for all $i \in \mathbb{I}_{\geq 0}$ where $\phi(i; x, \mathbf{w}_i)$ is the solution at time $i$ if the initial state at time 0 is $x$ and the disturbance sequence is $\mathbf{w}_i := \{w(0), w(1), \ldots, w(i-1)\}$.

The next section describes how DP may be used, in principle, to achieve robust receding horizon control (RHC). The purpose of this section is to provide some insight into the problem of robust control;

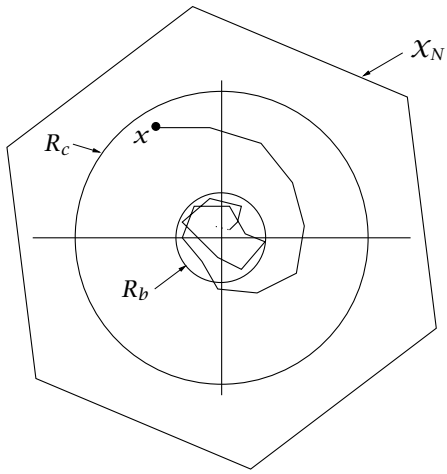



**Figure 3.2:** The sets $\mathcal{X}_N$, $R_b$, and $R_c$.

the section does not show how to obtain robust model predictive controllers that are implementable. Readers whose main concern is implementable robust MPC may prefer to proceed directly to Section 3.4.

## 3.3 Dynamic Programming Solution

### 3.3.1 Introduction

In this section we show how robust RHC may be obtained, in principle, using DP. Our concern is to use DP to gain insight. The results we obtain here are not of practical use for complex systems, but reveal the nature of the problem and show what the ideal optimal control problem solved online should be.

In Section 3.2 we examined the inherent robustness of an asymptotically stable system. If uncertainty is present, and it always is, it is preferable to design the controller to be *robust*, i.e., able to cope with some uncertainty. In this section we discuss the design of a robust controller for the system

$$x^+ = f(x, u, w) \tag{3.16}$$

in which a bounded disturbance input $w$ models the uncertainty. The disturbance is assumed to satisfy $w \in \mathbb{W}$ where $\mathbb{W}$ is compact convex, and contains the origin in its interior. The controlled system is

required to satisfy the same state and control constraints as above, namely $x \in \mathbb{X}$ and $u \in \mathbb{U}$, as well as a terminal constraint $x(N) \in \mathbb{X}_f$. The solution at time $k$ of (3.16) with control and disturbance sequences $\mathbf{u} = \{u(0), \ldots, u(N-1)\}$ and $\mathbf{w} = \{w(0), \ldots, w(N-1)\}$ if the initial state is $x$ at time 0 is $x(k; x, \mathbf{u}, \mathbf{w})$. Similarly, the solution at time $k$ due to feedback policy $\boldsymbol{\mu}$ and disturbance sequence $\mathbf{w}$ is denoted by $x(k; x, \boldsymbol{\mu}, \mathbf{w})$. As discussed previously, the cost may be taken to be that of the nominal trajectory, or the average, or maximum taken over all possible realizations of the disturbance sequence. Here we employ, as is common in the literature, the maximum over all realizations of the disturbance sequence $\mathbf{w}$, and define the cost due to policy $\boldsymbol{\mu}$ with initial state $x$ to be

$$V_N(x, \boldsymbol{\mu}) := \max_{\mathbf{w}} \{J_N(x, \boldsymbol{\mu}, \mathbf{w}) \mid \mathbf{w} \in \mathcal{W}\} \tag{3.17}$$

in which $\mathcal{W} = \mathbb{W}^N$ is the set of admissible disturbance sequences, and $J_N(x, \boldsymbol{\mu}, \mathbf{w})$ is the cost due to an individual realization $\mathbf{w}$ of the disturbance process and is defined by

$$J_N(x, \boldsymbol{\mu}, \mathbf{w}) := \sum_{i=0}^{N-1} \ell(x(i), u(i), w(i)) + V_f(x(N)) \tag{3.18}$$

in which $\boldsymbol{\mu} = \{u(0), \mu_1(\cdot), \ldots, \mu_{N-1}(\cdot)\}$, $x(i) = \phi(i; x, \boldsymbol{\mu}, \mathbf{w})$, and $u(i) = \mu_i(x(i))$. Let $\mathcal{M}(x)$ denote the set of feedback policies $\boldsymbol{\mu}$ that for a given initial state $x$ satisfy: the state and control constraints, and the terminal constraint for every admissible disturbance sequence $\mathbf{w} \in \mathcal{W}$. The first element $u(0)$ in $\boldsymbol{\mu}$ is a control action rather than a control law because the initial state $x$ is known, whereas future states are uncertain. Thus $\mathcal{M}(x)$ is defined by

$$\begin{aligned} \mathcal{M}(x) := \{\boldsymbol{\mu} \mid & u(0) \in \mathbb{U} \\ & \phi(i; x, \boldsymbol{\mu}, \mathbf{w}) \in \mathbb{X},\ \mu_i(\phi(i; x, \boldsymbol{\mu}, \mathbf{w})) \in \mathbb{U} \quad \forall i \in \mathbb{I}_{0:N-1} \\ & \phi(N; x, \boldsymbol{\mu}, \mathbf{w}) \in \mathbb{X}_f \ \forall \mathbf{w} \in \mathcal{W}\} \end{aligned}$$

The robust optimal control problem is

$$\mathbb{P}_N(x): \quad \inf_{\boldsymbol{\mu}} \{V_N(x, \boldsymbol{\mu}) \mid \boldsymbol{\mu} \in \mathcal{M}(x)\} \tag{3.19}$$

The solution to $\mathbb{P}_{\mathbb{N}}(x)$, if it exists, is the policy $\boldsymbol{\mu}^0(x)$

$$\boldsymbol{\mu}^0(x) = \{u^0(0; x), \mu_1^0(\cdot; x), \ldots, \mu_{N-1}^0(\cdot, x)\}$$

and the value function is $V_N^0(x) = V_N(x, \boldsymbol{\mu}^0(x))$. As in conventional MPC, the control applied to the system if the state is $x$ is $u^0(0;x)$, the first element in $\boldsymbol{\mu}^0(x)$; the implicit model predictive feedback control law is $\kappa_N(\cdot)$ defined by

$$\kappa_N(x) := u^0(0;x)$$

### 3.3.2 Preliminary Results

As in conventional MPC, the value function and implicit control law may, in principle, be obtained by DP. But DP is, in most cases, impossible to use because of its large computational demands. There are, of course, important exceptions such as $H_2$ and $H_\infty$ optimal control for unconstrained linear systems with quadratic cost functions. DP also can be used for low dimensional constrained optimal control problems when the system is linear, the constraints are affine, and the cost is affine or quadratic. Even when DP is computationally prohibitive, however, it remains a useful tool because of the insight it provides. Because of the cost definition, min-max DP is required. For each $i \in \{0, 1, \ldots, N\}$, let $V_i^0(\cdot)$ and $\kappa_i(\cdot)$ denote, respectively, the partial value function and the optimal solution to the optimal control problem $\mathbb{P}_i$ defined by (3.19) with $i$ replacing $N$. The DP recursion equations for computing these functions are

$$\begin{aligned}
V_i^0(x) &= \min_{u\in\mathbb{U}} \max_{w\in\mathbb{W}} \{\ell(x,u,w) + V_{i-1}^0(f(x,u,w)) \mid f(x,u,\mathbb{W}) \subseteq \mathcal{X}_{i-1}\} \\
\kappa_i(x) &= (\arg\min_{u\in\mathbb{U}}) \max_{w\in\mathbb{W}} \{\ell(x,u,w) + V_{i-1}^0(f(x,u)) \mid f(x,u,\mathbb{W}) \subseteq \mathcal{X}_{i-1}\} \\
\mathcal{X}_i &= \{x \in \mathbb{X} \mid \exists\, u \in \mathbb{U} \text{ such that } f(x,u,\mathbb{W}) \subseteq \mathcal{X}_{i-1}\}
\end{aligned}$$

with boundary conditions

$$V_0^0(x) = V_f(x) \qquad \mathcal{X}_0 = \mathbb{X}_f$$

In these equations, the subscript $i$ denotes the time to go. For each $i$, $\mathcal{X}_i$ is the domain of $V_i^0(\cdot)$ (and $\kappa_i(\cdot)$) and is therefore the set of states $x$ for which a solution to problem $\mathbb{P}_i(x)$ exists. Thus $\mathcal{X}_i$ is the set of states that can be *robustly* steered by state feedback, i.e., by a policy $\boldsymbol{\mu} \in \mathcal{M}(x)$, to $\mathbb{X}_f$ in $i$ steps or less satisfying all constraints for all disturbance sequences. It follows from these definitions that

$$V_i^0(x) = \max_{w\in\mathbb{W}} \{\ell(x, \kappa_i(x), w) + V_{i-1}^0(f(x, \kappa_i(x), w))\} \tag{3.20}$$

as discussed in Exercise 3.1.

As in the deterministic case studied in Chapter 2, we are interested in obtaining sufficient conditions that ensure that the RHC law $\kappa_N(\cdot)$ is stabilizing. We wish to replace the stabilizing Assumptions 2.12 and 2.13 in Section 2.4.3 of Chapter 2 by conditions appropriate to the robust control problem. The presence of a disturbance requires us to generalize some earlier definitions; we therefore define the terms *robust control invariant* and *robust positive invariant* that generalize our previous definitions of *control invariant* and *positive invariant* respectively.

**Definition 3.6** (Robust control invariance)**.** A set $X \subseteq \mathbb{R}^n$ is *robust control invariant* for $x^+ = f(x,u,w)$, $w \in \mathbb{W}$ if, for every $x \in X$, there exists a $u \in \mathbb{U}$ such that $f(x,u,\mathbb{W}) \subseteq X$.

**Definition 3.7** (Robust positive invariance)**.** A set $X$ is *robust positive invariant* for $x^+ = f(x,w)$, $w \in \mathbb{W}$ if, for every $x \in X$, $f(x,\mathbb{W}) \subseteq X$.

Stabilizing conditions are imposed on the ingredients $\ell(\cdot), V_f(\cdot)$ and $\mathbb{X}_f$ of the optimal control problem to ensure that the resultant controlled system has desirable stability properties. Our generalization of the stabilizing Assumptions 2.12 and 2.13 that we wish to employ, at least for certain problems, are the following Assumptions 3.8 and 3.9.

**Assumption 3.8** (Basic stability assumption; robust case)**.**

(a) For all $x \in \mathbb{X}_f$

$$\min_{u\in\mathbb{U}} \max_{w\in\mathbb{W}} [\Delta V_f + \ell](x,u,w) \leq 0$$

in which $\Delta V_f(x,u,w) = V_f(f(x,u,w)) - V_f(x)$.[4]

(b) $\mathbb{X}_f \subseteq \mathbb{X}$.

Assumption 3.8 implicitly requires that for each $x \in \mathbb{X}_f$, there exists a $u \in \mathbb{U}$ such that $f(x,u,\mathbb{W}) \subseteq \mathbb{X}_f$, i.e., Assumption 3.8 implicitly implies Assumption 3.9.

**Assumption 3.9** (Implied stability assumption; robust case)**.** The set $\mathbb{X}_f$ is robust control invariant for $x^+ = f(x,u,w)$, $w \in \mathbb{W}$.

Before proceeding to analyze stability, we should ask if there are any examples that satisfy these conditions. There is at least one important

[4] Generalizing, for any real-valued function $V(\cdot)$, $\Delta V(x,u,w)$ is defined to be $\Delta V(x,u,w) := V(f(x,u,w)) - V(x)$.

example. Assume that $f(x,u,w) = Ax + Bu + Gw$ is linear and the cost is

$$\ell(x,u) = (1/2)(\,|x|_Q^2 + |u|_R^2 - \rho^2\,|w|^2\,) \tag{3.21}$$

in which $Q = C'C$, $R$ is positive definite and $|x|_Q^2$ and $|u|_R^2$ denote, respectively, $x'Qx$ and $u'Ru$. In the absence of constraints, problem $\mathbb{P}_\infty(x)$ becomes a standard infinite horizon, linear quadratic $H_\infty$ optimal control problem. If $(A,B,C)$ has no zeros on the unit circle, which is the case if $Q$ and $R$ are positive definite, the conditions required in Appendix B of Green and Limebeer (1995) for the full information case are satisfied so that there exists a $\widetilde{\rho} > 0$ such that a positive definite solution $P_f$ to the associated (generalized) $H_\infty$ algebraic Riccati equation exists for all $\rho > \widetilde{\rho}$. Suppose $\rho_f > \widetilde{\rho}$ and that $P_f$ is the solution of the $H_\infty$ algebraic Riccati equation, then the associated optimal control and disturbance laws are $u = K_u x$ and $w = K_w x$, respectively, and the matrices $A_f := A + BK_u$ and $A_c := A + BK_u + GK_w$ are both stable. We define the terminal cost function $V_f(\cdot)$ by

$$V_f(x) := (1/2)|x|_{P_f}^2$$

The terminal cost function $V_f(\cdot)$ is the infinite horizon value function, defined globally in $\mathbb{R}^n$ and satisfying $V_f(x) = \max_w\{\ell(x,K_u x,w) + V_f(f(x,K_u x,w))\}$, so that

$$[\Delta V_f + \ell](x,K_u x,w) \le 0$$

for all $(x,w)$. Hence Assumptions 3.8 and 3.9 are satisfied with $\mathbb{X}_f$ chosen to be any robust positive invariant set for $x^+ = (A+BK_u)x + Gw$, $w \in \mathbb{W}$, that satisfies $\mathbb{X}_f \subseteq \mathbb{X}$ and $K_u\mathbb{X}_f \subseteq \mathbb{U}$, provided such a set exists. Since a positive invariant set for $x^+ = (A+BK_u)x + Gw$ increases with $\mathbb{W}$, and since $\{0\}$ is positive invariant if $\mathbb{W} = \{0\}$, a suitable $\mathbb{X}_f$ exists if $\mathbb{W}$ is sufficiently "small." A similar result can be obtained for a nonlinear system $x^+ = f(x,u,w)$ with $\ell(\cdot)$ defined as in (3.21), provided that $f(\cdot)$ is continuously differentiable, and $A := f_x(0,0,0)$, $B := f_u(0,0,0)$ and $G := f_w(0,0,0)$; see Section 2.5.3.2 of Chapter 2.

Since Assumptions 3.8 and 3.9 appear to be similar to Assumptions 2.12 and 2.13, we would expect to obtain stability results analogous to those obtained in Chapter 2. We do obtain preliminary results that are similar, but the stability properties of the closed-loop system are quite different. Before stating the preliminary results, we note that Assumptions 3.8 and 3.9 imply the existence of a *terminal control law* $\kappa_f : \mathbb{X}_f \to \mathbb{U}$ with the following four properties: (i)

$[\Delta V_f + \ell](x, \kappa_f(x), w) \leq 0$ for all $x \in \mathbb{X}_f$, all $w \in \mathbb{W}$, (ii) $\mathbb{X}_f$ is robust positive invariant for $x^+ = f(x, \kappa_f(x), w)$, (iii) $\mathbb{X}_f \subseteq \mathbb{X}$, and (iv) $\kappa_f(\mathbb{X}_f) \subseteq \mathbb{U}$.

**Theorem 3.10** (Recursive feasibility of control policies)**.** *Suppose Assumptions 3.8 and 3.9 hold. Then:*

*(a)* $\mathcal{X}_N \supseteq \mathcal{X}_{N-1} \supseteq \ldots \supseteq \mathcal{X}_1 \supseteq \mathcal{X}_0 = \mathbb{X}_f$.

*(b)* $\mathcal{X}_i$ *is robust control invariant for* $x^+ = f(x, u, w)$ $\forall i \in \{0, 1, \ldots, N\}$.

*(c)* $\mathcal{X}_i$ *is robust positive invariant for* $x^+ = f(x, \kappa_i(x), w) \quad \forall i \in \{0, 1, \ldots, N\}$.

*(d)* $V_i^0(x) \leq V_{i-1}^0(x) \quad \forall x \in \mathcal{X}_{i-1} \quad \forall i \in \{1, \ldots, N\}$.

*(e)* $V_N^0(x) \leq V_f(x) \quad \forall\ x \in X_f$.

*(f)* $[\Delta V_N^0 + \ell](x, \kappa_N(x), w) \leq [V_N^0 - V_{N-1}^0](f(x, \kappa_N(x), w)) \leq 0$ $\forall (x, w) \in \mathcal{X}_N \times \mathbb{W}$.

*(g) For any* $x \in \mathcal{X}_N$, $\{\kappa_N(x), \kappa_{N-1}(\cdot), \ldots, \kappa_1(\cdot), \kappa_f(\cdot)\}$ *is a feasible policy for* $\mathbb{P}_{N+1}(x)$*, and, for any* $x \in \mathcal{X}_{N-1}$, $\{\kappa_{N-1}(x), \kappa_{N-2}(\cdot), \ldots, \kappa_1(\cdot), \kappa_f(\cdot)\}$ *is a feasible policy for* $\mathbb{P}_N(x)$.

*Proof.*

(a)–(c) Suppose, for some $i$, $\mathcal{X}_i$ is robust control invariant so that any point $x \in \mathcal{X}_i$ can be robustly steered into $\mathcal{X}_i$. By construction, $\mathcal{X}_{i+1}$ is the set of all points $x$ that can be robustly steered into $\mathcal{X}_i$. Hence $\mathcal{X}_{i+1} \supseteq \mathcal{X}_i$ and $\mathcal{X}_{i+1}$ is robust control invariant. But $\mathcal{X}_0 = \mathbb{X}_f$ is robust control invariant. Both (a) and (b) follow by induction. Part (c) follows from (b).

(d) Assume $V_i^0(x) \leq V_{i-1}^0(x)$ for all $x \in \mathcal{X}_{i-1}$. Then from (3.20) we have

$$\begin{aligned}
[V_{i+1}^0 - V_i^0](x) &= \max_{w \in \mathbb{W}} \{\ell(x, \kappa_{i+1}(x), w) + V_i^0(f(x, \kappa_{i+1}(x), w))\} \\
&\quad - \max_{w \in \mathbb{W}} \{\ell(x, \kappa_i(x), w) + V_{i-1}^0(f(x, \kappa_i(x), w))\} \\
&\leq \max_{w \in \mathbb{W}} \{\ell(x, \kappa_i(x), w) + V_i^0(f(x, \kappa_i(x), w))\} \\
&\quad - \max_{w \in \mathbb{W}} \{\ell(x, \kappa_i(x), w) + V_{i-1}^0(f(x, \kappa_i(x), w))\}
\end{aligned}$$

for all $x \in \mathcal{X}_i$ since $\kappa_i(\cdot)$ may *not* be optimal for problem $\mathbb{P}_{i+1}(x)$. We now use the fact that $\max_w\{a(w)\} - \max_w\{b(w)\} \leq \max_w\{a(w) -$

$b(w)\}$, which is discussed in Exercise 3.2, to obtain

$$[V_{i+1}^0 - V_i^0](x) \le \max_{w\in\mathbb{W}}\{[V_i^0 - V_{i-1}^0](f(x,\kappa_i(x),w))\}$$

for all $(x,w) \in X_i \times \mathbb{W}$. Also, for all $x \in \mathcal{X}_0 = \mathbb{X}_f$,

$$\begin{aligned}[V_1^0 - V_0^0](x) &= \min_{u\in\mathbb{U}}\max_{w\in\mathbb{W}}\{\ell(x,u,w) + V_f(f(x,u,w)) - V_f(x)\}\\ &= \min_{u\in\mathbb{U}}\max_{w\in\mathbb{W}}[\Delta V_f + \ell](x,u,w)\\ &\le 0\end{aligned}$$

in which the last inequality follows from Assumption 3.8. By induction, $V_i^0(x) \le V_{i-1}^0(x)\ \forall x \in \mathcal{X}_{i-1},\ \forall i \in \{1,\dots,N\}$; this is the monotonicity property of the value function for a constrained min-max optimal control problem.

(e) This result is a direct consequence of (a) and (d).

(f) For all $x \in \mathcal{X}_N$, for all $w \in \mathbb{W}$.

$$\begin{aligned}[\Delta V_N^0 + \ell](x,\kappa_N(x),w) &= V_N^0(f(x,\kappa_N(x),w)) - V_N^0(x) + \ell(x,\kappa_N(x),w)\\ &\le V_N^0(f(x,\kappa_N(x),w) + \ell(x,\kappa_N(x),w))\\ &\quad - \ell(x,\kappa_N(x),w) - V_{N-1}^0(f(x,\kappa_N(x),w))\\ &= [V_N^0 - V_{N-1}^0](f(x,\kappa_N(x),w))\\ &\le 0\end{aligned}$$

in which the last inequality follows from (d) since $f(x,\kappa_N(x),w) \in \mathcal{X}_{N-1}$. The result clearly holds with $N$ replaced by any $i \in \{1,\dots,N\}$.

(g) Suppose $x \in \mathcal{X}_N$. Then $\boldsymbol{\mu}^0(x) = \{\kappa_N(x), \kappa_{N-1}(\cdot),\dots,\kappa_1(\cdot)\}$ is a feasible and optimal policy for problem $\mathbb{P}_N(x)$, and steers every trajectory emanating from $x$ into $\mathcal{X}_0 = \mathbb{X}_f$ in $N$ time steps. Because $\mathbb{X}_f$ is positive invariant for $x^+ = f(x,\kappa_f(x),w),\ w \in \mathbb{W}$, the policy $\{\kappa_N(x), \kappa_{N-1}(\cdot),\dots,\kappa_1(\cdot),\kappa_f(\cdot)\}$ is feasible for problem $\mathbb{P}_{N+1}(x)$. Similarly, the policy $\{\kappa_{N-1}(x), \kappa_{N-2}(\cdot),\dots,\kappa_1(\cdot)\}$ is feasible and optimal for problem $\mathbb{P}_{N-1}(x)$, and steers every trajectory emanating from $x \in \mathcal{X}_{N-1}$ into $\mathcal{X}_0 = \mathbb{X}_f$ in $N-1$ time steps. Therefore the policy $\{\kappa_{N-1}(x), \kappa_{N-2}(\cdot),\dots,\kappa_1(\cdot),\kappa_f(\cdot)\}$ is feasible for $\mathbb{P}_N(x)$ for any $x \in \mathcal{X}_{N-1}$. ■

### 3.3.3 Stability of Min-Max Receding Horizon Control

We consider in this subsection the stability properties of min-max RHC for the system $x^+ = f(x,u,w)$ with $\mathbb{P}_N(x)$ defined in (3.19) with

$\ell(x,u,w) := (1/2)(|x|_Q^2 + |u|_R^2) - (\rho^2/2)|w|^2$ and $V_f(x) := (1/2)|x|_{P_f}^2$ where $Q$, $R$ and $P_f$ are positive definite. In Section 3.3.2, we showed that $\max_{w\in\mathbb{W}}[\Delta V_N^0 + \ell](x,\kappa_N(x),w) \le 0$ for all $x \in \mathcal{X}_N$ provided that Assumption 3.8 holds. We used the condition $[\Delta V_N^0 + \ell](x,\kappa_N(x),w) \le 0$ to establish asymptotic stability of the origin for a deterministic system in Chapter 2. Can we do so for the problem considered here? The answer is no; the disturbance $w$ prevents convergence of state trajectories to the origin.

The obstacle appears in theoretical analysis as follows. Our usual conditions for establishing asymptotic stability of the origin for this problem are the existence of a Lyapunov function $V(\cdot)$ satisfying for all $x \in \mathcal{X}_N$

(a) $V(x) \ge \alpha_1(|x|)$

(b) $V(x) \le \alpha_2(|x|)$

(c) $\max_{w\in\mathbb{W}} \Delta V(x,\kappa_N(x),w) \le -\alpha_3(|x|)$

in which $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ are $\mathcal{K}_\infty$ functions and $\alpha_3(\cdot)$ is a positive definite, continuous function.

Choosing $V(\cdot)$ to be the value function $V_N^0(\cdot)$, we see that (a) is satisfied because $V_N^0(x) = \min_{\boldsymbol{\mu}} \max_{\mathbf{w}} J_N(x,\boldsymbol{\mu},\mathbf{w}) \ge J_N(x,\boldsymbol{\mu}^0(x),\mathbf{0}) \ge \ell(x,\kappa_N(x),0) \ge (1/2)|x|_Q^2 \ge \alpha_1(|x|)$ for some $\alpha_1(\cdot) \in \mathcal{K}_\infty$ where $\mathbf{0} = \{0,0,\ldots,0\}$ is a sequence of zeros and $Q$ is positive definite. Also (b) is satisfied for all $x \in \mathbb{X}_f$ because $V_N^0(x) \le V_f(x) = (1/2)|x|_{P_f}^2$ where $P_f$ is positive definite, yielding $V_N^0(x) \le \alpha_2(|x|)$ for all $x \in \mathbb{X}_f$, some $\alpha_2(\cdot) \in \mathcal{K}_\infty$. The region of validity may be extended, as in Chapter 2, to $x \in \mathcal{X}_N$ if $\mathcal{X}_N$ is bounded. The stumbling block is condition (c). We have

$$\Delta V_N^0(x,\kappa_N(x),w) \le -\ell(x,\kappa_N(x),w)$$

for all $(x,w) \in \mathcal{X}_N \times \mathbb{W}$. Thus $V_N^0(\cdot)$ has the following properties; there exist $\mathcal{K}_\infty$ functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ such that

$$V_N^0(x) \ge \alpha_1(|x|)$$
$$V_N^0(x) \le \alpha_2(|x|)$$
$$\Delta V_N^0(x,\kappa_N(x),w) \le -\ell(x,\kappa_N(x),w) \le -\alpha_1(|x|) + (\rho^2/2)|w|^2$$

for all $(x,w) \in \mathcal{X}_N \times \mathbb{W}$ if $\mathcal{X}_N$ is bounded. The last property, because of the term $(\rho^2/2)|w|^2$, prevents us from establishing asymptotic stability of the origin: the disturbance $w$ prevents convergence of $x$ to the origin. We have to employ alternative notions of stability.

**Finite $\ell_2$ gain.** Suppose Assumptions 3.8 and 3.9 hold. It follows from Theorem 3.10 that

$$\Delta V_N^0(x, \kappa_N(x), w) \le -\ell(x, \kappa_N(x), w) \tag{3.22}$$

for all $(x, w) \in \mathcal{X}_N \times \mathbb{W}$. Let $\mathbf{x} = \{x(0), x(1), x(2), \ldots\}$, $x(0) = x \in \mathcal{X}_N$, denote any infinite sequence (state trajectory) of the closed-loop system with receding horizon control; $\mathbf{x}$ satisfies

$$x(i+1) = f(x(i), \kappa_N(x(i)), w(i))$$

for some admissible disturbance sequence $\mathbf{w} = \{w(0), w(1), \ldots\}$ in which $w(i) \in \mathbb{W}$ for all $i$. Using (3.22), which implies $V_N^0(x(i+1)) \le V_N^0(x(i)) - \ell(x(i), \kappa_N(x(i)), w(i))$ for all $i$, we deduce that for any positive integer $M > 0$

$$V_N^0(x(M)) \le V_N^0(x(0)) - \sum_{i=0}^{M-1} \ell(x(i), \kappa_N(x(i)), w(i))$$

If we express $\ell(\cdot)$ in the form

$$\ell(x, u, w) = (1/2)|y|^2 - (\rho^2/2)w^2 \qquad y := \begin{bmatrix} Cx \\ Du \end{bmatrix}$$

in which $Q = C'C$ and $R = D'D$, we obtain

$$\sum_{i=0}^{M-1} |y(i)|^2 \le \rho^2 \sum_{i=0}^{M-1} |w(i)|^2 + 2V_N^0(x)$$

for any positive integer $M$. If $\mathbf{w} \in \ell_2$ ($\sum_{i=1}^{\infty} |w(i)|^2 < \infty$), then

$$\sum_{i=0}^{\infty} |y(i)|^2 \le \rho^2 \sum_{i=0}^{\infty} |w(i)|^2 + 2V_N^0(x)$$

and the closed-loop system $x^+ = f(x, \kappa_N(x), w)$ has finite $\ell_2$ gain from $w$ to $y$.

We showed above that there exist $\mathcal{K}_\infty$ functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ such that

$$\begin{aligned} V_N^0(x) &\ge \alpha_1(|x|) \\ V_N^0(x) &\le \alpha_2(|x|) \\ \Delta V_N^0(f(x, \kappa_N(x), w)) &\le -\alpha_1(|x|) + (\rho^2/2)|w|^2 \end{aligned}$$

for all $x \in \mathcal{X}_N$, if $\mathcal{X}_N$ is bounded, and all $w \in \mathbb{W}$. Since $w \mapsto (\rho^2/2)|w|^2$ is a $\mathcal{K}_\infty$ function, the closed-loop system $x^+ = f(x, \kappa_N(x), w)$ is also ISS with $w$ as the input as discussed in Lemma B.38.

**Asymptotic stability of the origin.** As noted previously, the presence of a bounded disturbance prevents trajectories of the closed-loop system $x^+ = f(x, \kappa_N(x), w)$ from converging to the origin. We show next that asymptotic stability of the origin is possible, however, if the controller $\kappa_N(\cdot)$ is determined on the basis that the disturbance is bounded ($w \in \mathbb{W}$), but the disturbance is either zero or converges to zero as the state tends to the origin.

We showed previously that the value function $V_N^0(\cdot)$ obtained on the basis that $w \in \mathbb{W}$ satisfies

$$
\begin{aligned}
&V_N^0(x) \geq \alpha_1(|x|) \\
&V_N^0(x) \leq \alpha_2(|x|) \\
&\Delta V_N^0(f(x, \kappa_N(x), w)) \leq -\alpha_1(|x|) + (\rho^2/2)|w|^2
\end{aligned}
$$

for all $x \in \mathcal{X}_N$, if $\mathcal{X}_N$ is bounded, all $w \in \mathbb{W}$ where $\alpha_1(\cdot) := (1/2)|x|_Q^2$ and $\alpha_2(\cdot)$ are $\mathcal{K}_\infty$ functions. Since $\Delta V_N^0(x, \kappa_N(x), w)$ is not necessarily negative, the origin is not necessarily asymptotically stable. If, however, the disturbance $w$ is identically zero, then

$$
\Delta V_N^0(f(x, \kappa_N(x), w)) \leq -\alpha_1(|x|)
$$

for all $x \in \mathcal{X}_N$. This condition, together with the lower and upper bounds on $V_N^0(\cdot)$, is sufficient to establish asymptotic stability of the origin with a domain of attraction $\mathcal{X}_N$.

As the condition $\Delta V_N^0(f(x, \kappa_N(x), w)) \leq -\alpha_1(|x|) + (\rho^2/2)|w|^2$ suggests, however, it is possible for the origin to be asymptotically stable even if some disturbance is present, providing that it decays sufficiently rapidly to zero as the state tends to the origin. We recall that

$$
\ell(x, u, w) = (1/2)|x|_Q^2 + (1/2)|u|_R^2 - (\rho^2/2)|w|^2
$$

Suppose that $w$ satisfies $(\rho^2/2)|w|^2 \leq |x|_Q^2/4$, or $|w| \leq |x|_Q/(\rho\sqrt{2})$ for all $x \in \mathcal{X}_N$. Then

$$
\Delta V_N^0(f(x, \kappa_N(x), w)) \leq -\alpha_1(|x|)/2
$$

for all $x \in \mathcal{X}_N$, all $w \in \mathbb{W}$ satisfying $|w| \leq |x|_Q/(\rho\sqrt{2})$. Since $\alpha_1/2$ is a $\mathcal{K}_\infty$ function, asymptotic stability of the origin with a domain of attraction $\mathcal{X}_N$ follows.

**Asymptotic stability of an invariant set.** In the deterministic case the origin is control invariant since there exists a control, namely $u = 0$, such that $x^+ = f(0,0) = 0$. When bounded disturbances are present, asymptotic stability of the origin must, in general, be replaced by asymptotic stability of an invariant set $\mathcal{O}$ that replaces the origin. Hence, when bounded disturbances are present, we make the following assumption:

**Assumption 3.11** (Existence of robust control invariant set)**.**

(a) There exists a compact set $\mathcal{O} \subseteq \mathbb{X}$ that contains the origin and is robust control invariant for $x^+ = f(x,u,w)$ so that for all $x \in \mathcal{O}$ there exists a $u \in \mathbb{U}$ such that $f(x,u,w) \in \mathcal{O}$ for all $w \in \mathbb{W}$.

(b) $\rho = 0$.

Assumption 3.11(a) implies the existence of a control law $\kappa_{\mathcal{O}} : \mathcal{O} \to \mathbb{U}$ such that $\mathcal{O}$ is robust positive invariant for $x^+ = f(x, \kappa_{\mathcal{O}}(x), w)$, i.e., $f(x, \kappa_{\mathcal{O}}(x), w) \in \mathcal{O}$ and $\kappa_{\mathcal{O}}(x) \in \mathbb{U}$ for all $x \in \mathcal{O}$, all $w \in \mathbb{W}$. We assume $\rho = 0$ for simplicity; the term $-\rho^2|w|^2$ in $\ell(\cdot)$ is needed in unconstrained problems to make maximization with respect to the disturbance sequence well defined and is not needed when the constraint $w \in \mathbb{W}$ is present. Accordingly, we replace $\ell(x,u,w)$ by $\ell(x,u)$. Returning to the discussion in Section 3.3.2, we now assume that $\mathcal{O}$ has properties analogous to those of the origin in the deterministic case. Specifically we assume:

**Assumption 3.12** (Properties of robust control invariant set)**.**

(a) $\mathcal{O} \subseteq \mathbb{X}_f$.

(b) $V_f(x) = 0$ for all $x \in \mathcal{O}$.

(c) $\ell(x, \kappa_{\mathcal{O}}(x)) = 0$ for all $x \in \mathcal{O}$.

(d) $\kappa_f(x) = \kappa_{\mathcal{O}}(x)$ for all $x \in \mathcal{O}$.

(e) There exists a $\mathcal{K}_\infty$ function $\alpha_1(\cdot)$ such that $\ell(x,u) \geq \alpha_1(|x|_{\mathcal{O}})$ for all $(x,u) \in \mathbb{X} \times \mathbb{U}$.

Since Theorem 3.10 remains true when $\rho = 0$, it is possible to demonstrate, under Assumption 3.12, the assumptions of Section 3.3.2, and the assumption that $\mathcal{X}_N$ is bounded, the existence of $\mathcal{K}_\infty$ functions

$\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ such that

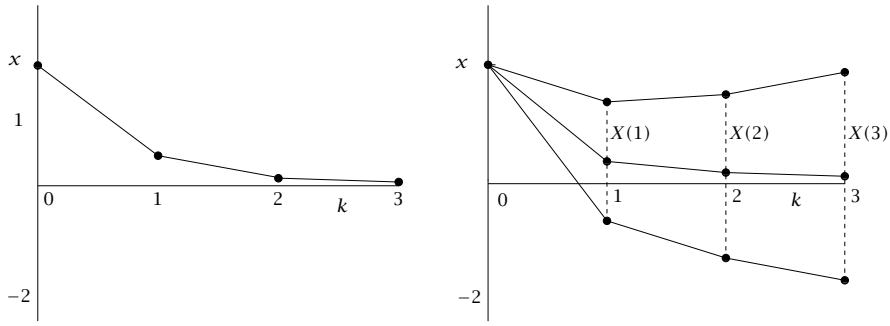
$$V_N^0(x) \geq \alpha_1(|x|_{\mathcal{O}})$$
$$V_N^0(x) \leq \alpha_2(|x|_{\mathcal{O}})$$
$$\Delta V_N^0(f(x, \kappa_N(x), w)) \leq -\alpha_1(|x|_{\mathcal{O}})$$

for all $x \in \mathcal{X}_N, w \in \mathbb{W}$. These bounds differ from those in Proposition 2.19 of Chapter 2 in that $|x|$ is replaced by $|x|_{\mathcal{O}}$ and the term $(\rho^2/2)$ in the last bound is absent. It follows from these bounds that, as shown in Theorem B.23 of Appendix B, the invariant set $\mathcal{O}$ is asymptotically stable for $x^+ = f(x, \kappa_N(x), w)$, $w \in \mathbb{W}$ with a region of attraction $\mathcal{X}_N$.

### 3.3.4 "Feedback" MPC

The DP solution yields the receding horizon control law $\kappa_N(\cdot)$ but requires extensive computation. In the deterministic case discussed in Chapter 2, $\kappa_N(x)$, the MPC action for a given state $x$ (usually the current state), can be obtained by solving an open-loop optimal control problem. For a given state $x$, the solutions obtained by DP and by solving the open-loop optimal control problem are identical, in which "open-loop" means the decision variable is the control sequence $\mathbf{u} = \{u(0), u(1), \ldots, u(N-1)\}$. Our first task is to find out if there is a similar relationship when uncertainty is present. DP may again be used to determine the receding horizon control law $\kappa_N(\cdot)$ as shown in Section 3.3. The question arises: does there exist an optimal control problem $\mathbb{P}_N(x)$, parameterized by the state $x$, the solution of which yields $\kappa_N(x)$, the value of the control law at $x$? The answer is "yes," but the problem is, unfortunately, no longer an open-loop optimal control problem.

In the deterministic case when $x^+ = f(x, u)$, the decision variable is $\mathbf{u} = \{u(0), u(1), \ldots, u(N-1)\}$, a sequence of control actions, and, if $x$ is the initial state at time 0, a state sequence $\mathbf{x} = \{x(0), x(1), \ldots, x(N)\}$, where $x(0) = x$ and $x(i) = \phi(i; x, \mathbf{u})$, is generated. In the uncertain case when $x^+ = f(x, u, w)$, the decision variable is a control policy $\boldsymbol{\mu} = \{u(0), \mu_1(\cdot)., \ldots, \mu_{N-1}(\cdot)\}$; if $x$ is the initial state, the policy $\boldsymbol{\mu}$ generates a state *tube* $\mathbf{X}(x, \boldsymbol{\mu}) = \{X(0; x), X(1; x, \boldsymbol{\mu}), \ldots, X(N; x, \boldsymbol{\mu})\}$ where $X(0; x) = \{x\}$ and, for all $i \in \mathbb{I}_{\geq 0}$, $X(i; x, \boldsymbol{\mu}) = \{\phi(i; x, \boldsymbol{\mu}, \mathbf{w}) \mid \mathbf{w} \in \mathcal{W}\}$. The tube $\mathbf{X}(x, \boldsymbol{\mu})$ is a bundle of state trajectories, one for each admissible disturbance sequence $\mathbf{w}$; see Figure 3.3. In Figure 3.3(b), the central trajectory corresponds to the disturbance sequence $\mathbf{w} = \{0, 0, 0\}$. The




(a) Deterministic case. (b) Uncertain case.

**Figure 3.3:** State trajectory and state tube.

tube $\mathbf{X}$ may be regarded as the solution of the *set* difference equation

$$X(i+1) = F(X(i), \mu_i(\cdot)) \qquad X(0) = \{x\}$$

in which $F(X, \mu_i(\cdot)) := \{f(x, \mu_i(x), w) \mid x \in X, w \in \mathbb{W}\}$.

If we define $V_N(\cdot)$ and $J_N(\cdot)$ as in (3.17) and (3.18), respectively, then the MPC problem $\mathbb{P}_N(x)$ at state $x$ is, as before

$$\mathbb{P}_N(x): \quad \inf_{\boldsymbol{\mu}} \{V_N(x, \boldsymbol{\mu}) \mid \boldsymbol{\mu} \in \mathcal{M}(x)\}$$

in which $\mathcal{M}(x)$ is the set of feedback policies $\boldsymbol{\mu} = \{u(0), \mu_1(\cdot), \ldots, \mu_{N-1}(\cdot)\}$ that, for a given initial state $x$, satisfy the state and control constraints $u(0) \in \mathbb{U}$, $\phi(i; x, \boldsymbol{\mu}, \mathbf{w}) \in \mathbb{X}$, $\phi(N; x, \boldsymbol{\mu}, \mathbf{w}) \in \mathbb{X}_f$, and $\mu_i(\phi(i; x, \boldsymbol{\mu}, \mathbf{w})) \in \mathbb{U}$, for all $i \in \{1, \ldots, N-1\}$ and every admissible disturbance sequence $\mathbf{w} \in \mathcal{W}$. This is precisely the problem solved by DP in Section 3.3. So the solution obtained by solving $\mathbb{P}_N(x)$ for the given state $x$, rather than for every state $x \in \mathcal{X}_N$ as provided by DP, is indeed the DP solution restricted to the sets $X^0(i; x) := \{\phi(i; x, \boldsymbol{\mu}^0(x), \mathbf{w}) \mid \mathbf{w} \in \mathcal{W}\}$, $i \in \{0, 1, \ldots, N\}$. More precisely, the DP solution yields, for each $i \in \{0, 1, \ldots, N\}$, the value function $V_i^0(z)$ and optimal control law $\kappa_i(z)$ for each $z \in \mathcal{X}_i$, whereas the solution to the MPC problem $\mathbb{P}_N(x)$ yields, for each $i \in \{0, 1, \ldots, N\}$, the value function $V_i^0(z)$ and optimal control law $\kappa_i(z)$ for each $z \in X^0(i; x)$.

While it is satisfying to know that one may pose an MPC problem for a given initial state $x$ whose solution is identical to a restriction of the DP solution, this result is of theoretical interest only because, unlike in the deterministic case where the MPC problem is simple enough to solve

online, in the uncertain case $\mathbb{P}_N(x)$ is much too complex. One reason for the complexity is that optimization of a bundle of trajectories is required in which each trajectory must satisfy all constraints. A second, even more important, reason is the complexity of the decision variable $\boldsymbol{\mu}$ which is infinite dimensional because it is a sequence of control *laws*.

## 3.4 Tube-Based Robust MPC

### 3.4.1 Introduction

To proceed realistically we need to sacrifice optimality for simplicity. Many methods for doing so have been proposed in the literature. We outline next one procedure that achieves this objective and that yields robust MPC by solving online an optimal control problem that has the same order of complexity as that employed for conventional MPC. We simplify the decision variable that, ideally, is a policy by replacing it with a finite-dimensional parameterization that consists of an open-loop control sequence and a simple local feedback controller. In addition, we replace the tube, whose exact determination is difficult, by a simply determined *outer-bounding* tube. The underlying idea is quite simple. We generate the "center" of the tube by using conventional MPC with tighter constraints on the nominal system, and restrict the "size" of the tube by using local feedback that attempts to steer all trajectories of the uncertain system to the central trajectory. The resultant controller may be regarded as a "two degrees of freedom" controller. The local feedback around the nominal trajectory is the inner loop and attenuates disturbances while MPC is used in the outer loop.

In this section we address robust MPC of constrained linear systems. To do so, we make use of some concepts in set algebra. Given two subsets $A$ and $B$ of $\mathbb{R}^n$, we define set addition, set subtraction (sometimes called Minkowski or Pontryagin set subtraction), set multiplication and Hausdorff distance between two sets as follows.

**Definition 3.13** (Set algebra and Hausdorff distance)**.**

(a) Set addition: $A \oplus B := \{a + b \mid a \in A, b \in B\}$.

(b) Set subtraction: $A \ominus B := \{x \in \mathbb{R}^n \mid \{x\} \oplus B \subseteq A\}$.

(c) Set multiplication: Let $K \in \mathbb{R}^{m \times n}$. Then $KA := \{Ka \mid a \in A\}$.

(d) The Hausdorff distance $d_H(\cdot)$ between two subsets $A$ and $B$ of $\mathbb{R}^n$

is defined by

$$d_H(A,B) := \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\}$$

in which $d(x,S)$ denotes the distance of a point $x \in \mathbb{R}^n$ from a set $S \subset \mathbb{R}^n$ and is defined by

$$d(x,S) := \inf_y\{d(x,y) \mid y \in S\} \qquad d(x,y) := |x-y|$$

In these definitions, $\{x\}$ denotes the set consisting of a single point $x$ and $\{x\} \oplus B$ therefore denotes the set $\{x+b \mid b \in B\}$; the set $A \ominus B$ is the largest set $C$ such that $B \oplus C \subseteq A$. A sequence $\{x(i)\}$ is said to converge to a set $S$ if $d(x(i),S) \to 0$ as $i \to \infty$. If $d_H(A,B) \le \varepsilon$, then the distance of every point $a \in A$ from $B$ is less than or equal to $\varepsilon$ and that the distance of every point $b \in B$ from $A$ is less than or equal to $\varepsilon$. We say that the sequence of sets $\{A(i)\}$ converges, in the Hausdorff metric, to the set $B$ if $d_H(A(i),B) \to 0$ as $i \to \infty$.

Our first task is to generate an outer-bounding tube. An excellent background for the following discussion is provided in Kolmanovsky and Gilbert (1998).

### 3.4.2 Outer-Bounding Tubes for Linear Systems with Additive Disturbances

Consider the following linear system

$$x^+ = Ax + Bu + w$$

in which $w \in \mathbb{W}$, a compact convex subset of $\mathbb{R}^n$ containing the origin. We assume that either $\mathbb{W}$ contains the origin in its interior, or, if not, $w = G\xi$ where $\xi \in \mathbb{R}^p$, $p < n$ lies in the compact convex set $\Xi$ that contains the origin in its interior and $(A,G)$ is controllable. Let $\phi(i;x,\mathbf{u},\mathbf{w})$ denote the solution of $x^+ = Ax + Bu + w$ at time $i$ if the initial state at time 0 is $x$, and the control and disturbance sequences are, respectively, $\mathbf{u}$ and $\mathbf{w}$.

Let the nominal system be described by

$$z^+ = Az + Bu$$

and let $\bar{\phi}(i;z,\mathbf{u})$ denote the solution of $z^+ = Az + Bu$ at time $i$ if the initial state at time 0 is $z$. Then $e := x - z$, the deviation of the actual state $x$ from the nominal state $z$, satisfies the difference equation

$$e^+ = Ae + w$$

so that

$$e(i) = A^i e(0) + \sum_{j=0}^{i-1} A^j w(j)$$

in which $e(0) = x(0) - z(0)$. If $e(0) = 0$, $e(i) \in S(i)$ where the set $S(i)$ is defined by

$$S(i) := \sum_{j=0}^{i-1} A^j \mathbb{W} = \mathbb{W} \oplus A\mathbb{W} \oplus \ldots \oplus A^{i-1}\mathbb{W}$$

in which $\sum$ and $\oplus$ denote set addition. It follows from our assumptions on $\mathbb{W}$ that $S(i)$ contains the origin in its interior for all $i \geq n$. Let us first consider the tube $\mathbf{X}(x, \mathbf{u})$ generated by the open-loop control sequence $\mathbf{u}$ when $x(0) = z(0) = x$, and $e(0) = 0$. It is easily seen that $\mathbf{X}(x, \mathbf{u}) = \{X(0; x), X(1; x, \mathbf{u}), \ldots, X(N; x, \mathbf{u})\}$ where

$$X(0; x) = \{x\} \qquad X(i; x, \mathbf{u}) := \{\phi(i; x, \mathbf{u}, \mathbf{w}) \mid \mathbf{w} \in \mathcal{W}\} = \{z(i)\} \oplus S(i)$$

and $z(i) = \bar{\phi}(i; x, \mathbf{u})$, the state at time $i$ of the nominal system, is the center of the tube. So it is relatively easy to obtain the exact tube generated by an open-loop control if the system is linear and has a bounded additive disturbance, provided that one can compute the sets $S(i)$. If $\mathbb{W} = G\mathbb{V}$ where $\mathbb{V}$ is convex, then $S(i)$ is convex for all $i \in \mathbb{I}_{\geq 0}$. If, in addition, $\mathbb{V}$ contains the origin in its interior and $(A, G)$ is controllable, then $S(i)$ contains the origin in its interior for all $i \in \mathbb{I}_{\geq n}$.

If $A$ is stable, then, as shown in Kolmanovsky and Gilbert (1998), $S(\infty) := \sum_{j=0}^{\infty} A^j \mathbb{W}$ exists and is positive invariant for $x^+ = Ax + w$, i.e., $x \in S(\infty)$ implies that $Ax + w \in S(\infty)$ for all $w \in \mathbb{W}$; also $S(i) \to S(\infty)$ in the Hausdorff metric as $i \to \infty$. The set $S(\infty)$ is known to be the minimal robust positive invariant set[5] for $x^+ = Ax + w$, $w \in \mathbb{W}$. Also $S(i) \subseteq S(i+1) \subseteq S(\infty)$ for all $i \in \mathbb{I}_{\geq 0}$ so that the tube $\hat{\mathbf{X}}(x, \mathbf{u})$ defined by

$$\hat{\mathbf{X}}(x, \mathbf{u}) := \{\hat{X}(0), \hat{X}(1; x, \mathbf{u}), \ldots, \hat{X}(N; x, \mathbf{u})\}$$

in which

$$\hat{X}(0) = \{x\} \qquad \hat{X}(i; x, \mathbf{u}) = \{z(i)\} \oplus S$$

in which $S = S(\infty)$ is an outer-bounding tube with constant "cross-section" $S$ for the exact tube $\mathbf{X}(x, \mathbf{u})$ ($X(i; x, \mathbf{u}) \subseteq \hat{X}(i; x, \mathbf{u})$ for all $i \in \mathbb{I}_{\geq 0}$). It is sometimes more convenient to use the "constant cross-section" outer-bounding tube $\hat{\mathbf{X}}(x, \mathbf{u})$ in place of the exact tube $\mathbf{X}(x, \mathbf{u})$.

[5] Every other robust positive invariant set $X$ satisfies $X \supseteq S_\infty$.

If we restrict attention to the interval $[0, N]$ as we do in computing the MPC action, then setting $S = S(N)$ yields a less conservative, constrained cross-section, outer-bounding tube for this interval.

While the exact tube $\mathbf{X}(x, \mathbf{u})$, and the outer-bounding tube $\hat{\mathbf{X}}(x, \mathbf{u})$, are easily obtained, their use may be limited for reasons discussed earlier—the sets $S(i)$ may be unnecessarily large simply because an open-loop control sequence rather than a feedback policy was employed to generate the tube. For example, if $\mathbb{W} = [-1, 1]$ and $x^+ = x + u + w$, then $S(i) = (i+1)\mathbb{W}$ increases without bound as time $i$ increases. We must introduce feedback to contain the size of $S(i)$, but wish to do so in a simple way because optimizing over arbitrary policies is prohibitive. The feedback policy we propose is

$$u = v + K(x - z)$$

in which $x$ is the current state of the system $x^+ = Ax + Bu + w$, $z$ is the current state of a nominal system defined below, and $v$ is the current input to the nominal system. With this feedback policy, the state $x$ satisfies the difference equation

$$x^+ = Ax + Bv + BKe + w$$

in which $e := x - z$ is the deviation of the actual state from the nominal state. Let $\phi(i; x, \mathbf{v}, \mathbf{e}, \mathbf{w})$ denote the solution at time $i$ of $x^+ = Ax + Bv + BKe + w$ if its initial state is $x$ at time 0, the control sequence is $\mathbf{v}$, the disturbance sequence is $\mathbf{w}$, and the error sequence is $\mathbf{e}$. The nominal system corresponding to the uncertain system $x^+ = Ax + Bv + BKe + w$ is

$$z^+ = Az + Bv$$

The deviation $e$ now satisfies the difference equation

$$e^+ = A_K e + w \qquad A_K := A + BK$$

which is the same equation used previously except that $A$, which is possibly unstable, is replaced by $A_K$, which is stable by design. If $K$ is chosen so that $A_K$ is stable, then the corresponding uncertainty sets $S_K(i)$ defined by

$$S_K(i) := \sum_{j=0}^{i-1} A_K^j \mathbb{W}$$

can be expected to be smaller than the original uncertainty sets $S(i)$, $i \in \mathbb{I}_{\geq 0}$, considerably smaller if $A$ is unstable and $i$ is large, but not

necessarily much smaller if $A$ is strongly stable. Our assumptions on $\mathbb{W}$ imply that $S_K(i)$, like $S(i)$, contains the origin in its interior for each $i$. Since $A_K$ is stable, the set $S_K(\infty) := \sum_{j=0}^{\infty} A_K^j \mathbb{W}$ exists and is positive invariant for $e^+ = A_K e + w$; also, $S_K(i) \to S_K(\infty)$ in the Hausdorff metric as $i \to \infty$. Since $K$ is fixed, the feedback policy $u = K(x - z) + v$ is simply parameterized by the open-loop control sequence $\mathbf{v}$. If $x(0) = z(0) = x$, the tube generated by the feedback policy $\boldsymbol{\mu}$ is $\mathbf{X}(x, \mathbf{v}) = \{X(0), X(1; x, \mathbf{v}), \ldots, X(N; x, \mathbf{v})\}$ where

$$X(0) = \{x\} \qquad X(i; x, \mathbf{v}) := \{\phi_K(i; x, \mathbf{v}, \mathbf{w}) \mid \mathbf{w} \in \mathcal{W}\} = \{z(i)\} \oplus S_K(i)$$

in which $z(i)$ is the solution of the nominal system $z^+ = Az + Bv$ at time $i$ if the initial state is $z(0) = x$, and the control sequence is $\mathbf{v}$. For given initial state $x$ and control sequence $\mathbf{v}$, the solution of $x^+ = Ax + B(v + Ke) + w$ lies in the tube $\mathbf{X}(x, \mathbf{v})$ for every admissible disturbance sequence $\mathbf{w}$. As before, $S_K(i)$ may be replaced by $S_K(\infty)$ to get an outer-bounding tube. If attention is confined to the interval $[0, N]$, $S_K(i)$ may be replaced by $S_K(N)$ to obtain a less conservative outer-bounding tube. If we consider again our previous example, $\mathbb{W} = [-1, 1]$ and $x^+ = x + u + w$, and choose $K = -(1/2)$, then $A_K = 1/2$, $S_K(i) = (1 + 0.5 + \ldots + 0.5^{i-1})\mathbb{W}$, and $S_K(\infty) = 2\mathbb{W} = [-2, 2]$. In contrast, $S(i) \to [-\infty, \infty]$ as $i \to \infty$.

In the preceding discussion, we required $x(0) = z(0)$ so that $e(0) = 0$ in order to ensure $e(i) \in S(i)$ or $e(i) \in S_K(i)$. When $A_K$ is stable, however, it is possible to relax this restriction. This follows from the previously stated fact that $S_K(\infty)$ exists and is robust positive invariant for $e^+ = A_K e + w$, i.e., $e \in S_K(\infty)$ implies $e^+ \in S_K(\infty)$ for all $e^+ \in \{A_K e\} \oplus \mathbb{W}$. Hence, if $e(0) \in S_K(\infty)$, then $e(i) \in S_K(\infty)$ for all $i \in \mathbb{I}_{\geq 0}$, all $\mathbf{w} \in \mathbb{W}^i$.

In tube-based MPC, discussed next, we ensure that $z(i) \to 0$ as $i \to \infty$, so that $x(i)$, which lies in the set $\{z(i)\} \oplus S_K(i)$, converges to the set $S_K(\infty)$ as $i \to \infty$. Even though $S_K(\infty)$ is difficult to compute, this is a useful theoretical property of the controlled system. The controller is required to ensure that state and control constraints are not transgressed. To do this, knowledge of $S_K(\infty)$ is not required. If we know that $e(0) \in S_K(\infty)$ because, for example, $z(0)$ is chosen to satisfy $z(0) = x(0)$, then $x(i)$ lies in $\{z(i)\} \oplus S_K(\infty)$ for all $i$. All that is then required, in the nominal optimal control problem $\bar{\mathbb{P}}_N$, is knowledge of a set $S$ that is an outer approximation of $S_K(\infty)$. If $x(i)$ lies in $\{z(i)\} \oplus S_K(\infty)$, it certainly lies in $\{z(i)\} \oplus S$. And if $\{z(i)\} \oplus S \subseteq \mathbb{X}$ for all $i$, then $x(i) \in \{z(i)\} \oplus S_K(\infty)$ certainly satisfies the state constraint

$x(i) \in \mathbb{X}$ for all $i$ and all admissible disturbance sequences. Of course, choosing a large outer-approximating set $S$ results in a degree of conservatism; the choice of $S$ is a tradeoff between simplicity and conservatism. The closer the set $S$ approximates $S_K(\infty)$, the less conservative but more complex $S$ is. If we wish to allow freedom in the choice of $z(0)$, we can choose $S$ to be a robust positive invariant outer approximation of $S_K(\infty)$; then $x(i) \in \{z(i)\} \oplus S$ for all $i$ if $x(0) \in \{z(0)\} \oplus S$.

Consider then the tube $X_\infty(z, i)$ defined by

$$X(z, \mathbf{v}) := \{X_0(z, \mathbf{v}), X_1(z, \mathbf{v}), \ldots, X_N(z, \mathbf{v})\}$$

in which, for each $i \in \{0, 1, \ldots, N\}$,

$$X_i(z, \mathbf{v}) := \{z(i)\} \oplus S \qquad z(i) := \bar{\phi}(i; z, \mathbf{v})$$

and $S$ is an outer approximation of $S_K(\infty)$ ($S_K(\infty) \subseteq S$). It follows from the previous discussion that if $x(0) \in \{z(0)\} \oplus S_K(\infty)$ and $S$ is merely an outer approximation of $S_K(\infty)$ *or* if $x(0) \in \{z(0)\} \oplus S$ where $S$ is a robust positive invariant outer approximation of $S_K(\infty)$ (Raković, Kerrigan, Kouramas, and Mayne, 2005a), then $e(i)$ lies in $S$ for all $i \in \mathbb{I}_{\geq 0}$, and every state trajectory $\{x(i)\}$ of $x^+ = Ax + B(v + Ke) + w$, $w \in \mathbb{W}$. In other words, each trajectory corresponding to an admissible realization of $\mathbf{w}$, lies in the tube $X(z, \mathbf{v})$, as shown in Figure 3.4. An obvious choice for $z(0)$ that ensures $e(0) \in S_K(\infty)$ is $z(0) = x(0)$. Similarly every control trajectory $\{u(i)\}$ of the uncertain system lies in the tube $\{\{v(i)\} \oplus KS_K(\infty)\}$ or in the tube $\{\{v(i)\} \oplus KS\}$.

The fact that the state and control trajectories of the uncertain system lie in known neighborhoods of the state and control trajectories, $\{z(i)\}$ and $\{v(i)\}$ respectively, is the basis for tube-based MPC described subsequently. It follows from this fact that if $\{z(i)\}$ and $\{v(i)\}$ are chosen to satisfy $\{z(i)\} \oplus S_K(\infty) \subseteq \mathbb{X}$ and $\{v(i)\} \oplus KS_K(\infty) \subseteq \mathbb{V}$ for all $i \in \mathbb{I}_{\geq 0}$, then $x(i) \in \mathbb{X}$ and $u(i) \in \mathbb{U}$ for all $i \in \mathbb{I}_{\geq 0}$. Thus $\{z(i)\}$ and $\{v(i)\}$ should be chosen to satisfy the *tighter* constraints $z(i) \in \mathbb{Z}$ and $v(i) \in \mathbb{V}$ for all $i \in \mathbb{I}_{\geq 0}$ where $\mathbb{Z} := X \ominus S$ and $\mathbb{V} := U \ominus KS$ in which $S = S_K(\infty)$ or is an outer approximation of $S_K(\infty)$. If $K = 0$, because $A$ is strongly stable, $\mathbb{X}$ and $\mathbb{V}$ should be chosen to satisfy $\mathbb{Z} = X \ominus S$ and $\mathbb{V} = U$, i.e., there is no need to tighten the constraint on $v$. It may seem that it is necessary to compute $S_K(\infty)$, or a robust positive invariant outer approximation $S$, which is known to be difficult, in order to employ this approach. This is not the case, however; we show

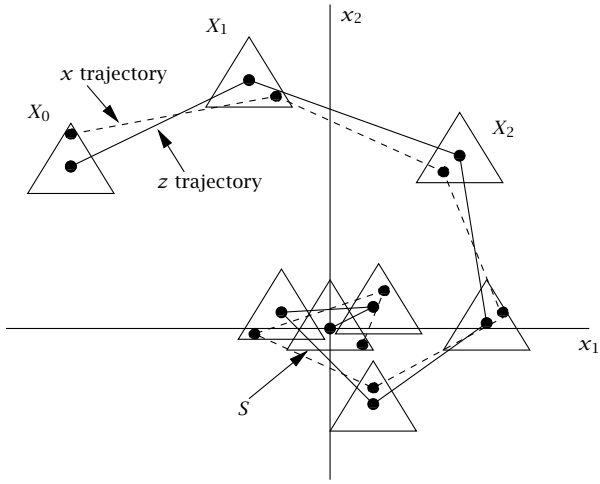



**Figure 3.4:** Outer-bounding tube $\mathbf{X}(z, \mathbf{v})$; $X_i = \{z(i)\} \oplus S$.

later that the tighter constraint sets $\mathbb{Z}$ and $\mathbb{V}$ may be relatively simply determined.

### 3.4.3 Tube-Based MPC of Linear Systems with Additive Disturbances

**Introduction.** Now that we have shown how to contain *all* the trajectories of an uncertain system emanating from the current state within a tube $X(z, \mathbf{v})$ where $z$ is the initial state of the nominal system and $\mathbf{v}$ is an open-loop control sequence, we show how this tool may be used to obtain robust control. We restrict attention in this subsection to constrained linear systems with a bounded additive disturbance. In later sections we consider alternative forms of uncertainty such as parametric uncertainty as well as constrained robust control of constrained nonlinear systems. Our goal is to develop forms of robust MPC that are only marginally more complex than nominal MPC despite the uncertainty.

In this subsection, we discuss first how to formulate an optimal control problem, the solution of which yields a control policy that minimizes a cost, and ensures that the state and control satisfy the given state and control constraints for all admissible bounded disturbances. The basic idea is simple. Choose a trajectory $\{z(i)\}$ for the nominal trajectory depending on the initial state $z$ and the control sequence $\mathbf{v}$, such that each trajectory $\{x(i)\}$ of the system being controlled sat-

isfies the state constraint $x(i) \in \mathbb{X}$ for all $i \in \{0, 1, \ldots, N\}$, and the actual control sequence $\{u(i)\}$ satisfies the control constraint for all $i \in \{0, 1, \ldots, N-1\}$. Recalling that the state satisfies $x(i) \in \{z(i)\} \oplus S$ for all $i \in \{0, 1, \ldots, N\}$ and all admissible disturbance sequences, the state constraint is satisfied if $\{z(i)\} \oplus S \in \mathbb{X}$ or, equivalently, if $z(i) \in \mathbb{X} \ominus S$ for all $i \in \{0, 1, \ldots, N\}$. Similarly the control constraint is satisfied if $u(i) = v(i) + Ke(i) \in \mathbb{U}$ for all $e(i) \in S$ or, equivalently, if $v(i) \in \mathbb{U} \ominus KS$ for all $i \in \{0, 1, \ldots, N-1\}$. These assertions only make sense if the disturbance set $\mathbb{W}$ is sufficiently small to ensure that Assumption 3.14 is satisfied where:

**Assumption 3.14** (Restricted disturbances for constraint satisfaction)**.** $S \subset \mathbb{X}$ and $KS \subset \mathbb{U}$.

We suppose Assumption 3.14 holds in the sequel. An assumption like this is not uncommon in robust control; if $\mathbb{W}$ is too large, there is no possibility of satisfying the constraints for all realizations of the disturbance sequence $\mathbf{w}$. Summarizing, the state and control constraints, $x(i) \in \mathbb{X}$ and $u(i) \in \mathbb{U}$, are satisfied at each time $i$ if the time-invariant control law $u = v + K(x - z)$, is employed, and the nominal system $z^+ = Az + Bv$ satisfies the *tighter* constraints

$$z(i) \in \mathbb{Z} := \mathbb{X} \ominus S \tag{3.23}$$

$$v(i) \in \mathbb{V} := \mathbb{U} \ominus KS \tag{3.24}$$

for all relevant $i$ and if, in addition,

$$x(0) \in \{z(0)\} \oplus S \qquad e(0) \in S$$

in which $S$ is robust positive invariant for $e^+ = A_K e + w$, $w \in \mathbb{W}$. Satisfaction of the state constraint at time $N$, i.e., satisfaction of $x(N) \in \mathbb{X}$, is ensured if the nominal system satisfies the terminal constraint

$$z(N) \in \mathbb{Z}_f \qquad \mathbb{Z}_f \subseteq \mathbb{Z} \tag{3.25}$$

**Tube-based robust predictive controller.** The first requirement for the simple tube-based model predictive controller is a suitable nominal trajectory. To obtain this, we define a finite horizon optimal control $\bar{\mathbb{P}}_N(z)$ in which $z$ is the current state of the nominal system. The optimal control problem is minimization of a cost function $\bar{V}_N(z, \mathbf{v})$ in which

$$\bar{V}_N(z, \mathbf{v}) := \sum_{k=0}^{N-1} \ell(z(k), v(k)) + V_f(z(N))$$

subject to satisfaction, by the state sequence $\mathbf{z} = \{z(0), z(1), \ldots, z(N)\}$ and the control sequence $\mathbf{v} = \{v(0), v(1), \ldots, v(N-1)\}$, of the nominal difference equation $z^+ = Az + Bv$ and the constraints (3.23)–(3.25). The nominal optimal control problem is, therefore

$$\bar{\mathbb{P}}_N(z): \quad \bar{V}_N^0(z) = \min_{\mathbf{v}} \{\bar{V}_N(z, \mathbf{v}) \mid \mathbf{v} \in \mathcal{V}_N(z)\}$$

in which the constraint set $\mathcal{V}_N(z)$, which depends, as the notation implies, on the parameter $z$, is defined by

$$\mathcal{V}_N(z) := \{\mathbf{v} \mid v(k) \in \mathbb{V}, \quad \bar{\phi}(k; z, \mathbf{v}) \in \mathbb{Z} \ \forall k \in \{0, 1, \ldots, N-1\}, \\ \bar{\phi}(N; z, \mathbf{v}) \in \mathbb{Z}_f\} \quad (3.26)$$

In (3.26), $\mathbb{Z}_f \subseteq \mathbb{Z}$ is the terminal constraint set. Solution of $\bar{\mathbb{P}}_N(z)$ yields the minimizing control sequence $\mathbf{v}^0(z) = \{v^0(0; z), v^0(1; z), \ldots, v^0(N-1; z)\}$. The model predictive control applied to the nominal system at state $z$ is $v^0(0; z)$, the first control action in the minimizing control sequence. The implicit nominal MPC control law is, therefore, $\bar{\kappa}_N(\cdot)$, defined by

$$\bar{\kappa}_N(z) := v^0(0; z)$$

Let $\mathcal{Z}_N$ denote the domain of $\bar{V}_N^0(\cdot)$, and of $\bar{\kappa}_N(\cdot)$,

$$\mathcal{Z}_N := \{z \in \mathbb{Z} \mid \mathcal{V}_N(z) \neq \emptyset\}$$

We propose to control the uncertain system $x^+ = Ax + Bu + w$ by constraining it to lie in a tube whose center is the solution of the nominal system obtained using the implicit nominal MPC control law $\bar{\kappa}_N(\cdot)$. The control applied to the system being controlled is $u = \kappa_N(x, z)$ in which $x$ is the current state of the system being controlled, $z$ is the current state of the nominal system, and $\kappa_N(\cdot)$ is defined by

$$\kappa_N(x, z) := \bar{\kappa}_N(z) + K(x - z)$$

The composite closed-loop system plus controller therefore satisfy

$$x^+ = Ax + B\kappa_N(x, z) + w \quad (3.27)$$

$$z^+ = Az + B\bar{\kappa}_N(z) \quad (3.28)$$

with initial state $(x, x)$. The center of the tube is the sequence $\mathbf{z} = \{z(0), z(1), \ldots\}$ obtained by solving (3.28) with initial state $z(0) = x$, i.e., for each $i \in \mathbb{I}_{\geq 0}$, $z(i) = \bar{\phi}(i; x, \bar{\kappa}_N(\cdot))$. Since the difference equation

$z^+ = Az + B\bar{\kappa}_N(z)$ is autonomous, the solution $\mathbf{z}$ may be computed beforehand—at least up to a finite number of time steps. The control $u(i)$ applied to the system at time $i$ is, then

$$u(i) = \kappa_N(x(i), z(i)) = v(i) + K[x(i) - z(i)]$$

in which $v(i) = \bar{\kappa}_N(z(i))$. The state sequence $\mathbf{x} = \{x(0), x(1), \ldots\}$ therefore satisfies

$$x(i+1) = Ax(i) + B\kappa_N(x(i), z(i)) + w(i) \qquad x(0) = x$$

To analyze stability of the closed-loop system, we have to consider, since the controller is a dynamic system with state $z$, the composite system whose state is $(x, z)$ or the equivalent system whose state is $(e, z)$. Since $(e, z)$ and $(x, z)$ are related by an invertible transformation

$$\begin{bmatrix} e \\ z \end{bmatrix} = T \begin{bmatrix} x \\ z \end{bmatrix} \qquad T := \begin{bmatrix} I & -I \\ 0 & I \end{bmatrix}$$

the two systems are equivalent. The composite system whose state is $(x, z)$ satisfies, as shown previously

$$x^+ = Ax + B\kappa_N(x, z) + w \tag{3.29}$$

$$z^+ = Az + B\bar{\kappa}_N(z) \tag{3.30}$$

with initial state $(x(0), z(0)) = (x, x)$ whereas the composite system whose state is $(e, z)$, $e := x - z$, satisfies

$$e^+ = A_K e + w \tag{3.31}$$

$$z^+ = Az + B\bar{\kappa}_N(z) \tag{3.32}$$

with initial state $(e(0), z(0)) = (0, x)$. The latter system is easier to analyze. So one way to proceed is to establish exponential stability of $S_K(\infty) \times \{0\}$ with region of attraction $S_K(\infty) \times \mathcal{Z}_N$ of the composite system described by (3.31) and (3.32); we leave this as Exercise 3.6.

Instead we consider the original system described by (3.29) and (3.30). We know, from the discussion above, that $e(i) \in S_K(\infty) \subseteq S$ and $x(i) \in \{z(i)\} \oplus S_K(\infty) \subseteq \{z(i)\} \oplus S$ for all $k \in \mathbb{I}_{\geq 0}$ if $e(0) \in S_K(\infty)$, and $K$ is such that $A_K$ is stable. Also, we know from Chapter 2 that if the stability Assumptions 2.12 and 2.13 are satisfied for the nominal optimal control problem $\bar{\mathbb{P}}_N(z)$, then the value function $\bar{V}_N^0(\cdot)$ satisfies

$$\bar{V}_N^0(z) \geq \ell(z, \bar{\kappa}_N(z)) \qquad \forall z \in \mathcal{Z}_N \tag{3.33}$$

$$\Delta\bar{V}_N^0(z) \leq -\ell(z, \bar{\kappa}_N(z)) \qquad \forall z \in \mathcal{Z}_N \tag{3.34}$$

$$\bar{V}_N^0(z) \leq V_f(z) \qquad \forall z \in \mathbb{Z}_f \tag{3.35}$$

in which $\Delta\bar{V}_N^0(z) = \bar{V}_N^0(z^+) - \bar{V}_N^0(z)$ with $z^+ = Az + B\bar{\kappa}_N(z)$.

If: (i) $\ell(z,v) = (1/2)|z|_Q^2 + (1/2)|v|_R^2$ in which $Q$ and $R$ are positive definite, (ii) the terminal cost $V_f(z) = (1/2)|z|_{P_f}^2$ in which $P_f$ is positive definite, (iii) Assumption 3.14 holds, and (iv) the terminal cost $V_f(\cdot)$ and terminal constraint set $\mathbb{Z}_f$ satisfy the stability Assumptions 2.12 and 2.13, and (v) $\mathcal{Z}_N$ is compact, then there exist constants $c_1$ and $c_2$ such that (3.33)–(3.35) become

$$\bar{V}_N^0(z) \geq c_1|z|^2 \qquad \forall z \in \mathcal{Z}_N \tag{3.36}$$

$$\Delta\bar{V}_N^0(z) \leq -c_1|z|^2 \qquad \forall z \in \mathcal{Z}_N \tag{3.37}$$

$$\bar{V}_N^0(z) \leq c_2|z|^2 \qquad \forall z \in \mathcal{Z}_N \tag{3.38}$$

Hence the origin is exponentially stable for the nominal system $z^+ = Az + B\bar{\kappa}_N(z)$ with a region of attraction $\mathcal{Z}_N$, i.e., there exists a $c > 0$ and a $\gamma \in (0,1)$ such that $|z(i)| \leq c|z(0)|\gamma^i$ for all $i \in \mathbb{I}_{\geq 0}$. Since $x(i) = z(i) + e(i)$ where $e(i) \in S_K(\infty)$

$$|x(i)|_{S_K(\infty)} = d(z(i) + e(i), S_K(\infty)) \leq d(z(i) + e(i), e(i)) = |z(i)|$$

Hence, for all $i \in \mathbb{I}_{\geq 0}$,

$$|x(i)|_{S_K(\infty)} \leq c|z(0)|\gamma^i$$

Let $\mathcal{A} \subset \mathbb{R}^n \times \mathbb{R}^n$ be defined as follows

$$\mathcal{A} := S_K(\infty) \times \{0\}$$

so that, with $|(x,z)| := |x| + |z|$,

$$|(x,z)|_{\mathcal{A}} = |x|_{S_K(\infty)} + |z|$$

It follows from the previous discussion that the state $(x,z)$ of the composite system satisfies

$$|(x(i),z(i))|_{\mathcal{A}} = |x(i)|_{S_K(\infty)} + |z(i)| \leq 2c|z(0)|\gamma^i \leq 2c\gamma^i|(x(0),z(0))|_{\mathcal{A}}$$

for all $i \in \mathbb{I}_{\geq 0}$ since $|z(0)| \leq |x(0)|_{S_K(\infty)} + |z(0)| = |(x(0),z(0))|_{\mathcal{A}}$. We have proved:

**Proposition 3.15** (Exponential stability of tube-based MPC)**.** *The set* $\mathcal{A} := S_K(\infty) \times \{0\}$ *is exponentially stable with a region of attraction* $(\mathcal{Z}_N \oplus S_K(\infty)) \times \mathcal{Z}_N$ *for the composite system* (3.29) *and* (3.30)*.*

Proposition 3.15 remains true if $S_K(\infty)$ is replaced by $S$ where $S \supset S_K(\infty)$ is robust positive invariant for $e^+ = A_K e + w$, $w \in \mathbb{W}$. The tube-based model predictive controller is formally described by the following algorithm in which $i$ denotes current time.

**Tube-based model predictive controller.**

**Initialization:** At time $i = 0$, set $x = z = x(0)$ in which $x(0)$ is the current state.

**Step 1 (Compute control):** At time $i$ and current state $(x, z)$, solve the nominal optimal control problem $\bar{\mathbb{P}}_N(z)$ to obtain the nominal control action $v = \bar{\kappa}_N(z)$ and the control action $u = v + K(x - z)$.

**Step 2 (Check):** If $\bar{\mathbb{P}}_N(z)$ is infeasible, adopt safety/recovery procedure.

**Step 3 (Apply control):** Apply the control $u$ to the system being controlled.

**Step 4 (Update):** Measure the successor state $x^+$ of the system being controlled and compute the successor state $z^+ = f(z, v)$ of the nominal system.

**Step 5:** Set $(x, z) = (x^+, z^+)$, set $i = i + 1$, and go to Step 1.

In this algorithm, $\bar{\kappa}_N(z)$ is, of course, the first element in the control sequence $\mathbf{v}^0(z)$ obtained by solving the nominal optimal control problem $\bar{\mathbb{P}}_N(z)$. Step 2, the check step, is not activated if the assumptions made previously are satisfied and, therefore, is ignored in our analysis.

**Computation of $\mathbb{Z}$ and $\mathbb{V}$.** To implement the tube-based controller, we need inner approximations $\mathbb{Z}$ and $\mathbb{V}$ to be, respectively, the sets $\hat{\mathbb{Z}} := \mathbb{X} \ominus S_K(\infty)$ and $\hat{\mathbb{V}} := \mathbb{U} \ominus KS_K(\infty)$; computation of the set $S_K(\infty)$, a difficult task, is not necessary. Suppose we have a single state constraint

$$\mathcal{Y} := c'x \leq d$$

so that $\mathbb{X} = \{x \in \mathbb{R}^n \mid c'x \leq d\}$. Then, since, for all $i \in \mathbb{I}_{\geq 0}$, $x(i) = z(i) + e(i)$ where $e(i) \in S_K(\infty)$ if $e(0) \in S_K(\infty)$, it follows that $c'x(i) \leq d$ if

$$c'z(i) \leq d - \max\{c'e \mid e \in S_K(\infty)\}$$

Let $\phi_\infty$ be defined as follows

$$\phi_\infty := \max_e \{c'e \mid e \in S_K(\infty)\}$$

Hence

$$\hat{\mathbb{Z}} = \{z \in \mathbb{R}^n \mid c'z \leq d - \phi_\infty\}$$

is a suitable constraint for the nominal system, i.e., $z \in \hat{\mathbb{Z}}$ implies $c'x = c'z + c'e \leq d$ or $x \in \mathbb{X}$ for all $e \in S_K(\infty)$. To obtain $\hat{\mathbb{Z}}$, we

need to compute $\phi_\infty$. But computing $\phi_\infty$ requires solving an infinite dimensional optimization problem, which is impractical. We can obtain an inner approximation to $\hat{\mathbb{Z}}$, which is all we need to implement robust MPC, by computing an upper bound to $\phi_\infty$. We now show how this may be done (Raković, Kerrigan, Kouramas, and Mayne, 2003). We require the following assumption.

**Assumption 3.16** (Compact convex disturbance set)**.** The compact convex set $\mathbb{W}$ contains the origin in its interior.

For each $i \in \mathbb{I}_{\geq 0}$, let $\phi_i$ be defined as follows

$$\phi_i := \max_e \{c'e \mid e \in S_K(i)\}$$

It can be shown that

$$\phi_N = \max_{\{w_i\}} \{c' \sum_{i=0}^{N-1} A_K^i w_i \mid w_i \in \mathbb{W}\}$$

and that

$$\phi_\infty = \max_{\{w_i\}} \{c' \sum_{i=0}^{\infty} A_K^i w_i \mid w_i \in \mathbb{W}\}$$

Suppose now we choose the feedback matrix $K$ and the horizon $N$ so that

$$A_K^N w \in \alpha \mathbb{W} \ \forall w \in \mathbb{W}$$

where $\alpha \in (0,1)$. Because $A_K$ is stable and $\mathbb{W}$ contains the origin in its interior, this choice is always possible. It follows from the definitions of $\phi_\infty$ and $\phi_N$ that

$$\begin{aligned}
\phi_\infty &= \phi_N + \max_{\{w_i\}} \{c' \sum_{i=N}^{\infty} A_K^i w_i \mid w_i \in \mathbb{W}\} \\
&= \phi_N + \max_{\{w_i\}} \{c'(A_K^N w_0 + A_K A_K^N w_1 + A_K^2 A_K^N w_2 + \ldots) \mid w_i \in \mathbb{W}\} \\
&\leq \phi_N + \max_{\{w_i\}} \{c'(\alpha w_0 + A_K \alpha w_1 + A_K^2 \alpha w_2 + \ldots) \mid w_i \in \mathbb{W}\}
\end{aligned}$$

where the last line follows from the fact that $A_K^N w \in \alpha W$ if $w \in \mathbb{W}$. It follows that

$$\phi_\infty \leq \phi_N + \alpha \phi_\infty$$

or

$$\phi_\infty \leq (1-\alpha)^{-1} \phi_N$$

Hence an upper bound for $\phi_\infty$ may be obtained by determining $\phi_N$, i.e., by solving a linear program. The constant $(1-\alpha)^{-1}$ may be made as close as desired to 1 by choosing $\alpha$ suitably small. The set $\mathbb{Z}$ defined by

$$\mathbb{Z} := \{z \in \mathbb{R}^n \mid c'z \le d - (1-\alpha)^{-1}\phi_N\} \subseteq \hat{\mathbb{Z}}$$

is a suitable constraint set for the robust controller. If there are several state constraints

$$y_j := c_j'x \le d_j \ \forall j \in \mathcal{J}$$

and $K$ and $N$ are chosen as previously to satisfy $A_K^N w \in \alpha\mathbb{W}$ for all $w \in \mathbb{W}$ and some $\alpha \in (0,1)$, then a suitable constraint set for the controller is the set

$$\mathbb{Z} := \{z \in \mathbb{R}^n \mid c_j'z \le d_j - (1-\alpha)^{-1}\phi_N^j, \ \forall j \in \mathcal{J}\} \subseteq \hat{\mathbb{Z}}$$

in which, for each $j \in \mathcal{J}$,

$$\phi_N^j := \max_{\{w_i\}}\{c_j'e \mid e \in S_K(i)\} = \max_{\{w_i\}}\{c_j' \sum_{i=0}^{N-1} A_K^i w_i \mid w_i \in \mathbb{W}\}$$

A similar procedure may be used to obtain a suitable constraint set $\mathbb{V} \subseteq \hat{\mathbb{V}} = \mathbb{U} \ominus KS_K(\infty)$. Suppose $\mathbb{U}$ is described by

$$\mathbb{U} := \{u \in \mathbb{R}^m \mid a_j'u \le b_j \ \forall j \in \mathcal{I}\}$$

If $K$ and $N$ are chosen as above, then a suitable constraint set $\mathbb{V}$ for the nominal system is

$$\mathbb{V} := \{v \in \mathbb{R}^m \mid a_j'v \le b_j - (1-\alpha)^{-1}\theta_N^j, \ j \in \mathcal{I}\}$$

in which, for each $j \in \mathcal{I}$,

$$\theta_N^j := \max_{\{w_i\}}\{a_j'Ke \mid e \in S_K(i)\} = \max_{\{w_i\}}\{a_j'K \sum_{i=0}^{N-1} A_K^i w_i \mid w_i \in \mathbb{W}\}$$

**Critique.** A feature of the robust controller that may appear strange is the fact that the nominal state trajectory $\{z(i)\}$ is completely independent of the state trajectory $\{x(i)\}$ of the uncertain system. Although the control $u = \kappa_N(x,z)$ applied to the uncertain system depends on the state of both systems, the control $v = \bar{\kappa}_N(z)$ applied to the nominal system depends only on the state $z$ of the nominal system. This feature arises because we are considering a very specific problem: determination of a control that steers an uncertain linear system robustly from a

known initial state $x$ to the neighborhood $\{x_f\} \oplus S_K(\infty)$ of a desired final state $x_f$; $x_f = 0$ in the previous analysis. More generally, the target state $x_f$ and a slowly varying external disturbance $d$ will vary with time, and the control $u = \kappa_N(x, z, x_f, d)$ will depend on these variables.

A form of feedback from $x$ to $v$ and, hence, to $z$ is easily added. Step 1 in the controller algorithm presented previously may be changed as follows.

**Step 1 (Compute control):** At time $i$ and current state $(x, z)$, solve the nominal optimal control problems $\bar{\mathbb{P}}_N(x)$ and $\bar{\mathbb{P}}_N(z)$ to obtain $\bar{\kappa}_N(z)$ and $\bar{\kappa}_N(x)$ . If $\bar{V}^0_N(x) \leq \bar{V}^0_N(z)$ and $x \in \mathbb{Z}$, set $z = x$ and $u = v = \bar{\kappa}_N(x)$. Otherwise set $v = \bar{\kappa}_N(z)$ and $u = v + K(x - z)$.

Since the modified controller produces, at state $(x, z)$, either a nominal cost $\bar{V}^0_N(z)$ or $\bar{V}^0_N(x) \leq \bar{V}^0_N(z)$ where $x \in \mathbb{Z}$ becomes the updated value of $z$, the analysis and conclusions above remain valid. The modification provides improved performance. From an alternative viewpoint, the modified controller may be regarded as an improved version of nominal MPC in which the nominal control $\bar{\kappa}_N(x)$ is replaced by a safe control $\bar{\kappa}_N(z)$ if $\bar{\kappa}_N(x)$ does not lead to a cost reduction because of the disturbance $w$.

As pointed out previously, the nominal controller $\bar{\kappa}_N(\cdot)$ steers the nominal state $z$ to the desired final state, the origin in our analysis, while the feedback controller $K$ keeps the state $x$ of the uncertain system close to the nominal state $z$. Hence the feedback controller $K$ should be chosen to reduce the effect of the additive disturbance; its choice depends, therefore, on the nature of the disturbance as shown in the examples in Section 3.6 that illustrate the fact that the feedback control $u = v + K(x - z)$ may even have a higher sampling rate than the nominal control $v = \bar{\kappa}_N(z)$ in order to attenuate more effectively high frequency disturbances.

### 3.4.4 Improved Tube-Based MPC of Linear Systems with Additive Disturbances

In this section we describe a version of the tube-based model predictive controller that has pleasing theoretical properties and that does not require computation of a nominal trajectory. It is, however, more difficult to implement since it requires knowledge of $S_K(\infty)$ or of a robust positive invariant outer approximation $S$. This section should therefore be omitted by readers interested only in easily implementable controllers.

We omitted, in Section 3.4.3, to make use of an additional degree of freedom available to the controller, namely $z(0)$, the initial state of the nominal system. Previously we set $z(0) = x(0) = x$. It follows from the discussion at the end of Section 3.4.3 that every trajectory of the system $x^+ = Ax + Bv + BKe + w$ emanating from an initial state $x$ lies in the tube $X(z, \mathbf{v})$, provided that the initial state $x$ of the closed-loop system and the initial state $z$ of the nominal system satisfy

$$x \in \{z\} \oplus S$$

in which $S$ is either $S_K(\infty)$ or a robust positive invariant set for $e^+ = A_K e + w$ that is an outer approximation of $S_K(\infty)$. So, we may optimize the choice of the initial state $z$ of the nominal system, provided we satisfy the constraints $x \in \{z\} \oplus S$ and $z \in \mathbb{Z}$. But we can go further. We can optimize the choice of $z$ at every time $i$ because, if the current state $x$ of the closed-loop system and the current state $z$ of the nominal system satisfy $x \in \{z\} \oplus S$, and the input to the system being controlled is $v + K(x - z)$ where $v$ is the input to the nominal system, then the subsequent states $x^+$ and $z^+$ satisfy $x^+ \in \{z^+\} \oplus S$. To this end, we define a new finite horizon optimal control problem $\mathbb{P}_N^*(x)$, to be solved online, that reduces the cost $\bar{V}_N^0(z)$ obtained in Section 3.4.3

$$\begin{aligned}\mathbb{P}_N^*(x): \quad V_N^*(x) &= \min_z \{\bar{V}_N^0(z) \mid x \in \{z\} \oplus S,\ z \in \mathbb{Z}\} \\ &= \min_{\mathbf{v},z} \{\bar{V}_N(z, \mathbf{v}) \mid \mathbf{v} \in \mathcal{V}_N(z),\ x \in \{z\} \oplus S,\ z \in \mathbb{Z}\}\end{aligned}$$

Because of the extra freedom provided by varying $z$, the domain of the value function $V_N^*(\cdot)$ is $\mathcal{X}_N := \mathcal{Z}_N \oplus S$ where $\mathcal{Z}_N$ is the domain of $\bar{V}_N^0(\cdot)$. The solution to problem $\mathbb{P}_N^*(x)$ is $z^*(x)$ and $\mathbf{v}^*(x) = \mathbf{v}^0(z^*(x))$; optimizing with respect to $z$ means that $z$ in $\bar{\mathbb{P}}_N(z)$ is replaced by $z^*(x)$. It follows that

$$V_N^*(x) = \bar{V}_N^0(z^*(x)) \tag{3.39}$$

for all $x \in \mathcal{X}_N$. The control applied to the system $x^+ = Ax + Bu + w$ at state $x$ is $\kappa_N(x)$ defined by

$$\kappa_N(x) := \kappa_N^*(x) + K(x - z^*(x))$$

in which $\kappa_N^*(x) = \bar{\kappa}_N(z^*(x))$ is the first element in the sequence $\mathbf{v}^*(x) = \mathbf{v}^0(z^*(x))$. The main change from the simple tube-based model predictive controller is that $z$ is replaced by $z^*(x)$. A theoretical advantage is that the applied control $\kappa_N(x)$ depends only on the current state $x$ and not on the composite state $(x, z)$ as in the simple controller.

It follows from (3.36), (3.37), (3.38) and (3.39) that the value function $V_N^*(\cdot)$ satisfies

$$V_N^*(x) = \bar{V}_N^0(z^*(x)) \geq c_1|z^*(x)|^2 \tag{3.40}$$

$$V_N^*(x) = \bar{V}_N^0(z^*(x)) \leq c_2|z^*(x)|^2 \tag{3.41}$$

$$\Delta V_N^*(x,w) \leq \Delta \bar{V}_N^0(z^*(x)) \leq -c_1|z^*(x)|^2 \tag{3.42}$$

for all $(x,w) \in \mathcal{X}_N \times \mathbb{W}$ in which the last line follows from the fact that

$$\begin{aligned}\Delta V_N^*(x,w) &:= V_N^*(x^+) - V_N^*(x) = \bar{V}_N^0(z^*(x^+)) - \bar{V}_N^0(z^*(x)) \\ &\leq \bar{V}_N^0((z^*(x))^+) - \bar{V}_N^0(z^*(x)) = \Delta \bar{V}_N^0(z^*(x)) \leq -c_1|z^*(x)|^2\end{aligned}$$

with $x^+ = Ax + B\kappa_N(x) + w$ and $(z^*(x))^+ = Az^*(x) + B\bar{\kappa}_N(z^*(x))$. Next we note that

$$V_N^*(x) = 0 \ \forall x \in S$$

This equality follows from the fact that for all $x \in S$, the constraint $x \in \{z\} \oplus S$ in problem $\mathbb{P}_N^*(x)$ is satisfied by $z = 0$ since $0 \in S$. Because $\bar{V}_N^0(0) = 0$, it follows that $V_N^*(x) = \bar{V}_N^0(z^*(x)) \leq \bar{V}_N^0(0) = 0$; since $V_N^*(x) \geq 0$ we deduce that $V_N^*(x) = 0$. It also follows that $z^*(x) = 0$ for all $x \in S$ so that $z^*(x)$ is a "measure" of how far $x$ is from the set $S$.

For each $i \in \mathbb{I}_{\geq 0}$, let $x(i) := \phi(i; x(0), \kappa_N(\cdot), \mathbf{w})$, the solution of $x^+ = Ax + B\kappa_N(x) + w$ at time $i$ if the initial state at time 0 is $x(0)$. It follows from (3.40)–(3.42) that $V_N^*(x(i)) \leq \gamma^i V_N^*(x(0))$ where $\gamma := 1 - c_1/c_2 \in (0,1)$. Hence there exist $c > 0$ and $\delta = \sqrt{\gamma}$ such that

$$|z^*(x(i))| \leq c\delta^i |z^*(x(0))| \tag{3.43}$$

for all $i \in \mathbb{I}_{\geq 0}$. For all $i$, $x(i) = z^*(x(i)) + e(i)$ where $e(i) \in S$ so that $|x(i)|_S = d(z^*(x(i)) + e(i), S) \leq d(z^*(x(i)) + e(i), e(i)) = |z^*(x(i))|$. In fact, though this is harder to show, $d(\{z\} \oplus S, S) = |z|$. Hence

$$|x(i)|_S \leq |z^*(x(i))| \leq c|z^*(x(0))|\delta^i$$

so that $x(i)$ converges robustly exponentially fast to $S$ but $S$ is not necessarily robustly exponentially stable for $x^+ = Ax + B\kappa_N(x) + w$, $w \in \mathbb{W}$.

We define the sets $X(i)$ for $i \in \mathbb{I}_{\geq 0}$ by

$$X(i) := \{z^*(x(i))\} \oplus S \tag{3.44}$$

The Hausdorff distance between $X(i)$ and $S$ satisfies

$$d_H(X(i), S) = |z^*(x(i))| \leq c\delta^i |z^*(x(0))| = c\delta^i d_H(X(0), S)$$

for all $i \in \mathbb{I}_{\geq 0}$. Exercise 3.4 shows that $d_H(\{z\} \oplus S, S) = |z|$. We have therefore proved the following.

**Proposition 3.17** (Exponential stability of tube-based MPC without nominal trajectory)**.** *The set $S$ is exponentially stable with a region of attraction $\mathcal{Z}_N \oplus S$ for the set difference equation*

$$X^+ = F(X, \mathbb{W})$$

*in which $F : 2^{\mathbb{X}} \to 2^{\mathbb{X}}$ is defined by*

$$F(X) := \{Ax + B\kappa_N(x) + w \mid x \in X, w \in \mathbb{W}\}$$

Robust exponential stability of $S$ for $X^+ = F(X, \mathbb{W})$ is not as strong as robust exponential stability of $S$ for $x^+ = Ax + B\kappa_N(x) + w$, $w \in \mathbb{W}$. To establish the latter, we would have to show that for some $c > 0$ and all $i \in \mathbb{I}_{\geq 0}$, $|x(i)|_S \leq c\delta^i |x(0)|_S$. Instead we have merely shown that $|x(i)|_S \leq c\delta^i |z^*(x(0))|$.

## 3.5 Tube-Based MPC of Linear Systems with Parametric Uncertainty

**Introduction.** Section 3.4 shows how it is possible to construct bounding tubes and, consequently, tube-based model predictive controllers when the uncertainty in the system takes the form of a bounded additive disturbance $w$. For this kind of uncertainty, the tube has a constant cross-section $S$ or a cross-section $S_k$ that increases with time $k$ and converges to $S$.

Here we consider a different form of uncertainty, parametric uncertainty in linear constrained systems. More specifically, we consider here robust control of the system

$$x^+ = Ax + Bu$$

in which the parameter $p := (A, B)$ can, at any time, take any value in the convex set $\mathcal{P}$ defined by

$$\mathcal{P} := \text{co}\{(A_j, B_j) \mid j \in \mathcal{J}\}$$

in which $\mathcal{J} := \{1, 2, \ldots, J\}$. We make the following assumption.

**Assumption 3.18** (Quadratic stabilizability)**.** The system $x^+ = Ax + Bu$ is quadratically stabilizable, i.e., there exists a positive definite function $V_f : x \mapsto (1/2)x'P_f x$, a feedback control law $u = Kx$, and a positive constant $\varepsilon$ such that

$$V_f((A + BK)x) - V_f(x) \le -\varepsilon|x|^2 \tag{3.45}$$

for all $x \in \mathbb{R}^n$ and all $p = (A, B) \in \{(A_j, B_j) \mid j \in \mathcal{J}\}$. The origin is globally exponentially stable for $x^+ = A_K x := (A + BK)x$ for all $(A, B) \in \{(A_j, B_j) \mid j \in \mathcal{J}\}$.

The feedback matrix $K$ and the positive definite matrix $P_f$ may be determined using linear matrix inequalities. Because $\mathcal{P}$ is convex and $V_f(\cdot)$ is strictly convex, (3.45) is satisfied for all $x \in \mathbb{R}^n$ and all $(A, B) \in \mathcal{P}$. The system is subject to the same constraints as before

$$x \in \mathbb{X} \qquad u \in \mathbb{U}$$

in which $\mathbb{X}$ and $\mathbb{U}$ are assumed, for simplicity, to be compact and polytopic; each set contains the origin in its interior. We define the nominal system to be

$$z^+ = \bar{A}z + \bar{B}v$$

in which

$$\bar{A} := (1/J)\sum_{j=1}^{J} A_j \qquad \bar{B} := (1/J)\sum_{j=1}^{J} B_j$$

The origin is globally exponentially stable for $x^+ = \bar{A}_K x := (\bar{A} + \bar{B}K)x$. The difference equation $x^+ = Ax + Bu$ of the system being controlled may be expressed in the form

$$x^+ = \bar{A}x + \bar{B}u + w$$

in which the disturbance[6] $w = w(x, u, p)$ is defined by

$$w := (A - \bar{A})x + (B - \bar{B})u$$

Hence, the disturbance $w$ lies in the set $\mathbb{W}$ defined by

$$\mathbb{W} := \{(A - \bar{A})x + (B - \bar{B})u \mid (A, B) \in \mathcal{P},\ (x, u) \in \mathbb{X} \times \mathbb{U}\}$$

Clearly $\mathbb{W}$ is polytopic. The state and control constraint sets, $\mathbb{Z}$ and $\mathbb{V}$ for the nominal optimal control problem, defined in Section 3.4, whose

[6]The controller "regards" $w$ as a disturbance and "assumes" that the system being controlled is $x^+ = \bar{A}x + \bar{B}u + w$.

solution yields implicitly the nominal control law $\bar{\kappa}_N(\cdot)$ are chosen to satisfy

$$\mathbb{Z} \oplus S_K(\infty) \subset \mathbb{X} \qquad \mathbb{V} \oplus KS_K(\infty) \subset \mathbb{U}$$

in which, as before,

$$S_K(\infty) := \sum_{i=0}^{\infty} (\bar{A}_K)^i \mathbb{W}$$

The origin is exponentially stable for the nominal system $z^+ = \bar{A}z + \bar{B}\bar{\kappa}_N(z)$ with a region of attraction $\mathcal{Z}_N$. We know from Section 3.4 that the control law $\kappa_N(x,z) = \bar{\kappa}_N(z) + K(x-z)$ results in satisfaction of the state and control constraints $x \in \mathbb{X}$ and $u \in \mathbb{U}$ for all admissible disturbance sequences provided that the initial state $(x(0), z(0))$ of the composite satisfies $(x(0), z(0)) \in \mathcal{M}_N := \{(x,z) \mid x \in \{z\} \oplus S_K(\infty), z \in \mathcal{Z}_N\} \subseteq \mathbb{X} \times \mathcal{Z}_N$. The set $S_K(\infty) \times \{0\}$ is robustly exponentially stable for the composite controlled system with a region of attraction $\mathcal{M}_N$.

Unlike the robust control problem studied in Section 3.4, the disturbance $w$ now depends on $x$ and $u$. In the sequel, we make use of the fact that $w \to 0$ uniformly in $p \in \mathcal{P}$ as $(x,u) \to 0$ to prove, under some assumptions, that the origin is robustly asymptotically stable for the composite system $x^+ = \bar{A}x + \bar{B}\kappa_N(x,z) + w$, $z^+ = \bar{A}z + \bar{B}\bar{\kappa}_N(z)$ with a region of attraction $\mathcal{M}_N$. We choose $\kappa_N(x,z)$ as in Section 3.4.3, to ensure that the origin is exponentially stable for $z^+ = \bar{A}z + \bar{B}\bar{\kappa}_N(z)$ with a region of attraction $\mathcal{Z}_N$ and $\kappa_N(x,z) := \bar{\kappa}_N(z) + K(x-z)$. The approach we adopt to establish that the origin is robustly asymptotically stable for the composite system may be summarized as follows. We consider, for the purpose of analysis, two sequences of nested sets $\{\mathbb{X}_i \mid i \in \mathbb{I}_{\geq 0}\}$ and $\{\mathbb{U}_i \mid i \in \mathbb{I}_{\geq 0}\}$ where, for each $i$, $\mathbb{X}_i := (1/2)^i \mathbb{X}$ and $\mathbb{U}_i := (1/2)^i \mathbb{U}$. For all $(x,u) \in \mathbb{X}_i \times \mathbb{U}_i$, all $i$, $w \in \mathbb{W}_i := (1/2)^i \mathbb{W}$. Clearly $\mathbb{W}_i$ is polytopic for all $i \in \mathbb{I}_{\geq 0}$. Let $S_0 := S_K(\infty)$ and, for each $i \in \mathbb{I}_{\geq 0}$, let $S_i \subset \mathbb{R}^n$ be defined as follows

$$S_i := (1/2)^i S_0$$

Clearly $S_i = \sum_{j=0}^{\infty} (\bar{A}_K)^j \mathbb{W}_i$ for each $i$. For each $i$, the set $S_i$ is robust positive invariant for $e^+ = \bar{A}_K e + w$, $w \in \mathbb{W}_i$. We now make the assumption:

**Assumption 3.19** (Restricted parameter uncertainty)**.** The set $\mathcal{P}$ is sufficiently small to ensure that $\mathbb{W} = \mathbb{W}_0$ satisfies

$$S_0 \subset (1/4)\mathbb{X}_0 \qquad KS_0 \subset (1/4)\mathbb{U}_0$$

where $\mathbb{X}_0 = \mathbb{X}$ and $\mathbb{U}_0 = \mathbb{U}$.

Consequently $S_i \subset (1/2)\mathbb{X}_{i+1}$ and $KS_i \subset (1/2)U_{i+1}$ for all $i \in \mathbb{I}_{\geq 0}$. Consider now the solution $(x(i), z(i))$ at time $i$ of the composite system $x^+ = \bar{A}x + \bar{B}\kappa_N(x,z) + w$, $z^+ = \bar{A}z + \bar{B}\bar{\kappa}_N(z)$ at time $i$ if the initial state is $(x,z) \in \mathcal{M}_N$ so that $x \in \mathbb{X}_0$ and $e := x - z \in S_0$. Hence $e(i) \in S_0$ for all $i \geq 0$. Since $x(i) = z(i) + e(i)$ and $u(i) = v(i) + Ke(i)$, $z(i) \to 0$ and $v(i) \to 0$ as $i \to \infty$ and $e(i) \in S_0$ for all $i$, it follows that there exists a finite time $i_0$ such that $x(i) = z(i) + e(i) \in 2S_0 \subset \mathbb{X}_1$ and $u(i) = v(i) + Ke(i) \in 2KS_0 \subset \mathbb{U}_1$ for all $i \geq i_0$ and every admissible disturbance sequence. Thus, for all $i \geq i_0$, $w(i) \in \mathbb{W}_1$ so that $e(i) \to S_1$ as $i \to \infty$. Since $z(i) \to 0$, $v(i) \to 0$ and $e(i) \to S_1$ as $i \to \infty$, there exists a finite time $i_1 > i_0$ such that $x(i) = z(i) + e(i) \in 2S_1 \subset \mathbb{X}_2$ and $u(i) = v(i) + Ke(i) \in 2KS_1 \subset \mathbb{U}_2$ for all $i \geq i_i$ and every admissible disturbance sequence. Proceeding similarly, we deduce that for all $j \in \mathbb{I}_{\geq 0}$, there exists a finite time $i_j$ such that $x(i) \in \mathbb{X}_{j+1}$ and $u(i) \in \mathbb{U}_{j+1}$ for all $i \geq i_j$ and every admissible disturbance sequence. Hence the initial state $(x,z)$ is robustly steered to the origin. Since $(x,z)$ is an arbitrary point in $\mathcal{M}_N$, the origin is robustly attractive for the composite system with a region of attraction $\mathcal{M}_N$.

To prove stability of the origin for the system

$$\begin{aligned} x^+ &= \bar{A}x + \bar{B}\kappa_N(x,z) + w \\ z^+ &= \bar{A}z + \bar{B}\bar{\kappa}_N(z) \end{aligned}$$

we consider the equivalent system,

$$\begin{aligned} e^+ &= \bar{A}_K e + w \\ z^+ &= \bar{A}z + \bar{B}\bar{\kappa}_N(z) \end{aligned}$$

The disturbance $w$ lies in a set that gets smaller as $(x,u)$ approaches the origin; indeed $x \in \epsilon\mathbb{X}$ and $u \in \epsilon\mathbb{U}$ implies $w \in \epsilon\mathbb{W}$. The states $(x,z)$ and $(e,z)$ of these two composite systems are related by

$$\begin{bmatrix} e \\ z \end{bmatrix} = T \begin{bmatrix} x \\ z \end{bmatrix}, \quad T := \begin{bmatrix} I & -I \\ 0 & I \end{bmatrix}$$

since $e := x - z$. Since $T$ is invertible, the two composite systems are equivalent and stability for one system implies stability for the other. We assume that the value function $\bar{V}_N^0(\cdot)$ for the nominal optimal control problem has the usual properties:

$$\begin{aligned} \bar{V}_N^0(z) &\geq c_1|z|^2 + c_1|\bar{\kappa}_N(z)|^2 \\ \bar{V}_N^0(f(z, \bar{\kappa}_N(z))) &\leq \bar{V}_N^0(z) - c_1|z|^2 \\ \bar{V}_N^0(z) &\leq c_2|z|^2 \end{aligned}$$

for all $z \in \mathcal{Z}_N$; these properties arise when the stage cost is quadratic and positive definite, $\mathcal{Z}_N$ is bounded, and an appropriate terminal cost and constraint set are employed. The first inequality, which is a minor extension of the inequality normally employed, follows from the definition of $\bar{V}_N^0(\cdot)$ and of $\ell(\cdot)$. It follows from these conditions that there exists a $c > 0$ such that

$$|\bar{\kappa}_N(z)| \leq c|z|$$

for all $z \in \mathcal{Z}_N$. For all $\alpha \geq 0$, let $\text{lev}_\alpha V$ denote the sublevel set of $\bar{V}_N^0(\cdot)$ defined by

$$\text{lev}_\alpha V := \{z \in \mathcal{Z}_N \mid \bar{V}_N^0(z) \leq \alpha\}$$

and let $S := S_K(\infty)$; $S$ is robust positive invariant for $e^+ = A_K e + w$, $w \in \mathbb{W}$ and, from Assumption 3.19, $S \subset (1/4)\mathbb{X}$ and $KS \subset (1/4)\mathbb{U}$.

We show below that, for all $\epsilon \in (0, 3/4]$, there exists a $\delta > 0$ such that $(z(0), e(0)) \leq \delta$ implies $z(i) \in \epsilon(3/4)\mathbb{X}$ and $e(i) \in \epsilon S$ for all $i \in \mathbb{I}_{\geq 0}$ thereby establishing robust stability of the origin for the composite system. The upper limit of $3/4$ on $\epsilon$ is not a limitation since the analysis shows that, for every $\epsilon \geq 3/4$, there exists a $\delta > 0$ such that $(z(0), e(0)) \leq \delta$ implies $z(i) \in \epsilon^* \mathbb{X}$ and $e(i) \in \epsilon^* S$ for all $i \in \mathbb{I}_{\geq 0}$ where $\epsilon^* = 3/4 \leq \epsilon$.

Let $\epsilon \in (0, 3/4]$ be arbitrary. From the properties of $\bar{V}_N^0(\cdot)$ and $\bar{\kappa}_N(\cdot)$, we may deduce the existence of an $\alpha > 0$ such that $\text{lev}_\alpha V \subseteq \epsilon(3/4)\mathbb{X}$ and $\bar{\kappa}_N(\text{lev}_\alpha V) \subseteq \epsilon(3/4)\mathbb{U}$. Hence there exists a $\delta \in (0, \epsilon)$ such that $\delta\mathcal{B} \subseteq \text{lev}_\alpha V \cap \epsilon S$ so that $|(z(0), e(0))| < \delta$ implies $z(0) \in \text{lev}_\alpha V$ and $e(0) \in \epsilon S$.

Suppose next that $z(i) \in \text{lev}_\alpha V$ and $e(i) \in \epsilon S$. Then $x(i) = z(i) + e(i) \in \epsilon(3/4)\mathbb{X} \oplus \epsilon(1/4)\mathbb{X} = \epsilon\mathbb{X}$. Similarly, $u(i) = \bar{\kappa}_N(z(i)) + Ke(i) \subseteq \bar{\kappa}_N(\text{lev}_\alpha V) \oplus \epsilon KS \subseteq \epsilon(3/4)\mathbb{U} \oplus \epsilon(1/4)\mathbb{U} = \epsilon\mathbb{U}$. Hence $w(i) \in \epsilon\mathbb{W}$. Since $\text{lev}_\alpha V$ is positive invariant for $z^+ = Az + B\bar{\kappa}_N(z)$, it follows that $z(i+1) \in \text{lev}_\alpha V$. Since $\epsilon S$ is robust positive invariant for $e^+ = A_K e + w$, $w \in \epsilon\mathbb{W}$, it follows that $e(i+1) \in \epsilon S$. By induction, $z(i) \in \epsilon\mathbb{X}$ and $e(i) \in \epsilon S$ for all $i \in \mathbb{I}_{\geq 0}$. We have proved:

**Proposition 3.20** (Asymptotic stability of tube-based MPC)**.** *The origin is asymptotically stable with a region of attraction $\mathcal{M}_N$ for the composite controlled system.*

This result shows that the standard tube-based model predictive system has a degree of robustness against parametric uncertainty, provided that we can bound the disturbance $w$ resulting from model error so that it lies in some compact set $\mathbb{W}_0$ that is sufficiently small. The

controller described previously is conservative since the nominal system is designed on the basis that the disturbance $w$ lies in $\mathbb{W}_0$. A less conservative design would exploit the fact that $w \in \mathbb{W}_i = (1/2)^i \mathbb{W}_0$ when $(x, u) \in (1/2)^i (\mathbb{X} \times \mathbb{U})$.

## 3.6 Tube-Based MPC of Nonlinear Systems

Satisfactory control in the presence of uncertainty requires feedback. As shown in Section 3.3.4, MPC of uncertain systems ideally requires optimization over control policies rather than control sequences, resulting in an optimal control problem that is usually impossibly complex. Practicality demands simplification; hence, in tube-based MPC of constrained linear systems we replace the control policy $\boldsymbol{\mu} = \{\mu_0(\cdot), \mu_1(\cdot), \ldots, \mu_{N-1}(\cdot)\}$, in which each element $\mu_i(\cdot)$ is an arbitrary function, by the simpler policy $\boldsymbol{\mu}$ in which each element has the simple form $\mu_i(x) = v(i) + K(x - z(i))$ in which $v(i)$ and $z(i)$, the control and state of the nominal system at time $i$, are determined using conventional MPC. The feedback gain $K$, which defines the local control law, is determined offline; it can be chosen so that all possible trajectories of the uncertain system lie in a tube centered on the nominal trajectory $\{z(0), z(1), \ldots\}$. The "cross-section" of the tube is a constant set $S$ so that every possible state of the uncertain system at time $i$ lies in the set $\{z(i)\} \oplus S$. This enables the nominal trajectory to be determined using MPC, to ensure that all possible trajectories of the uncertain system satisfy the state and control constraints, and that all trajectories converge to an invariant set centered on the origin.

It would be desirable to extend this methodology to the control of constrained nonlinear systems, but we face some formidable challenges. It is possible to define a nominal system and, as shown in Chapter 2, to determine, using MPC with "tightened" constraints, a nominal trajectory that can serve as the center of a tube. But it seems to be prohibitively difficult to determine a local control law that steers all trajectories of the uncertain system toward the nominal trajectory, and of a set centered on the nominal trajectory in which these trajectories can be guaranteed to lie.

We overcome these difficulties by employing *two* model predictive controllers. The first uses MPC with tightened constraints to determine, as before, a nominal trajectory; the second, the ancillary controller, uses MPC to steer the state of the uncertain system toward the nominal trajectory. We avoid the difficult task of determining, a priori, a local

control law by employing MPC that merely determines a suitable control action for the current state.

The system to be controlled is described by a nonlinear difference equation

$$x^+ = f(x,u) + w \tag{3.46}$$

in which the additive disturbance is assumed to lie in the compact set $\mathbb{W}$ that contains the origin. The state $x$ and the control $u$ are required to satisfy the constraints

$$x \in \mathbb{X} \qquad u \in \mathbb{U}$$

The solution of (3.46) at time $i$ if the initial state at time 0 is $x$ and the control is generated by policy $\boldsymbol{\mu}$ is $\phi(i;x,\boldsymbol{\mu},\mathbf{w})$, in which $\mathbf{w}$ denotes, as usual, the disturbance sequence $\{w(0),w(1),\ldots\}$. Similarly, $\phi(i;x,\kappa,\mathbf{w})$ denotes the solution of (3.46) at time $i$ if the initial state at time 0 is $x$ and the control is generated by a time invariant control law $\kappa(\cdot)$.

The nominal system is obtained by neglecting the disturbance $w$ and is therefore described by

$$z^+ = f(z,v)$$

Its solution at time $i$ if its initial state is $z$ is denoted by $\bar{\phi}(i;z,\mathbf{v})$, in which $\mathbf{v} := \{v(0),v(1),\ldots\}$ is the nominal control sequence. The deviation between the actual and nominal state is $e := x - z$ and satisfies

$$e^+ = f(x,u) - f(z,v) + w$$

Because $f(\cdot)$ is nonlinear, this difference equation cannot be simplified as in the linear case where $e^+$ is independent of $x$ and $z$, depending only on their difference $e = x - z$ and on $w$.

### 3.6.1 The Central Path

The central path is a feasible trajectory for the nominal system that is sufficiently far from the boundaries of the original constraints to enable the ancillary controller for the uncertain system to satisfy these constraints. It is generated by the solution to a nominal optimal control problem $\bar{\mathbb{P}}_N(z)$ where $z$ is the state of the nominal system. The cost function $\bar{V}_N(\cdot)$ for the nominal optimal control problem is defined by

$$\bar{V}_N(z,\mathbf{v}) := \sum_{k=0}^{N-1} \ell(z(k),v(k)) + V_f(z(N)) \tag{3.47}$$

in which $z(k) = \bar{\phi}(k; z, \mathbf{v})$ and $z$ is the initial state. The function $\ell(\cdot)$ is defined by

$$\ell(z, v) := (1/2)\left( |z|_Q^2 + |v|_R^2 \right)$$

in which $Q$ and $R$ are positive definite, $|z|_Q^2 := z^T Q z$, and $|v|_R^2 := v^T Q v$. We impose the following state and control constraints on the nominal system

$$z \in \mathbb{Z} \qquad v \in \mathbb{V}$$

in which $\mathbb{Z} \subset \mathbb{X}$ and $\mathbb{V} \subset \mathbb{U}$. The choice of $\mathbb{Z}$ and $\mathbb{V}$ is more difficult than in the linear case because it is difficult to bound the deviation $e = x - z$ of the state $x$ of the uncertain system from the state $z$ of the nominal system. We assume that these two constraint sets are compact. The terminal cost function $V_f(\cdot)$ together with the terminal constraint set $\mathbb{Z}_f \subseteq \mathbb{X}$ for the nominal system are chosen as described in Chapter 2 and Section 3.4 to satisfy the usual "stability axioms." The state and control constraints, and the terminal constraint $z(N) \in \mathbb{Z}_f$ impose a parametric constraint $\mathbf{v} \in \mathcal{V}_N(z)$ on the nominal control sequence in which $\mathcal{V}_N(z)$ is defined by

$$\mathcal{V}_N(z) := \{\mathbf{v} \mid v(k) \in \mathbb{V}, \quad \bar{\phi}(k; z, \mathbf{v}) \in \mathbb{Z} \quad \forall k \in \mathbb{I}_{0:N-1}, \\ \bar{\phi}(N; z, \mathbf{v}) \in \mathbb{Z}_f\}$$

For each $z$, the set $\mathcal{V}_N(z)$ is compact; it is bounded because of the assumptions on $\mathbb{V}$, and closed because of the continuity of $\bar{\phi}(\cdot)$. The nominal optimal control problem $\bar{\mathbb{P}}_N(z)$ is defined by

$$\bar{\mathbb{P}}_N(z): \quad \bar{V}_N^0(z) = \min_{\mathbf{v}} \{\bar{V}_N(z, \mathbf{v}) \mid \mathbf{v} \in \mathcal{V}_N(z)\}$$

A solution exists if $z$ is feasible for $\bar{\mathbb{P}}_N(z)$ because $\bar{V}_N(\cdot)$ is continuous and $\mathcal{V}_N(z)$ is compact. Let $\mathcal{Z}_N := \{z \mid \mathcal{V}_N(z) \neq \varnothing\}$ denote the domain of $\bar{V}_N^0(z)$, the set of feasible states for $\bar{\mathbb{P}}_N(z)$. By virtue of our assumptions, the set $\mathcal{Z}_N$ is bounded. The solution of $\bar{\mathbb{P}}_N(z)$ is the minimizing control sequence

$$\mathbf{v}^0(z) = \{v^0(0; z), v^0(1; z), \ldots, v^0(N-1; z)\}$$

which we assume is unique, and the associated optimal state sequence is

$$\mathbf{z}^0(z) = \{z, z^0(1; z), \ldots, z^0(N; z)\}$$

The first element $v^0(0; z)$ of $\mathbf{v}^0(z)$ is the control that is applied in MPC. The implicit MPC control law is, therefore, $\bar{\kappa}_N(\cdot)$ defined by

$$\bar{\kappa}_N(z) := v^0(0; z)$$

The nominal system under MPC satisfies

$$z^+ = f(z, \bar{\kappa}_N(z))$$

The central path that defines the ancillary control problem defined in the next subsection consists of the state trajectory

$$\mathbf{z}^*(z) := \{z^*(0;z), z^*(1;z), \ldots\}$$

and the control trajectory

$$\mathbf{v}^*(z) := \{v^*(0;z), v^*(1;z), \ldots\}$$

in which $z$ is the initial state of the nominal system. These trajectories are the solutions of the controlled nominal system described by

$$z^+ = f(z, \bar{\kappa}_N(z))$$

so that for all $i$

$$z^*(i;z) = \bar{\phi}(i;z,\bar{\kappa}_N) \qquad v^*(i;z) = \bar{\kappa}_N(z^*(i;z)) \tag{3.48}$$

If the terminal cost function $V_f(\cdot)$ and terminal constraint set $\mathbb{Z}_f$ are chosen to satisfy the usual stability assumptions, which we assume to be the case, and $\mathcal{Z}_N$ is bounded, there exist $c_1 > 0$ and $\bar{c}_2 > c_1$ such that

$$\begin{aligned}
\bar{V}_N^0(z) &\geq c_1|z|^2 \\
\bar{V}_N^0(z) &\leq \bar{c}_2|z|^2 \\
\Delta\bar{V}_N^0(z) &\leq -c_1|z|^2
\end{aligned}$$

for all $z \in \mathcal{Z}_N$ in which

$$\Delta\bar{V}_N^0(z) := \bar{V}_N^0(f(z, \bar{\kappa}_N(z))) - \bar{V}_N^0(z)$$

It follows that the origin is exponentially stable with a region of attraction $\mathcal{Z}_N$ for the system $z^+ = f(z, \bar{\kappa}_N(z))$. The state of the controlled nominal system converges to the origin exponentially fast.

### 3.6.2 Ancillary Controller

The purpose of the ancillary controller is to maintain the state of the uncertain system $x^+ = f(x,u) + w$ close to the trajectory of the nominal system $z^+ = f(z, \bar{\kappa}_N(z))$. The ancillary controller replaces the

controller $u = v + K(x - z)$ employed in the linear case. To obtain $u$ in the nonlinear control, we determine a control sequence that minimizes the cost of the deviation between the trajectories of the two systems, $x^+ = f(x,u)$ and $z^+ = f(z, \bar{\kappa}_N(z))$, with initial states $x$ and $z$, respectively, and choose $u$ to be the first element of this sequence. If the optimal control problem is properly posed, the resultant control $u$ steers the state of the deterministic system $x^+ = f(x,u)$ toward the nominal trajectory, and, hence, as in the linear case, tends to keep the trajectory of the uncertain system $x^+ = f(x,u) + w$ close to the nominal trajectory.

The ancillary controller is, therefore, based on the composite system

$$x^+ = f(x,u) \tag{3.49}$$

$$z^+ = f(z, \bar{\kappa}_N(z)) \tag{3.50}$$

The cost $V_N(x,z,\mathbf{u})$ that measures the distance between the trajectories of these two systems is defined by

$$V_N(x,z,\mathbf{u}) := \sum_{i=0}^{N-1} \ell\big(x(i) - z^*(i;z), u(i) - v^*(i;z)\big) \tag{3.51}$$

in which, for each $i$, $x(i) := \bar{\phi}(i;x,\mathbf{u})$ is the solution of (3.49) at time $i$ if the initial state is $x$ and the control input sequence is $\mathbf{u}$; $z^*(i;z)$ and $v^*(i;z)$ are defined in (3.48). For the purpose of analysis it is convenient to suppose that the entire infinite sequences $\mathbf{z}^*(z)$ and $\mathbf{v}^*(z)$ have been precalculated. In practice, apart from initialization, generation of the sequences used in (3.51) require only one solution of $\mathbb{P}_N$ at each iteration. It is not necessary for the cost function $\ell(\cdot)$ in (3.51) to be the same function as in (3.47) that defines the cost for the nominal controller. Indeed, as we show subsequently, it is not even necessary for the ancillary controller to have the same sample time as the nominal controller. The ancillary control problem is the minimization of $V_N(x,z,\mathbf{u})$ with respect to $\mathbf{u}$ subject to merely one state constraint, the terminal equality constraint $x(N) = z^*(N;z)$. The tube-based controller implicitly satisfies the state and input constraints. The terminal constraint is chosen for simplicity to ensure stability. Hence, the ancillary control problem $\mathbb{P}_N(x,z)$ is defined by

$$V_N^0(x,z) = \min_{\mathbf{u}} \{V_N(x,z,\mathbf{u}) \mid \mathbf{u} \in \mathcal{U}_N(x,z)\}$$

$$\mathcal{U}_N(x,z) := \{\mathbf{u} \in \mathbb{U}^N \mid \bar{\phi}(N;x,\mathbf{u}) = z^*(N;z)\}$$

in which $\mathcal{U}_N(x,z)$ is the constraint set. For each $(x,z)$, the set $\mathcal{U}_N(x,z)$ is compact. There is no terminal cost and the terminal constraint set is the single state $z^*(N;z) = \bar{\phi}(N;z,\bar{\kappa}_N(\cdot))$

$$\mathbb{X}_f(z) = \{z^*(N;z)\}$$

For each $z \in \mathcal{Z}_N$, the domain of the value function $V_N^0(\cdot,z)$ and of the minimizer is the set $X_N(z)$ defined by

$$X_N(z) := \{x \in \mathbb{X} \mid \mathcal{U}_N(x,z) \neq \emptyset\}$$

For each $z \in \mathcal{Z}_N$, the set $X_N(z)$ is bounded. For future reference, let the set $\mathcal{M}_N \subset \mathbb{R}^n \times \mathbb{R}^n$ be defined by

$$\mathcal{M}_N := \{(x,z) \mid z \in \mathcal{Z}_N,\ x \in X_N(z)\}$$

The set $\mathcal{M}_N$ is bounded. For any $(x,z) \in \mathcal{M}_N$, the minimizing control sequence is $\mathbf{u}^0(x,z) = \{u^0(0;x,z), u^0(1;x,z), \ldots, u^0(N-1;x,z)\}$, and the control applied to the system is $u^0(0;x,z)$, the first element in this sequence. The corresponding optimal state sequence is $\mathbf{x}^0(x,z) = \{x, x^0(1;x,z), \ldots, x^0(N;x,z)\}$. The implicit ancillary control law is, therefore, $\kappa_N(\cdot)$ defined by

$$\kappa_N(x,z) := u^0(0;x,z)$$

The composite uncertain system then satisfies

$$x^+ = f(x, \kappa_N(x,z)) + w \tag{3.52}$$

$$z^+ = f(x, \bar{\kappa}_N(z)) \tag{3.53}$$

If $x = z$, then, as is easily verified, $V_N^0(x,z) = 0$ and

$$u^0(i;x,z) = v^*(i;z),\ i = 0,1,\ldots,N-1$$

so that the control and state trajectories of the two systems (3.49) and (3.50) are identical. In particular

$$\kappa_N(z,z) = \bar{\kappa}_N(z)$$

If some controllability assumptions are satisfied, the value function $V_N^0(\cdot)$ has properties analogous to those of $\bar{V}_N^0(\cdot)$, except that the bounds are $\mathcal{K}_\infty$ functions of $x - z$ rather than of $x$

$$V_N^0(x,z) \geq c_1|x-z|^2 \tag{3.54}$$

$$V_N^0(x,z) \leq c_2|x-z|^2 \tag{3.55}$$

$$\Delta V_N^0(x,z) \leq -c_1|x-z|^2 \tag{3.56}$$

for all $(x, z) \in \mathcal{M}_N$ in which, now

$$\Delta V_N^0(x, z) := V_N^0(f(z, \kappa_N(x, z)), f(x, \bar{\kappa}_N(z))) - V_N^0(x, z)$$

Note that $\Delta V_N^0(x, z)$ is the change in the value as $x$ changes to $x^+ = f(x, \kappa_N(x, z))$ *and* $z$ changes to $z^+ = f(x, \bar{\kappa}_N(z))$; the effect of the disturbance $w$ is ignored in this expression. It follows from (3.54)-(3.56) that

$$V_N^0(f(x, \kappa_N(x, z)), f(z, \bar{\kappa}_N(z))) \leq \gamma V_N^0(x, z)$$

in which $\gamma := 1 - c_1/c_2 \in (0, 1)$ and, hence, that the origin is exponentially stable with a region of attraction $\mathcal{M}_N$ for the composite *deterministic* system $x^+ = f(x, \kappa_N(x, z))$, $z^+ = f(z, \bar{\kappa}_N(z))$. This property is sufficient to bound $e = x - z$ for the composite uncertain system $x^+ = f(x, \kappa_N(x, z)) + w$, $z^+ = f(z, \bar{\kappa}_N(z))$ and allows us, as shown subsequently, to determine suitable tightened constraint sets $\mathbb{Z}$ and $\mathbb{V}$. Assuming these sets have been determined, a robust MPC algorithm for nonlinear systems can be proposed; we do this next.

### 3.6.3 Controller Algorithm

Suppose $\mathbb{Z}$ and $\mathbb{V}$ have been chosen. In the following algorithm, $\mathbf{v}^*$ denotes the control sequence $\{v^*(0), v^*(1), \ldots, v^*(N-1)\}$, and $\mathbf{z}^*$ denotes the state sequence $\{z^*(0), z^*(1), \ldots, z^*(N)\}$. The controller algorithm is:

**Robust control algorithm.**

**Initialization:** At time 0, set $i = 0$, $x = x(0)$, and $z = x$. Solve $\tilde{\mathbb{P}}_N$ for $N$ time steps to obtain the nominal closed-loop state and control sequences $\mathbf{v}^* = \mathbf{v}^*(z) = \{v^*(0; z), v^*(1; z), \ldots, v^*(N-1; z)\}$ and $\mathbf{z}^* = \mathbf{z}^*(z) = \{z^*(0; z), z^*(1; z), \ldots, z^*(N; z)\}$, and set $u = \bar{\kappa}_N(z) = v^*(0; z)$.[7]

**Step 1 (Compute control):** At time $i$, compute $u = \kappa_N(x, z)$ by solving $\mathbb{P}_N(x, z)$.

**Step 2 (Control):** Apply $u$ to the system being controlled.

**Step 3 (Update $x$):** Set $x = x^+$ where $x^+ = f(x, u) + w$ is the successor state.

[7] Recall $z^*(0; z) = z$ and $v^*(0; z) = \bar{\kappa}_N(z)$.

**Step 4 (Update $z$, $\mathbf{v}^*$, and $\mathbf{z}^*$):** Compute $v^* = \bar{\kappa}_N(z^*(N))$ and $z^* = f(z^*(N), v^*)$ by solving $\mathbb{P}_N(z^*(N))$. Set $z = z^*(1)$. Set $\mathbf{v}^* = \{v^*(1), \ldots, v^*(N-1), v^*\}$ and set $\mathbf{z}^* = \{z^*(1), \ldots, z^*(N), z^*\}$.

**Step 5 (Repeat):** Set $i = i + 1$. Go to Step 1.

A check step may be incorporated as done previously to safeguard against unanticipated events.

### 3.6.4 Analysis

Because of the nonlinearity and the terminal equality constraint in problem $\mathbb{P}_N(x, z)$, analysis is technical and requires use of the implicit function theorem. Full details appear on the website `www.che.wisc.edu/~jbraw/mpc` and the references cited there. Here we give an outline of the analysis. Ideally we would like, as in the linear case, to have a constant set $S$ such that given $z(i)$, the state of the nominal system at time $i$, we could assert that the state $x(i)$ of the uncertain system lies in $\{z(i)\} \oplus S$. Instead, as we show subsequently, for each state $z(i)$ of the nominal system, the state $x(i)$ of the uncertain system lies, for some $d > 0$, in the set $S_d(z(i))$ where the set-valued $S_d(\cdot)$ is defined, for all $z \in \mathcal{Z}_N$

$$S_d(z) := \{x \in \mathbb{R}^n \mid V_N^0(x, z) \le d\}$$

The set $S_d(z)$ is a sublevel set of the function $x \mapsto V_N^0(x, z)$. Since $S_0(z) = \{z\}$, the set $S_d(z)$ is a neighborhood of $z$. The set $S_d(z)$, that varies with $z$, replaces the set $\{z\} \oplus S$ employed in Section 3.4 because of the following important property that holds under certain controllability and differentiability assumptions:

**Proposition 3.21** (Existence of tubes for nonlinear systems)**.** *There exists a $d > 0$ such that if the state $(x, z)$ of the composite system* (3.52) *and* (3.53) *lies in $\mathcal{M}_N$ and satisfies $x \in S_d(z)$, then the successor state $(x^+, z^+)$ satisfies $x^+ \in S_d(z^+)$, i.e.,*

$$x^+ = f(x, \kappa_N(x, z)) + w \in S_d(z^+) \qquad z^+ = f(z, \bar{\kappa}_N(z))$$

*for all $w$ satisfying $|w| \le (1 - \gamma)d/k(z)$ where $k(z)$ is a local Lipschitz constant for $x \mapsto V_N^0(x, z)$.*

If $w \in \mathbb{W}$ implies $|w| \le (1 - \gamma)d/k$ where $k$ is an upper bound for $k(z)$ in $\mathcal{Z}_N$, then every solution of the system $x^+ = f(x, \kappa_N(x, z)) + w$, $w \in \mathbb{W}$ lies in the tube $\mathbf{S} := \{S_d(z), S_d(z^*(1; z)), S_d(z^*(2; z)), \ldots\}$ for

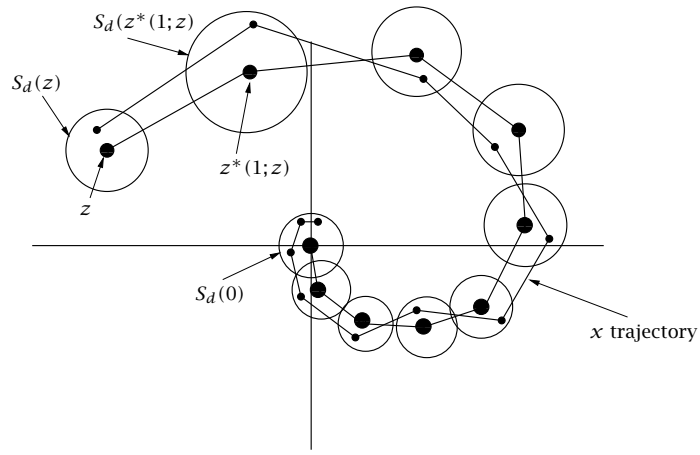



**Figure 3.5:** Tube for a nonlinear system.

all disturbance sequences $\{w(i)\}$ satisfying $w(i) \in \mathbb{W}$ for all $i \in \mathbb{I}_{\geq 0}$. Figure 3.5 illustrates this result and the fact that the cross-section of the tube varies with the state of the nominal system.

### 3.6.5 Choosing $\mathbb{Z}$ and $\mathbb{V}$

The tightened constraint sets $\mathbb{Z}$ and $\mathbb{V}$ may, in principle, be computed. Suppose there exists a single state constraint $c'x \leq e$. The tightened state constraint set is $\mathbb{Z} := \{x \mid c'x \leq f\}$ where $f < e$. Assuming that the constant $d$ is known, the tightened state constraint set is suitable provided that

$$\phi(z) := \max_x \{c'x \mid x \in S_d(z)\} \leq e$$

for all $z \in \mathbb{Z} \cap \mathcal{Z}_N$, i.e., for all $z \in \mathcal{Z}_N$ satisfying $c'z \leq f$. In practice, $\phi(z)$ could be computed for a finite number of representative points in $\mathbb{Z} \cap \mathcal{Z}_N$. Since $S_d(z) := \{x \mid V_N^0(x,z) \leq d\}$, $\phi(z)$ may be computed using

$$-\phi(z) = \min_x \{-c'x \mid V_N^0(x,z) \leq d\} = \min_{(x,\mathbf{u})} \{-c'x \mid V_N(x,z,\mathbf{u}) \leq d\}$$

Other state constraints may be similarly treated. The tightened control constraint set also may be computed.

An alternative is the following. If, as is often the case even for nonlinear systems, the sets $\mathbb{X}$ and $\mathbb{U}$ are polyhedral, we may choose tightened constraint sets $\mathbb{Z} = \alpha\mathbb{X}$ and $\mathbb{V} = \beta\mathbb{V}$ where $\alpha,\ \beta \in (0,1)$

by a simple modification of the defining inequalities. If, for example, $\mathbb{X} = \{x \mid Ax \le a\}$, then $\alpha\mathbb{X} = \{x \mid Ax \le \alpha a\}$. This choice may be tested by Monte Carlo simulation of the controlled system. If constraints are violated in the simulation, $\alpha$ and $\beta$ may be reduced; if the constraints are too conservative, $\alpha$ and $\beta$ may be increased. For each choice of $\alpha$ and $\beta$, the controller provides a degree of robustness that can be adjusted by modifying the "tuning" parameters $\alpha$ and $\beta$.

**Example 3.22: Robust control of an exothermic reaction**

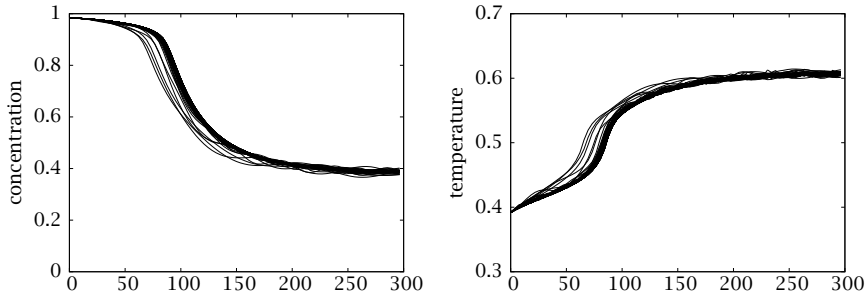
Consider the control of a continuous-stirred-tank reactor. We use a model derived in Hicks and Ray (1971) and modified by Kameswaran and Biegler (2006). The reactor is described by the second-order differential equation

$$\dot{x}_1 = (1/\theta)(1 - x_1) - kx_1 \exp(-M/x_2)$$
$$\dot{x}_2 = (1/\theta)(x_f - x_2) + kx_1 \exp(-M/x_2) - \alpha u(x_2 - x_c) + w$$

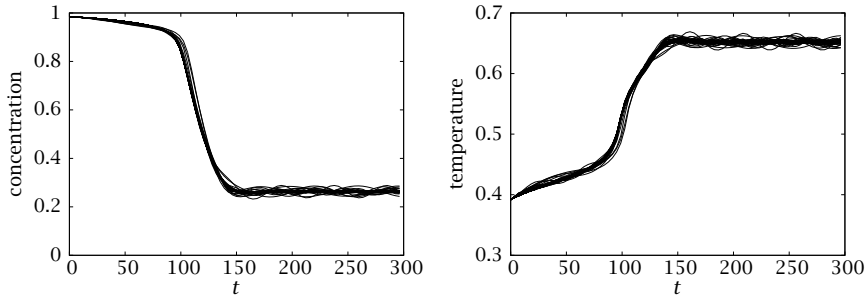
in which $x_1$ is the product concentration, $x_2$ is the temperature, and $u$ is the coolant flowrate. The model parameters are $\theta = 20$, $k = 300$, $M = 5$, $x_f = 0.3947$, $x_c = 0.3816$, and $\alpha = 0.117$. The state, control and disturbance constraint sets are

$$\mathbb{X} = \{x \in \mathbb{R}^2 \mid x_1 \in [0, 2],\ x_2 \in [0, 2]\}$$
$$\mathbb{U} = \{u \in \mathbb{R} \mid u \in [0, 2]\}$$
$$\mathbb{W} = \{w \in \mathbb{R} \mid w \in [-0.001, 0.001]\}$$

The controller is required to steer the system from a locally stable steady state $x(0) = (0.9831, 0.3918)$ at time 0, to a locally unstable steady state $z_e = (0.2632, 0.6519)$. Because the desired terminal state is $z_e$ rather than the origin, the stage cost $\ell(z, v)$ is replaced by $\ell(z - z_e, v - v_e)$ where $\ell(z, v) := (1/2)(|z|^2 + v^2)$ and $(z_e, v_e)$ is an equilibrium pair satisfying $z_e = f(z_e, v_e)$; the terminal constraint set $\mathbb{Z}_f$ is chosen to be $\{z_e\}$. The constraint sets for the nominal control problem are $\mathbb{Z} = \mathbb{X}$ and $\mathbb{V} = [0.02, 2]$. Since the state constraints are not activated, there is no need to tighten $\mathbb{X}$. The disturbance is chosen to be $w(t) = A\sin(\omega t)$ where $A$ and $\omega$ are independent uniformly distributed random variables, taking values in the sets $[0, 0.001]$ and $[0, 1]$, respectively. The horizon length is $N = 40$ and the sample time is $\Delta = 3$ giving a horizon time of 120. The ancillary controller uses $\ell_a(x, u) = (1/2)(|x|^2 + u^2)$ and the same horizon and sample time.




(a) Standard MPC.

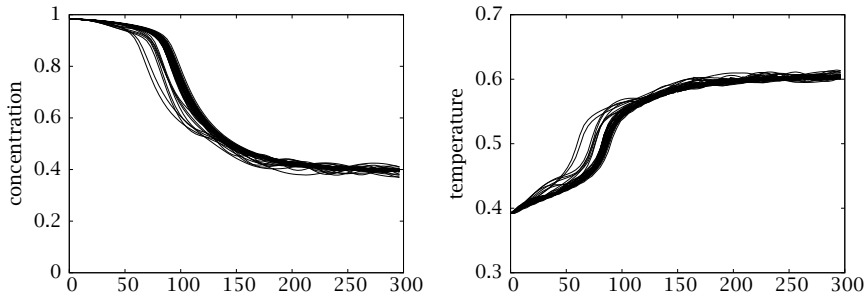



(b) Tube-based MPC.

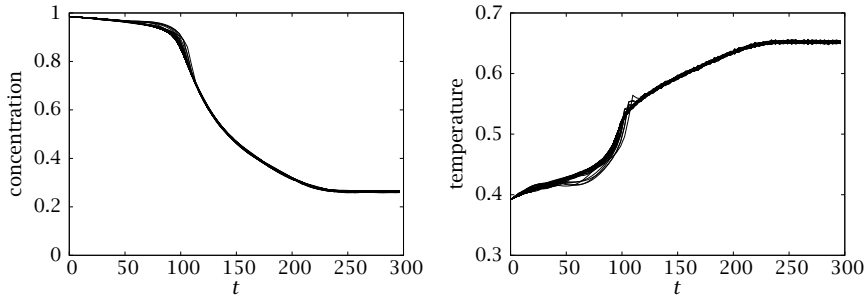
**Figure 3.6:** Comparison of 100 realizations of standard and tube-based MPC for the chemical reactor example.

For comparison, a standard MPC controller using the same stage cost $\ell(\cdot)$, and the same terminal constraint set $\mathbb{Z}_f$ employed in the central path controller is simulated. Figure 3.6(a) illustrates the performance standard MPC, and Figure 3.6(b) the performance of tube-based MPC for 100 realizations of the disturbance sequence. Tube-based MPC, as expected, has a smaller spread of trajectories than is the case for standard MPC. Because each controller has the same stage cost and terminal constraint, the spread of trajectories in the steady-state phase when $z(t) = z_e$ is the same for the two controllers. Because the control constraint set for the tube-based central controller is tighter than that for the standard controller, however, the tube-based controller is somewhat slower than the standard controller.

The ancillary controller may be tuned to reduce more effectively




(a) Standard MPC.




(b) Tube-based MPC.

**Figure 3.7:** Comparison of standard and tube-based MPC with an aggressive ancillary controller.

the spread of trajectories due to the external disturbance. It can be said that the main purpose of the central controller is to steer the system from one equilibrium state to another, while the purpose of the ancillary controller is to reduce the effect of the disturbance. These different objectives may require different stage costs. Our next simulation compares the performance of the standard and tube-based MPC when a more "aggressive" stage cost is employed for the ancillary controller. Figure 3.7 shows the performance of these two controllers when the nominal and standard MPC controller employ $\ell(z - z_e, v - v_e)$ with $\ell(z, v) := (1/2)|z|^2 + 5v^2$ and the ancillary controller employs $\ell_a(x, u) = 50|x|^2 + (1/20)u^2$. The tube-based MPC controller reduces the spread of the trajectories during both the transient *and* the steady state phases.

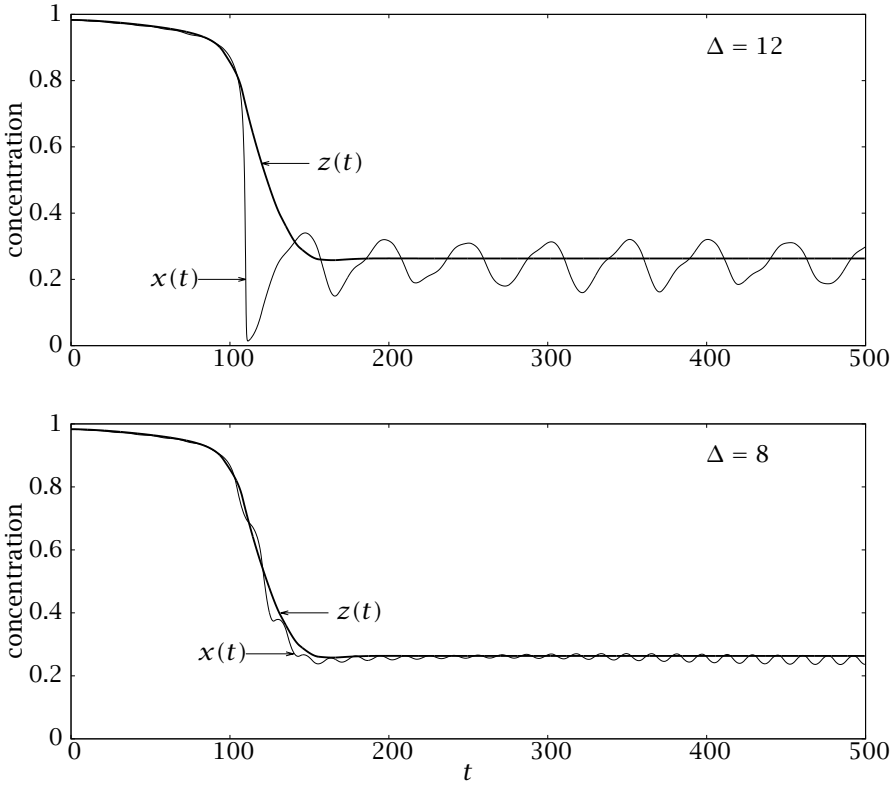



**Figure 3.8:** Concentration versus time for the ancillary controller for sample time 12 (top) and 8 (bottom).

It is also possible to tune the sample time of the ancillary controller. This feature may be useful when the disturbance frequency lies outside the pass band of the central path (nominal) controller. Figure 3.8 shows how concentration varies with time when the disturbance is $w(t) = 0.002\sin(0.4t)$, the sample time of the central path controller is 12 whereas the sample time of the ancillary controller is 12 (top figure) and 8 (bottom figure). The central path controller employs $\ell(z - z_e, v - v_e)$, and the ancillary controller employs $\ell(x, u)$ where $\ell(x, u) := (1/2)(|x|^2 + u^2)$. The ancillary controller with the smaller sample time is more effective in rejecting the disturbance. □

## 3.7 Notes

There is now a considerable volume of research on robust MPC; for a review of the literature up to 2000 see Mayne, Rawlings, Rao, and Scokaert (2000). Early literature examines robustness of nominal MPC under perturbations in Scokaert, Rawlings, and Meadows (1997), and robustness under model uncertainty in De Nicolao, Magni, and Scattolini (1996), and Magni and Sepulchre (1997). Sufficient conditions for robust stability of nominal MPC with modeling error are provided in Santos and Biegler (1999). Teel (2004) provides an excellent discussion of the interplay between nominal robustness and continuity of the Lyapunov function, and also presents some illuminating examples of nonrobust MPC.

The limitations of nominal MPC when uncertainty is present motivated the introduction of feedback, or closed-loop, MPC in which the decision variable is a *policy*, i.e., a sequence of control laws, rather than a sequence of control actions (Mayne, 1995; Kothare, Balakrishnan, and Morari, 1996; Mayne, 1997; Lee and Yu, 1997; Scokaert and Mayne, 1998). With this formulation, the implicit MPC control law can be the same as the receding horizon control law obtained by DP. See Section 3.3.4 and papers such as Magni, De Nicolao, Scattolini, and Allgöwer (2003), where a $H_\infty$ MPC control law is obtained. But such results are *conceptual* because the decision variable is infinite dimensional. Hence practical controllers employ suboptimal policies that are finitely parameterized, an extreme example being nominal MPC. To avoid constraint violation, suboptimal MPC often requires tightening of the constraints in the optimal control problem solved online (Michalska and Mayne, 1993; Chisci, Rossiter, and Zappa, 2001; Mayne and Langson, 2001). Of particular interest is the demonstration in Limón Marruedo, Álamo, and Camacho (2002) that using a sequence of nested constraint sets yields input-to-state stability of nominal MPC if the disturbance is sufficiently small. This procedure was extended in Grimm, Messina, Tuna, and Teel (2007), who do not require the value function to be continuous and do not require the terminal cost to be a control-Lyapunov function. The robust suboptimal controllers discussed in this chapter employ the concept of tubes introduced in the pioneering papers by Bertsekas and Rhodes (1971a,b), and developed for continuous time systems by Aubin (1991) and Khurzhanski and Valyi (1997) and, for linear systems, use a control parameterization proposed by Rossiter, Kouvaritakis, and Rice (1998). Robust positive

invariant sets are employed to construct tubes as shown in (Chisci et al., 2001) and (Mayne and Langson, 2001). Useful references are the surveys by Blanchini (1999) and Kolmanovsky and Gilbert (1995), as well as the recent book by Blanchini and Miani (2008). Kolmanovsky and Gilbert (1995) provide an extensive coverage of the theory and computation of minimal and maximal robust (disturbance) invariant sets. The computation of approximations to robust invariant sets that are themselves invariant is discussed in a series of papers by Raković and colleagues (Raković et al., 2003, 2005a; Raković, Mayne, Kerrigan, and Kouramas, 2005b; Kouramas, Raković, Kerrigan, Allwright, and Mayne, 2005). The tube-based controllers described previously are based on the papers (Langson, Chryssochoos, Raković, and Mayne, 2004; Mayne, Seron, and Raković, 2005).

Because robust MPC is still an active area of research, other methods for achieving robustness have been proposed. Diehl, Bock, and Kostina (2006) simplify the robust nonlinear MPC problem by using linearization, also employed in (Nagy and Braatz, 2004), and present some efficient numerical procedures to determine an approximately optimal control sequence. Goulart, Kerrigan, and Maciejowski (2006) propose a control that is an affine function of current and past states; the decision variables are the associated parameters. This method subsumes the tube-based controllers described in this chapter and has the advantage that a separate nominal trajectory is not required. A disadvantage is the increased complexity of the decision variable, although an efficient computational procedure that reduces computational time per iteration from $O(N^6)$ to $O(N^3)$ has been developed by Goulart, Kerrigan, and Ralph (2008).

Considerable attention has recently been given to input-to-state stability of uncertain systems. Thus Limon, Alamo, Raimondo, de la Peña, Bravo, and Camacho (2008) present the theory of input-to-state stability as a unifying framework for robust MPC, generalizes the tube-based MPC described in (Langson et al., 2004), and extends existing results on min-max MPC. Another example of research in this vein is the paper by Lazar, de la Peña, Hemeels, and Alamo (2008) that utilizes input-to-state practical stability to establish robust stability of feedback min-max MPC. A different approach is described by Angeli, Casavola, Franze, and Mosca (2008) where it is shown how to construct, for each time $i$, an ellipsoidal inner approximation $\mathcal{E}_i$ to the set $\mathcal{T}_i$ of states that can be robustly steered in $i$ steps to a robust control invariant set $\mathcal{T}$. All that is required from the online controller is the determination of

the minimum $i$ such that the current state $x$ lies in $\mathcal{E}_i$ and a control that steers $x \in \mathcal{E}_i$ into the set $\mathcal{E}_{i-1} \subset \mathcal{E}_i$.

## 3.8 Exercises

### Exercise 3.1: Removing the outer min in a min-max problem

Show that $V_i^0 : \mathcal{X}_i \to \mathbb{R}$ and $\kappa_i : \mathcal{X}_i \to \mathbb{U}$ defined by

$$V_i^0(x) = \min_{u\in\mathbb{U}} \max_{w\in\mathbb{W}} \{\ell(x,u,w) + V_{i-1}^0(f(x,u,w)) \mid f(x,u,\mathbb{W}) \subset \mathcal{X}_{i-1}\}$$

$$\kappa_i(x) = \arg\min_{u\in\mathbb{U}} \max_{w\in\mathbb{W}} \{\ell(x,u,w) + V_{i-1}^0(f(x,u,w)) \mid f(x,u,\mathbb{W}) \subset \mathcal{X}_{i-1}\}$$

$$\mathcal{X}_i = \{x \in \mathbb{X} \mid \exists u \in \mathbb{U} \text{ such that } f(x,u,\mathbb{W}) \subset \mathcal{X}_{i-1}\}$$

satisfy

$$V_i^0(x) = \max_{w\in\mathbb{W}} \{\ell(x,\kappa_i(x),w) + V_{i-1}^0(f(x,\kappa_i(x),w))\}$$

### Exercise 3.2: Maximizing a difference

Prove the claim used in the proof of Theorem 3.10 that

$$\max_w \{a(w)\} - \max_w \{b(w)\} \le \max_w \{a(w) - b(w)\}$$

Also show the following minimization version

$$\min_w \{a(w)\} - \min_w \{b(w)\} \ge \min_w \{a(w) - b(w)\}$$

### Exercise 3.3: Equivalent constraints

Assuming that $S$ is a polytope and, therefore, defined by linear inequalities, show that the constraint $x \in \{z\} \oplus S$ (on $z$ for given $x$) may be expressed as $Bz \le b + Bx$, i.e., $z$ must lie in a polytope. If $S$ is symmetric ($x \in S$ implies $-x \in S$), show that $x \in \{z\} \oplus S$ is equivalent to $z \in \{x\} \oplus S$.

### Exercise 3.4: Hausdorff distance between translated sets

Prove that the Hausdorff distance between two sets $\{x\} \oplus S$ and $S$, where $S$ is a compact subset of $\mathbb{R}^n$ and $x$ and $y$ are points in $\mathbb{R}^n$, is $|x - y|$.

### Exercise 3.5: Exponential convergence of $X(i)$

Prove that the sequence of sets $\{X(i)\}$ defined in (3.44) by $X(i) := \{z^*(x(i))\} \oplus S$ converges exponentially to the set $S$.

### Exercise 3.6: Exponential stability of composite system

Show that the set $S_K(\infty) \times \{0\}$ is exponentially stable with a region of attraction $S_K(\infty) \times Z_N$ for the composite system described by (3.31) and (3.32).

### Exercise 3.7: Simulating a robust MPC controller

This exercise explores robust MPC for linear systems with an additive bounded disturbance

$$x^+ = Ax + Bu + w$$

The first task, using the tube-based controller described in Section 3.4.3 is to determine state and control constraint sets $\mathbb{Z}$ and $\mathbb{V}$ such that if the nominal system $z^+ = Az + Bv$ satisfies $z \in \mathbb{Z}$ and $v \in \mathbb{V}$, then the actual system $x^+ = Ax + Bu + w$ with $u = v + K(x - z)$ where $K$ is such that $A + BK$ is strictly stable, satisfies the constraints $x \in \mathbb{X}$ and $u \in \mathbb{U}$.

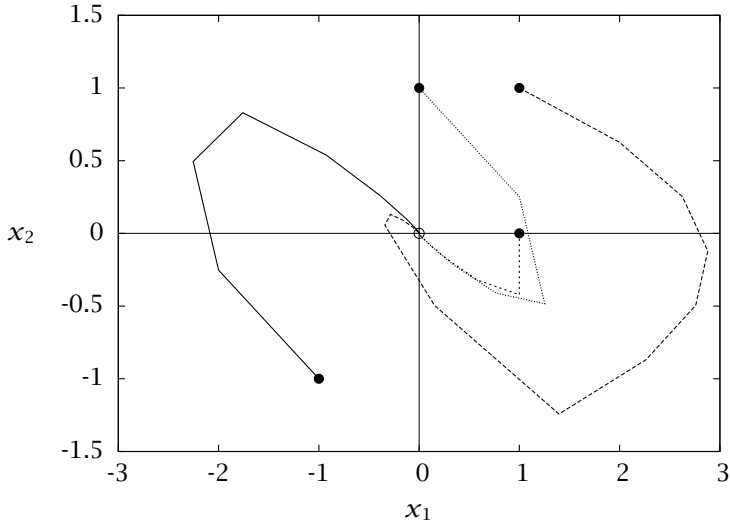



**Figure 3.9:** Closed-loop robust MPC state evolution with $|w| \leq 0.1$ from four different $x_0$.

(a) To get started, consider the scalar system

$$x^+ = x + u + w$$

with constraint sets $\mathbb{X} = \{x \mid x \leq 2\}$, $\mathbb{U} = \{u \mid |u| \leq 1\}$ and $\mathbb{W} = \{w \mid |w| \leq 0.1\}$. Choose $K = -(1/2)$ so that $A_K = 1/2$. Determine $\mathbb{Z}$ and $\mathbb{V}$ so that if the nominal system $z^+ = z + v$ satisfies $z \in \mathbb{Z}$ and $v \in \mathbb{V}$, the uncertain system $x^+ = Ax + Bu + w$, $u = v + K(x - z)$ satisfies $x \in \mathbb{X}$, $u \in \mathbb{U}$.

(b) Repeat part (a) for the following uncertain system

$$x^+ = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + w$$

with the constraint sets $\mathbb{X} = \{x \in \mathbb{R}^2 \mid x_1 \leq 2\}$, $\mathbb{U} = \{u \in \mathbb{R} \mid |u| \leq 1\}$ and $\mathbb{W} = [-0.1, 0.1]$. Choose $K = \begin{bmatrix} -0.4 & -1.2 \end{bmatrix}$.

(c) Determine a model predictive controller for the nominal system and constraint sets $\mathbb{Z}$ and $\mathbb{V}$ used in (b).

(d) Implement robust MPC for the uncertain system and simulate the closed-loop system for a few initial states and a few disturbance sequences for each initial state. The phase plot for initial states $[-1, -1]$, $[1, 1]$, $[1, 0]$ and $[0, 1]$ should resemble Figure 3.9.

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# 4

# State Estimation

## 4.1 Introduction

We now turn to the general problem of estimating the state of a noisy dynamic system given noisy measurements. We assume that the system generating the measurements is given by

$$\begin{aligned} x^+ &= f(x, w) \\ y &= h(x) + v \end{aligned} \tag{4.1}$$

in which the process disturbance, $w$, measurement disturbance, $v$, and system initial state, $x(0)$, are independent random variables with stationary probability densities. One of our main purposes is to provide a state estimate to the MPC regulator as part of a feedback control system, in which case the model changes to $x^+ = f(x, u, w)$ with both process disturbance $w$ and control input $u$. But state estimation is a general technique that is often used in monitoring applications without any feedback control. In Chapter 5, we discuss the combined use of state estimation with MPC regulation. In this chapter we consider state estimation as an independent subject. For notational convenience, we often neglect the control input $u$ as part of the system model in this chapter.

## 4.2 Full Information Estimation

Of all the estimators considered in this chapter, full information estimation will prove to have the best theoretical properties in terms of stability and optimality. Unfortunately, it will also prove to be computationally intractable except for the simplest cases, such as a linear system model. Its value therefore lies in clearly defining what is *desirable* in a state estimator. One method for practical estimator design

| | System variable | Decision variable | Optimal decision |
|---|---|---|---|
| state | $x$ | $\chi$ | $\hat{x}$ |
| process disturbance | $w$ | $\omega$ | $\hat{w}$ |
| measured output | $y$ | $\eta$ | $\hat{y}$ |
| measurement disturbance | $v$ | $\nu$ | $\hat{v}$ |

**Table 4.1:** System and state estimator variables.

therefore is to come as close as possible to the properties of full information estimation while maintaining a tractable online computation. This design philosophy leads directly to moving horizon estimation (MHE).

First we define some notation necessary to distinguish the system variables from the estimator variables. We have already introduced the system variables $(x, w, y, v)$. In the estimator optimization problem, these have corresponding decision variables, which we denote $(\chi, \omega, \eta, \nu)$. The *optimal* decision variables are denoted $(\hat{x}, \hat{w}, \hat{y}, \hat{v})$ and these optimal decisions are the estimates provided by the state estimator. This notation is summarized in Table 4.1. Next we summarize the relationships between these variables

$$\begin{aligned} x^+ &= f(x, w) & y &= h(x) + v \\ \chi^+ &= f(\chi, \omega) & y &= h(\chi) + \nu \\ \hat{x}^+ &= f(\hat{x}, \hat{w}) & y &= h(\hat{x}) + \hat{v} \end{aligned}$$

Notice that it is always the system measurement $y$ that appears in the second column of equations. We can also define the decision variable output, $\eta = h(\chi)$, but notice that $\nu$ measures the fitting error, $\nu = y - h(\chi)$, and we must use the system measurement $y$ and not $\eta$ in this relationship. Therefore, we do not satisfy a relationship like $\eta = h(\chi) + \nu$, but rather

$$\begin{aligned} y &= h(\chi) + \nu & \eta &= h(\chi) \\ y &= h(\hat{x}) + \hat{v} & \hat{y} &= h(\hat{x}) \end{aligned}$$

We begin with a reasonably general definition of the full information estimator that produces an estimator that is *stable*, which we also shall

define subsequently. The full information objective function is

$$V_T(\chi(0), \boldsymbol{\omega}) = \ell_x(\chi(0) - \overline{x}_0) + \sum_{i=0}^{T-1} \ell_i(\omega(i), \nu(i)) \tag{4.2}$$

subject to

$$\chi^+ = f(\chi, \omega) \qquad \mathcal{Y} = h(\chi) + \nu$$

in which $T$ is the current time, $\mathcal{Y}(i)$ is the measurement at time $i$, and $\overline{x}_0$ is the prior information on the initial state.[1] Because $\nu = \mathcal{Y} - h(\chi)$ is the error in fitting the measurement $\mathcal{Y}$, $\ell_i(\omega, \nu)$ costs the model disturbance and the fitting error. These are the two error sources we reconcile in all state estimation problems.

The full information estimator is then defined as the solution to

$$\min_{\chi(0), \boldsymbol{\omega}} V_T(\chi(0), \boldsymbol{\omega}) \tag{4.3}$$

We denote the solution as $\hat{x}(0|T), \hat{w}(i|T), 0 \le i \le T-1, T \ge 1$, and the optimal cost as $V_T^0$. We also use $\hat{x}(T) := \hat{x}(T|T)$ to simplify the notation. The optimal solution and cost also depend on the measurement sequence $\mathbf{y}$, and the prior $\overline{x}_0$, but this dependency is made explicit only when necessary. The choice of stage costs $\ell_x(\cdot)$ and $\ell_i(\cdot)$ is made after we define the class of disturbances affecting the system.

The next order of business is to decide what class of systems to consider if the goal is to obtain a stable state estimator. A standard choice in most nonlinear estimation literature is to assume system observability. The drawback with this choice is that it is overly restrictive for even linear systems. As discussed in Chapter 1, for linear systems we require only detectability for stable estimation (Exercise 1.33). We therefore start instead with an assumption of detectability that is appropriate for nonlinear systems. First we require the definition of i-IOSS (Sontag and Wang, 1997)

**Definition 4.1** (i-IOSS)**.** The system $x^+ = f(x, w), \mathcal{Y} = h(x)$ is *incrementally input/output-to-state stable* (i-IOSS) if there exists some $\beta(\cdot) \in \mathcal{KL}$ and $\gamma_1(\cdot), \gamma_2(\cdot) \in \mathcal{K}$ such that for every two initial states $z_1$ and

[1] Notice we have dropped the final measurement $\mathcal{Y}(T)$ compared to the problem considered in Chapter 1 to formulate the prediction form rather than the filtering form of the state estimation problem. This change is purely for notational convenience, and all results developed in this chapter can also be expressed in the filtering form of MHE.

$z_2$, and any two disturbance sequences $\mathbf{w}_1$ and $\mathbf{w}_2$

$$|x(k;z_1,\mathbf{w}_1) - x(k;z_2,\mathbf{w}_2)| \leq \beta(|z_1 - z_2|, k) + \gamma_1(\|\mathbf{w}_1 - \mathbf{w}_2\|_{0:k-1}) + \gamma_2(\|\mathbf{y}_{z_1,\mathbf{w}_1} - \mathbf{y}_{z_2,\mathbf{w}_2}\|_{0:k})$$

The notation $x(k;x_0,\mathbf{w})$ denotes the solution to $x^+ = f(x,w)$ satisfying initial condition $x(0) = x_0$ with disturbance sequence $\mathbf{w} = \{w(0), w(1), \ldots\}$. We also require the system with an "initial" condition at a time $k_1$ other than $k_1 = 0$, and use the notation $x(k;x_1,k_1,\mathbf{w})$ to denote the solution to $x^+ = f(x,w)$ satisfying the condition $x(k_1) = x_1$ with disturbance sequence $\mathbf{w} = \{w(0), w(1), \ldots\}$.

One of the most important and useful implications of the i-IOSS property is the following proposition.

**Proposition 4.2** (Convergence of state under i-IOSS)**.** *If system $x^+ = f(x,w)$, $y = h(x)$ is i-IOSS, $w_1(k) \to w_2(k)$ and $y_1(k) \to y_2(k)$ as $k \to \infty$, then*

$$x(k;z_1,\mathbf{w}_1) \to x(k;z_2,\mathbf{w}_2) \quad \text{as } k \to \infty \quad \text{for all } z_1, z_2$$

The proof of this proposition is discussed in Exercise 4.3.

The class of disturbances $(w,v)$ affecting the system is defined next. Often we assume these are random variables with stationary probability densities, and often zero-mean normal densities. When we wish to establish estimator stability, however, we wish to show that if the disturbances affecting the measurement converge to zero, then the estimate error also converges to zero. So here we restrict attention to *convergent* disturbances.

**Assumption 4.3** (Convergent disturbances)**.** The sequence $(w(k), v(k))$ for $k \in \mathbb{I}_{\geq 0}$ are bounded and converge to zero as $k \to \infty$.

**Remark 4.4** (Summable disturbances)**.** If the disturbances satisfy Assumption 4.3, then there exists a $\mathcal{K}$-function $\gamma_w(\cdot)$ such that the disturbances are summable

$$\sum_{i=0}^{\infty} \gamma_w(|(w(i), v(i))|) \quad \text{is bounded}$$

See Sontag (1998b, Proposition 7) for a statement and proof of this result.[2]

[2] This result is also useful in establishing the converse Lyapunov function theorem for asymptotic stability as discussed in Exercise B.4 of Appendix B.

Given this class of disturbances, the estimator stage cost is chosen to satisfy the following property.

**Assumption 4.5** (Positive definite stage cost)**.** The initial state cost and stage costs are continuous functions and satisfy the following inequalities for all $x \in \mathbb{R}^n$, $w \in \mathbb{R}^g$, and $v$ in $\mathbb{R}^p$

$$\underline{\gamma}_x(|x|) \leq \ell_x(x) \quad \leq \gamma_x(|x|) \tag{4.4}$$

$$\underline{\gamma}_w(|(w,v)|) \leq \ell_i(w,v) \leq \gamma_w(|(w,v)|) \qquad i \geq 0 \tag{4.5}$$

in which $\underline{\gamma}_x, \underline{\gamma}_w, \gamma_x, \gamma_w \in \mathcal{K}_\infty$ and $\gamma_w$ is defined in Remark 4.4.

Notice that if we change the class of disturbances affecting the system, we may also have to change the stage cost in the state estimator to satisfy $\ell_i(w,v) \leq \gamma_w(|(w,v)|)$ in (4.5). The standard stage cost is the quadratic function, but slowly decaying disturbances in the data require "stronger" than quadratic stage costs to ensure summability. An interaction between anticipated disturbances affecting the system and choice of stage cost in the state estimator is hardly surprising, but Remark 4.4 and Assumption 4.5 make the requirements explicit.

Next we define estimator stability. Again, because the system is nonlinear, we must define stability of a solution. Consider the zero estimate error solution for all $k \geq 0$. This solution arises when the system's initial state is equal to the estimator's prior and there are zero disturbances, $x_0 = \overline{x}_0$, $(w(i), v(i)) = 0$ all $i \geq 0$. In this case, the optimal solution to the full information problem is $\hat{x}(0|T) = \overline{x}_0$ and $\hat{w}(i|T) = 0$ for all $0 \leq i \leq T$, $T \geq 1$, which also gives perfect agreement of estimate and measurement $h(\hat{x}(i|T)) = y(i)$ for $0 \leq i \leq T$, $T \geq 1$. The perturbation to this solution are: the system's initial state (distance from $\overline{x}_0$), and the process and measurement disturbances. We next define stability properties so that *asymptotic stability* considers the case $x_0 \neq \overline{x}_0$ with zero disturbances, and *robust stability* considers the case in which $(w(i), v(i)) \neq 0$.

**Definition 4.6** (Global asymptotic stability)**.** The estimate is based on the *noise-free* measurement $\mathbf{y} = h(\mathbf{x}(x_0, 0))$. The estimate is (nominally) globally asymptotically stable (GAS) if there exists a $\mathcal{KL}$-function $\beta(\cdot)$ such that for all $x_0, \overline{x}_0$ and $k \in \mathbb{I}_{\geq 0}$

$$|x(k; x_0, 0) - \hat{x}(k)| \leq \beta(|x_0 - \overline{x}_0|, k)$$

It bears mentioning that the standard definition of estimator stability for *linear systems* is consistent with Definition 4.6.

**Definition 4.7** (Robust global asymptotic stability)**.** The estimate is based on the *noisy* measurement $\mathbf{y} = h(\mathbf{x}(x_0, \mathbf{w})) + \mathbf{v}$. The estimate is robustly GAS if for all $x_0$ and $\overline{x}_0$, and $(\mathbf{w}, \mathbf{v})$ satisfying Assumption 4.3, the following hold.

(a) The estimate converges to the state; as $k \to \infty$

$$\hat{x}(k) \to x(k; x_0, \mathbf{w})$$

(b) For every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\gamma_x(|x_0 - \overline{x}_0|) + \sum_{i=0}^{\infty} \gamma_w(\,|(w(i), v(i))|\,) \le \delta \tag{4.6}$$

implies $|x(k; x_0, \mathbf{w}) - \hat{x}(k)| \le \varepsilon$ for all $k \in \mathbb{I}_{\ge 0}$.

The first part of the definition ensures that converging disturbances lead to converging estimates. The second part provides a bound on the transient estimate error given a bound on the disturbances. Note also that robust GAS implies GAS (see also Exercise 4.9). With the pieces in place, we can state the main result of this section.

**Theorem 4.8** (Robust GAS of full information estimates)**.** *Given an i-IOSS (detectable) system and measurement sequence generated by* (4.1) *with disturbances satisfying Assumption 4.3, then the full information estimate with stage cost satisfying Assumption 4.5 is robustly GAS.*

*Proof.*

(a) First we establish that the full information cost is bounded for all $T \ge 1$ including $T = \infty$. Consider a candidate set of decision variables

$$\chi(0) = x_0 \qquad \omega(i) = w(i) \quad 0 \le i \le T - 1$$

The full information cost for this choice is

$$V_T(\chi(0), \boldsymbol{\omega}) = \ell_x(x_0 - \overline{x}_0) + \sum_{i=0}^{T-1} \ell_i(w(i), v(i))$$

From Remark 4.4, the sum is bounded for all $T$ including the limit $T = \infty$. Therefore, let $V_\infty$ be an upper bound for the right-hand side. The optimal cost exists for all $T \ge 0$ because $V_T$ is a continuous function and goes to infinity as any of its arguments goes to infinity due to the lower bounds in Assumption 4.5. Next we show that the optimal cost

sequence converges. Evaluate the cost at time $T-1$ using the optimal solution from time $T$. We have that

$$V_{T-1}(\hat{x}(0|T), \hat{\mathbf{w}}_T) = V_T^0 - \ell_T(\hat{w}(T|T), \hat{v}(T|T))$$

Optimization at time $T-1$ can only improve the cost giving

$$V_T^0 \geq V_{T-1}^0 + \ell_T(\hat{w}(T|T), \hat{v}(T|T))$$

and we see that the optimal sequence $\{V_T^0\}$ is nondecreasing and bounded above by $V_\infty$. Therefore the sequence converges and the convergence implies

$$\ell_T(\hat{w}(T|T), \hat{v}(T|T)) \to 0$$

as $T \to \infty$. The lower bound in (4.5) then gives that $\hat{v}(T) = y(T) - h(\hat{x}(T|T)) \to 0$ and $\hat{w}(T|T) \to 0$ as $T \to \infty$. Since the measurement satisfies $y = h(x) + v$, and $v(T)$ converges to zero, we have that

$$h(x(T)) - h(\hat{x}(T|T)) \to 0 \qquad \hat{w}(T|T) \to 0 \qquad T \to \infty$$

Because the system is i-IOSS, we have the following inequality for all $x_0$, $\hat{x}(0|k)$, $\mathbf{w}$, $\hat{\mathbf{w}}_k$, and $k \geq 0$,

$$\begin{aligned} \left| x(k; x_0, \mathbf{w}) - x(k; \hat{x}(0|k), \hat{\mathbf{w}}_k) \right| \leq \beta(|x_0 - \hat{x}(0|k)|, k) + \\ \gamma_1(\|\mathbf{w} - \hat{\mathbf{w}}_k\|_{0:k-1}) + \gamma_2(\|h(\mathbf{x}) - h(\hat{\mathbf{x}}_k)\|_{0:k}) \end{aligned} \quad (4.7)$$

Since $w(k)$ converges to zero, $w(k) - \hat{w}(k)$ converges to zero, and $h(x(k)) - h(\hat{x}(k))$ converges to zero. From Proposition 4.2 we conclude that $\left| x(k; x_0, \mathbf{w}) - x(k; \hat{x}(0|k), \hat{\mathbf{w}}_k) \right|$ converges to zero. Since the state estimate is $\hat{x}(k) := x(k; \hat{x}(0|k), \hat{\mathbf{w}}_k)$ and the state is $x(k) = x(k; x_0, \mathbf{w})$, we have that

$$\hat{x}(k) \to x(k) \qquad k \to \infty$$

and the estimate converges to the system state. This establishes part (a) of the robust GAS definition.[3]

(b) Assume that (4.6) holds for some arbitrary $\delta > 0$. This gives immediately an upper bound on the optimal full information cost function for all $T$, $0 \leq T \leq \infty$, i.e, $V_\infty = \delta$. We then have the following bounds on the initial state estimate for all $k \geq 0$, and the initial state

$$\underline{\gamma}_x(|\hat{x}(0|k) - \overline{x}_0|) \leq \delta \qquad \gamma_x(|x_0 - \overline{x}_0|) \leq \delta$$

[3] It is not difficult to extend this argument to conclude $\hat{x}(i|k) \to x(i; x_0, \mathbf{w})$ as $k \to \infty$ for $k - N \leq i \leq k$ and any finite $N \geq 0$.

These two imply a bound on the initial estimate error, $|x_0 - \hat{x}(0|k)| \leq \underline{\gamma}_x^{-1}(\delta) + \gamma_x^{-1}(\delta)$. The process disturbance bounds are for all $k \geq 0$, $0 \leq i \leq k$

$$\underline{\gamma}_w(|\hat{w}(i|k)|) \leq \delta \qquad \gamma_w(|w(k)|) \leq \delta$$

and we have that $|w(i) - \hat{w}(i|k)| \leq \underline{\gamma}_w^{-1}(\delta) + \gamma_w^{-1}(\delta)$. A similar argument gives for the measurement disturbance $|v(i) - \hat{v}(i|k)| \leq \underline{\gamma}_w^{-1}(\delta) + \gamma_w^{-1}(\delta)$. Since $-(v(i) - \hat{v}(i|k)) = h(x(i)) - h(\hat{x}(i|k))$, we have that

$$|h(x(i)) - h(\hat{x}(i|k))| \leq \underline{\gamma}_w^{-1}(\delta) + \gamma_w^{-1}(\delta)$$

We substitute these bounds in (4.7) and obtain for all $k \geq 0$

$$|x(k) - \hat{x}(k)| \leq \overline{\beta}\big(\underline{\gamma}_x^{-1}(\delta) + \gamma_x^{-1}(\delta)\big) + (\gamma_1 + \gamma_2)\big(\underline{\gamma}_w^{-1}(\delta) + \gamma_w^{-1}(\delta)\big)$$

in which $\overline{\beta}(s) := \beta(s, 0)$ is a $\mathcal{K}$-function. Finally we choose $\delta$ such that the right-hand side is less than $\varepsilon$, which is possible since the right-hand side defines a $\mathcal{K}$-function, which goes to zero with $\delta$. This gives for all $k \geq 0$

$$|x(k) - \hat{x}(k)| \leq \varepsilon$$

and part (b) of the robust GAS definition is established. ■

### 4.2.1 State Estimation as Optimal Control of Estimate Error

Given the many structural similarities between estimation and regulation, the reader may wonder why the stability analysis of the full information estimator presented in the previous section looks rather different than the zero-state regulator stability analysis presented in Chapter 2. To provide some insight into essential *differences*, as well as similarities, between estimation and regulation, consider again the estimation problem in the simplest possible setting with a linear time invariant model and Gaussian noise

$$\begin{aligned} x^+ &= Ax + Gw \qquad & w &\sim N(0, Q) \\ y &= Cx + v \qquad & v &\sim N(0, R) \end{aligned} \tag{4.8}$$

and random initial state $x(0) \sim N(\overline{x}(0), P^-(0))$. In full information estimation, we define the objective function

$$V_T(\chi(0), \boldsymbol{\omega}) = \frac{1}{2}\Big( |\chi(0) - \overline{x}(0)|^2_{(P^-(0))^{-1}} + \sum_{i=0}^{T-1} |\omega(i)|^2_{Q^{-1}} + |\nu(i)|^2_{R^{-1}} \Big)$$

subject to $\chi^+ = A\chi + G\omega$, $y = C\chi + v$. Denote the solution to this optimization as

$$(\hat{x}(0|T), \hat{\mathbf{w}}_T) = \arg \min_{\chi(0),\boldsymbol{\omega}} V_T(\chi(0), \boldsymbol{\omega})$$

and the trajectory of state estimates comes from the model $\hat{x}(i+1|T) = A\hat{x}(i|T) + G\hat{w}(i|T)$. We define estimate error as $\tilde{x}(i|T) = x(i) - \hat{x}(i|T)$ for $0 \le i \le T-1$, $T \ge 1$.

Because the system is *linear*, the estimator is stable if and only if it is stable with zero process and measurement disturbances. So analyzing stability is equivalent to the following simpler question. If noise-free data are provided to the estimator, $(w(i), v(i)) = 0$ for all $i \ge 0$ in (4.8), is the estimate error asymptotically stable as $T \to \infty$ for all $x_0$? We next make this statement precise. First we note that the noise-free measurement satisfies $y(i) - C\hat{x}(i|T) = C\tilde{x}(i|T), 0 \le i \le T$ and the initial condition term can be written in estimate error as $\hat{x}(0) - \overline{x}(0) = -(\tilde{x}(0) - a)$ in which $a = x(0) - \overline{x}(0)$. For the noise-free measurement we can therefore rewrite the cost function as

$$V_T(a, \tilde{x}(0), \mathbf{w}) = \frac{1}{2}\left( |\tilde{x}(0) - a|^2_{(P^-(0))^{-1}} + \sum_{i=0}^{T-1} |C\tilde{x}(i)|^2_{R^{-1}} + |w(i)|^2_{Q^{-1}} \right) \tag{4.9}$$

in which we list explicitly the dependence of the cost function on parameter $a$. For estimation we solve

$$\min_{\tilde{x}(0),\mathbf{w}} V_T(a, \tilde{x}(0), \mathbf{w}) \tag{4.10}$$

subject to $\tilde{x}^+ = A\tilde{x} + Gw$. Now consider problem (4.10) as an optimal control problem using $w$ as manipulated variable and minimizing an objective that measures size of estimate error $\tilde{x}$ and control $w$. We denote the optimal solution as $\tilde{x}^0(0; a)$ and $\mathbf{w}^0(a)$. Substituting these into the model equation gives optimal estimate error $\tilde{x}^0(j|T; a), 0 \le j \le T, 0 \le T$. Parameter $a$ denotes how far $x(0)$, the system's initial state generating the measurement, is from $\overline{x}(0)$, the prior. If we are lucky and $a = 0$, the optimal solution is $(\tilde{x}^0, \mathbf{w}^0) = 0$, and we achieve zero cost in $V_T^0$ and zero estimate error $\tilde{x}(j|T)$ at all time in the trajectory $0 \le j \le T$ for all time $T \ge 1$. The stability analysis in estimation is to show that the origin for $\tilde{x}$ is asymptotically stable. In other words, we wish to show there exists a $KL$-function $\beta$ such that $\left|\tilde{x}^0(T; a)\right| \le \beta(|a|, T)$ for all $T \in \mathbb{I}_{\ge 0}$.

We note the following differences between standard regulation and the estimation problem (4.10). First we see that (4.10) is slightly nonstandard because it contains an extra decision variable, the initial state, and an extra term in the cost function, (4.9). Indeed, without this extra term, the regulator could choose $\widetilde{x}(0) = 0$ to zero the estimate error immediately, choose $\mathbf{w} = 0$, and achieve zero cost in $V_T^0(a)$ for all $a$. The nonstandard regulator allows $\widetilde{x}(0)$ to be manipulated as a decision variable, but penalizes its distance from $a$. Next we look at the stability question. The stability analysis is to show there exists $KL$-function $\beta$ such that $\left|\widetilde{x}^0(T;a)\right| \leq \beta(|a|, T)$ for all $T \in \mathbb{I}_{\geq 0}$. Here convergence is a question about the terminal state in a sequence of *different* optimal control problems with increasing horizon length $T$. That is also not the standard regulator convergence question, which asks how the state trajectory evolves using the optimal control law. In standard regulation, we inject the optimal first input and ask whether we are successfully moving the system to the origin as time increases. In estimation, we do not inject anything into the system; we are provided more information as time increases and ask whether our explanation of the data is improving (terminal estimate error is decreasing) as time increases.

Because stability is framed around the behavior of the terminal state, we would not choose *backward* dynamic programming (DP) to solve (4.10), as in standard regulation. We do not seek the optimal first control move as a function of a known initial state. Rather we seek the optimal terminal state $\widetilde{x}^0(T;a)$ as a function of the parameter $a$ appearing in the cost function. This problem is better handled by *forward* DP as discussed in Sections 1.3.2 and 1.4.3 of Chapter 1 when solving the full information state estimation problem. Exercise 4.12 discusses how to solve (4.10); we obtain the following recursion for the optimal terminal state

$$\widetilde{x}^0(k+1;a) = (A - \widetilde{L}(k)C)\,\widetilde{x}^0(k;a) \tag{4.11}$$

for $k \geq 0$. The initial condition for the recursion is $\widetilde{x}^0(0;a) = a$. The time-varying gains $\widetilde{L}(k)$ and associated cost matrices $P^-(k)$ required are

$$\begin{aligned} P^-(k+1) &= GQG' + AP^-(k)A' \\ &\quad - AP^-(k)C'(CP^-(k)C' + R)^{-1}CP^-(k)A \end{aligned} \tag{4.12}$$

$$\widetilde{L}(k) = AP^-(k)C'(CP^-(k)C' + R)^{-1} \tag{4.13}$$

in which $P^-(0)$ is specified in the problem. As expected, these are

the standard estimator recursions developed in Chapter 1. Asymptotic stability of the estimate error can be established by showing that $V(k,\tilde{x}) := (1/2)\tilde{x}'P(k)^{-1}\tilde{x}$ is a Lyapunov function for (4.11) (Jazwinski, 1970, Theorem 7.4). Notice that this Lyapunov function is *not* the optimal cost of (4.10) as in a standard regulation problem. The optimal cost of (4.10), $V_T^0(a)$, is an *increasing* function of $T$ rather than a decreasing function of $T$ as required for a Lyapunov function. Also note that the argument used in Jazwinski (1970) to establish that $V(k,x)$ is a Lyapunov function for the *linear* system is *more complicated* than the argument used in Section 4.2 to prove stability of full information estimation for the *nonlinear* system. Although one can find Lyapunov functions valid for estimation, they do not have the same simple connection to optimal cost functions as in standard regulation problems, even in the linear, unconstrained case. Stability arguments based instead on properties of $V_T^0(a)$ are simpler and more easily adapted to cover new situations arising in research problems. If a Lyapunov function is required for further analysis, a converse theorem guarantees its existence.

### 4.2.2 Duality of Linear Estimation and Regulation

For linear systems, the estimate error $\tilde{x}$ in full information and state $x$ in regulation to the origin display an interesting duality that we summarize briefly here. Consider the following steady-state estimation and infinite horizon regulation problems.

**Estimator problem.**

$$x(k+1) = Ax(k) + Gw(k)$$
$$y(k) = Cx(k) + v(k)$$

$$R > 0 \qquad Q > 0 \qquad (A,C) \text{ detectable} \qquad (A,G) \text{ stabilizable}$$

$$\tilde{x}(k+1) = \left(A - \tilde{L}C\right)\tilde{x}(k)$$

**Regulator problem.**

$$x(k+1) = Ax(k) + Bu(k)$$
$$y(k) = Cx(k)$$

$$R > 0 \qquad Q > 0 \qquad (A,B) \text{ stabilizable} \qquad (A,C) \text{ detectable}$$

$$x(k+1) = (A + BK)\,x(k)$$

| Regulator | Estimator |
|---|---|
| $A$ | $A'$ |
| $B$ | $C'$ |
| $C$ | $G'$ |
| $k$ | $l = N - k$ |
| $\Pi(k)$ | $P^-(l)$ |
| $\Pi(k-1)$ | $P^-(l+1)$ |
| $\Pi$ | $P^-$ |
| $Q$ | $Q$ |
| $R$ | $R$ |
| $P_f$ | $P^-(0)$ |
| $K$ | $-\widetilde{L}'$ |
| $A + BK$ | $(A - \widetilde{L}C)'$ |
| $x$ | $\widetilde{x}'$ |

| Regulator | Estimator |
|---|---|
| $R > 0, \quad Q > 0$ | $R > 0, \quad Q > 0$ |
| $(A, B)$ stabilizable | $(A, C)$ detectable |
| $(A, C)$ detectable | $(A, G)$ stabilizable |

**Table 4.2:** Duality variables and stability conditions for linear quadratic regulation and least squares estimation.

In Appendix A, we derive the dual dynamic system following the approach in Callier and Desoer (1991), and obtain the duality variables in regulation and estimation listed in Table 4.2.

We also have the following result connecting controllability of the original system and observability of the dual system

**Lemma 4.9** (Duality of controllability and observability)**.** *$(A, B)$ is controllable (stabilizable) if and only if $(A', B')$ is observable (detectable).*

This result can be established directly using the Hautus lemma and is left as an exercise. This lemma and the duality variables allows us to translate stability conditions for infinite horizon regulation problems into stability conditions for full information estimation problems and vice versa. For example, the following is a basic theorem covering convergence of Riccati equations in the form that is useful in establishing exponential stability of regulation as discussed in Chapter 1.

**Theorem 4.10** (Riccati iteration and regulator stability)**.** *Given $(A, B)$ stabilizable, $(A, C)$ detectable, $Q > 0$, $R > 0$, $P_f \geq 0$, and the discrete*

*Riccati equation*

$$\begin{aligned}\Pi(k-1) &= C'QC + A'\Pi(k)A - \\ &\qquad A'\Pi(k)B(B'\Pi(k)B+R)^{-1}B'\Pi(k)A, \quad k = N,\dots,1 \\ \Pi(N) &= P_f\end{aligned}$$

*Then*

*(a) There exists* $\Pi \geq 0$ *such that for every* $P_f \geq 0$

$$\lim_{k\to-\infty} \Pi(k) = \Pi$$

*and* $\Pi$ *is the unique solution of the steady-state Riccati equation*

$$\Pi = C'QC + A'\Pi A - A'\Pi B(B'\Pi B + R)^{-1}B'\Pi A$$

*among the class of positive semidefinite matrices.*

*(b) The matrix* $A + BK$ *in which*

$$K = -(B'\Pi B + R)^{-1}B'\Pi A$$

*is a stable matrix.*

Bertsekas (1987, pp.59–64) provides a proof for a slightly different version of this theorem. Exercise 4.13 explores translating this theorem into the form that is useful for establishing exponential convergence of full information estimation.

## 4.3 Moving Horizon Estimation

As displayed in Figure 1.4 of Chapter 1, in MHE we consider only the $N$ most recent measurements, $\mathbf{y}_N(T) = \{y(T-N), y(T-N+1), \dots y(T-1)\}$. For $T > N$, the MHE objective function is given by

$$\hat{V}_T(\chi(T-N), \boldsymbol{\omega}) = \Gamma_{T-N}(\chi(T-N)) + \sum_{i=T-N}^{T-1} \ell_i(\omega(i), v(i))$$

subject to $x^+ = f(x,\omega)$, $y = h(x) + v$. The MHE problem is defined to be

$$\min_{\chi(T-N),\boldsymbol{\omega}} \hat{V}_T(\chi(T-N), \boldsymbol{\omega}) \tag{4.14}$$

in which $\boldsymbol{\omega} = \{\omega(T-N), \dots, \omega(T-1)\}$. The designer chooses the prior weighting $\Gamma_k(\cdot)$ for $k > N$. Until the data horizon is full, i.e., for times $T \leq N$, we generally *define* the MHE problem to be the full information problem.

### 4.3.1 Zero Prior Weighting

Here we discount the early data completely and choose $\Gamma_i(\cdot) = 0$ for all $i \geq N$. Because it discounts the past data completely, this form of MHE must be able to asymptotically reconstruct the state using only the most recent $N$ measurements. The first issue is establishing existence of the solution. Unlike the full information problem, in which the positive definite initial penalty guarantees that the optimization takes place over a bounded (compact) set, here there is zero initial penalty. So we must restrict the system further than i-IOSS to ensure solution existence. We show next that observability is sufficient for this purpose.

**Definition 4.11** (Observability)**.** The system $x^+ = f(x,w), y = h(x)$ is *observable* if there exist finite $N_o \in \mathbb{I}_{\geq 1}$, $\gamma_1(\cdot)$, $\gamma_2(\cdot) \in \mathcal{K}$ such that for every two initial states $z_1$ and $z_2$, and any two disturbance sequences $\mathbf{w}_1, \mathbf{w}_2$, and all $k \geq N_o$

$$|z_1 - z_2| \leq \gamma_1(\|\mathbf{w}_1 - \mathbf{w}_2\|_{0:k-1}) + \gamma_2(\|\mathbf{y}_{z_1,\mathbf{w}_1} - \mathbf{y}_{z_2,\mathbf{w}_2}\|_{0:k})$$

At any time $T \geq N$ consider decision variables $\chi(T-N) = x(T-N)$ and $\omega(i) = w(i)$ for $T-N \leq i \leq T-1$. For these decision variables the cost function has the value

$$\hat{V}_T(\chi(T-N), \boldsymbol{\omega}) = \sum_{i=T-N}^{T-1} \ell_i(w(i), v(i)) \tag{4.15}$$

which is less than $V_\infty$ defined in the full information problem. Observability then ensures that for all $k \geq N \geq N_o$

$$|x(k-N) - \hat{x}(k-N|k)| \leq \gamma_2(\|\mathbf{v}\|_{k-N:k})$$

Since $v(k)$ is bounded for all $k \geq 0$ by Assumption 4.3, observability has bounded the distance between the initial estimate in the horizon and the system state for all $k \geq N$. That along with continuity of $\hat{V}_T(\chi, \boldsymbol{\omega})$ ensures existence of the solution to the MHE problem by the Weierstrass theorem (Proposition A.7). But the solution does not have to be unique.

We show next that final-state observability (FSO) is the natural system requirement for MHE with zero prior weighting to provide stability and convergence.

**Definition 4.12** (Final-state observability)**.** The system $x^+ = f(x,w)$, $y = h(x)$ is *final-state observable* (FSO) if there exist finite $N_o \in \mathbb{I}_{\geq 1}$,

$\overline{\gamma}_1(\cdot)$, $\overline{\gamma}_2(\cdot) \in \mathcal{K}$ such that for every two initial states $z_1$ and $z_2$, and any two disturbance sequences $\mathbf{w}_1, \mathbf{w}_2$, and all $k \geq N_0$

$$|x(k; z_1, \mathbf{w}_1) - x(k; z_2, \mathbf{w}_2)| \leq \overline{\gamma}_1(\, \|\mathbf{w}_1 - \mathbf{w}_2\|_{0:k-1}\,) + \overline{\gamma}_2(\, ||\mathbf{y}_{z_1,\mathbf{w}_1} - \mathbf{y}_{z_2,\mathbf{w}_2}||_{0:k}\,)$$

Notice that FSO is not the same as observability. It is weaker than observability and stronger than i-IOSS (detectability) as discussed in Exercise 4.10. Consider two equal disturbance sequences, $\mathbf{w}_1 = \mathbf{w}_2$, and two equal measurement sequences $\mathbf{y}_1 = \mathbf{y}_2$. FSO implies that for every pair $z_1$ and $z_2$, $x(N_0; z_1, \mathbf{w}_1) = x(N_0; z_2, \mathbf{w}_1)$; we know the *final* states at time $k = N_0$ are equal. FSO does not imply that the *initial* states are equal as required by observability. We can of course add the nonnegative term $\beta(|z_1 - z_2|, k)$ to the right-hand side of the FSO inequality and obtain the i-IOSS inequality, so FSO implies i-IOSS. Exercise 4.11 treats observability, FSO, and detectability of the linear time-invariant system, which can be summarized compactly in terms of the eigenvalues of the partitioned state transition matrix corresponding to the unobservable modes.

Next we show that the MHE cost function converges to zero as $T \to \infty$ for all $x_0$ and converging disturbances $(w(i), v(i))$. Since $(w(i), v(i))$ converges to zero, (4.15) implies that $\hat{V}_T$ converges to zero as $T \to \infty$. The optimal cost at $T$, $\hat{V}_T^0$, is bounded above by $\hat{V}_T$ so $\hat{V}_T^0$ also converges to zero. The optimal cost is

$$\hat{V}_T^0 = \sum_{i=T-N}^{T-1} \ell_i(\hat{w}(i|T), y(i) - h(\hat{x}(i|T)))$$

in which $(\hat{x}(i|T), \hat{w}(i|T))$ are the optimal decisions for $T - N \leq i \leq T - 1$, $T \geq N$. Since $\hat{V}_T^0$ converges to zero, we have

$$y(i) - h(\hat{x}(i|T)) \to 0 \qquad \hat{w}(i|T) \to 0$$

as $T \to \infty$. Since $y = h(x) + v$ and $v(i)$ converges to zero, and $w(i)$ converges to zero, we also have

$$h(x(i)) - h(\hat{x}(i|T)) \to 0 \qquad w(i) - \hat{w}(i|T) \to 0 \tag{4.16}$$

for $T - N \leq i \leq T - 1$, $T \geq N$.

We have the following theorem for this estimator.

**Theorem 4.13** (Robust GAS of MHE with zero prior weighting)**.** *Consider an observable system and measurement sequence generated by* (4.1)

*with disturbances satisfying Assumption 4.3. The MHE estimate with zero prior weighting, $N \geq N_o$, and stage cost satisfying* (4.5)*, is robustly GAS.*

*Proof.* We establish the two parts of Definition 4.7

(a) Consider the system to be at state $x(k-N)$ at time $k-N$ and subject to disturbances $\mathbf{w}_k = \{w(k-N), \dots w(k-1)\}$. At time $k$, the estimator has initial state $\hat{x}(k-N|k)$ and disturbance sequence $\hat{w}_k$. We have that $x(k; x(k-N), k-N, \mathbf{w}_k) = x(k)$ and $x(k; \hat{x}(k-N|k), k-N, \hat{\mathbf{w}}_k) = \hat{x}(k)$, and the FSO property gives for $k \geq N \geq N_o$

$$|x(k) - \hat{x}(k)| \leq \overline{\gamma}_1(\,||\mathbf{w}_k - \hat{\mathbf{w}}_k||_{k-N:k-1}\,) + \overline{\gamma}_2(\,||h(\mathbf{x}_k) - h(\hat{\mathbf{x}}_k)||_{k-N:k}\,) \quad (4.17)$$

By (4.16) the right-hand side converges to zero as $k \to \infty$, which gives

$$\hat{x}(k) \to x(k)$$

as $k \to \infty$ for all $x_0$ and measurement sequence generated by (4.1) with disturbances satisfying Assumption 4.3.

(b) For $k \leq N$, MHE is equivalent to full information estimation, and Theorem 4.8 applies. So we consider $k > N$. Assume (4.6) holds for some $\delta > 0$. This implies

$$||\mathbf{w}_k - \hat{\mathbf{w}}_k||_{k-N:k-1} \leq \underline{\gamma}_w^{-1}(\delta) + \gamma_w^{-1}(\delta)$$
$$||h(\mathbf{x}_k) - h(\hat{\mathbf{x}}_k)||_{k-N:k} \leq \underline{\gamma}_w^{-1}(\delta) + \gamma_w^{-1}(\delta)$$

Using these bounds in (4.17) gives for $k > N$

$$|x(k) - \hat{x}(k)| \leq (\overline{\gamma}_1 + \overline{\gamma}_2)(\underline{\gamma}_w^{-1}(\delta) + \gamma_w^{-1}(\delta))$$

Choose an $\varepsilon > 0$. Since the right-hand side defines a $\mathcal{K}$-function, we can choose $\delta > 0$ small enough to meet the bound $|x(k) - \hat{x}(k)| \leq \varepsilon$ for all $k > N$. Coupled with Theorem 4.8 to cover $k \leq N$, we have established part (b) of robust GAS. ■

### 4.3.2 Nonzero Prior Weighting

The two drawbacks of zero prior weighting are: the system had to be assumed *observable* rather than detectable to ensure existence of the solution to the MHE problem; and a large horizon $N$ may be required to obtain performance comparable to full information estimation. We

address these two disadvantages by using nonzero prior weighting. To get started, we use forward DP, as we did in Chapter 1 for the unconstrained linear case, to decompose the full information problem exactly into the MHE problem (4.14) in which $\Gamma(\cdot)$ is chosen as arrival cost.

**Definition 4.14** (Full information arrival cost)**.** The full information arrival cost is defined as

$$Z_T(p) = \min_{\chi(0),\boldsymbol{\omega}} V_T(\chi(0),\boldsymbol{\omega}) \tag{4.18}$$

subject to

$$\chi^+ = f(\chi,\omega) \qquad y = h(\chi) + v \qquad \chi(T;\chi(0),\boldsymbol{\omega}) = p$$

We have the following equivalence.

**Lemma 4.15** (MHE and full information estimation)**.** *The MHE problem* (4.14) *is equivalent to the full information problem* (4.3) *for the choice* $\Gamma_k(\cdot) = Z_k(\cdot)$ *for all* $k > N$ *and* $N \geq 1$.

The proof is left as an exercise. This lemma is the essential insight provided by the DP recursion. But notice that evaluating arrival cost in (4.18) has the same computational complexity as solving a full information problem. So next we generate an MHE problem that has simpler computational requirements, but retains the excellent stability properties of full information estimation. We proceed as follows.

**Definition 4.16** (MHE arrival cost)**.** The MHE arrival cost $\hat{Z}(\cdot)$ is defined for $T > N$ as

$$\begin{aligned}\hat{Z}_T(p) &= \min_{z,\boldsymbol{\omega}} \hat{V}_T(z,\boldsymbol{\omega}) \\ &= \min_{z,\boldsymbol{\omega}} \Gamma_{T-N}(z) + \sum_{i=T-N}^{T-1} \ell_i(\omega(i),v(i))\end{aligned} \tag{4.19}$$

subject to

$$\chi^+ = f(\chi,\omega) \qquad y = h(\chi) + v \qquad \chi(T;z,T-N,\boldsymbol{\omega}) = p$$

For $T \leq N$ we usually define the MHE problem to be the full information problem, so $\hat{Z}_T(\cdot) = Z_T(\cdot)$ and $\hat{V}_T^0 = V_T^0$. Notice from the second equality in the definition that the MHE arrival cost at $T$ is defined in terms of the prior weighting at time $T-N$.

We next show that choosing a prior weighting that *underbounds* the MHE arrival cost is the key sufficient condition for stability and convergence of MHE.

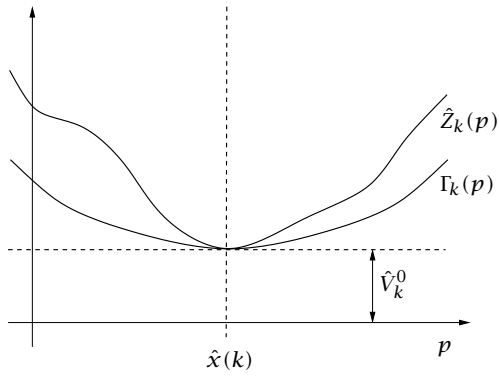


**Figure 4.1:** MHE arrival cost $\hat{Z}_k(p)$, underbounding prior weighting $\Gamma_k(p)$, and MHE optimal value $\hat{V}_k^0$; for all $p$ and $k > N$, $\hat{Z}_k(p) \geq \Gamma_k(p) \geq \hat{V}_k^0$, and $\hat{Z}_k(\hat{x}(k)) = \Gamma_k(\hat{x}(k)) = \hat{V}_k^0$.

**Assumption 4.17** (Prior weighting)**.** We assume that $\Gamma_k(\cdot)$ is continuous and satisfies the following inequalities for all $k > N$

(a) Upper bound

$$\Gamma_k(p) \leq \hat{Z}_k(p) = \min_{z,\boldsymbol{\omega}} \Gamma_{k-N}(z) + \sum_{i=k-N}^{k-1} \ell_i(\omega(i), \nu(i)) \tag{4.20}$$

subject to $\chi^+ = f(\chi, \omega), y = h(\chi) + \nu, \chi(k; z, k-N, \boldsymbol{\omega}) = p$.

(b) Lower bound

$$\Gamma_k(p) \geq \hat{V}_k^0 + \underline{\gamma}_p(|\, p - \hat{x}(k)\,|) \tag{4.21}$$

in which $\underline{\gamma}_p \in \mathcal{K}_\infty$.

This assumption is depicted in Figure 4.1.

To establish convergence of the MHE estimates, it will prove useful to have an upper bound for the MHE optimal cost. Next we establish the stronger result that the MHE arrival cost is bounded above by the full information arrival cost as stated in the following proposition.

**Proposition 4.18** (Arrival cost of full information greater than MHE)**.**

$$\hat{Z}_T(\cdot) \leq Z_T(\cdot) \qquad T \geq 1 \tag{4.22}$$

*Proof.* We know this result holds for $T \in \mathbb{I}_{1:N}$ because MHE is equivalent to full information for these $T$. Next we show that the inequality at $T$

implies the inequality at $T+N$. Indeed, we have by the definition of the arrival costs

$$\hat{Z}_{T+N}(p) = \min_{z,\boldsymbol{\omega}} \Gamma_T(z) + \sum_{i=T}^{T+N-1} \ell_i(\omega(i), \nu(i))$$
$$Z_{T+N}(p) = \min_{z,\boldsymbol{\omega}} Z_T(z) + \sum_{i=T}^{T+N-1} \ell_i(\omega(i), \nu(i))$$

in which both optimizations are subject to the same constraints $\chi^+ = f(\chi,\omega)$, $\mathcal{Y} = h(\chi)+\nu$, $\chi(k;z,k-N,\boldsymbol{\omega}) = p$. From (4.20) $\Gamma_T(\cdot) \le \hat{Z}_T(\cdot)$, and $\hat{Z}_T(\cdot) \le Z_T(\cdot)$ by assumption. Together these imply the optimal values satisfy $\hat{Z}_{T+N}(p) \le Z_{T+N}(p)$ for all $p$, and we have established $\hat{Z}_{T+N}(\cdot) \le Z_{T+N}(\cdot)$. Therefore we have extended (4.22) from $T \in \mathbb{I}_{1:N}$ to $T \in \mathbb{I}_{1:2N}$. Continuing this recursion establishes (4.22) for $T \in \mathbb{I}_{\ge 1}$. ■

Given (4.22) we also have the analogous inequality for the optimal costs of MHE and full information

$$\hat{V}_T^0 \le V_T^0 \qquad T \ge 1 \tag{4.23}$$

**Definition 4.19** (MHE detectable system)**.** We say a system $x^+ = f(x,w)$, $\mathcal{Y} = h(x)$ is *MHE detectable* if the system augmented with an extra disturbance $w_2$

$$x^+ = f(x,w_1) + w_2 \qquad \mathcal{Y} = h(x)$$

is i-IOSS with respect to the augmented disturbance $(w_1, w_2)$.

Note that MHE detectable is stronger than i-IOSS (detectable) but weaker than observable and FSO. See also Exercise 4.10.

**Theorem 4.20** (Robust GAS of MHE)**.** *Consider an MHE detectable system and measurement sequence generated by* (4.1) *with disturbances satisfying Assumption 4.3. The MHE estimate defined by* (4.14) *using the prior weighting function* $\Gamma_k(\cdot)$ *satisfying Assumption 4.17 and stage cost satisfying Assumption 4.5 is robustly GAS.*

*Proof.* The MHE solution exists for $T \le N$ by the existence of the full information solution, so we consider $T > N$. For disturbances satisfying Assumption 4.3, we established in the proof of Theorem 4.8 for the full information problem that $V_T^0 \le V_\infty$ for all $T \ge 1$ including $T = \infty$. From Proposition 4.18 and (4.23), we have that the MHE optimal cost also has the upper bound $\hat{V}_T^0 \le V_\infty$ for all $T \ge 1$ including $T = \infty$. Since we have assumed $f(\cdot)$ and $h(\cdot)$ are continuous, $\Gamma_i(\cdot)$ is continuous for

$i > N$, and $\ell_i(\cdot)$ is continuous for all $i \geq 0$, the MHE cost function $\hat{V}_T(\cdot)$ is continuous for $T > N$. The lower bound on $\Gamma_i$ for $i > N$ and $\ell_i$ for all $i \geq 0$ imply that for $T > N$, $\hat{V}_T(\chi(T-N), \boldsymbol{\omega})$ goes to infinity as either $\chi(T-N)$ or $\boldsymbol{\omega}$ goes to infinity. Therefore the MHE optimization takes place over a bounded, closed set for $T > N$, and the the solution exists by the Weierstrass theorem.

(a) Consider the solution to the MHE problem at time $T$, $(\hat{x}(T-N|T), \hat{\mathbf{w}}_T)$. We have that

$$\hat{V}_T^0 = \Gamma_{T-N}(\hat{x}(T-N|T)) + \sum_{i=T-N}^{T-1} \ell_i(\hat{w}(i|T), \hat{v}(i|T))$$

From (4.21) we have

$$\Gamma_{T-N}(\hat{x}(T-N|T)) \geq \hat{V}_{T-N}^0 + \underline{\gamma}_p(|\hat{x}(T-N|T) - \hat{x}(T-N|T-N)|)$$

Using this inequality in the previous equation we have

$$\hat{V}_T^0 \geq \hat{V}_{T-N}^0 + \underline{\gamma}_p(|\hat{x}(T-N|T) - \hat{x}(T-N|T-N)|) + \sum_{i=T-N}^{T-1} \ell_i(\hat{w}(i|T), \hat{v}(i|T)) \quad (4.24)$$

and we have established that the sequence $\{\hat{V}_{T+iN}^0\}$ is a nondecreasing sequence in $i = 1, 2, \ldots$ for any fixed $T \geq 1$. Since $\hat{V}_k^0$ is bounded above for all $k \geq 1$, the sequence $\hat{V}_{T+iN}^0$ converges as $i \to \infty$ for any $T \geq 1$. This convergence gives as $T \to \infty$

$$\underline{\gamma}_p(\,|\hat{x}(T-N|T) - \hat{x}(T-N|T-N)|\,) \to 0 \quad (4.25)$$

$$\sum_{i=T-N}^{T-1} \ell_i(\hat{w}(i|T), \hat{v}(i|T)) \to 0 \quad (4.26)$$

Next we create a single estimate sequence by concatenating MHE sequences from times $N, 2N, 3N, \ldots$. This gives the state sequence and corresponding $\overline{\mathbf{w}}_1$ and $\overline{\mathbf{w}}_2$ sequences listed in the following table so that

$$\overline{x}^+ = f(\overline{x}, \overline{w}_1) + \overline{w}_2 \quad \text{for } k \geq 0 \qquad y = h(\overline{x}) + \overline{v}$$

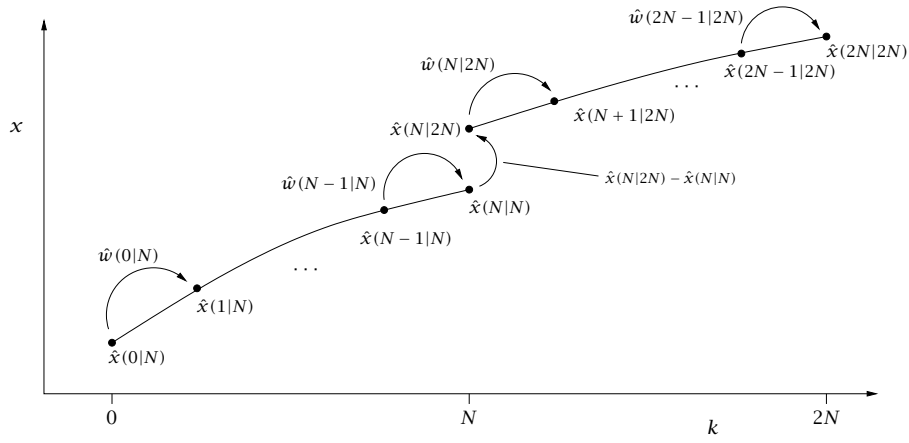


**Figure 4.2:** Concatenating two MHE sequences to create a single state estimate sequence from time 0 to $2N$.

| $\overline{\mathbf{x}}$ | $\overline{\mathbf{w}}_1$ | $\overline{\mathbf{w}}_2$ | $\overline{\mathbf{v}}$ |
|---|---|---|---|
| $\hat{x}(0\|N)$ | $\hat{w}(0\|N)$ | 0 | $\hat{v}(0\|N)$ |
| $\hat{x}(1\|N)$ | $\hat{w}(1\|N)$ | 0 | $\hat{v}(1\|N)$ |
| $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |
| $\hat{x}(N-1\|N)$ | $\hat{w}(N-1\|N)$ | $\hat{x}(N\|2N) - \hat{x}(N\|N)$ | $\hat{v}(N-1\|N)$ |
| $\hat{x}(N\|2N)$ | $\hat{w}(N\|2N)$ | 0 | $\hat{v}(N\|2N)$ |
| $\hat{x}(N+1\|2N)$ | $\hat{w}(N+1\|2N)$ | 0 | $\hat{v}(N+1\|2N)$ |
| $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |
| $\hat{x}(2N-1\|2N)$ | $\hat{w}(2N-1\|2N)$ | $\hat{x}(2N\|3N) - \hat{x}(2N\|2N)$ | $\hat{v}(2N-1\|2N)$ |
| $\hat{x}(2N\|3N)$ | $\hat{w}(2N\|3N)$ | 0 | $\hat{v}(2N\|3N)$ |
| $\hat{x}(2N+1\|3N)$ | $\hat{w}(2N+1\|3N)$ | 0 | $\hat{v}(2N+1\|3N)$ |
| $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |

Notice that every $N$ rows in the array, there is a nonzero entry in the $\mathbf{w}_2$ column. That disturbance is required to move from one MHE sequence to the next as shown in Figure 4.2. But (4.25) implies that $\overline{w}_2(k) \to 0$ as integer $k \to \infty$, and (4.26) implies that $\overline{w}_1(k) \to 0$ as $k \to \infty$. Therefore $|(w_1(k), 0) - (\overline{w}_1(k), \overline{w}_2(k))| \to 0$ as $k \to \infty$. We also have from (4.26) that $h(x(k)) - h(\overline{x}(k)) = \overline{v}(k) - v(k) \to 0$ as $k \to \infty$. Next we apply the MHE-detectability assumption to the $\mathbf{x}$ and $\overline{\mathbf{x}}$ sequences, to obtain the inequality

$$|x(k) - \overline{x}(k)| \le \beta(|x(0) - \hat{x}(0|N)|, k) + \gamma_1(\|(\mathbf{w}_1, 0) - (\overline{\mathbf{w}}_1, \overline{\mathbf{w}}_2)\|_{0:k-1}) + \gamma_2(\|h(\mathbf{x}) - h(\overline{\mathbf{x}})\|_{0:k}) \quad (4.27)$$

From Proposition 4.2 we conclude that $\overline{x}(k) \to x(k)$ as $k \to \infty$. Therefore we have that $\hat{x}(iN+j|(i+1)N) \to x(iN+j)$ for all $j = 0, 1, \ldots, N-1$ as integer $i \to \infty$. Note that only $\hat{x}(iN|iN)$ is missing from this argument. But $x(iN) = f(x(iN-1), w(iN-1))$ and $\hat{x}(iN|iN) = f(\hat{x}(iN-1|iN), \hat{w}(iN-1|iN))$. Since $\hat{x}(iN-1|iN) \to x(iN-1)$, $\hat{w}(iN-1|iN) \to w(iN-1)$, and $f(\cdot)$ is continuous, we have that $\hat{x}(iN|iN) \to x(iN)$ as well. We can repeat this concatenation construction using the MHE sequences $N+j, 2N+j, 3N+j, \ldots$ for $j = 1, \ldots, N-1$ to conclude that $\hat{x}(k) \to x(k)$ as $k \to \infty$, and convergence is established.

(b) As previously, assume the following holds for some $\delta > 0$

$$\gamma_x(|x_0 - \overline{x}_0|) + \sum_{i=0}^{\infty} \gamma_w(\,|(w(i), v(i))|\,) \le \delta$$

We wish to show that for every $\varepsilon > 0$ there exists $\delta > 0$ such that this equation implies $|x(k; x_0, \mathbf{w}) - \hat{x}(k)| \le \varepsilon$ for all $k \ge 0$. We know such a $\delta$ exists for $k \le N$ from Theorem 4.8. We therefore consider $k > N$. The optimal MHE cost is bounded above by the optimal full information, which is bounded above by $\delta$, $\hat{V}_T^0 \le V_T^0 \le \delta$ for $T \ge 0$. So we have using $T = N$,

$$\underline{\gamma}_x(|\hat{x}(0|N) - \overline{x}_0|) \le \delta \qquad \gamma_x(|x_0 - \overline{x}_0|) \le \delta$$

which gives $|x(0) - \hat{x}(0|N)| \le (\underline{\gamma}_x^{-1} + \gamma_x^{-1})(\delta)$. From (4.24) and the fact that $\hat{V}_T^0 \le \delta$, we know that

$$\underline{\gamma}_p(\,|\hat{x}((i+1)N|iN) - \hat{x}(iN|iN)|\,) \le \delta$$
$$|\hat{x}((i+1)N|iN) - \hat{x}(iN|iN)| \le \underline{\gamma}_p^{-1}(\delta)$$

which implies $|\overline{w}_2(k)| \le \underline{\gamma}_p^{-1}(\delta)$ for all $k \ge 0$. Examining the terms in the $\overline{\mathbf{w}}_1$ column, we conclude as before that $|w(k) - \overline{w}_1(k)| \le (\underline{\gamma}_p^{-1} + \underline{\gamma}_w^{-1})(\delta)$ for all $k \ge 0$. The $\overline{\mathbf{v}}$ column gives the bound

$$|v(k) - \overline{v}(k)| = |h(x(k)) - h(\overline{x}(k))| \le (\underline{\gamma}_w^{-1} + \gamma_w^{-1})(\delta)$$

We also have the bounds

$$\begin{aligned}
\|(\mathbf{w}_1, 0) - (\overline{\mathbf{w}}_1, \overline{\mathbf{w}}_2)\| &= \max_{k \ge 0} |(w_1(k), 0) - (\overline{w}_1(k), \overline{w}_2(k))| \\
&= \max_{k \ge 0} |(w_1(k) - \overline{w}_1(k), -\overline{w}_2(k))| \\
&\le \max_{k \ge 0} |w_1(k) - \overline{w}_1(k)| + |\overline{w}_2(k)| \\
&\le (2\underline{\gamma}_p^{-1} + \underline{\gamma}_w^{-1})(\delta)
\end{aligned}$$

Substituting these into (4.27) gives

$$|x(k) - \overline{x}(k)| \le \overline{\beta}\big((\underline{\gamma}_x^{-1} + \gamma_x^{-1})(\delta)\big) + \gamma_1\big((2\underline{\gamma}_p^{-1} + \underline{\gamma}_w^{-1})(\delta)\big) + \gamma_2\big((\underline{\gamma}_w^{-1} + \gamma_w^{-1})(\delta)\big)$$

Recall $\overline{\beta}(s) := \beta(s,0)$, which is a $\mathcal{K}$-function, and the right-hand side therefore defines a $\mathcal{K}$-function, so we can make $|x(k) - \overline{x}(k)|$ as small as desired for all $k > N$. This gives a bound for $|x(iN) - \hat{x}(iN + j|iN)|$ for all $i \ge 1$ and $j$ satisfying $0 \le j \le N-1$. Next we use the continuity of $f(\cdot)$ to make $|x(iN) - \hat{x}(iN|iN)|$ small for all $i \ge 0$. Finally we repeat the concatenation construction using the MHE sequences $N + j, 2N + j, 3N + j, \ldots$ for $j = 1, \ldots, N-1$ to make $|x(k) - \hat{x}(k)|$ as small as desired for all $k > N$, and part (b) of the robust GAS definition is established. ■

Satisfying the prior weighting *inequality* (4.20) is computationally less complex than satisfying the equality in the MHE arrival cost recursion (Definition 4.16), as we show subsequently in the constrained, linear case. But for the general nonlinear case, ensuring satisfaction of even (4.20) remains a key technical challenge for MHE research.

### 4.3.3 Constrained Estimation

Constraints in estimation may be a useful way to add information to the estimation problem. We may wish to enforce physically known facts such as: concentrations of impurities, although small, must be nonnegative, fluxes of mass and energy must have the correct sign given temperature and concentration gradients, and so on. Unlike the regulator, the estimator has no way to enforce these constraints on the *system*. Therefore, it is important that any constraints imposed on the estimator are satisfied by the system generating the measurements. Otherwise we may prevent convergence of the estimated state to the system state. For this reason, care should be used in adding constraints to estimation problems.

Because we have posed state estimation as an optimization problem, it is straightforward to add constraints to the formulation. We assume that the system generating the data satisfy the following constraints.

**Assumption 4.21** (Estimator constraint sets)**.**

(a) For all $k \in \mathbb{I}_{\ge 0}$, the sets $\mathbb{W}_k$, $\mathbb{X}_k$, and $\mathbb{V}_k$ are nonempty and closed, and $\mathbb{W}_k$ and $\mathbb{V}_k$ contain the origin.

(b) For all $k \in \mathbb{I}_{\geq 0}$, the disturbances and state satisfy

$$x(k) \in \mathbb{X}_k \qquad w(k) \in \mathbb{W}_k \qquad v(k) \in \mathbb{V}_k$$

(c) The prior satisfies $\overline{x}_0 \in \mathbb{X}_0$.

**Constrained full information.** The constrained full information estimation objective function is

$$V_T(\chi(0), \boldsymbol{w}) = \ell_x(\chi(0) - \overline{x}_0) + \sum_{i=0}^{T-1} \ell_i(w(i), v(i)) \tag{4.28}$$

subject to

$$\begin{aligned} \chi^+ &= f(\chi, w) \qquad y = h(\chi) + v \\ \chi(i) \in \mathbb{X}_i \qquad w(i) \in \mathbb{W}_i \qquad v(i) \in \mathbb{V}_i \qquad i \in \mathbb{I}_{0:T-1} \end{aligned}$$

The constrained full information problem is

$$\min_{\chi(0), \boldsymbol{w}} V_T(\chi(0), \boldsymbol{w}) \tag{4.29}$$

**Theorem 4.22** (Robust GAS of constrained full information)**.** *Consider an i-IOSS (detectable) system and measurement sequence generated by* (4.1) *with constrained, convergent disturbances satisfying Assumptions 4.3 and 4.21. The constrained full information estimator* (4.29) *with stage cost satisfying Assumption 4.5 is robustly GAS.*

**Constrained MHE.** The constrained moving horizon estimation objective function is

$$\hat{V}_T(\chi(T-N), \boldsymbol{w}) = \Gamma_{T-N}(\chi(T-N)) + \sum_{i=T-N}^{T-1} \ell_i(w(i), v(i)) \tag{4.30}$$

subject to

$$\begin{aligned} \chi^+ &= f(\chi, w) \qquad y = h(\chi) + v \\ \chi(i) \in \mathbb{X}_i \qquad w(i) \in \mathbb{W}_i \qquad v(i) \in \mathbb{V}_i \qquad i \in \mathbb{I}_{T-N:T-1} \end{aligned}$$

The constrained MHE is given by the solution to the following problem

$$\min_{\chi(T-N), \boldsymbol{w}} \hat{V}_T(\chi(T-N), \boldsymbol{w}) \tag{4.31}$$

**Theorem 4.23** (Robust GAS of constrained MHE)**.** *Consider an MHE detectable system and measurement sequence generated by* (4.1) *with convergent, constrained disturbances satisfying Assumptions 4.3 and 4.21. The constrained MHE estimator* (4.31) *using the prior weighting function* $\Gamma_k(\cdot)$ *satisfying Assumption 4.17 and stage cost satisfying Assumption 4.5 is robustly GAS.*

Because the *system* satisfies the state and disturbance constraints due to Assumption 4.21, both full information and MHE optimization problems are feasible at all times. Therefore the proofs of Theorems 4.22 and 4.23 closely follow the proofs of their respective unconstrained versions, Theorems 4.8 and 4.20, and are omitted.

### 4.3.4 Smoothing and Filtering Update

We next focus on *constrained linear systems*

$$x^+ = Ax + Gw \qquad y = Cx + v \tag{4.32}$$

We proceed to strengthen several results of the previous sections for this special case. First, the i-IOSS assumption of full information estimation and the MHE detectability assumption both reduce to the assumption that $(A, C)$ *is detectable* in this case. We usually choose a constant quadratic function for the estimator stage cost for all $i \in \mathbb{I}_{\geq 0}$

$$\ell_i(w, v) = (1/2)(|w|^2_{Q^{-1}} + |v|^2_{R^{-1}}) \qquad Q, R > 0 \tag{4.33}$$

In the unconstrained linear problem, we can of course find the full information arrival cost exactly; it is

$$Z_k(z) = V_k^0 + (1/2)\,|z - \hat{x}(k)|_{(P^-(k))^{-1}} \qquad k \geq 0$$

in which $P^-(k)$ satisfies the recursion (4.12) and $\hat{x}(k)$ is the full information estimate at time $k$. We use this quadratic function for the MHE prior weighting.

**Assumption 4.24** (Prior weighting for linear system)**.**

$$\Gamma_k(z) = \hat{V}_k^0 + (1/2)\,|z - \hat{x}(k)|_{(P^-(k))^{-1}} \qquad k > N \tag{4.34}$$

in which $\hat{V}_k^0$ is the optimal MHE cost at time $k$.

Because the unconstrained arrival cost is available, we usually choose it to be the prior weighting in MHE, $\Gamma_k(\cdot) = Z_k(\cdot)$, $k \geq 0$. This choice

implies robust GAS of the MHE estimator also for the *constrained case* as we next demonstrate. To ensure the form of the estimation problem to be solved online is a quadratic program, we specialize the constraint sets to be polyhedral regions.

**Assumption 4.25** (Polyhedral constraint sets)**.** For all $k \in \mathbb{I}_{\geq 0}$, the sets $\mathbb{W}_k$, $\mathbb{X}_k$, and $\mathbb{V}_k$ in Assumption 4.21 are nonempty, closed polyhedral regions containing the origin.

**Corollary 4.26** (Robust GAS of constrained MHE)**.** *Consider a detectable linear system and measurement sequence generated by* (4.32) *with convergent, constrained disturbances satisfying Assumptions 4.3 and 4.25. The constrained MHE estimator* (4.31) *using prior weighting function satisfying* (4.34) *and stage cost satisfying* (4.33) *is robustly GAS.*

This corollary follows as a special case of Theorem 4.23.

The MHE approach discussed to this point uses at all time $T > N$ the MHE estimate $\hat{x}(T-N)$ and prior weighting function $\Gamma_{T-N}(\cdot)$ derived from the unconstrained arrival cost as shown in (4.34). We call this approach a "filtering update" because the prior weight at time $T$ is derived from the solution of the MHE "filtering problem" at time $T-N$, i.e., the estimate of $\hat{x}(T-N) := \hat{x}(T-N|T-N)$ given measurements up to time $T-N-1$. For implementation, this choice requires storage of a window of $N$ prior filtering estimates to be used in the prior weighting functions as time progresses.

Next we describe a "smoothing update" that can be used instead. In the smoothing update we wish to use $\hat{x}(T-N|T-1)$ (instead of $\hat{x}(T-N|T-N)$) for the prior and wish to find an appropriate prior weighting based on this choice. For the linear *unconstrained* problem we can find an exact prior weighting that gives an equivalence to the full information problem. When constraints are added to the problem, however, the smoothing update provides a different MHE than the filtering update. Like the filtering prior, the smoothing prior weighting does give an underbound for the constrained full information problem, and therefore maintains the excellent stability properties of MHE with the filtering update. As mentioned previously the unconstrained full information arrival cost is given by

$$Z_{T-N}(z) = V^0_{T-N} + (1/2)\,|z - \hat{x}(T-N)|^2_{(P^-(T-N))^{-1}} \qquad T > N \quad (4.35)$$

in which $\hat{x}(T-N)$ is the optimal estimate for the unconstrained full information problem. Next we consider using $\hat{x}(T-N|T-2)$ in place of

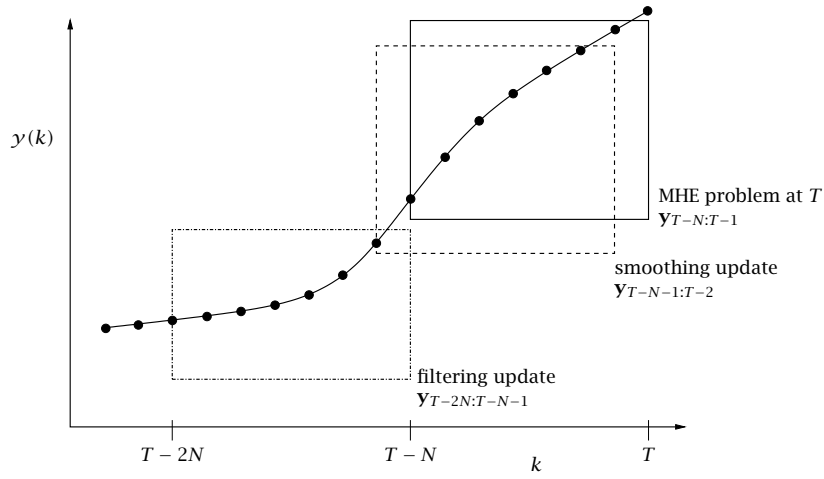


**Figure 4.3:** Smoothing update.

$\hat{x}(T-N) := \hat{x}(T-N|T-1)$. We might guess that the proper weight for this prior estimate would be the smoothed covariance $P(T-N|T-2)$ instead of $P^-(T-N) := P(T-N|T-1)$, and that guess is correct, but not complete. Notice that the smoothed prior $\hat{x}(T-N|T-2)$ is influenced by the measurements $\mathbf{y}_{0:T-2}$. But the sum of stage costs in the MHE problem at time $T$ depends on measurements $\mathbf{y}_{T-N:T-1}$, so we have to adjust the prior weighting so we do not double count the data $\mathbf{y}_{T-N:T-2}$. The correct prior weighting for the smoothing update has been derived by Rao, Rawlings, and Lee (2001), which we summarize next. The following notation is useful; for any square matrix $R$ and integer $k \geq 1$, define $\text{diag}_k(R)$ to be the following

$$\text{diag}_k(R) := \underbrace{\begin{bmatrix} R & & \\ & \ddots & \\ & & R \end{bmatrix}}_{k \text{ times}} \qquad \mathcal{O}_k = \begin{bmatrix} 0 & & & \\ C & & & \\ CA & C & & \\ \vdots & \vdots & \ddots & \\ CA^{k-2} & CA^{k-3} & \cdots & C \end{bmatrix}$$

$$W_k = \text{diag}_k(R) + \mathcal{O}_k(\text{diag}_k(Q))\mathcal{O}_k'$$

We require the smoothed covariance $P(T-N|T-2)$, which we can obtain from the following recursion (Rauch, Tung, and Striebel, 1965;

Bryson and Ho, 1975)

$$P(k|T) = P(k) +$$
$$P(k)A'(P^-(k+1))^{-1}\Big(P(k+1|T) - P^-(k+1)\Big)(P^-(k+1))^{-1}AP(k)$$

We iterate this equation backwards $N-1$ times starting from the known value $P(T-1|T-2) := P^-(T-1)$ to obtain $P(T-N|T-2)$. The smoothing arrival cost is then given by

$$\widetilde{Z}_{T-N}(z) = \hat{V}^0_{T-1} + (1/2)\,|z - \hat{x}(T-N|T-2)|^2_{(P(T-N|T-2))^{-1}} - (1/2)\,|\mathbf{y}_{T-N:T-2} - \mathcal{O}_{N-1}z|^2_{(W_{N-1})^{-1}} \qquad T > N$$

See Rao et al. (2001) and Rao (2000, pp.80–93) for a derivation that shows $\widetilde{Z}_T(\cdot) = Z_T(\cdot)$ for $T > N$.[4] Examining this alternative expression for arrival cost we see that the second term accounts for the use of the smoothed covariance and the smoothed estimate, and the third term subtracts the effect of the measurements that have been double counted in the MHE objective as well as the smoothed prior estimate. Setting the prior weighting $\Gamma_{T-N}(\cdot) = Z_{T-N}(\cdot)$ from (4.35) or $\Gamma_{T-N}(\cdot) = \widetilde{Z}_{T-N}(\cdot)$ from (4.4) give the same results as the Kalman filter for the unconstrained linear problem. But the two arrival costs are approximations of the true arrival cost and give different results once constraints are added to the problem or we use a nonlinear system model. Since the unconstrained arrival cost $\widetilde{Z}_k(\cdot)$ is also an underbound for the constrained arrival cost, MHE based on the smoothing update also provides a robustly GAS estimator for constrained linear systems satisfying the conditions of Theorem 4.26.

## 4.4 Extended Kalman Filtering

The extended Kalman filter (EKF) generates estimates for *nonlinear* systems by first linearizing the nonlinear system, and then applying the linear Kalman filter equations to the linearized system. The approach can be summarized in a recursion similar in structure to the Kalman filter (Stengel, 1994, pp.387–388)

$$\begin{aligned}\hat{x}^-(k+1) &= f(\hat{x}(k), 0)\\ P^-(k+1) &= \overline{A}(k)P(k)\overline{A}(k)' + \overline{G}(k)Q\overline{G}(k)'\\ \hat{x}^-(0) &= \overline{x}_0 \qquad P^-(0) = Q_0\end{aligned}$$

[4] Note that Rao et al. (2001) and Rao (2000) contain some minor typos in the smoothed covariance recursion and the formula for $W_k$.

The mean and covariance after measurement are given by

$$\begin{aligned}\hat{x}(k) &= \hat{x}^-(k) + L(k)(y(k) - h(\hat{x}^-(k)))\\ L(k) &= P^-(k)\overline{C}(k)'(R + \overline{C}(k)P^-(k)\overline{C}(k)')^{-1}\\ P(k) &= P^-(k) - L(k)\overline{C}(k)P^-(k)\end{aligned}$$

in which the following linearizations are made

$$\overline{A}(k) = \left.\frac{\partial f(x,w)}{\partial x}\right|_{(\hat{x}(k),0)} \quad \overline{G}(k) = \left.\frac{\partial f(x,w)}{\partial w}\right|_{(\hat{x}(k),0)} \quad \overline{C}(k) = \left.\frac{\partial h(x)}{\partial x}\right|_{\hat{x}^-(k)}$$

The densities of $w, v$ and $x_0$ are assumed to be normal. Many variations on this theme have been proposed, such as the iterated EKF and the second-order EKF (Gelb, 1974, 190–192). Of the nonlinear filtering methods, the EKF method has received the most attention due to its relative simplicity and demonstrated effectiveness in handling some nonlinear systems. Examples of implementations include estimation for the production of silicon/germanium alloy films (Middlebrooks and Rawlings, 2006), polymerization reactions (Prasad, Schley, Russo, and Bequette, 2002), and fermentation processes (Gudi, Shah, and Gray, 1994). The EKF is at best an *ad hoc* solution to a difficult problem, however, and hence there exist many pitfalls to the practical implementation of EKFs (see, for example, (Wilson, Agarwal, and Rippin, 1998)). These problems include the inability to accurately incorporate physical state constraints and the naive use of linearization of the nonlinear model.

Until recently, few properties regarding the stability and convergence of the EKF have been established. Recent research shows bounded estimation error and exponential convergence for the continuous and discrete EKF forms given observability, small initial estimation error, small noise terms, and no model error (Reif, Günther, Yaz, and Unbehauen, 1999; Reif and Unbehauen, 1999; Reif, Günther, Yaz, and Unbehauen, 2000). Depending on the system, however, the bounds on initial estimation error and noise terms may be unrealistic. Also, initial estimation error may result in bounded estimate error but not exponential convergence, as illustrated by Chaves and Sontag (2002).

Julier and Uhlmann (2004a) summarize the status of the EKF as follows.

> The extended Kalman filter is probably the most widely used estimation algorithm for nonlinear systems. However, more than 35 years of experience in the estimation community

> has shown that it is difficult to implement, difficult to tune, and only reliable for systems that are almost linear on the time scale of the updates.

We seem to be making a transition from a previous era in which new approaches to nonlinear filtering were criticized as overly complex because "the EKF works," to a new era in which researchers are demonstrating ever simpler examples in which the EKF fails completely. The unscented Kalman filter is one of the methods developed specifically to overcome the problems caused by the naive linearization used in the EKF.

## 4.5 Unscented Kalman Filtering

The linearization of the nonlinear model at the current state estimate may not accurately represent the dynamics of the nonlinear system behavior even for one sample time. In the EKF prediction step, the mean propagates through the full nonlinear model, but the covariance propagates through the linearization. The resulting error is sufficient to throw off the correction step and the filter can diverge even with a perfect model. The unscented Kalman filter (UKF) avoids this linearization at a single point by sampling the nonlinear response at several points. The points are called sigma points, and their locations and weights are chosen to satisfy the given starting mean and covariance (Julier and Uhlmann, 2004a,b).[5] Given $\hat{x}$ and $P$, choose sample points, $z^i$, and weights, $w^i$, such that

$$\hat{x} = \sum_i w^i z^i \qquad P = \sum_i w^i (z^i - \hat{x})(z^i - \hat{x})'$$

Similarly, given $w \sim N(0, Q)$ and $v \sim N(0, R)$, choose sample points $n^i$ for $w$ and $m^i$ for $v$. Each of the sigma points is propagated forward at each sample time using the nonlinear system model. The locations and weights of the transformed points then update the mean and covariance

$$\begin{aligned} z^i(k+1) &= f(z^i(k), n^i(k)) \\ \eta^i &= h(z^i) + m^i \quad \text{all } i \end{aligned}$$

[5] Note that this idea is fundamentally different than the idea of particle filtering, which is discussed subsequently. The sigma points are chosen deterministically, for example as points on a selected covariance contour ellipse or a simplex. The particle filtering points are chosen by random sampling.

From these we compute the forecast step

$$\hat{x}^- = \sum_i w^i z^i \qquad \hat{y}^- = \sum_i w^i \eta^i$$
$$P^- = \sum_i w^i (z^i - \hat{x}^-)(z^i - \hat{x}^-)'$$

After measurement, the EKF correction step is applied after first expressing this step in terms of the covariances of the innovation and state prediction. The output error is given as $\tilde{y} := y - \hat{y}^-$. We next rewrite the Kalman filter update as

$$\hat{x} = \hat{x}^- + L(y - \hat{y}^-)$$
$$L = \underbrace{\mathcal{E}((x - \hat{x}^-)\tilde{y}')}_{P^-C'} \underbrace{\mathcal{E}(\tilde{y}\tilde{y}')^{-1}}_{(R+CP^-C')^{-1}}$$
$$P = P^- - L \underbrace{\mathcal{E}((x - \hat{x}^-)\tilde{y}')'}_{CP^-}$$

in which we approximate the two expectations with the sigma point samples

$$\mathcal{E}((x - \hat{x}^-)\tilde{y}') \approx \sum_i w^i (z^i - \hat{x}^-)(\eta^i - \hat{y}^-)'$$
$$\mathcal{E}(\tilde{y}\tilde{y}') \approx \sum_i w^i (\eta^i - \hat{y}^-)(\eta^i - \hat{y}^-)'$$

See Julier, Uhlmann, and Durrant-Whyte (2000); Julier and Uhlmann (2004a); van der Merwe, Doucet, de Freitas, and Wan (2000) for more details on the algorithm. An added benefit of the UKF approach is that the partial derivatives $\partial f(x,w)/\partial x, \partial h(x)/\partial x$ are not required. See also (Nørgaard, Poulsen, and Ravn, 2000) for other derivative-free nonlinear filters of comparable accuracy to the UKF. See (Lefebvre, Bruyninckx, and De Schutter, 2002; Julier and Uhlmann, 2002) for an interpretation of the UKF as a use of statistical linear regression.

The UKF has been tested in a variety of simulation examples taken from different application fields including aircraft attitude estimation, tracking and ballistics, and communication systems. In the chemical process control field, Romanenko and coworkers have compared the EKF and UKF on a strongly nonlinear exothermic chemical reactor (Romanenko and Castro, 2004), and a pH system (Romanenko, Santos, and Afonso, 2004). The reactor has nonlinear dynamics and a linear measurement model, i.e., a subset of states is measured. In this case, the

UKF performs significantly better than the EKF when the process noise is large. The pH system has linear dynamics but a strongly nonlinear measurement, i.e., the pH measurement. In this case, the authors show a modest improvement in the UKF over the EKF.

## 4.6 Interlude: EKF, UKF, and MHE Comparison

One nice feature enjoyed by the EKF and UKF formulations is the recursive update equations. One-step recursions are computationally efficient, which may be critical in online applications with short sample times. The MHE computational burden may be reduced by shortening the length of the moving horizon, $N$. But use of short horizons may produce inaccurate estimates, especially after an unmodeled disturbance. This unfortunate behavior is the result of the system nonlinearity. As we saw in Sections 1.4.3–1.4.4, for *linear systems*, the full information problem and the MHE problem are identical to a one-step recursion using the appropriate state penalty coming from the filtering Riccati equation. Losing the equivalence of a one-step recursion to full information or a finite moving horizon problem brings into question whether the one-step recursion can provide equivalent estimator performance. We show in the following example that the EKF and the UKF do not provide estimator performance comparable to MHE.

**Example 4.27: EKF and UKF**

Consider the following set of reversible reactions taking place in a well-stirred, isothermal, gas-phase batch reactor

$$\mathrm{A} \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} \mathrm{B} + \mathrm{C} \qquad 2\mathrm{B} \underset{k_{-2}}{\overset{k_2}{\rightleftharpoons}} \mathrm{C}$$

The material balance for the reactor is

$$\frac{d}{dt}\begin{bmatrix} c_A \\ c_B \\ c_C \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} k_1 c_A - k_{-1} c_B c_C \\ k_2 c_B^2 - k_{-2} c_C \end{bmatrix}$$

$$\frac{dx}{dt} = f_c(x)$$

with states and measurement

$$x = \begin{bmatrix} c_A & c_B & c_C \end{bmatrix}' \qquad y = RT \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} x$$

in which $c_j$ denotes the concentration of species $j$ in mol/L, $R$ is the gas constant, and $T$ is the reactor temperature in K. The measurement is the reactor pressure in atm, and we use the ideal gas law to model the pressure. The model is nonlinear because of the two second-order reactions. We model the system plus disturbances with the following discrete time model

$$x^+ = f(x) + w$$
$$y = Cx + v$$

in which $f$ is the solution of the ODEs over the sample time, $\Delta$, i.e, if $s(t, x_0)$ is the solution of $dx/dt = f_c(x)$ with initial condition $x(0) = x_0$ at $t = 0$, then $f(x) = s(\Delta, x)$. The state and measurement disturbances, $w$ and $v$, are assumed to be zero-mean independent normals with constant covariances $Q$ and $R$. The following parameter values are used in the simulations

$$RT = 32.84\,\text{mol} \cdot \text{atm/L}$$
$$\Delta = 0.25 \quad k_1 = 0.5 \quad k_{-1} = 0.05 \quad k_2 = 0.2 \quad k_{-2} = 0.01$$
$$C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} RT \qquad P(0) = (0.5)^2 I \qquad Q = (0.001)^2 I \qquad R = (0.25)^2$$
$$\overline{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} \qquad x(0) = \begin{bmatrix} 0.5 \\ 0.05 \\ 0 \end{bmatrix}$$

The prior density for the initial state, $N(\overline{x}(0), P(0))$, is deliberately chosen to poorly represent the actual initial state to model a large initial disturbance to the system. We wish to examine how the different estimators recover from this large unmodeled disturbance.

**Solution**

Figure 4.4 (top) shows a typical EKF performance for these conditions. Note that the EKF cannot reconstruct the state for this system and that the estimates converge to incorrect steady states displaying negative concentrations of A and B. For some realizations of the noise sequences, the EKF may converge to the correct steady state. Even for these cases, however, negative concentration estimates still occur during the transient, which correspond to physically impossible states. Figure 4.4 (bottom) presents typical results for the clipped EKF, in which negative values of the filtered estimates are set to zero. Note that although the estimates converge to the system states, this estimator gives pressure

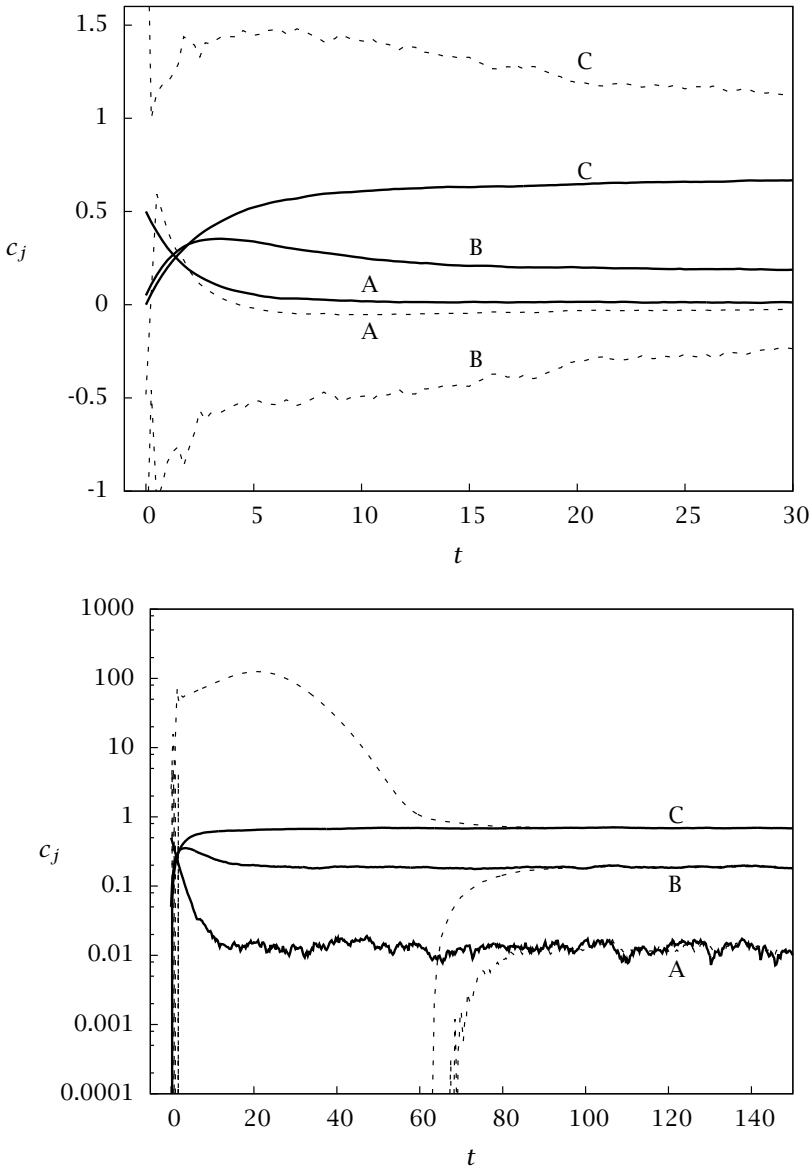


**Figure 4.4:** Evolution of the state (solid line) and EKF state estimate (dashed line). Top plot shows negative concentration estimates with the standard EKF. Bottom plot shows large estimate errors and slow convergence with the clipped EKF.

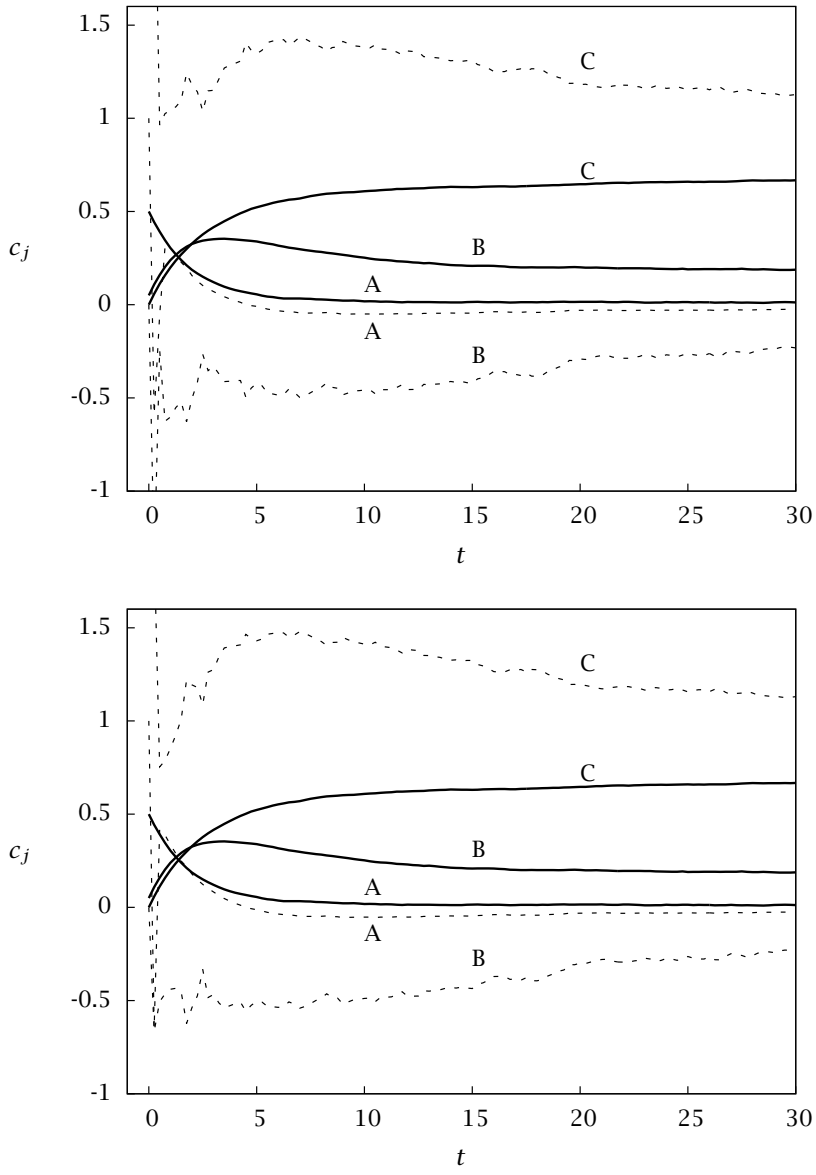


**Figure 4.5:** Evolution of the state (solid line) and UKF state estimate (dashed line). Top plot shows negative concentration estimates with the standard UKF. Bottom plot shows similar problems even if constraint scaling is applied.

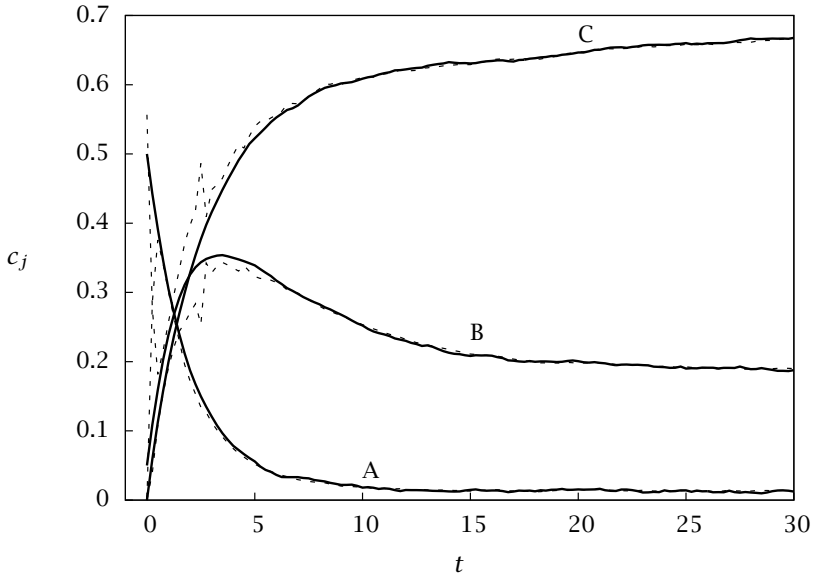


**Figure 4.6:** Evolution of the state (solid line) and MHE state estimate (dashed line).

estimates that are two orders of magnitude larger than the measured pressure before convergence is achieved.

The standard UKF achieves results similar to the EKF as shown in Figure 4.5 (top). Vachhani, Narasimhan, and Rengaswamy (2006) have proposed a modification to the UKF to handle constrained systems. In this approach, the sigma points that violate the constraints are scaled back to the feasible region boundaries and the sigma point weights are modified accordingly. If this constrained version of the UKF is applied to this case study, the estimates do not significantly improve as shown in Figure 4.5 (bottom). The UKF formulations used here are based on the algorithm presented by Vachhani et al. (2006, Sections 3 and 4) with the tuning parameter $\kappa$ set to $\kappa = 1$. Adjusting this parameter using other suggestions from the literature (Julier and Uhlmann, 1997; Qu and Hahn, 2009; Kandepu, Imsland, and Foss, 2008) and trial and error does not substantially improve the UKF estimator performance. Better performance is obtained in this example if the sigma points that violate the constraints are simply saturated rather than rescaled to the feasible

region boundaries. This form of clipping still does not prevent the occurrence of negative concentrations in this example, however. Negative concentration estimates are not avoided by either scaling or clipping of the sigma points. As a solution to this problem, the use of constrained optimization for the sigma points is proposed (Vachhani et al., 2006; Teixeira, Tôrres, Aguirre, and Bernstein, 2008). If one is willing to perform online optimization, however, MHE with a short horizon is likely to provide more accurate estimates at similar computational cost compared to approaches based on optimizing the locations of the sigma points.

Finally, Figure 4.6 presents typical results of applying constrained MHE to this example. For this simulation we choose $N = 10$ and the smoothing update for the arrival cost approximation. Note that MHE recovers well from the poor initial prior. Comparable performance is obtained if the filtering update is used instead of the smoothing update to approximate the arrival cost. The MHE estimates are also insensitive to the choice of horizon length $N$ for this example. □

The EKF, UKF, and all one-step recursive estimation methods, suffer from the "short horizon syndrome" by *design*. One can try to reduce the harmful effects of a short horizon through tuning various other parameters in the estimator, but the basic problem remains. Large initial state errors lead to inaccurate estimation and potential estimator divergence. The one-step recursions such as the EKF and UKF can be viewed as one extreme in the choice between speed and accuracy in that only a single measurement is considered at each sample. That is similar to an MHE problem in which the user chooses $N = 1$. Situations in which $N = 1$ lead to poor MHE performance often lead to unreliable EKF and UKF performance as well.

## 4.7 Particle Filtering

Particle filtering is a different approach to the state estimation problem in which statistical sampling is used to approximate the evolution of the conditional density of the state given measurements (Handschin and Mayne, 1969). This method also handles nonlinear dynamic models and can address nonnormally distributed random disturbances to the state and measurement.

**Sampled density.** Consider a smooth probability density, $p(x)$. In particle filtering we find it convenient to represent this smooth density

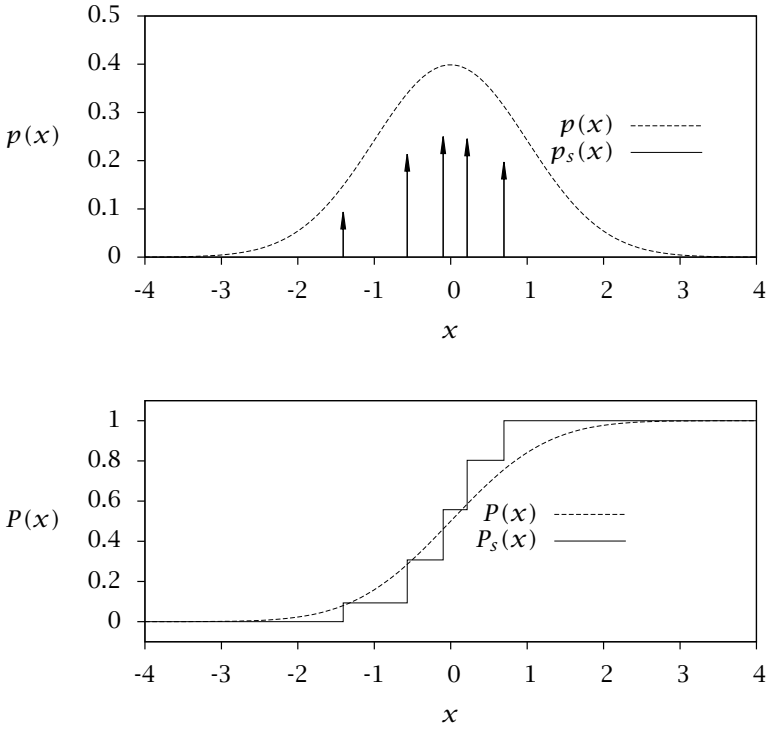


**Figure 4.7:** Top: exact density $p(x)$ and a sampled density $p_s(x)$ with five samples for $\xi \sim N(0,1)$. Bottom: corresponding exact $P(x)$ and sampled $P_s(x)$ cumulative distributions.

as a weighted, sampled density, $p_s(x)$

$$p(x) \approx p_s(x) := \sum_{i=1}^{s} w_i \delta(x - x_i)$$

in which $x_i, i = 1, \ldots s$ are the samples, $w_i$ are the weights. As an example, the top of Figure 4.7 displays a normally distributed scalar random variable represented by a sampled density with five samples. The sampled density is a series of impulses at the sample locations $x_i$. In this example, the weights $w_i$ are the values of $p(x_i)$, normalized to sum to unity. It may seem strange to represent a well-behaved function like $p(x)$ with such a "rough" function like $p_s(x)$, but we will

see the advantages shortly. Sometimes we may wish to study convergence of a sampled density to the original density as the number of samples becomes large. To define convergence of this representation of the probability distribution, we refer to the corresponding cumulative distribution rather than the density. From integration, the sampled cumulative distribution is

$$P_s(x) = \sum_{i \in \mathbb{I}_x} w_i \qquad \mathbb{I}_x = \{i | x_i \leq x\}$$

The bottom of Figure 4.7 shows the corresponding cumulative sampled distribution for the sampled density with five samples. The cumulative sampled distribution is a staircase function with steps of size $w_i$ at the sample locations $x_i$. We can then measure convergence of $P_s(x)$ to $P(x)$ as $s \to \infty$ in any convenient function norm. We delay further discussion of convergence until Section 4.7.2 in which we present some of the methods for choosing the samples and the weights.

In the sequel, we mostly drop the subscript $s$ on sampled densities and cumulative distributions when it is clear from context that we are referring to this type of representation of a probability distribution. We can conveniently calculate the expectation of any function of a random variable having a sampled density by direct integration to obtain

$$\begin{aligned} \mathcal{E}(f(\xi)) &= \int p_s(x) f(x) dx \\ &= \int \sum_i w_i \delta(x - x_i) f(x) dx \\ &= \sum_i w_i f(x_i) \end{aligned}$$

For example, we often wish to evaluate the mean of the sampled density, which is

$$\mathcal{E}(\xi) = \sum_i w_i x_i$$

The convenience of integrating the sampled density is one of its main attractive features. If we create a new function (not necessarily a density) by multiplication of $p(x)$ by another function $g(x)$

$$\overline{p}(x) = g(x) p(x)$$

we can easily obtain the sampled function $\overline{p}$. We simply adjust the

weights and leave the samples where they are

$$\begin{aligned}\overline{p}(x) &= g(x)p(x)\\ &= \sum_i g(x)w_i\delta(x-x_i)\\ &= \sum_i w_i g(x_i)\delta(x-x_i)\\ \overline{p}(x) &= \sum_i \overline{w}_i\delta(x-x_i) \qquad \overline{w}_i = w_i g(x_i)\end{aligned} \tag{4.36}$$

### 4.7.1 The Sampled Density of a Transformed Random Variable

Given the random variable $\xi$, assume we have a sampled density for its density $p_\xi(x)$

$$p_\xi(x) = \sum_{i=1}^{s} w_i\delta(x-x_i)$$

Define a new random variable $\eta$ by an invertible, possibly nonlinear transformation

$$\eta = f(\xi) \qquad \xi = f^{-1}(\eta)$$

We wish to find a sampled density for the random variable $\eta$, $p_\eta(y)$. Denote the sampled density for $\eta$ as

$$p_\eta(y) = \sum_{i=1}^{s} \overline{w}_i\delta(y-y_i)$$

We wish to find formulas for $\overline{w}_i$ and $y_i$ in terms of $w_i, x_i$ and $f$. We proceed as in the development of equation (A.30) in Appendix A. We wish to have an equivalence for every function $g(x)$

$$\begin{aligned}\mathcal{E}_{p_\xi}(g(\xi)) &= \mathcal{E}_{p_\eta}(g(f^{-1}(\eta)))\\ \int p_\xi(x)g(x)dx &= \int p_\eta(y)g(f^{-1}(y))dy \qquad \text{for all } g(\cdot)\end{aligned}$$

Using the sampled densities on both sides of the equation

$$\sum_i w_i g(x_i) = \sum_i \overline{w}_i g(f^{-1}(y_i))$$

One solution to this equation that holds for every $g$ is the simple choice

$$y_i = f(x_i) \qquad \overline{w}_i = w_i \tag{4.37}$$

We see that for the transformed sampled density, we transform the samples and use the weights of the original density.

**Example 4.28: Sampled density of the lognormal**

The random variable $\eta$ is distributed as a lognormal if its logarithm is distributed as a normal. Let $\xi \sim N(0, P)$ and consider the transformation

$$\eta = e^{\xi} \qquad \xi = \log(\eta) \quad \eta > 0$$

Represent $p_\xi$ as a sampled density, use (4.37) to find a sampled density of $p_\eta$, and plot histograms of the two sampled densities. Compare the sampled density of $p_\eta$ to the lognormal density. The two densities are given by

$$p_\xi(x) = \frac{1}{\sqrt{2\pi P}}\, e^{-x^2/2P}$$
$$p_\eta(y) = \frac{1}{y\sqrt{2\pi P}}\, e^{-\log^2(y)/2P}$$

**Solution**

First we take samples $x_i$ from $N(0, 1)$ for $\xi$. Figure 4.8 shows the histogram of the sampled density for 5000 samples. Next we compute $y_i = e_i^x$ to generate the samples of $\eta$. The histogram of this sampled density is shown in Figure 4.9. Notice the good agreement between the sampled density and the lognormal density, which is shown as the continuous curve in Figure 4.9. □

**Noninvertible transformations.** Next consider $\eta$ to be a noninvertible transformation of $\xi$

$$\eta = f(\xi) \qquad f \text{ not invertible}$$

Let $\xi$'s sampled density be given by $\{x_i, w_i\}$. The sampled density $\{f(x_i), w_i\}$ remains a valid sampled density for $\eta$, which we show next

$$\xi \sim \{x_i, w_i\} \qquad \eta \sim \{f(x_i), w_i\}$$

We wish to show that

$$\mathcal{E}_{p_\eta}(g(\eta)) = \mathcal{E}_{p_\xi}(g(f(\xi))) \qquad \text{for all } g(\cdot)$$

Taking the expectations

$$\int p_\eta(y) g(y) dy = \int p_\xi(x) g(f(x)) dx$$

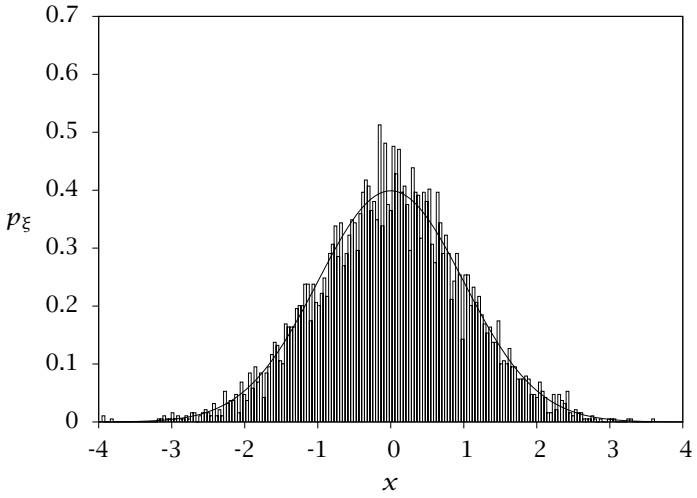


**Figure 4.8:** Sampled and exact probability densities for $\xi \sim N(0,1)$; 5000 samples.

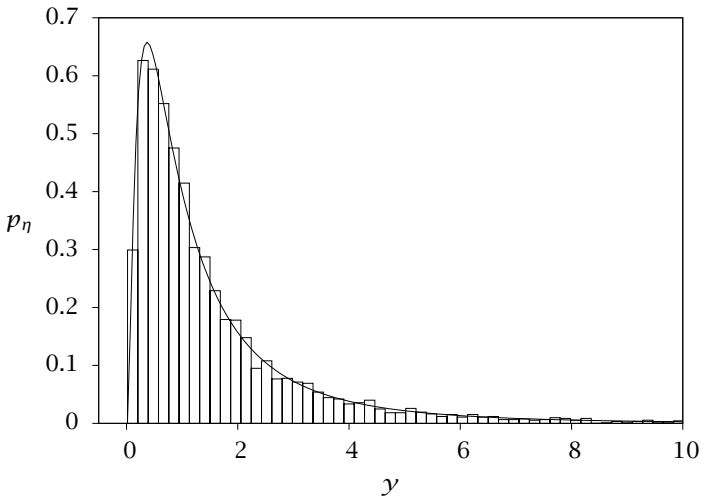


**Figure 4.9:** Sampled and exact probability densities for nonlinear transformation $\eta = e^{\xi}$; 5000 samples. The exact density of $\eta$ is the lognormal, shown as the continuous curve.

Letting $\eta$'s sampled density be $\{y_i, \overline{w}_i\}$, and using $\xi$'s sampled density give

$$\sum_{i=1}^{s} \overline{w}_i g(y_i) = \sum_{i=1}^{s} w_i g(f(x_i))$$

and setting $\overline{w}_i = w_i, y_i = f(x_i), i = 1, \ldots, s$ achieves equality for all $g(\cdot)$, and we have established the result. The difference between the noninvertible and invertible cases is that we do not have a method to obtain samples of $\xi$ from samples of $\eta$ in the noninvertible case. We can transform the sampled density in only one direction, from $p_\xi$ to $p_\eta$.

### 4.7.2 Sampling and Importance Sampling

Consider a random variable $\xi$ with a smooth probability density $p(x)$. Assume one is able to draw samples $x_i$ of $\xi$ with probability

$$p_{\text{sa}}(x_i) = p(x_i) \tag{4.38}$$

in which $p_{\text{sa}}(x_i)$ denotes the probability of drawing a sample with value $x_i$. In this case, if one draws $s$ samples, a sampled density for $\xi$ is given by

$$p_s(x) = \sum_i w_i \delta(x - x_i) \qquad w_i = 1/s, \quad i = 1, \ldots, s \tag{4.39}$$

and the weights are all equal to $1/s$.

**Convergence of sampled densities.** It is instructive to examine how a typical sampled density converges with sample size to the density from which the samples are drawn. Consider a set of $s$ samples. When drawing multiple samples of a density, we assume the samples are mutually independent

$$p_{\text{sa}}(x_1, x_2, \ldots, x_s) = p_{\text{sa}}(x_1) p_{\text{sa}}(x_2) \cdots p_{\text{sa}}(x_s)$$

We denote the cumulative distribution of the sampled density as

$$P_s(x; s) = \sum_{i \in \mathbb{I}_x} w_i \qquad \mathbb{I}_x = \{i | x_i \leq x\}$$

in which the second argument $s$ is included to indicate $P_s$'s dependence on the sample size. The value of $P_s$ is itself a random variable because it is determined by the sample values $x_i$ and weights $w_i$. We consider

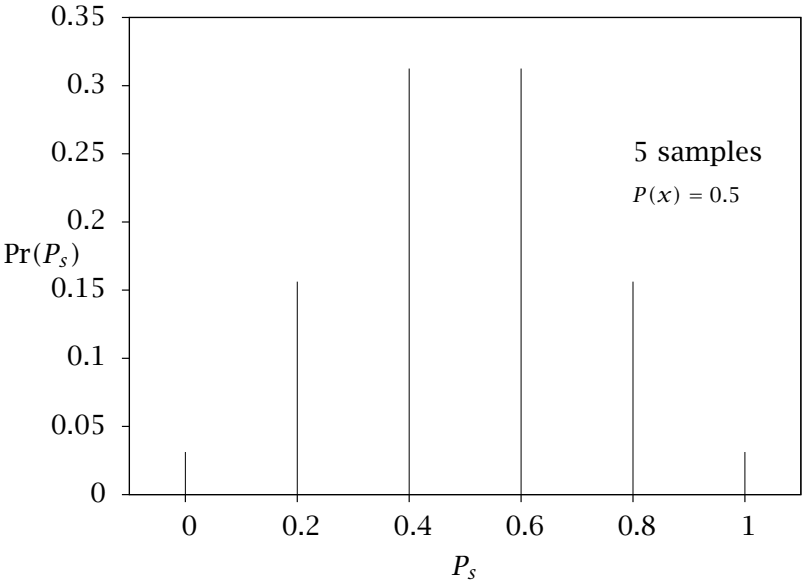


**Figure 4.10:** Probability density $\Pr(P_s(x;s))$ for $x$ corresponding to $P(x) = 0.5$ and $s = 5$ samples. The distribution is centered at correct value, $P(x)$, but the variance is large.

the case with equal sample weights $w_i = 1/s$ and study the $P_s$ values as a function of $s$ and scalar $x$. They take values in the range

$$P_s \in \left\{0, \frac{1}{s}, \ldots, \frac{s-1}{s}, 1\right\} \qquad s \geq 1 \quad -\infty < x < \infty$$

Given the sampling process we can readily evaluate the probability of $P_s$ over this set

$$\Pr(P_s(x;s)) = \begin{cases} \binom{s}{i} P(x)^i (1-P(x))^{s-i}, & P_s = \frac{i}{s}, \quad i = 0, \ldots, s \\ 0, & \text{otherwise} \end{cases}$$
$$-\infty < x < \infty \quad (4.40)$$

These probabilities are calculated as follows. For $P_s$ to take on value zero, for example, all of the samples $x_i$ must be greater than $x$. The probability that any $x_i$ is greater than $x$ is $1-P(x)$. Because the samples are mutually independent, the probability that all $s$ samples are greater than $x$ is $(1-P(x))^s$, which is the $i = 0$ entry of (4.40). Similarly, for $P_s$

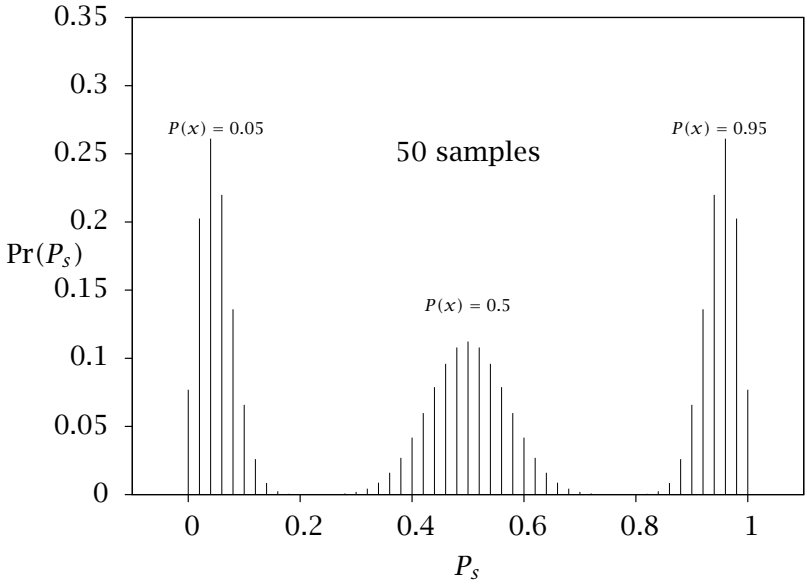


**Figure 4.11:** Probability density $\Pr(P_s(x;s))$ for three different $x$ corresponding to $P(x) = 0.05, 0.5, 0.95$ and $s = 50$ samples. The three distributions are centered at the correct values, $P(x)$, and the variance is much reduced compared to Figure 4.10.

to have value $i/s$, $i$ samples must be less than $x$ and $s-i$ samples must be greater than $x$. This probability is given by $\binom{s}{i}P(x)^i(1-P(x)^{s-i})$, in which $P(x)^i(1-P(x)^{s-i})$ is the probability of having a sample with $i$ values less than $x$ and $s-i$ values greater than $x$, and $\binom{s}{i}$ accounts for the number of ways such a sample can be drawn from a set of $s$ samples. Figure 4.10 shows the distribution of $P_s$ for a sample size $s = 5$ at the mean, $P(x) = 0.5$. Notice the maximum probability occurs near the value $P_s = P(x)$ but the probability distribution is fairly wide with only 5 samples. The number of samples is increased to 50 in Figure 4.11, and three different $x$ values are shown, at which $P(x) = 0.05, 0.5, 0.95$. The peak for each $P_s$ distribution is near the value $P(x)$, and the distribution is much narrower for 50 samples. The sampled density $P_s(x;s)$ becomes arbitrarily sharply distributed with value $P(x)$

as the sample size $s$ increases.

$$\lim_{s\to\infty} \Pr(P_s(x;s)) = \begin{cases} 1 & P_s = P(x) \\ 0 & \text{otherwise} \end{cases} \qquad -\infty < x < \infty$$

The convergence is often not uniform in $x$. Achieving a given variance in $P_s(x;s)$ generally requires larger sample sizes for $x$ values in the tails of the density $p(x)$ compared to the sample sizes required to achieve this variance for $x$ values in regions of high density. The nonuniform convergence is perhaps displayed more clearly in Figures 4.12 and 4.13. We have chosen the beta distribution for $P(x)$ and show the spread in the probability of $P_s$ for three $x$ values, corresponding to $P(x) = \{0.1, 0.5, 0.9\}$. Given $s = 25$ samples in Figure 4.12, we see a rather broad probability distribution for the sampled distribution $P_s(x)$. Turning up the number of samples to $s = 250$ gives the tighter probability distribution shown in Figure 4.13.

Finally, we present a classic sampling error distribution result due to Kolmogorov. The measure of sampling error is defined to be

$$D_s = \sup_x |P_s(x;s) - P(x)|$$

and we have the following result on the distribution of $D_s$ for large sample sizes.

**Theorem 4.29** (Kolmogoroff (1933))**.** [6] *Suppose that $P(x)$ is continuous. Then for every fixed $z \geq 0$ as $s \to \infty$*

$$\Pr\left(D_s \leq z s^{-1/2}\right) \to L(z) \tag{4.41}$$

*in which $L(z)$ is the cumulative distribution function given for $z > 0$ by*

$$L(z) = \sqrt{2\pi} z^{-1} \sum_{\nu=1}^{\infty} e^{-(2\nu-1)^2\pi^2/8z^2} \tag{4.42}$$

*and $L(z) = 0$ for $z \leq 0$.*

One of the significant features of results such as this one is that the limiting distribution is independent of the details of the sampled distribution $P(x)$ itself. Feller (1948) provides a proof of this theorem and discussion of this and other famous sampling error distribution results due to Smirnov (1939).

[6] Kolmogorov's theorem on sampling error was published in an Italian journal with the spelling Kolmogoroff.

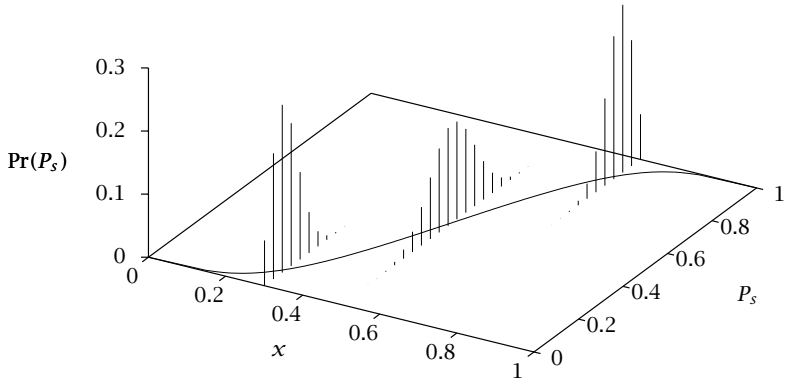


**Figure 4.12:** Probability density $\Pr(P_s(x;s))$ for $s = 25$ samples at three different $x$.

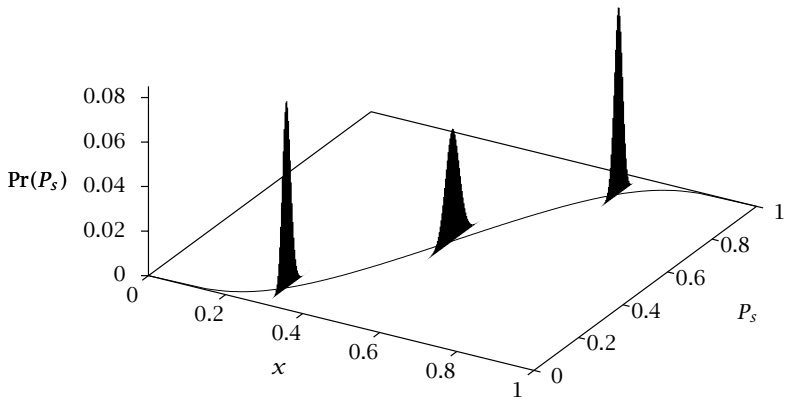


**Figure 4.13:** Probability density $\Pr(P_s(x;s))$ for $s = 250$ samples. Note the variance is much reduced compared to Figure 4.12.

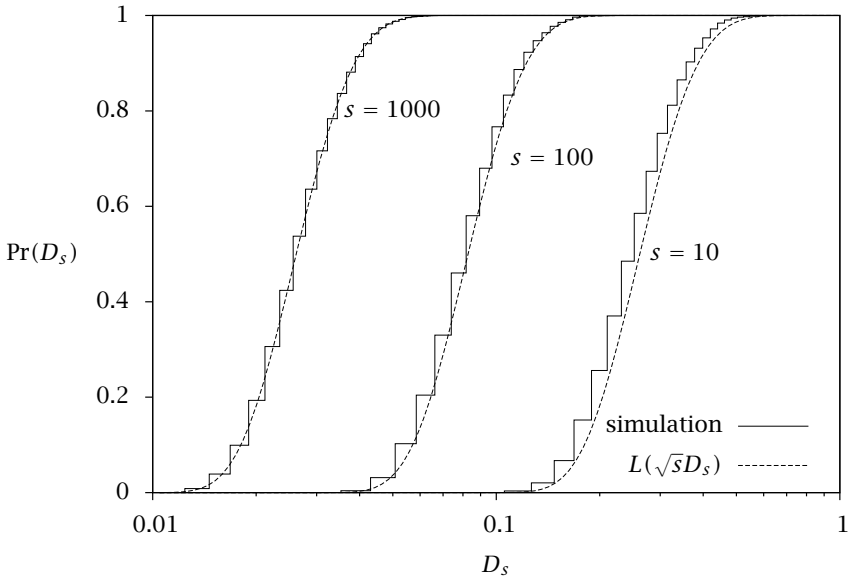


**Figure 4.14:** Cumulative distribution for the sampling error $\Pr(D_s)$ for three different sample sizes, $s = 10, 100, 1000$. Distribution from simulation using 5000 realizations (solid) and Kolmogorov limiting distribution (dashed).

**Example 4.30: Sampling error distribution for many samples**

Plot the actual and limiting distributions for $D_s$ for $s = 10, 100, 1000$ when sampling a normal distribution with unit variance. How close is the limiting sampling error distribution to the actual sampling error distribution for these three sample sizes?

**Solution**

Figure 4.14 displays the result using 5000 realizations of the sampling process to approximate the actual distribution of $D_s$. Notice that for the small sample size, we can see a slight difference between the Kolmogorov limiting distribution and the one obtained from simulation. This difference is not noticeable for samples sizes greater than $s = 100$. From the argument scaling given in (4.41) we see that the mean of the sampling error decreases by a factor of $\sqrt{10}$ for each factor of 10 increase in sample size (on the log scale, the distribution of $D_s$ is trans-

lated to the left by $\sqrt{10}$). Exercise 4.20 discusses this example further. □

**Unbiasedness of sampled densities.** A sampled density is *unbiased* if it possesses the following property

$$\mathcal{E}_{\text{sa}}(P_s(x;s)) = P(x) \qquad 1 \le s, \quad -\infty < x < \infty$$

in which the expectation is taken over the probability density of $P_s$ considered as a random variable as discussed previously. As we discuss subsequently, some sampling procedures are unbiased for all $s$, while others are only asymptotically unbiased as $s$ becomes large. A convenient test for unbiasedness is the following

$$\mathcal{E}_{\text{sa}}\left(\int p_s(x)g(x)dx\right) = \int p(x)g(x)dx \qquad \text{for all } g(\cdot) \tag{4.43}$$

In other words, the *expectation over the sampling process* of integrals of any function $g$ with the sampled density should be equal to the integral of $g$ with the exact density. If the sampling process has the probability given by (4.38), we can verify (4.43) as follows

$$\begin{aligned}
\mathcal{E}_{\text{sa}}\left(\int p_s(x)g(x)dx\right) &= \mathcal{E}_{\text{sa}}\left(\sum_i w_i g(x_i)\right) \\
&= \int p_{\text{sa}}(x_1,\ldots,x_s)\sum_i w_i g(x_i)dx_1\cdots dx_s \\
&= \int p_{\text{sa}}(x_1)\cdots p_{\text{sa}}(x_s)\sum_i w_i g(x_i)dx_1\cdots dx_s \\
&= \int p(x_1)\cdots p(x_s)\sum_i w_i g(x_i)dx_1\cdots dx_s \\
&= \frac{1}{s}\sum_i \int p(x_i)g(x_i)dx_i \prod_{j\neq i}\int p(x_j)dx_j \\
&= \frac{1}{s}\sum_i \int p(x_i)g(x_i)dx_i \\
\mathcal{E}_{\text{sa}}\left(\int p_s(x)g(x)dx\right) &= \int p(x)g(x)dx
\end{aligned}$$

**Example 4.31: Sampling independent random variables**

Consider two independent random variables $\xi$, $\eta$, whose probability density satisfies

$$p_{\xi,\eta}(x,y) = p_\xi(x)p_\eta(y)$$

and assume we have samples of the two marginals

$$\xi \sim \{x_i, w_{xi}\} \qquad w_{xi} = 1/s_x, \quad i = 1, \ldots, s_x$$
$$\eta \sim \{y_j, w_{yj}\} \qquad w_{yj} = 1/s_y, \quad j = 1, \ldots, s_y$$

We have many valid options for creating samples of the joint density. Here are three useful ones.

(a) Show the following is a valid sample of the joint density

$$\{(x_i, y_j), w_{ij}\} \qquad w_{ij} = 1/(s_x s_y), \quad i = 1, \ldots, s_x, \quad j = 1, \ldots, s_y$$

Notice we have $s_x s_y$ total samples of the joint density.

(b) If $s_x = s_y = s$, show the following is a valid sample of the joint density

$$\{(x_i, y_i), w_i\} \qquad w_i = 1/s, \quad i = 1, \ldots, s$$

Notice we have $s$ total samples of the joint density unlike the previous case in which we would have $s^2$ samples.

(c) If we have available (or select) only a single sample of $\xi$'s marginal, $s_x = 1$ and $s_y = s$ samples of $\eta$'s marginal, show the following is a valid sample of the joint density

$$\{(x_1, y_i), w_{yi}\} \qquad w_{yi} = 1/s, \quad i = 1, \ldots, s$$

Here we have generated again $s$ samples of the joint density, but we have allowed unequal numbers of samples of the two marginals.

**Solution**

Because the two random variables are independent, the probability of drawing a sample with values $(x_i, y_j)$ is given by

$$p_{sa}(x_i, y_j) = p_{sa}(x_i) p_{\text{sa}}(y_j) = p_\xi(x_i) p_\eta(y_j) = p_{\xi,\eta}(x_i, y_j)$$

Denote the samples as $z_k = (x_{i(k)}, y_{j(k)})$. We have for all three choices

$$p_{\text{sa}}(z_k) = p_{\xi,\eta}(z_k) \qquad k = 1, \ldots, s \tag{4.44}$$

(a) For this case,

$$i(k) = \text{mod}(k-1, s_x) + 1 \qquad j(k) = \text{ceil}(k/s_x)$$
$$w_k = \frac{1}{s_x s_y}, \quad k = 1, \ldots, s_x s_y$$

in which $\text{ceil}(x)$ is the smallest integer not less than $x$.

(b) For this case

$$i(k) = k \qquad j(k) = k \qquad w_k = 1/s, \quad k = 1, \ldots, s$$

(c) For this case

$$i(k) = 1 \qquad j(k) = k \qquad w_k = 1/s, \quad k = 1, \ldots, s$$

Because all three cases satisfy (4.44) and the weights are equal to each other in each case, these are all valid samples of the joint density. □

If we arrange the $s_x$ $\xi$ samples and $s_y$ $\eta$ samples in a rectangle, the first option takes all the points in the rectangle, the second option takes the diagonal (for a square), and the third option takes one edge of the rectangle. See Exercise 4.19 for taking a single point in the rectangle. In fact, any set of points in the rectangle is a valid sample of the joint density.

**Example 4.32: Sampling a conditional density**

The following result proves useful in the later discussion of particle filtering. Consider conditional densities satisfying the following property

$$p(a, b, c|d, e) = p(a|b, d)p(b, c|e) \tag{4.45}$$

We wish to draw samples of $p(a, b, c|d, e)$ and we proceed as follows. We draw samples of $p(b, c|e)$. Call these samples $(b_i, c_i), i = 1, \ldots, s$. Next we draw for each $i = 1, \ldots, s$, one sample of $p(a|b_i, d)$. Call these samples $a_i$. We assemble the $s$ samples $(a_i, b_i, c_i)$ and claim they are samples of the desired density $p(a, b, c|d, e)$ with uniform weights $w_i = 1/s$. Prove or disprove this claim.

**Solution**

The claim is true, and to prove it we need to establish that the probability of drawing a sample with value $(a_i, b_i, c_i)$ is equal to the desired density $p(a_i, b_i, c_i|d, e)$. We proceed as follows. From the definition of conditional density, we know

$$p_{\text{sa}}(a_i, b_i, c_i|d, e) = p_{\text{sa}}(a_i|b_i, c_i, d, e)p_{\text{sa}}(b_i, c_i|d, e)$$

For the selection of $a_i$ described previously, we know

$$p_{\text{sa}}(a_i|b_i, c_i, d, e) = p(a_i|b_i, d)$$

The values of $c_i$ and $e$ are irrelevant to the sampling procedure generating the $a_i$. For the $(b_i, c_i)$ samples, the sampling procedure gives

$$p_{\text{sa}}(b_i, c_i | d, e) = p(b_i, c_i | e)$$

and the value of $d$ is irrelevant to the procedure for generating the $(b_i, c_i)$ samples. Combining these results, we have for the $(a_i, b_i, c_i)$ samples

$$p_{\text{sa}}(a_i, b_i, c_i | d, e) = p(a_i | b_i, d) p(b_i, c_i | e)$$

Equation (4.45) then gives

$$p_{\text{sa}}(a_i, b_i, c_i | d, e) = p(a_i, b_i, c_i | d, e)$$

We conclude the sampling procedure is selecting $(a_i, b_i, c_i)$ samples with the desired probability, and as shown in (4.39), the weights are all equal to $1/s$ under this kind of sampling. □

**Importance sampling.** Consider next the case in which we have a smooth density $p(x)$ that is easy to *evaluate* but difficult to *sample* with probability given by (4.38). This situation is not unusual. In fact, it arises frequently in applications for the following reason. Many good algorithms are available for generating samples of the uniform density. One simple method to sample an arbitrary density for a scalar random variable is the following. First compute $P(x)$ from $p(x)$ by integration. Let $u_i$ be the samples of the uniform density on the interval $[0, 1]$. Then samples of $x_i$ of density $p(x)$ are given by

$$x_i = P^{-1}(u_i) \qquad u_i = P(x_i)$$

Figures 4.15 and 4.16 give a graphical display of this procedure. We briefly verify that the samples $x_i$ have the claimed density. We show that if $\mu$ is a uniform random variable and $\xi$ is defined by the invertible transformation given previously, $\mu = P(\xi)$, then $\xi$ has density $p_\xi(x) = dP(x)/dx$. From (A.30) we have

$$p_\xi(x) = p_\mu(P(x)) \left| \frac{dP(x)}{dx} \right|$$

Since $\mu$ is uniformly distributed, $p_\mu = 1$, and $dP(x)/dx \geq 0$, we have

$$p_\xi(x) = \frac{dP(x)}{dx}$$

and the samples have the desired density. But notice this procedure for generating samples of $p(x)$ uses $P(x)$, which requires integration,

as well as evaluating $P^{-1}(x)$, which generally requires solving nonlinear equations. Importance sampling is a method for sampling $p(x)$ without performing integration or solving nonlinear equations.

The following idea motivates importance sampling. Consider the random variable $\xi$ to be distributed with density $p$. Consider a new random variable $\eta$ to be distributed with density $q$

$$\xi \sim p(x) \qquad \eta \sim q(x)$$

The density $q(x)$, known as the importance function, is any density that can be readily sampled according to (4.38) and has the same support as $p$. Examples of such $q$ are uniforms for bounded intervals, lognormals and exponentials for semi-infinite intervals, and normals for infinite intervals. For any function $g(x)$, we have

$$\begin{aligned}
\mathcal{E}_p(g(\xi)) &= \int g(x)p(x)dx \\
&= \int \left[ g(x)\frac{p(x)}{q(x)} \right] q(x)dx \\
\mathcal{E}_p(g(\xi)) &= \mathcal{E}_q\left( g(\eta)\frac{p(\eta)}{q(\eta)} \right) \qquad \text{for all } g(\cdot)
\end{aligned}$$

When we can sample $p$ directly, we use for the sampled density

$$p_s = \left\{ x_i, \quad w_i = \frac{1}{s} \right\} \qquad p_{sa}(x_i) = p(x_i)$$

So when we cannot conveniently sample $p$ but can sample $q$, we use instead

$$\overline{p}_s = \left\{ x_i, \quad w_i = \frac{1}{s}\frac{p(x_i)}{q(x_i)} \right\} \qquad p_{is}(x_i) = q(x_i)$$

Given $s$ samples $x_i$ from $q(x)$, denote the sampled density of $q$ as $q_s$, and we have defined the importance-sampled density $\overline{p}_s(x)$ as

$$\overline{p}_s(x) = q_s(x)\frac{p(x)}{q(x)}$$

We next show that $\overline{p}_s(x)$ converges to $p(x)$ as sample size increases (Smith and Gelfand, 1992). Using the fact that $q_s$ converges to $q$ gives

$$\lim_{s\to\infty} \overline{p}_s(x) = \lim_{s\to\infty} q_s(x)\frac{p(x)}{q(x)} = p(x)$$

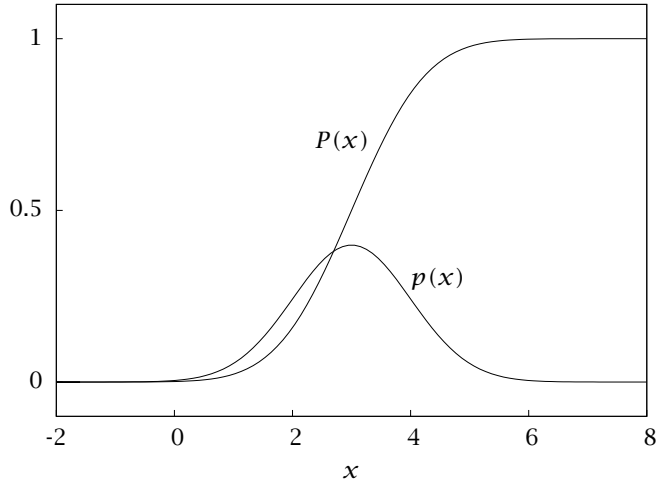


**Figure 4.15:** Probability density $p(x)$ to be sampled and the corresponding cumulative distribution $P(x)$.

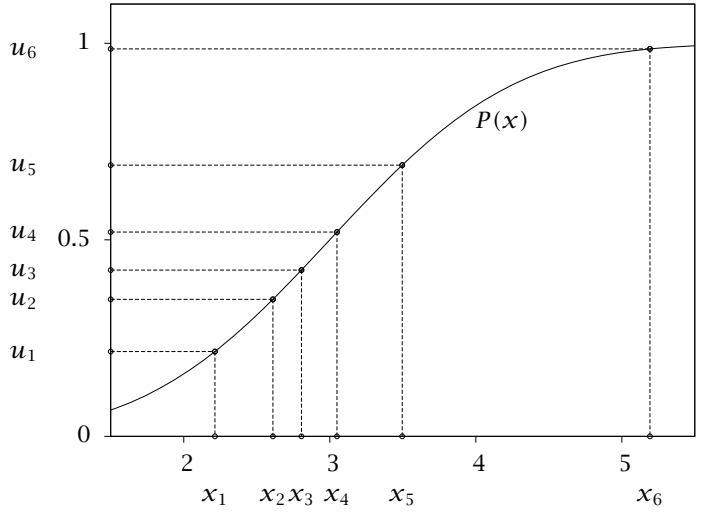


**Figure 4.16:** Six samples of the uniform density on $[0,1]$, $u_i$, and the corresponding samples of $p(x)$, $x_i$. The samples satisfy $x_i = P^{-1}(u_i)$.

The weighted sample of $p$ is also unbiased for all sample sizes, which we can verify as follows

$$\begin{aligned}
\mathcal{E}_{\text{is}}\left(\overline{p}_s(x)\right) &= \mathcal{E}_{\text{is}}\left(\sum_i w_i \delta(x - x_i)\right) \\
&= \int p_{\text{is}}(x_1, \ldots, x_s) \sum_i w_i \delta(x - x_i) dx_1 \cdots dx_s \\
&= \int q(x_1) \cdots q(x_s) \sum_i w_i \delta(x - x_i) dx_1 \cdots dx_s \\
&= \sum_i \int q(x_i) w_i \delta(x - x_i) dx_i \prod_{j \neq i} \int q(x_j) dx_j \\
&= \sum_i \int q(x_i) \frac{1}{s} \frac{p(x_i)}{q(x_i)} \delta(x - x_i) dx_i \\
&= \frac{1}{s} \sum_i p(x) \\
\mathcal{E}_{\text{is}}\left(\overline{p}_s(x)\right) &= p(x)
\end{aligned}$$

Notice this result holds for all $s \geq 1$.

Using the same development, we can represent any function $h(x)$ (not necessarily a density) having the same support as $q(x)$ as a sampled function

$$\begin{aligned}
\overline{h}_s(x) &= \sum_{i=1}^{s} w_i \delta(x - x_i) \qquad w_i = \frac{1}{s} \frac{h(x_i)}{q(x_i)} \\
\lim_{s \to \infty} \overline{h}_s(x) &= h(x)
\end{aligned} \tag{4.46}$$

The next example demonstrates using importance sampling to generate samples of a multimodal density.

**Example 4.33: Importance sampling of a multimodal density**

Given the following bimodal distribution

$$p(x) = \frac{1}{2\sqrt{2\pi P_1}} e^{-(1/2)(x-m_1)^2/P_1} + \frac{1}{2\sqrt{2\pi P_2}} e^{-(1/2)(x-m_2)^2/P_2}$$
$$m_1 = -4 \quad m_2 = 4 \quad P_1 = P_2 = 1$$

generate samples using the following unimodal importance function

$$q(x) = \frac{1}{\sqrt{2\pi P}} e^{-(1/2)(x-m)^2/P} \qquad m = 0 \quad P = 4$$

**Solution**

Figure 4.17 shows the exact and sampled density of the importance function $q(x)$ using 5000 samples. The weighted density for $p(x)$ is shown in Figure 4.18. We obtain a good representation of the bimodal distribution with 5000 samples. Notice also that one should use a broad density for $q(x)$ to obtain sufficient samples in regions where $p(x)$ has significant probability. Using $q(x)$ with variance of $P = 1$ instead of $P = 4$ would require many more samples to obtain an accurate representation of $p(x)$. Of course we cannot choose $q(x)$ too broad or we sample the region of interest too sparsely. Choosing an appropriate importance function for an unknown $p(x)$ is naturally a significant challenge in many applications. □

**Importance sampling when the density cannot be evaluated.** In many applications we have a density $p(x)$ that is difficult to evaluate directly, but it can be expressed as

$$p(x) = \frac{h(x)}{\int h(x)dx} \qquad p(x) \propto h(x)$$

in which $h(x)$ is readily evaluated. We wish to avoid the task of integration of $h$ to find the normalizing constant. Importance sampling can still be used to sample $p$ in this case, but, as we discuss next, we lose the unbiased property of the sampled density for finite sample sizes. In this case, define the candidate sampled density as

$$\overline{p}_s(x) = \frac{q_s(x)}{d(s)}\frac{h(x)}{q(x)} \qquad d(s) = \frac{1}{s}\sum_j \frac{h(x_j)}{q(x_j)} \tag{4.47}$$

in which the samples are again chosen from the importance function $q(x)$. Summarizing, the candidate sampled density is

$$\overline{p}_s(x) = \sum_i w_i \delta(x - x_i)$$

$$p_{\text{is}}(x_i) = q(x_i) \qquad w_i = \frac{h(x_i)/q(x_i)}{\sum_j h(x_j)/q(x_j)} \quad i = 1, \ldots, s \tag{4.48}$$

Notice the weights are normalized in the case when we do not know the normalizing constant to convert from $h(x)$ to $p(x)$. We next show that $\overline{p}_s(x)$ converges to $p(x)$ as sample size increases (Smith and Gelfand,

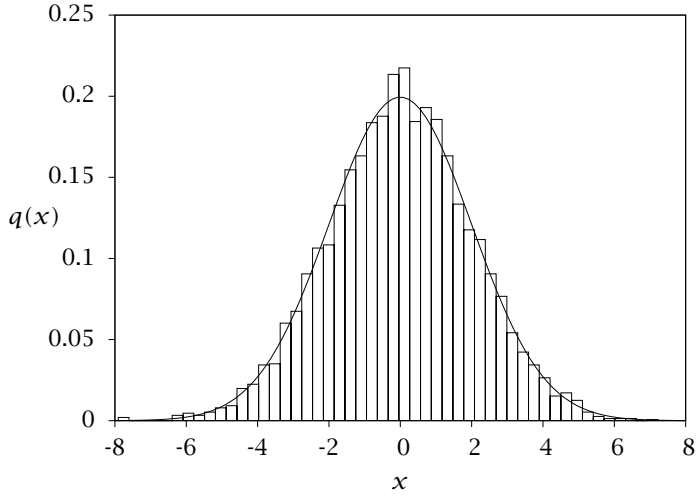


**Figure 4.17:** Importance function $q(x)$ and its histogram based on 5000 samples.

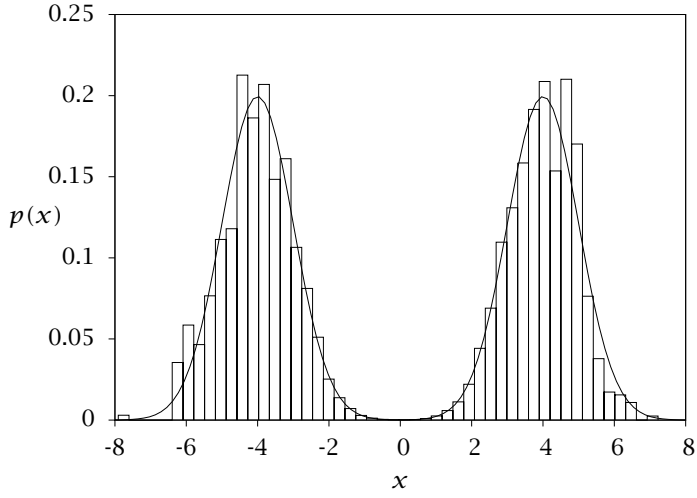


**Figure 4.18:** Exact density $p(x)$ and its histogram based on 5000 importance samples.

1992). First we express $d(s)$ as

$$\begin{aligned} d(s) &= \frac{1}{s}\sum_j \frac{h(x_j)}{q(x_j)} \\ &= \int_{-\infty}^{\infty} \frac{1}{s}\sum_j \frac{h(x_j)}{q(x_j)}\delta(x - x_j)dx \\ &= \int_{-\infty}^{\infty} \frac{1}{s}\sum_j \frac{h(x)}{q(x)}\delta(x - x_j)dx \\ d(s) &= \int_{-\infty}^{\infty} h_s(x)dx \end{aligned}$$

Exchanging the order of limit and integral and using (4.46) give

$$\lim_{s\to\infty} d(s) = \int_{-\infty}^{\infty} \lim_{s\to\infty} h_s(x)dx = \int_{-\infty}^{\infty} h(x)dx$$

Next we take the limit in (4.47) to obtain

$$\begin{aligned} \lim_{s\to\infty} \overline{p}_s(x) &= \lim_{s\to\infty} \frac{q_s(x)}{d(s)}\frac{h(x)}{q(x)} \\ &= \frac{\lim_{s\to\infty} q_s(x)}{\lim_{s\to\infty} d(s)}\frac{h(x)}{q(x)} \\ &= \frac{q(x)}{\int h(x)dx}\frac{h(x)}{q(x)} \\ &= \frac{h(x)}{\int h(x)dx} \\ \lim_{s\to\infty} \overline{p}_s(x) &= p(x) \end{aligned}$$

Notice the unbiased property no longer holds for a finite sample size. We can readily show

$$\mathcal{E}_{\text{is}}\left(\overline{p}_s(x)\right) \neq p(x) \qquad \text{for finite } s \tag{4.49}$$

For example, take $s = 1$. We have from (4.48) that $w_1 = 1$, and therefore

$$\begin{aligned} \mathcal{E}_{\text{is}}\left(\overline{p}_s(x)\right) &= \int p_{\text{is}}(x_1)w_1\delta(x - x_1)dx_1 \\ &= \int q(x_1)\delta(x - x_1)dx_1 \\ \mathcal{E}_{\text{is}}\left(\overline{p}_s(x)\right) &= q(x) \end{aligned}$$

and we see that the expectation of the sampling process with a single sample gives back the importance function $q(x)$ rather than the desired $p(x)$. Obviously we should choose many more samples than $s = 1$ for this case to reduce this bias. Consider the next example in which we use a large number of samples.

**Example 4.34: Importance sampling of a multimodal function**

We revisit Example 4.33 but use the following function $h(x)$

$$h(x) = e^{-(1/2)(x-m_1)^2/P_1} + e^{-(1/2)(x-m_2)^2/P_2}$$
$$m_1 = -4, m_2 = 4, P_1 = P_2 = 1$$

and we do not have the normalization constant available. We again generate samples using the following importance function

$$q(x) = \frac{1}{\sqrt{2\pi P}} e^{-(1/2)(x-m)^2/P} \qquad m = 0, P = 4$$

**Solution**

The exact and sampled density of the importance function $q(x)$ using 5000 samples is the same as Figure 4.17. The weighted density for $p(x)$ is shown in Figure 4.19. Comparing Figure 4.19 to Figure 4.18 shows the representation of the bimodal distribution with 5000 samples using $h(x)$ is of comparable quality to the one using $p(x)$ itself. The bias is not noticeable using 5000 samples. □

**Weighted importance sampling.** In applications of importance sampling to state estimation, the importance function is often available as a *weighted* sample in which the weights are not all equal. Therefore, as a final topic in importance sampling, we consider the case in which a weighted sample of the importance function is available

$$q_s(x) = \sum_{i=1}^{s} w_i^- \delta(x - x_i) \qquad w_i^- \geq 0$$

We have the two cases of interest covered previously.

(a) We can evaluate $p(x)$. For this case we define the sampled density for $p(x)$ as

$$\overline{p}_s(x) = \sum_{i=1}^{s} w_i \delta(x - x_i) \qquad w_i = w_i^- \frac{p(x_i)}{q(x_i)}$$

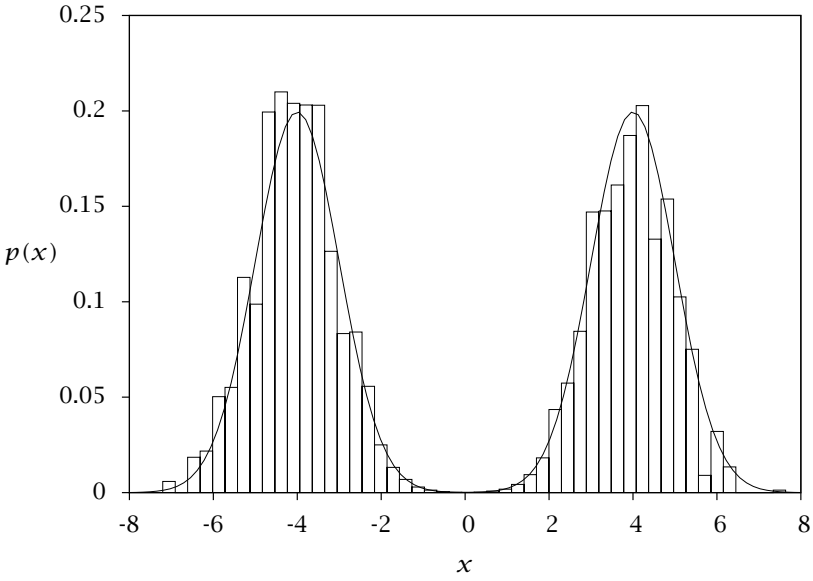


**Figure 4.19:** Exact density $p(x)$ and its histogram based on 5000 importance samples evaluating $h(x)$ in place of $p(x) = h(x)/\int h(x)dx$.

For this case, the sampled density is unbiased for all sample sizes and converges to $p(x)$ as the sample size increases.

(b) We cannot evaluate $p(x)$, but can evaluate only $h(x)$ with $p(x) = h(x)/\int h(x)dx$. For this case, we define the sampled density as

$$\overline{p}_s(x) = \sum_{i=1}^{s} \overline{w}_i \delta(x - x_i)$$

$$w_i = w_i^- \frac{h(x_i)}{q(x_i)} \qquad \overline{w}_i = \frac{w_i}{\sum_j w_j} \quad (4.50)$$

For this case, the sampled density is biased for all finite sample sizes, but converges to $p(x)$ as the sample size increases.

The proofs of these properties are covered in Exercises 4.21 and 4.22.

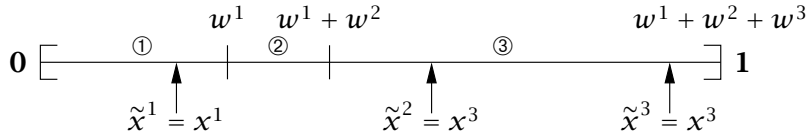


**Figure 4.20:** Interval $[0,1]$ partitioned by original sample weights, $w_i$. The arrows depict the outcome of drawing three uniformly distributed random numbers. For the case depicted here, the new samples are $\widetilde{x}_1 = x_1$, $\widetilde{x}_2 = x_3$, $\widetilde{x}_3 = x_3$ because the first arrow falls into the first interval and the other two arrows both fall into the third interval. Sample $x_2$ is discarded and sample $x_3$ is repeated twice in the resample. The new sample's weights are simply $\widetilde{w}^1 = \widetilde{w}^2 = \widetilde{w}^3 = 1/3$.

### 4.7.3 Resampling

Consider a set of samples at $x = x_i$, $i = 1, \ldots s$ and associated normalized weights $w_i$, $w_i \geq 0$, $\sum_{i=1}^{s} w_i = 1$. Define a probability density using these samples and weights by

$$p(x) = \sum_{i=1}^{s} w_i \delta(x - x_i)$$

Consider any function $f(x)$ defined on a set that contains the samples, $x_i$. Then the integral of $f$ using the defined density is

$$\int f(x)p(x)dx = \sum_{i=1}^{s} w_i f(x_i) = \sum_{i=1}^{s} w_i f_i$$

in which $f_i = f(x_i)$. We now consider a resampling procedure that produces a new set of samples $\widetilde{x}_i$ with new weights $\widetilde{w}_i$. The resampling procedure is depicted in Figure 4.20 for the case $s = 3$. We partition the interval $[0,1]$ into $s$ intervals using the original sample weights, $w_i$, as shown in Figure 4.20, in which the $i$th interval has width $w_i$. To choose $s$ resamples, we generate $s$ random numbers from a uniform distribution on $[0,1]$. Denote these random numbers as $u_i$, $i = 1, \ldots, s$. For each $i$, we find the interval in which the drawn random number falls. Denote the interval number as $m(i)$, defined by the relation

$$0 \leq w_1 + w_2 + \cdots + w_{m(i)-1} \leq u_i \leq w_1 + w_2 + \cdots + w_{m(i)} \leq 1$$

We then choose as resamples

$$\tilde{x}_i = x_{m(i)} \qquad i = 1, \ldots s$$

The resampling selects the new sample locations $\tilde{x}$ in regions of high density. We set all the $\tilde{w}$ weights equal to $1/s$. The result illustrated in Figure 4.20 is summarized in the following table

| Original sample | | Resample | |
|---|---|---|---|
| State | Weight | State | Weight |
| $x_1$ | $w_1 = \frac{3}{10}$ | $\tilde{x}_1 = x_1$ | $\tilde{w}_1 = \frac{1}{3}$ |
| $x_2$ | $w_2 = \frac{1}{10}$ | $\tilde{x}_2 = x_3$ | $\tilde{w}_2 = \frac{1}{3}$ |
| $x_3$ | $w_3 = \frac{6}{10}$ | $\tilde{x}_3 = x_3$ | $\tilde{w}_3 = \frac{1}{3}$ |

The properties of the resamples are summarized by

$$p_{\text{re}}(\tilde{x}_i) = \begin{cases} w_j & \tilde{x}_i = x_j \\ 0 & \tilde{x}_i \neq x_j \end{cases}$$
$$\tilde{w}_i = 1/s \quad \text{all } i$$

We can associate with each resampling a sampled probability density

$$\tilde{p}(x) = \sum_{i=1}^{s} \tilde{w}_i \delta(x - \tilde{x}_i)$$

The resampled density is clearly *not the same* as the original sampled density. It is likely that we have moved many of the new samples to places where the original density has large weights. But by resampling in the fashion described here, we have not introduced bias into the estimates.

Consider taking many such resamples. We can calculate for each of these resamples a value of the integral of $f$ as follows

$$\int f(x)\tilde{p}(x)dx = \sum_{i=1}^{s} \tilde{w}_i f(\tilde{x}_i)$$

To show this resampling procedure is valid, we show that the average over these values of the $f$ integrals with $\tilde{p}(x)$ is equal to the original value of the integral using $p(x)$. We state this result for the resampling procedure described previously as the following theorem.

**Theorem 4.35** (Resampling). *Consider a sampled density $p(x)$ with $s$ samples at $x = x_i$ and associated weights $w_i$*

$$p(x) = \sum_{i=1}^{s} w_i \delta(x - x_i) \qquad w_i \geq 0, \qquad \sum_{i=1}^{s} w_i = 1$$

*Consider the resampling procedure that gives a resampled density*

$$\tilde{p}(x) = \sum_{i=1}^{s} \tilde{w}_i \delta(x - \tilde{x}_i)$$

*in which the $\tilde{x}_i$ are chosen according to resample probability $p_{\text{re}}$*

$$p_{\text{re}}(\tilde{x}_i) = \begin{cases} w_j & \tilde{x}_i = x_j \\ 0 & \tilde{x}_i \neq x_j \end{cases}$$

*and with uniform weights $\tilde{w}_i = 1/s$. Consider a function $f(\cdot)$ defined on a set $X$ containing the points $x_i$.*

*With this resampling procedure, the expectation over resampling of any integral of the resampled density is equal to that same integral of the original density*

$$\mathcal{E}_{\text{re}}\left(\int f(x)\tilde{p}(x)dx\right) = \int f(x)p(x)dx \qquad \text{all } f$$

The proof of this theorem is discussed in Exercise 4.16. To get a feel for why this resampling procedure works, however, consider the case $s = 2$. There are four possible outcomes of $\tilde{x}_1, \tilde{x}_2$ in resampling. Because of the resampling procedure, the random variables $\tilde{x}_i$ and $\tilde{x}_j$, $j \neq i$ are independent, and their joint density is

$$p_{\text{re}}(\tilde{x}_1, \tilde{x}_2) = \begin{cases} w_1^2 & \tilde{x}_1 = x_1, \tilde{x}_2 = x_1 \\ w_1 w_2 & \tilde{x}_1 = x_1, \tilde{x}_2 = x_2 \\ w_2 w_1 & \tilde{x}_1 = x_2, \tilde{x}_2 = x_1 \\ w_2^2 & \tilde{x}_1 = x_2, \tilde{x}_2 = x_2 \end{cases}$$

The values of the integral of $f$ for each of these four outcomes is

$$\sum_{i=1}^{2} \tilde{w}_i f(\tilde{x}_i) = \begin{cases} \frac{1}{2}(f_1 + f_1) & \tilde{x}_1 = x_1, \tilde{x}_2 = x_1 \\ \frac{1}{2}(f_1 + f_2) & \tilde{x}_1 = x_1, \tilde{x}_2 = x_2 \\ \frac{1}{2}(f_2 + f_1) & \tilde{x}_1 = x_2, \tilde{x}_2 = x_1 \\ \frac{1}{2}(f_2 + f_2) & \tilde{x}_1 = x_2, \tilde{x}_2 = x_2 \end{cases}$$

Notice there are only three different values for the integral of $f$. Next, calculating the expectation over the resampling process gives

$$\begin{aligned}\mathcal{E}_{\text{re}}\left(\sum_{i=1}^{2}\widetilde{w}_i f(\widetilde{x}_i)\right) &= w_1^2 f_1 + w_1 w_2 (f_1 + f_2) + w_2^2 f_2^2 \\ &= (w_1^2 + w_1 w_2) f_1 + (w_1 w_2 + w_2^2) f_2 \\ &= w_1 (w_1 + w_2) f_1 + w_2 (w_1 + w_2) f_2 \\ &= w_1 f_1 + w_2 f_2 \\ &= \int f(x) p(x) dx\end{aligned}$$

and the conclusion of the theorem is established for $s = 2$.

One can also change the total number of samples in resampling without changing the conclusions of Theorem 4.35. Exercise 4.17 explores this issue in detail. In many applications of sampling, we use the resampling process to discard samples with excessively small weights in order to reduce the storage requirements and computational burden associated with a large number of samples.

To make this discussion explicit, consider again the bimodal distribution of Example 4.33 shown in Figure 4.18 that was sampled using importance sampling. Many of the samples are located in the interval $[-1, 1]$ because the importance function $q$ has large density in this interval. In fact, 1964 of the 5000 samples fall in this interval given the random sample corresponding to Figure 4.18. But notice the weights in this interval are quite small. If we resample $p$, we can retain the accuracy with many fewer samples as we show in the next example.

**Example 4.36: Resampling a bimodal density**

Consider the bimodal sampled density obtained in Example 4.33 using importance sampling. Resample this sampled density with 500 samples. Compare the accuracy to the original density with 5000 samples.

**Solution**

The histogram of the resampled density with 500 samples is shown in Figure 4.21. The weights in the resampled density are all equal to 1/500. Notice that the accuracy is comparable to Figure 4.18 with one tenth as many samples because most of the samples with small weights have been removed by the resampling process. In fact, none of the 500 resamples fall in the interval $[-1, 1]$. □

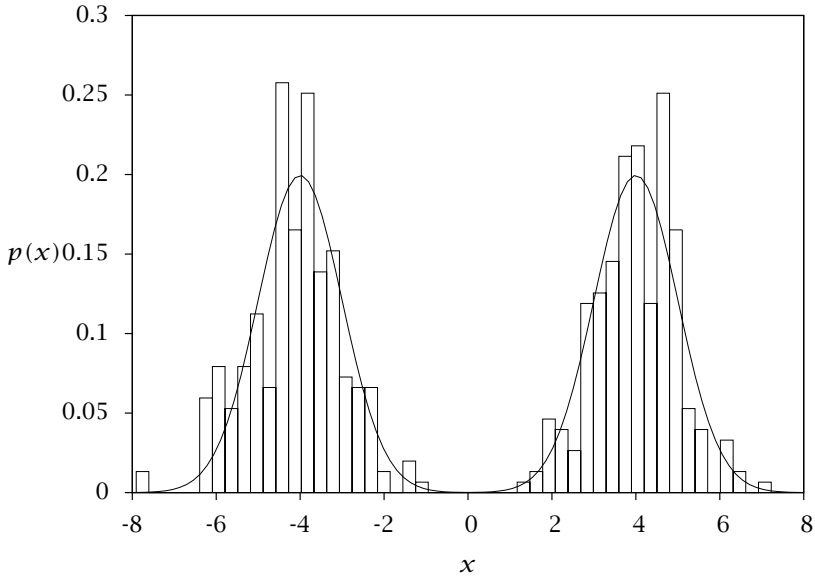


**Figure 4.21:** Resampled density of Example 4.33 using 500 samples. Compare to Figure 4.18 that uses 5000 samples.

### 4.7.4 The Simplest Particle Filter

Next we implement these sampling ideas for state estimation. This first version follows the approach given by Gordon, Salmond, and Smith (1993). In state estimation, the density $p(x(k)|\mathbf{y}(k))$ contains the information of most interest. The model is of the form

$$\begin{aligned} x(k+1) &= f(x(k), n(k)) \\ y(k) &= h(x(k), m(k)) \end{aligned}$$

in which $f$ is a possibly nonlinear function of the state and process noise, $n$, and $h$ is a possibly nonlinear function of the state and measurement noise, $m$. We assume that the densities of $m, n$ and $x(0)$ are available. To start things off, first assume the conditional density $p(x(k)|\mathbf{y}(k))$ is available as a sampled density

$$p(x(k)|\mathbf{y}(k)) = \sum_{i=1}^{s} w_i(k)\delta(x(k) - x_i(k))$$

and we wish to find samples for $p(x(k+1)|\mathbf{y}(k))$. The state evolution can be considered a noninvertible transformation from $x(k), n(k)$ to $x(k+1)$, in which $n(k)$ is statistically independent of $x(k)$ and $\mathbf{y}(k)$. We generate $s$ samples of $n(k)$, call these $n_i(k)$, and we have $s$ samples of the conditional density $p(x(k), n(k)|\mathbf{y}(k))$ given by $\{x_i(k), n_i(k)\}, i = 1, \ldots, s$. As shown in Section 4.7.1, the sampled density of $p(x(k+1)|\mathbf{y}(k))$ is given by

$$p(x(k+1)|\mathbf{y}(k)) = \{x_i(k+1), w_i^-(k+1)\}$$
$$x_i(k+1) = f(x_i(k), n_i(k)) \qquad w_i^-(k+1) = w_i(k)$$

Next, given the sampled density for the conditional density $p(x(k)|\mathbf{y}(k-1))$

$$p(x(k)|\mathbf{y}(k-1)) = \sum_{i=1}^{s} w_i^-(k)\delta(x(k) - x_i(k))$$

we add the measurement $y(k)$ to obtain the sampled density $p(x(k)|\mathbf{y}(k))$. Notice that $\mathbf{y}(k) = (y(k), \mathbf{y}(k-1))$ and use the relationship (see Exercise 1.47)

$$p_{A|B,C}(a|b,c) = p_{C|A,B}(c|a,b)\frac{p_{A|B}(a|b)}{p_{C|B}(c|b)}$$

to obtain

$$p(x(k)|\mathbf{y}(k)) = \frac{p(y(k)|x(k), \mathbf{y}(k-1))p(x(k)|\mathbf{y}(k-1))}{p(y(k)|\mathbf{y}(k-1))}$$

Because the process is Markov, $p(y(k)|x(k), \mathbf{y}(k-1)) = p(y(k)|x(k))$, and we have

$$p(x(k)|\mathbf{y}(k)) = \frac{p(y(k)|x(k))p(x(k)|\mathbf{y}(k-1))}{p(y(k)|\mathbf{y}(k-1))}$$

The density of interest is in the form

$$p(x(k)|\mathbf{y}(k)) = g(x(k))p(x(k)|\mathbf{y}(k-1))$$
$$g(x(k)) = \frac{p(y(k)|x(k))}{p(y(k)|\mathbf{y}(k-1))}$$

and we have a sampled density for $p(x(k)|\mathbf{y}(k-1))$. If we could conveniently evaluate $g$, then we could obtain a sampled density using the product rule given in (4.36)

$$p(x(k)|\mathbf{y}(k)) = \{x_i(k), \tilde{w}_i(k)\}$$

in which

$$\tilde{w}_i(k) = w_i^-(k)\frac{p(y(k)|x_i(k))}{p(y(k)|\mathbf{y}(k-1))} \tag{4.51}$$

This method would provide an unbiased sampled density, but it is inconvenient to evaluate the term $p(y(k)|\mathbf{y}(k-1))$. So we consider an alternative in which the available sampled density is used as a weighted importance function for the conditional density of interest. If we define the importance function $q(x(k)) = p(x(k)|\mathbf{y}(k-1))$, then the conditional density is of the form

$$p(x(k)|\mathbf{y}(k)) = \frac{h(x(k))}{\int h(x(k))dx(k)}$$
$$h(x(k)) = p(y(k)|x(k))p(x(k)|\mathbf{y}(k-1))$$

We then use weighted importance sampling and (4.50) to obtain

$$p(x(k)|\mathbf{y}(k)) = \{x_i(k), \overline{w}_i(k)\} \qquad w_i(k) = w_i^-(k)p(y(k)|x_i(k))$$
$$\overline{w}_i(k) = \frac{w_i(k)}{\sum_j w_j(k)}$$

By using this form of importance sampling, the sampled density is biased for all finite sample sizes, but converges to $p(x(k)|\mathbf{y}(k))$ as the sample size increases.

**Summary.** Starting with $s$ samples of $p(n(k))$ and $s$ samples of $p(x(0))$, we assume that we can evaluate $p(y(k)|x(k))$ using the measurement equation. The iteration for the simple particle filter is summarized by the following recursion.

$$\begin{array}{rcl}
p(x(0)) & = & \{x_i(0), w_i(0) = 1/s\} \\
p(x(k)|\mathbf{y}(k)) & = & \{x_i(k), \overline{w}_i(k)\} \\
w_i(k) & = & \overline{w}_i(k-1)p(y(k)|x_i(k)) \\
\overline{w}_i(k) & = & \dfrac{w_i(k)}{\sum_j w_j(k)} \\
p(x(k+1)|\mathbf{y}(k)) & = & \{x_i(k+1), \overline{w}_i(k)\} \\
x_i(k+1) & = & f(x_i(k), n_i(k))
\end{array}$$

The sampled density of the simplest particle filter converges to the conditional density $p(x(k)|\mathbf{y}(k))$ in the limit of large sample size. The sampled density is biased for all finite sample sizes.

**Analysis of the simplest particle filter.** The simplest particle filter has well-known weaknesses that limit its use as a practical method for state estimation. The variances in both the particle locations and the filter weights can increase without bound as time increases and more measurements become available. Consider first the particle locations. For even the simple linear model with Gaussian noise, we have

$$\begin{aligned} x_i(k+1) &= Ax_i(k) + Bu(k) + Gw_i(k) \\ x_i(0) \sim N(\overline{x}(0), Q_0) &\qquad w_i(k) \sim N(0, Q) \end{aligned}$$

which gives the following statistical properties for the particle locations

$$\begin{aligned} x_i(k) &\sim N(\overline{x}(k), \overline{P}(k)) \quad i = 1, \ldots, s \\ \overline{x}(k) &= A\overline{x}(k-1) + Bu(k) \\ \overline{P}(k) &= A\overline{P}(k-1)A' + GQG' \end{aligned} \tag{4.52}$$

If $A$ is not strictly stable, the variance of the samples locations, $P(k)$, increases without bound despite the availability of the measurement at every time. In this simplest particle filter, one is expecting the particle weights to carry all the information in the measurements. As we will see in the upcoming example, this idea does not work and after a few time iterations the resulting state estimates are useless.

To analyze the variance of the resulting particle weights, it is helpful to define the following statistical properties and establish the following identities. Consider two random variables $A$ and $B$. Conditional expectations of $A$ and functions of $A$ and conditional variance of $A$ are defined as

$$\begin{aligned} \mathcal{E}(A|B) &:= \int p_{A|B}(a|b)a\,da \\ \mathcal{E}(A^2|B) &:= \int p_{A|B}(a|b)a^2\,da \\ \mathcal{E}(g(A)|B) &:= \int p_{A|B}(a|b)g(a)\,da \\ \mathrm{var}(A|B) &:= \mathcal{E}(A^2|B) - \mathcal{E}^2(A|B) \end{aligned}$$

in which we assume as usual that $B$'s marginal is nonzero so the conditional density is well defined. We derive a first useful identity

$$E(E(g(A)|B)) = E(g(A)) \tag{4.53}$$

as follows

$$\begin{aligned}\mathcal{E}(\mathcal{E}(g(A)|B)) &= \int p_B(b) \int p_{A|B}(a|b) g(a) da db \\ &= \int p_B(b) \int \frac{p_{A,B}(a,b)}{p_B(b)} g(a) da db \\ &= \iint p_{A,B}(a,b) g(a) da db \\ &= \int p_A(a) g(a) da \\ &= \mathcal{E}(g(A))\end{aligned}$$

We require a second identity

$$\text{var}(A) = \mathcal{E}(\text{var}(A|B)) + \text{var}(\mathcal{E}(A|B)) \tag{4.54}$$

which is known as the conditional variance formula or the law of total variance. We establish this identity as follows. Starting with the definition of variance

$$\text{var}(A) = \mathcal{E}(A^2) - \mathcal{E}^2(A)$$

we use (4.53) to obtain

$$\text{var}(A) = \mathcal{E}(\mathcal{E}(A^2|B)) - \mathcal{E}^2(\mathcal{E}(A|B))$$

Using the definition of variance on the first term on the right-hand side gives

$$\begin{aligned}\text{var}(A) &= \mathcal{E}\left(\text{var}(A|B) + \mathcal{E}^2(A|B)\right) - \mathcal{E}^2(\mathcal{E}(A|B)) \\ &= \mathcal{E}(\text{var}(A|B)) + \mathcal{E}(\mathcal{E}^2(A|B)) - \mathcal{E}^2(\mathcal{E}(A|B))\end{aligned}$$

and using the definition of variance again on the last two terms on the right-hand side gives

$$\text{var}(A) = \mathcal{E}(\text{var}(A|B)) + \text{var}(\mathcal{E}(A|B))$$

which establishes the result. Notice that since variance is nonnegative, this result also implies the inequality

$$\text{var}(\mathcal{E}(A|B)) \leq \text{var}(A)$$

which shows that the conditional expectation of random variable $A$ has less variance than $A$ itself.

We proceed to analyze the simplest particle filter. Actually we analyze the behavior of the weights for the idealized, unbiased case given by (4.51)

$$w_i(k) = w_i(k-1)\frac{p(y(k)|x_i(k))}{p(y(k)|\mathbf{y}(k-1))}$$

in which we consider the random variable $w_i(k)$ to be a function of the random variables $y(k), x_i(k)$. We next consider the conditional density of the random variables $y(k), x_i(k)$ relative to the previous samples $x_i(k-1)$, and the data $\mathbf{y}(k-1)$. We have

$$\begin{aligned} &p(y(k), x_i(k)|\mathbf{y}(k-1), x_i(k-1)) \\ &\quad = p(y(k)|\mathbf{y}(k-1), x_i(k-1))p(x_i(k)|\mathbf{y}(k-1), x_i(k-1)) \\ &\quad = p(y(k)|\mathbf{y}(k-1))p(x_i(k)|x_i(k-1)) \end{aligned}$$

The first equation results from the statistical independence of $y(k)$ and $x_i(k)$, and the second results from the sampling procedure used to generate $x_i(k)$ given $x_i(k-1)$. Note that in the next section, we use a different sampling procedure in which $x_i(k)$ depends on both the new data $y(k)$ as well as the $x_i(k-1)$. Now we take the expectation of the weights at time $k$ conditional on the previous samples and previous measurement trajectory

$$\begin{aligned} &E(w_i(k)|x_i(k-1), \mathbf{y}(k-1)) \\ &\quad = \iint w_i(k)p(y(k), x_i(k)|x_i(k-1), \mathbf{y}(k-1))dx_i(k)dy(k) \\ &\quad = \iint w_i(k)p(y(k)|\mathbf{y}(k-1))p(x_i(k)|x_i(k-1))dx_i(k)dy(k) \end{aligned}$$

Substituting the weight recursion and simplifying yields

$$\begin{aligned} &E(w_i(k)|x_i(k-1), \mathbf{y}(k-1)) \\ &\qquad = \iint w_i(k-1)\frac{p(y(k)|x_i(k))}{p(y(k)|\mathbf{y}(k-1))} \\ &\qquad p(y(k)|\mathbf{y}(k-1))p(x_i(k)|x_i(k-1))dx_i(k)dy(k) \end{aligned}$$

$$\begin{aligned} &E(w_i(k)|x_i(k-1), \mathbf{y}(k-1)) \\ &\quad = \iint w_i(k-1)p(y(k)|x_i(k))p(x_i(k)|x_i(k-1))dx_i(k)dy(k) \end{aligned}$$

Taking $w_i(k-1)$ outside the integral and performing the integral over $x_i(k)$ and then $y(k)$ gives

$$E(w_i(k)|x_i(k-1),\mathbf{y}(k-1)) = w_i(k-1)\int p(y(k)|x_i(k-1))dy(k)$$
$$E(w_i(k)|x_i(k-1),\mathbf{y}(k-1)) = w_i(k-1)$$

Taking the variance of both sides and using the conditional variance formula (4.54) gives

$$\text{var}(E(w_i(k)|x_i(k-1),\mathbf{y}(k-1))) = \text{var}(w_i(k-1))$$
$$\text{var}(w_i(k)) - E(\text{var}(w_i(k)|x_i(k-1),\mathbf{y}(k-1))) = \text{var}(w_i(k-1))$$

Again, noting that variance is nonnegative gives the inequality

$$\text{var}(w_i(k)) \geq \text{var}(w_i(k-1))$$

and we see that the variance for the unbiased weights of the simplest particle filter increases with time.

Next we present two examples that show the serious practical limitations of the simplest particle filter and the simplest particle filter with resampling.

**Example 4.37: What's wrong with the simplest particle filter?**

Consider the following linear system with Gaussian noise.

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \quad \theta = 6 \qquad C = \begin{bmatrix} 0.5 & 0.25 \end{bmatrix} \qquad G = I \qquad B = I$$
$$\overline{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad Q_0 = \frac{1}{4}\begin{bmatrix} 7 & 5 \\ 5 & 7 \end{bmatrix} \qquad Q = 0.01\, I \qquad R = 0.01$$
$$u(0,1,\cdots,4) = \begin{bmatrix} 7 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

(a) Plot the particle locations versus time from $k = 0$ to $k = 5$. Plot also the 95% contour of the true conditional density $p(x(k)|\mathbf{y}(k))$. Discuss the locations of the particles using the simplest particle filter.

(b) Write out the recursions for the conditional density of the particle locations $p(x_i(k)|\mathbf{y}(k))$ as well as the true conditional density $p(x(k)|\mathbf{y}(k))$. Discuss the differences.

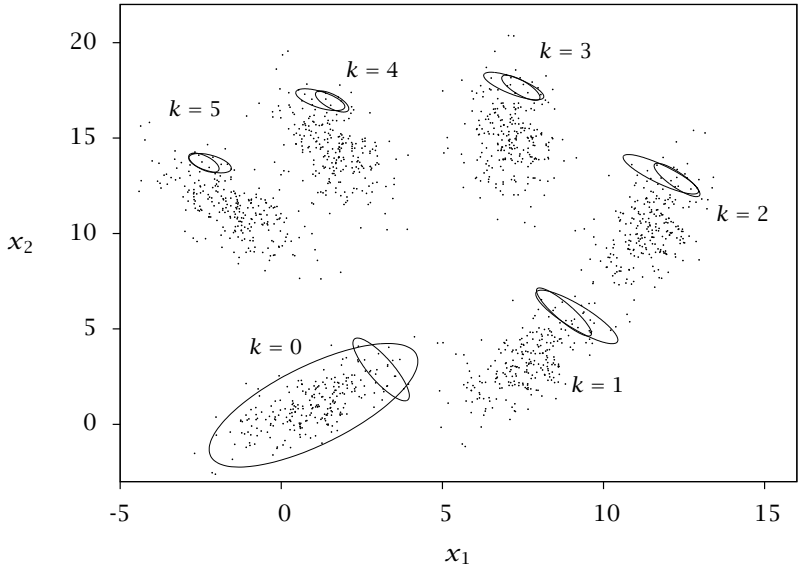


**Figure 4.22:** Particles' locations versus time for the simplest particle filter; 250 particles. Ellipses show the 95% contour of the true conditional densities before and after measurement.

**Solution**

(a) The samples and 95% conditional density contour are shown in Figure 4.22. The particles are located properly at $k = 0$ and about 95% of them are inside the state's initial density. But notice that the particles spread out quickly and few particles remain inside the 95% contour of the true conditional density after a few time steps.

(b) The true conditional density is the normal density given by the time-varying Kalman filter recursion. The conditional density of the particle location is given by (4.52) and the samples are identi-

cally distributed

$$
\begin{aligned}
p(x(k)|\mathbf{y}(k)) &\sim N(\hat{x}(k), P(k)) \\
\hat{x}(k+1) &= A\hat{x}(k) + Bu(k) + L(k)(y(k) - C(A\hat{x}(k) + Bu(k))) \\
P(k+1) &= AP(k)A' + GQG' - L(k+1)C(AP(k)A' + GQG') \\
p(x_i(k)|\mathbf{y}(k)) &\sim N(\overline{x}(k), \overline{P}(k)), \quad i = 1, \ldots, s \\
\overline{x}(k+1) &= A\overline{x}(k) + Bu(k) \\
\overline{P}(k+1) &= A\overline{P}(k)A' + GQG'
\end{aligned}
$$

The major differences are underlined. Notice that the mean of the particle samples is independent of $\mathbf{y}(k)$, which causes the samples to drift away from the conditional density's mean with time. Notice that the covariance does not have the reduction term present in the Kalman filter, which causes the variance of the particles to increase with time. Therefore, due to the missing underlined terms, the mean of the samples drifts and the variance increases with time. The particle weights cannot compensate for the inaccurate placement of the particles, and the state estimates from the simplest particle filter are not useful after a few time iterations.

□

**Example 4.38: Can resampling fix the simplest particle filter?**

Repeat the simulation of Example 4.37, but use resampling after each time step. Discuss the differences.

**Solution**

Applying the resampling strategy gives the results in Figure 4.23. Notice that resampling prevents the samples from drifting away from the mean of the conditional density. Resampling maintains a high concentration of particles in the 95% probability ellipse. If we repeat this simulation 500 times and compute the fraction of particles within the conditional density's 95% probability contour, we obtain the results shown in Figure 4.24. Notice the dramatic improvement. Without resampling, fewer than 10% of the particles are in the 95% confidence ellipse after only five time steps. With resampling, about 80% of the samples are inside the 95% confidence ellipse. There is one caution against resampling too frequently, however. If the measurement has a small covariance, then the weights computed from

$$w_i(k) = w_i(k-1)p(y(k)|x_i(i))$$

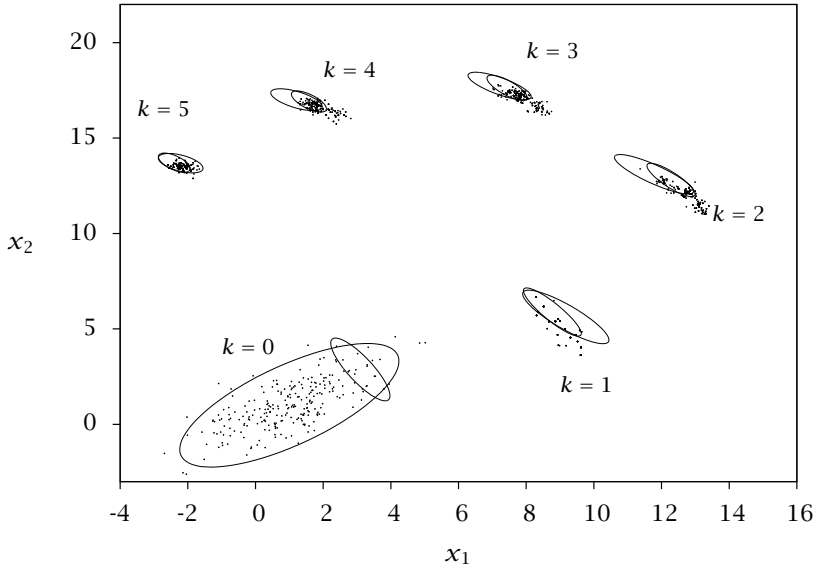


**Figure 4.23:** Particles' locations versus time for the simplest particle filter with resampling; 250 particles. Ellipses show the 95% contour of the true conditional densities before and after measurement.

will be dominated by only a few particles whose prediction of $y$ is closest to the measurement. Resampling in this situation gives only those few particles repeated many times in the resample. For a sufficiently small covariance, this phenomenon can produce a single $x_i$ value in the resample. This phenomenon is known as sample *impoverishment* (Doucet, Godsill, and Andrieu, 2000; Rawlings and Bakshi, 2006). □

### 4.7.5 A Particle Filter Based on Importance Sampling

Motivated by the drawbacks of the simplest particle filter of the previous section, researchers have developed alternatives based on a more flexible importance function (Arulampalam, Maskell, Gordon, and Clapp, 2002). We present this approach next. Rather than start with the statistical property of most interest, $p(x(k)|\mathbf{y}(k))$, consider instead the density of the entire *trajectory* of states conditioned on the measurements, $p(\mathbf{x}(k)|\mathbf{y}(k))$, as we did in moving horizon estimation. Our first

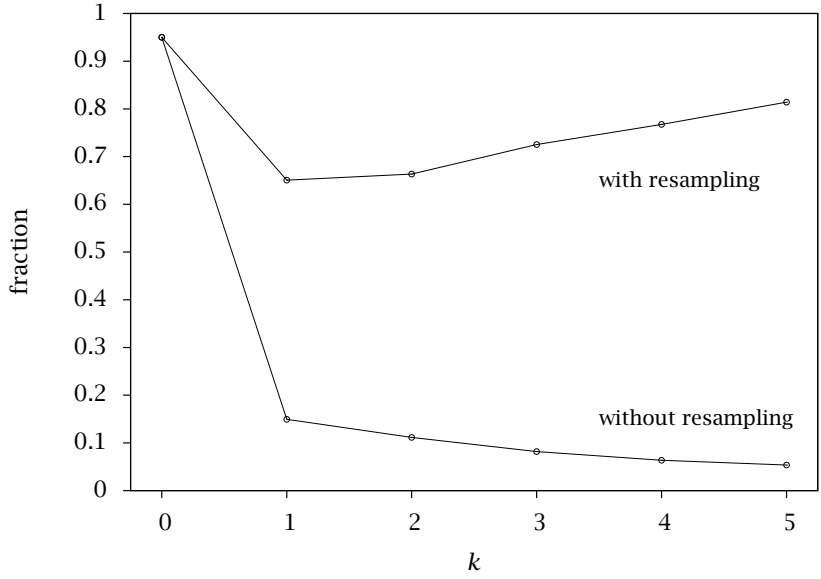


**Figure 4.24:** Fraction of particles inside the 95% contour of the true conditional density versus time; with and without resampling; average of 500 runs.

objective then is to obtain samples of $p(\mathbf{x}(k+1)|\mathbf{y}(k+1))$ from samples of $p(\mathbf{x}(k)|\mathbf{y}(k))$ and the model. We use importance sampling to accomplish this objective. Assume we have $s$ weighted samples of the trajectory conditioned on measurements up to time $k$

$$p(\mathbf{x}(k)|\mathbf{y}(k)) = \{\mathbf{x}_i(k), \overline{w}_i(k)\} \quad i = 1, \ldots, s$$

in which the samples have been drawn from an importance function $q$, whose properties will be chosen as we proceed further. The weights $\overline{w}_i(k)$ are given by

$$\begin{aligned} w_i(k) &= \frac{h(\mathbf{x}_i(k))}{q(\mathbf{x}_i(k)|\mathbf{y}(k))} \\ p(\mathbf{x}_i(k)|\mathbf{y}(k)) &= \frac{h(\mathbf{x}_i(k))}{\int h(\mathbf{x}_i(k))d\mathbf{x}_i(k)} \\ \overline{w}_i(k) &= \frac{w_i(k)}{\sum_j w_j(k)} \end{aligned}$$

Notice $\mathbf{x}_i(k)$ is a set of $ks$ $n$-vector samples, and, as in full information estimation, the storage requirements grow linearly with time. We remove this drawback subsequently, but for now we wish to obtain samples of $p(\mathbf{x}(k+1)|\mathbf{y}(k+1))$ in which $\mathbf{x}(k+1) = \{x(k+1), \mathbf{x}(k)\}$ and $\mathbf{y}(k+1) = \{y(k+1), \mathbf{y}(k)\}$. We start with

$$p(\mathbf{x}(k+1)|\mathbf{y}(k+1)) = \frac{p(y(k+1)|\mathbf{x}(k+1))p(\mathbf{x}(k+1)|\mathbf{y}(k))}{p(y(k+1)|\mathbf{y}(k))} \quad (4.55)$$

in which we have used the second identity in Exercise 1.47 and the Markov property, which implies

$$p(y(k+1)|\mathbf{x}(k+1), y(k)) = p(y(k+1)|\mathbf{x}(k+1))$$

Again, because the process is Markov $p(y(k+1)|\mathbf{x}(k+1)) = p(y(k+1)|x(k+1))$. We next use the identity $p_{A,B|C}(a,b|c) = p_{A|B,C}(a|b,c)p_{B|C}(b|c)$ (see Exercise 1.46) and obtain

$$p(\mathbf{x}(k+1)|\mathbf{y}(k)) = p(x(k+1)|\mathbf{x}(k), \mathbf{y}(k))p(\mathbf{x}(k)|\mathbf{y}(k))$$

Again using the Markov property in this equation, we know $p(x(k+1)|\mathbf{x}(k), \mathbf{y}(k)) = p(x(k+1)|x(k))$ and therefore

$$p(\mathbf{x}(k+1)|\mathbf{y}(k)) = p(x(k+1)|x(k))p(\mathbf{x}(k)|\mathbf{y}(k))$$

Substituting these relations into (4.55) gives

$$p(\mathbf{x}(k+1)|\mathbf{y}(k+1)) = \frac{p(y(k+1)|x(k+1))p(x(k+1)|x(k))}{p(y(k+1)|\mathbf{y}(k))} p(\mathbf{x}(k)|\mathbf{y}(k)) \quad (4.56)$$

We use importance sampling to sample this density. Notice the denominator does not depend on $\mathbf{x}(k+1)$ and is therefore not required when using importance sampling. We use instead

$$p(\mathbf{x}(k+1)|\mathbf{y}(k+1)) = \frac{h(\mathbf{x}(k+1))}{\int h(\mathbf{x}(k+1))d\mathbf{x}(k+1)}$$

$$h(\mathbf{x}(k+1)) = p(y(k+1)|x(k+1))p(x(k+1)|x(k))p(\mathbf{x}(k)|\mathbf{y}(k)) \quad (4.57)$$

Note also that using importance sampling here when we do not wish to evaluate the normalizing constant introduces bias for finite sample size

as stated in (4.49). We now state the two properties of $q$ that provide a convenient importance function

$$q(x(k+1)|\mathbf{x}(k), \mathbf{y}(k+1)) = q(x(k+1)|x(k), y(k+1))$$

$$q(\mathbf{x}(k+1)|\mathbf{y}(k+1)) = q(x(k+1)|x(k), y(k+1))\, q(\mathbf{x}(k)|\mathbf{y}(k)) \quad (4.58)$$

The first property of $q$ is satisfied also by the density $p$, so it is not unusual to pick an importance function to share this behavior. The second property is *not* satisfied by the density, however, and it is chosen strictly for convenience; it allows a recursive evaluation of $q$ at time $k+1$ from the value at time $k$. See Exercise 4.18 for further discussion of this point.

Next we need to generate the samples of $q(\mathbf{x}(k+1)|\mathbf{y}(k+1))$. Given the second property in (4.58), we have

$$\begin{aligned} q(\mathbf{x}(k+1)|\mathbf{y}(k+1)) &= q(x(k+1), x(k), \mathbf{x}(k-1)|y(k+1), \mathbf{y}(k)) \\ &= q(x(k+1)|x(k), y(k+1))\, q(x(k), \mathbf{x}(k-1)|\mathbf{y}(k)) \end{aligned}$$

which is of the form studied in Example 4.32 with the substitution

$$a = x(k+1) \qquad b = x(k) \qquad c = \mathbf{x}(k-1) \qquad d = y(k+1) \qquad e = \mathbf{y}(k)$$

Using the results of that example, our sampling procedure is as follows. We have available samples of $q(\mathbf{x}(k), \mathbf{y}(k)) = q(x(k), \mathbf{x}(k-1)|\mathbf{y}(k))$. Denote these samples by $(x_i(k), \mathbf{x}_i(k-1)), i = 1, \ldots, s$. Then we draw one sample from $q(x(k+1)|x_i(k), y(k+1))$ for each $i = 1, \ldots, s$. Denote these samples as $x_i(k+1)$. Then the samples of $q(\mathbf{x}(k+1)|\mathbf{y}(k+1))$ are given by $(x_i(k+1), x_i(k), \mathbf{x}_i(k-1)) = (x_i(k+1), \mathbf{x}_i(k))$. So we have

$$\mathbf{x}_i(k+1) = (x_i(k+1), \mathbf{x}_i(k)) \qquad i = 1, \ldots, s$$

Next we evaluate the weights for these samples

$$w_i(k+1) = \frac{h(\mathbf{x}_i(k+1)|\mathbf{y}(k+1))}{q(\mathbf{x}_i(k+1)|\mathbf{y}(k+1))}$$

Using (4.57) to evaluate $h$ and the second property of the importance

function to evaluate $q$ gives

$$w_i(k+1) = \frac{p(y(k+1)|x_i(k+1))p(x_i(k+1)|x_i(k))}{q(x_i(k+1)|x_i(k), y(k+1))} \frac{h(\mathbf{x}_i(k)|\mathbf{y}(k))}{q(\mathbf{x}_i(k)|\mathbf{y}(k))}$$

$$w_i(k+1) = \frac{p(y(k+1)|x_i(k+1))p(x_i(k+1)|x_i(k))}{q(x_i(k+1)|x_i(k), y(k+1))} w_i(k) \quad (4.59)$$

$$\overline{w}_i(k+1) = \frac{w_i(k+1)}{\sum_j w_j(k+1)}$$

Notice we obtain a convenient recursion for the weights that depends only on the values of the samples $x_i(k+1)$ and $x_i(k)$ and not the rest of the trajectory contained in the samples $\mathbf{x}_i(k)$. The trajectory's sampled density is given by

$$p(x(k+1), \mathbf{x}(k)|\mathbf{y}(k+1)) = \sum_{i=1}^{s} \overline{w}_i(k+1)\delta(x(k+1) - x_i(k+1))\delta(\mathbf{x}(k) - \mathbf{x}_i(k))$$

Integrating both sides over the $\mathbf{x}(k)$ variables gives the final result

$$p(x(k+1)|\mathbf{y}(k+1)) = \sum_{i=1}^{s} \overline{w}_i(k+1)\delta(x(k+1) - x_i(k+1))$$

Since we generate $x_i(k+1)$ from sampling $q(x(k+1)|x_i(k), y(k+1))$, the trajectory samples, $\mathbf{x}_i(k)$, and measurement trajectory, $\mathbf{y}(k)$, are not required at all, and the particle filter storage requirements do not grow with time. Notice also that if we choose the importance function

$$q(x_i(k+1)|x_i(k), y(k+1)) = p(x_i(k+1)|x_i(k))$$

which ignores the current measurement when sampling, we obtain for the weights

$$w_i(k+1) = w_i(k)\ p(y(k+1)|x_i(k+1))$$

This choice of importance function reduces to the simplest particle filter of the previous section, with its concomitant drawbacks.

**Summary.** We select an importance function $q(x(k+1)|x(k), y(k+1))$. We start with $s$ samples of $p(x(0))$. We assume that we can evaluate $p(y(k)|x(k))$ using the measurement equation and $p(x(k+$

$1)|x(k))$ using the model equation. The importance function particle filter is summarized by the following recursion

$$
\begin{aligned}
p(x(0)|y(0)) &= \{x_i(0), \overline{w}_i(0)\} \\
w_i(0) &= p(y(0)|x_i(0)) \qquad \overline{w}_i(0) = \frac{w_i(0)}{\sum_j w_j(0)} \\
p(x(k)|\mathbf{y}(k)) &= \{x_i(k), \overline{w}_i(k)\} \\
w_i(k+1) &= \overline{w}_i(k)\, \frac{p(y(k+1)|x_i(k+1))p(x_i(k+1)|x_i(k))}{q(x_i(k+1)|x_i(k), y(k+1))} \\
\overline{w}_i(k+1) &= \frac{w_i(k+1)}{\sum_j w_j(k+1)}
\end{aligned}
$$

and $x_i(k+1)$ is a sample of $q(x(k+1)|x_i(k), y(k+1)), \quad i = 1, \ldots, s$. The sampled density of the importance-sampled particle filter converges to the conditional density $p(x(k)|\mathbf{y}(k))$ in the limit of infinite samples. Because of the way importance sampling was used, the sampled density is biased for all finite sample sizes.

Exercise 4.23 provides the recursion for the weights in the unbiased particle filter; these weights require the evaluation of $p(y(k)|\mathbf{y}(k-1))$. Exercise 4.24 shows that the variance of the unbiased weights increases with time.

### 4.7.6 Optimal Importance Function

In this section we develop the so-called "optimal" importance function $q(x(k)|x_i(k-1), y(k))$. We start with the weight recursion for the importance function particle filter given in (4.59), repeated here with $k$ replacing $k+1$

$$
w_i(k) = w_i(k-1)\, \frac{p(y(k)|x_i(k))p(x_i(k)|x_i(k-1))}{q(x_i(k)|x_i(k-1), y(k))}
$$

We consider the $w_i(k)$ conditioned on the random variables $x_i(k-1), \mathbf{y}(k)$. The weight $w_i(k)$ is then a function of the random variable $x_i(k)$, which is sampled from the importance function $q(x(k)|x_i(k-$

$1), y(k))$. Taking the expectation gives

$$\begin{aligned}
&\mathcal{E}\left(w_i(k)|x_i(k-1), \mathbf{y}(k)\right) \\
&\qquad = \int w_i(k)\; q(x_i(k)|x_i(k-1), y(k))\; dx_i(k) \\
&\qquad = \int \frac{p(y(k)|x_i(k))p(x_i(k)|x_i(k-1))}{q(x_i(k)|x_i(k-1), y(k))} \\
&\qquad\qquad w_i(k-1)\; q(x_i(k)|x_i(k-1), y(k))\; dx_i(k) \\
&\qquad = \int p(y(k)|x_i(k))\; p(x_i(k)|x_i(k-1))\; w_i(k-1)\; dx_i(k) \\
&\qquad = w_i(k-1)\; p(y(k)|x_i(k-1))
\end{aligned}$$

Next we compute the conditional variance of the weights

$$\begin{aligned}
&\text{var}(w_i(k)|x_i(k-1), \mathbf{y}(k)) \\
&\qquad = \mathcal{E}(w_i^2(k)|x_i(k-1), \mathbf{y}(k)) - \mathcal{E}^2(w_i|x_i(k-1), \mathbf{y}(k))
\end{aligned}$$

Using the recursion in the first term and the expectation just derived in the second term gives

$$\begin{aligned}
&\text{var}(w_i(k)|x_i(k-1), \mathbf{y}(k)) = \\
&\qquad \int w_i^2(k)\; q(x_i(k)|x_i(k-1), y(k))\; dx_i(k) \\
&\qquad\quad - \left(w_i(k-1)\; p(y(k)|x_i(k-1))\right)^2
\end{aligned}$$

$$\begin{aligned}
&\text{var}(w_i(k)|x_i(k-1), \mathbf{y}(k)) \\
&= \int w_i^2(k-1) \frac{\left(p(y(k)|x_i(k))\; p(x_i(k)|x_i(k-1))\right)^2}{q^2(x_i(k)|x_i(k-1), y(k))} \\
&\quad q(x_i(k)|x_i(k-1), y(k))\; dx_i(k) - \left(w_i(k-1)\; p(y(k)|x_i(k-1))\right)^2 \\
&= w_i^2(k-1)\Big[ \int \frac{p^2(y(k)|x_i(k))\; p^2(x_i(k)|x_i(k-1))}{q(x_i(k)|x_i(k-1), y(k))}\; dx_i(k) \\
&\qquad - p^2(y(k)|x_i(k-1))\Big]
\end{aligned}$$

We can now optimize the choice of $q(x_i(k)|x_i(k-1), y(k))$ to minimize this conditional variance. Consider the choice

$$\boxed{q(x_i(k)|x_i(k-1), y(k)) = p(x_i(k)|x_i(k-1), y(k))} \tag{4.60}$$

which makes the samples at $k$ depend on current measurement $y(k)$ as well as the past samples. We know from Bayes's rule and the Markov property

$$\begin{aligned} q(x_i(k)|x_i(k-1), y(k)) &= p(x_i(k)|x_i(k-1), y(k)) \\ &= \frac{p(y(k)|x_i(k), x_i(k-1))p(x_i(k)|x_i(k-1))}{p(y(k)|x_i(k-1))} \\ q(x_i(k)|x_i(k-1), y(k)) &= \frac{p(y(k)|x_i(k))p(x_i(k)|x_i(k-1))}{p(y(k)|x_i(k-1))} \end{aligned}$$

Using this result we have for the integral term

$$\begin{aligned} &\int \frac{p^2(y(k)|x_i(k))\ p^2(x_i(k)|x_i(k-1))}{q(x_i(k)|x_i(k-1), y(k))}\, dx_i(k) \\ &= p(y(k)|x_i(k-1)) \int p(y(k)|x_i(k))\ p(x_i(k)|x_i(k-1))\, dx_i(k) \\ &= p^2(y(k)|x_i(k-1)) \end{aligned}$$

Substituting this result into the previous equation for conditional variance gives

$$\text{var}(w_i(k)|x_i(k-1), \mathbf{y}(k)) = 0$$

Since variance is nonnegative, the choice of importance function given in (4.60) is optimal for reducing the conditional variance of the weights. This choice has the important benefit of making the samples $x_i(k)$ more responsive to the measurement $y(k)$, which we show in the next example is a big improvement over the simplest particle filter.

**Example 4.39: Optimal importance function applied to a linear estimation problem**

Given the linear system of Example 4.37 and 250 particles, show the particles' locations for times $k = 0, 1, \ldots, 5$ along with the 95% elliptical contour of the true conditional density $p(x(k)|\mathbf{y}(k))$. Perform this calculation with and without resampling after every time step.

**Solution**

The optimal importance function is given in (4.60)

$$q(x_i(k)|x_i(k-1), y(k)) = p(x_i(k)|x_i(k-1), y(k))$$

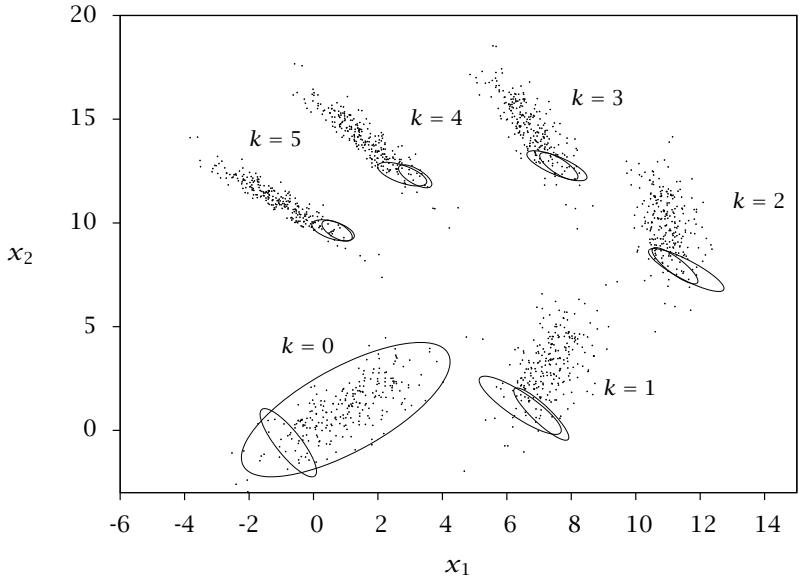


**Figure 4.25:** Particles' locations versus time using the optimal importance function; 250 particles. Ellipses show the 95% contour of the true conditional densities before and after measurement.

The conditional density on the right-hand side is given by

$$p(x_i(k)|x_i(k-1), y(k)) \sim N(\overline{x}(k), \overline{P})$$

$$\overline{x}(k) = \overline{P}\left(Q^{-1}(Ax_i(k-1) + Bu(k-1)) + C'R^{-1}y(k)\right)$$
$$\overline{P} = \left(Q^{-1} + C'R^{-1}C\right)^{-1}$$

Exercise 4.25 discusses establishing this result. So the $x_i(k)$ are generated by sampling this normal, and the results are shown in Figure 4.25. We see that the optimal importance function adds a $y(k)$ term to the evolution of the particle mean. This term makes the particles more responsive to the data and the mean particle location better tracks the conditional density's mean. Compare Figure 4.22 for the simplest particle filter with Figure 4.25 to see the improvement. Also the variance no longer increases with time as in the simplest particle filter so the particles do not continue to spread apart.

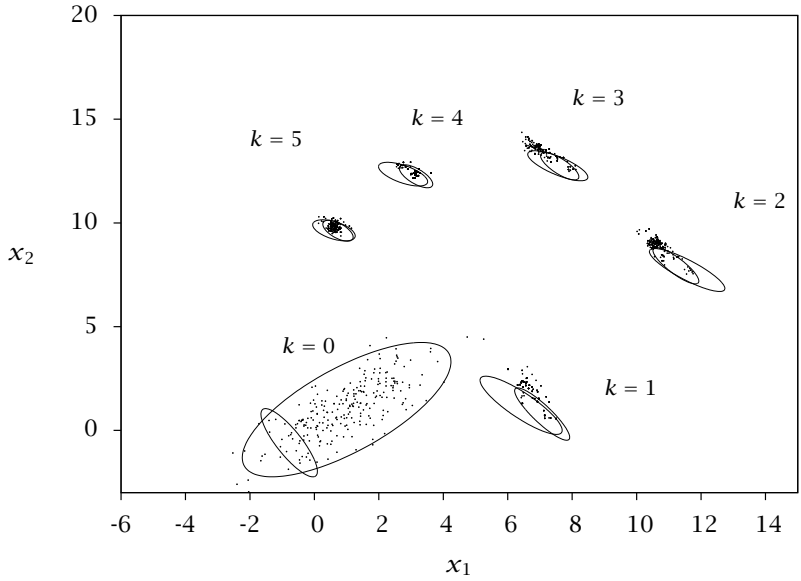


**Figure 4.26:** Particles' locations versus time using the optimal importance function with resampling; 250 particles. Ellipses show the 95% contour of the true conditional densities before and after measurement.

If we apply resampling at every time step, we obtain the results in Figure 4.26. As we saw in the case of the simplest particle filter, resampling greatly increases the number of samples inside the 95% probability ellipse of the conditional density.

If we rerun the simulation 500 times and plot versus time the fraction of particles that are inside the 95% contour of the true conditional density, we obtain the result shown in Figure 4.27. The optimal importance function is able to maintain about 20% of the particles in the 95% probability ellipse. With the optimal importance function and resampling, more than 90% of the particles are inside the 95% probability ellipse. The earlier warning about sample impoverishment applies here as well. □

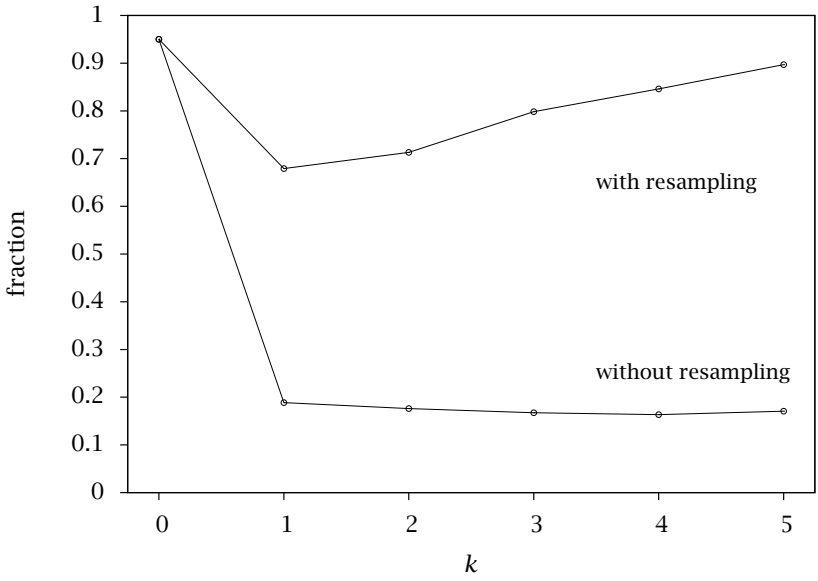


**Figure 4.27:** Fraction of particles inside the 95% contour of the true conditional density versus time; with and without resampling; average of 500 runs.

## 4.8 Combined MHE/Particle Filtering

We next propose a new state estimation method that combines some of the best elements of MHE and PF. This type of combination has several design parameters and can take different forms, and we use the general term combined MHE/PF to designate this entire class of state estimators. To motivate the design of MHE/PF, consider the strengths and weaknesses of pure MHE and pure PF. The main *strengths of MHE* are

1. MHE propagates the state using the full nonlinear model.
2. MHE uses optimization to find the most likely estimate. Physical constraints can be included in the optimization.
3. MHE employs a horizon of measurements.

Using the full nonlinear model prevents inaccurate model linearizations from interfering with the fitting of the model to the data. The

use of optimization produces the best state or state trajectory to describe the current snapshot of data. Optimization methods generally evaluate a small set of points in the state space to find the best estimate compared to exhaustive enumeration, gridding, and sampling strategies. That becomes a significant strength as the dimension of the state space model increases past $n \approx 2$–3. The use of a moving window of data provides some robustness to unmodeled disturbances entering the system. The goal in most recursive estimation is to consider measurements one at a time. That is often a valid goal, mainly because it allows faster computation of the current estimate given the current measurement. But unmodeled disturbances are often problematic when measurements are considered one at a time. No single measurement is sufficient to conclude that an unmodeled disturbance has shifted the state significantly from its current estimated value. Only when several sequential measurements are considered at once is the evidence sufficient to overturn the current state estimate and move the state a significant distance to better match all of the measurements. MHE has this capability built in.

The main *weaknesses of MHE* are

1. MHE may take significant computation time.
2. MHE uses local instead of global optimization.

Of course attempting global optimization is possible, but that exacerbates weakness 1 significantly and no guarantees of finding a global optimum are available for anything but the simplest nonlinear models. Note that for the special case of linear models, MHE finds the global optimum and weakness 2 is removed.

Particle filtering displays quite different characteristics than those of MHE. The main *strengths of PF* are

1. PF uses the full nonlinear model to propagate the samples.
2. The PF sampled density can represent a general conditional density.
3. PF is simple to program and executes quickly for small sample sizes.

As we have illustrated with simple examples, pure PF also demonstrates significant weaknesses, and these are not remedied by any suggestions in the research literature of which we are aware. The *weaknesses of PF* include

1. PF exhibits significant decrease in performance with increasing state dimension.

2. PF displays poor robustness to unmodeled disturbances.

The lack of robustness is a direct outcome of the sampling strategies. Sampling any of the proposed PF importance functions does not locate the samples close to the true state after a significant and unmodeled disturbance. Once the samples are in the wrong place with respect to the peak in the conditional density, they do not recover. If the samples are in the wrong part of the state space, the weights cannot carry the load and represent the conditional density. Resampling does not successfully reposition the particles if they are already out of place. An appeal to sampled density convergence to the true conditional density with increasing sample number is unrealistic. The number of samples required is simply too large for even reasonably small state dimensions considered in applications; $n > 50$ is not unusual in applications.

In constructing a class of combined methods we propose to

1. Use MHE to locate/relocate the samples.

2. Use PF to obtain fast recursive estimation between MHE optimizations.

We overcome the potentially expensive MHE optimization by using PF to process the measurements and provide rapid online estimates while an MHE computation is underway. We position the samples in regions of high conditional density after every run of the MHE optimization, which allows recovery from unmodeled disturbances as soon as an MHE computation completes. A challenge that is not addressed is the appearance of multiple peaks in the conditional density when using nonlinear models. Handling the multimodal conditional density remains a challenge for any online, and indeed offline, state estimation procedure.

Next we propose a specific state estimator in this general MHE/PF class and examine its performance with some simple computational examples. Because this class of estimators is new, we fully expect significant modifications and improvements to come along. At this early juncture we expect only to be able to illustrate some of the new capabilities of the approach.

Let $\hat{Z}_k(x)$ denote the MHE arrival cost function given in Definition 4.16. We let $\hat{V}_k^0$ denote the optimal cost and $\hat{x}(k)$ the optimal estimate of the last stage at time $k$. We consider the quadratic approximation of

$\hat{Z}_k(\cdot)$ at the optimum $\hat{x}(k)$

$$V(x) = V_k^0(\hat{x}(k)) + (1/2)(x - \hat{x}(k))' H (x - \hat{x}(k))$$

in which $H$ is the Hessian of $\hat{Z}_k(x)$ evaluated at the optimum $\hat{x}(k)$. We use this function as an importance function for sampling the conditional density. Notice that this procedure is not the same as assuming the conditional density itself is a normal distribution. We are using $N(\hat{x}(k), H^{-1})$ strictly as an importance function for sampling the unknown conditional density. The samples $x_i(k)$ are drawn from $N(\hat{x}(k), H^{-1})$. The weights are given by

$$w_i(k) = V(x_i(k)) \qquad \overline{w}_i(k) = \frac{w_i(k)}{\sum_j w_j(k)} \tag{4.61}$$

and the sampled density is given by

$$p_s(x) = \{x_i(k), \overline{w}_i(k)\}$$

If the conditional density is well represented by the normal approximation, then the normalized weights are all nearly equal to $1/s$. The MHE cost function modifies these ideal weights as shown in (4.61).

**Example 4.40: Comparison of MHE, PF, and combined MHE/PF**

Consider a well-mixed semibatch chemical reactor in which the following reaction takes place

$$2\text{A} \xrightarrow{k} \text{B} \qquad r = kc_A^2$$

The material balances for the two components are

$$\frac{dc_A}{dt} = -2kc_A^2 + \frac{Q_f}{V} c_{Af}$$
$$\frac{dc_B}{dt} = kc_A^2 + \frac{Q_f}{V} c_{Bf}$$

with constant parameter values

$$\frac{Q_f}{V} = 0.4 \qquad k = 0.16 \qquad c_{Af} = 1 \qquad c_{Bf} = 0$$

The scalar measurement is the total pressure, which is the sum of the two states. The sample time is $\Delta = 0.1$. The initial state is $x(0) = [3\ 1]'$ and the initial prior mean is $\hat{x}(0) = [0.1\ 4.5]'$. Moreover, the input

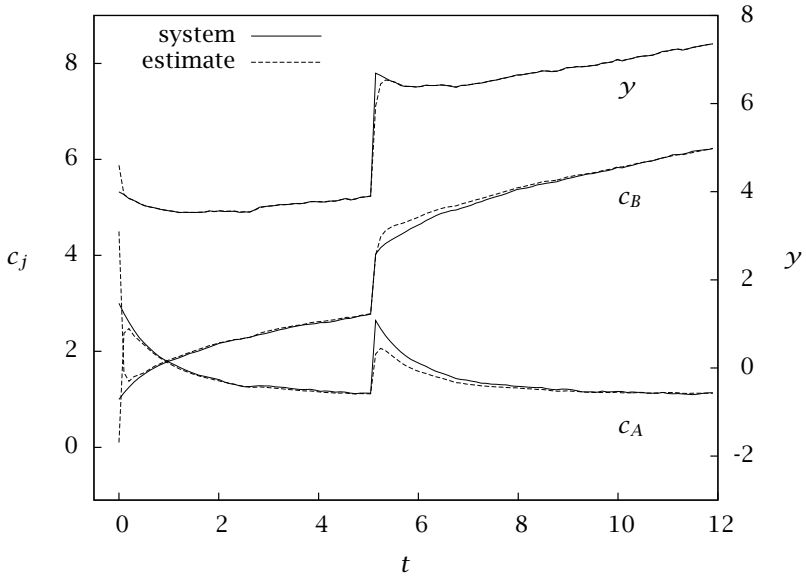


**Figure 4.28:** Pure MHE.

suffers an unmodeled step disturbance at $t = 5$ for two samples. So this example tests robustness of the estimator to initial state error and unmodeled disturbances.

First we apply MHE to the example and the results are displayed in Figure 4.28. The horizon is chosen as $N = 15$. The initial covariance is chosen to be $P_0 = 10I_2$ to reflect the poor confidence in the initial state. Notice that MHE is able to recover from the poor initial state prior in only 4 or 5 samples.

Next we apply pure particle filtering using 50 particles. We use the optimal importance function because the measurement equation is linear. The particles are initialized using the same initial density as used in the MHE estimator.

$$p_{x(0)}(x) = n(x, \hat{x}(0), P(0))$$

The results are shown in Figure 4.29. The figure shows the state and output mean versus time. We notice two effects. The particle filter is unable to recover from the poor initial samples. The measurement is predicted well but neither state is estimated accurately. The A concentration estimate is also negative, which is physically impossible. The

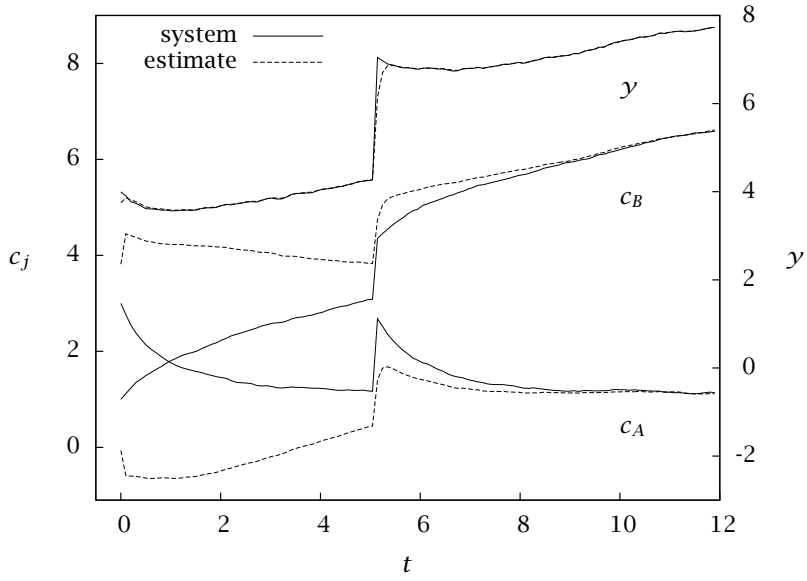


**Figure 4.29:** Pure PF with optimal importance function.

disturbance at $t = 5$ is fortuitous and helps the PF get back on track.

Next we assume that the MHE optimization cannot finish in one sample, but requires $M$ samples. If we attempt a pure MHE solution in this situation, the estimator falls hopelessly behind; an estimate using data $\mathbf{y}(k-M,k), k \geq M$ is not available until time $Mk$. Instead we use MHE/PF as follows.

1. At time $k$ run MHE on data $\mathbf{y}(k-M,k)$. This computation is assumed to finish at time $k+M$. For simplicity, assume $N$ large and a noninformative prior.

2. Draw samples from $N(\hat{x}(k), P(k))$. Run the particle filtering update from time $k$ to time $k+M$. For illustrative purposes, we assume this PF step finishes in one sample.

3. Update $k$ to $k+M$ and repeat.

For illustrative purposes, we choose $M = 10$ and apply the combination of MHE and PF with the simple importance function, also using 50 particles as before. The results are shown in Figure 4.30. Notice that again the poor initial samples lead to significant estimate error.

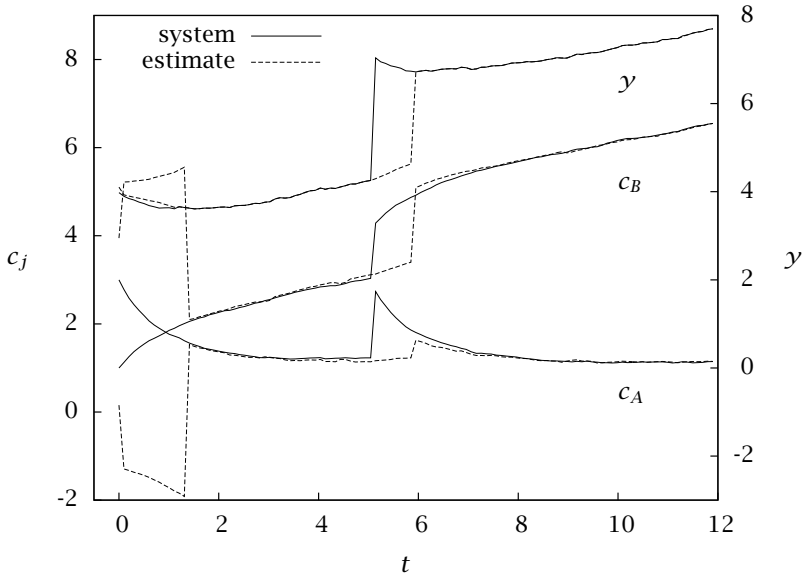


**Figure 4.30:** Combination MHE/PF with simple importance function.

But the inaccurate sample is repaired after $M = 10$ samples. The MHE calculation completes by about $t = 2$, and the samples are reinitialized from the MHE cost function at $t = 1$, and run forward from $t = 1$. These reinitialized samples converge to the true state shortly after $t = 1$.[7]

The disturbance at $t = 5$ also causes the PF samples with the simple importance function to be in the wrong locations. They do not recover and inaccurate estimates are produced by the PF. Another MHE calculation starts at $t = 5$ and finishes at $t = 6$, and the samples are reinitialized with the MHE cost function at $t = 5$ and run forward. After this resampling, the PF estimates again quickly converge to the true estimates after $t = 6$.

Next we use the combination of MHE and PF with the optimal importance function. These results are shown in Figure 4.31. We see as

[7]Even with only 50 particles, we find that particle filtering is not so much faster than MHE, that its computation time can be neglected as we have done here. The two computations take about the *same* time with 50 particles. The computational expense in PF arises from calling an ODE solver 50 times at each sample time. No attempt was made to tailor the ODE solver for efficiency by exploiting the fact that the sample time is small. Note, however, that tailoring the ODE solver would speed up MHE as well as PF.

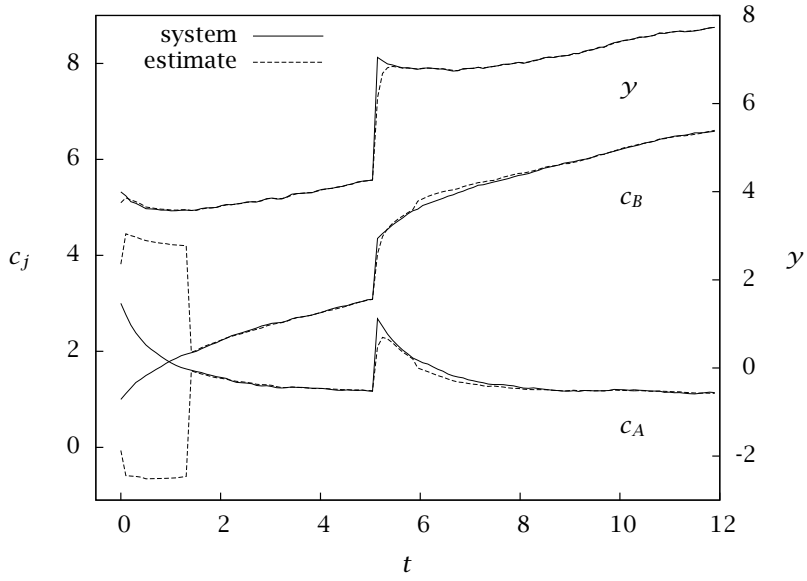


**Figure 4.31:** Combination MHE/PF with optimal importance function.

in the early part of Figure 4.29 that the samples cannot recover from the poor initial state prior and resampling from the MHE cost function takes place at $t = 1$ after the first MHE calculation finishes at $t = 2$. But as in the case of pure PF with the optimal importance function, the disturbance does not move the state so far from the samples that they are unable to recover and continue to provide accurate estimates. The MHE resampling that takes place at $t = 5$ after MHE finishes at $t = 6$ does not modify significantly the PF samples that are already well placed. □

Of course, the simulations shown in Figures 4.28–4.31 display the outcome of only single random realizations. A full characterization of the behavior of the four estimators is determined by running many such random simulations and computing the statistics of interest. We have not compiled these statistics because the single simulations are rather time consuming. After running several random simulations for each estimator, these single simulations were selected manually as representative behavior of the different estimators.

## 4.9 Notes

State estimation is a fundamental topic appearing in many branches of science and engineering, and has a large literature. A nice and brief annotated bibliography describing the early contributions to optimal state estimation of the *linear Gaussian* system is provided by Åström (1970, pp. 252-255). Kailath (1974) provides a comprehensive and historical review of *linear* filtering theory including the historical development of Wiener-Kolmogorov theory for filtering and prediction that preceded Kalman filtering (Wiener, 1949; Kolmogorov, 1941).

Jazwinski (1970) provides an early and comprehensive treatment of the optimal stochastic state estimation problem for linear and *nonlinear* systems. As mentioned in Section 4.2.1, Jazwinski (1970) proves stability of the optimal time-varying state estimator for the linear Gaussian case using $V(k,x) = x'P(k)^{-1}x$ as the Lyapunov function for the linear time-varying system governing estimate error. Note that this dynamic system is time-varying even if the model is time invariant because the optimal estimator gains are time varying. This choice of Lyapunov function has been used to establish estimator stability in many subsequent textbooks (Stengel, 1994, pp.474-475). Kailath (1974, p.152) remarks that the known proofs that the optimal filter is stable "are somewhat difficult, and it is significant that only a small fraction of the vast literature on the Kalman filter deals with this problem." Perhaps the stability analysis developed in Section 4.2 can alleviate the difficulties associated with developing Lyapunov function arguments in optimal estimation.

For establishing stability of the *steady-state* optimal linear estimator, simpler arguments suffice because the estimate error equation is time invariant. Establishing duality with the optimal regulator is a favorite technique for establishing estimator stability in this case. See, for example, Kwakernaak and Sivan (1972, Theorem 4.11) for a general steady-state stability theorem for the linear Gaussian case. This result is proved by establishing properties of the steady-state solution to the Riccati equation for regulation and, by duality, estimation.

Many of the full information and MHE results in this chapter are given by Rao (2000) and Rao, Rawlings, and Mayne (2003). The full information analysis given here is more general because (i) we assume nonlinear detectability rather than nonlinear observability, and (ii) we establish asymptotic stability under process and measurement disturbances, which were neglected in previous analysis.

Muske, Rawlings, and Lee (1993) and Meadows, Muske, and Rawlings (1993) apparently were the first to use the increasing property of the optimal cost to establish asymptotic stability for full information estimation for linear models with constraints. Robertson and Lee (2002) present the interesting statistical interpretation of MHE for the constrained linear system. Michalska and Mayne (1995) establish stability of moving horizon estimation with zero prior weighting for the continuous time nonlinear system.

## 4.10 Exercises

### Exercise 4.1: Input to state stability and convergence

Assume the nonlinear system

$$x^+ = f(x,u)$$

is input to state stable (ISS) so that for all $x_0 \in \mathbb{R}^n$, input sequences $\mathbf{u}$, and $k \geq 0$

$$|x(k;x_0,\mathbf{u})| \leq \beta(|x_0|,k) + \gamma(\|\mathbf{u}\|)$$

in which $x(k;x_0,\mathbf{u})$ is the solution to the system equation at time $k$ starting at state $x_0$ using input sequence $\mathbf{u}$, and $\gamma \in \mathcal{K}$ and $\beta \in \mathcal{KL}$.

(a) Show that the ISS property also implies

$$|x(k;x_0,\mathbf{u})| \leq \beta(|x_0|,k) + \gamma(\|\mathbf{u}\|_{0:k})$$

in which $\|\mathbf{u}\|_{0:k} = \max_{0\leq j\leq k} |u(j)|$.

(b) Show that the ISS property implies the "converging-input converging-state" property (Jiang and Wang, 2001), (Sontag, 1998a, p. 330), i.e., show that if the system is ISS, then $u(k) \to 0$ implies $x(k) \to 0$.

### Exercise 4.2: Output to state stability and convergence

Assume the nonlinear system

$$x^+ = f(x) \qquad y = h(x)$$

is output to state stable (OSS) so that for all $x_0 \in \mathbb{R}^n$ and $k \geq 0$

$$|x(k;x_0)| \leq \beta(|x_0|,k) + \gamma(\|\mathbf{y}\|_{0:k})$$

in which $x(k;x_0)$ is the solution to the system equation at time $k$ starting at state $x_0$, and $\gamma \in \mathcal{K}$ and $\beta \in \mathcal{KL}$.

Show that the OSS property implies the "converging-output converging-state" property (Sontag and Wang, 1997, p. 281) i.e., show that if the system is OSS, then $y(k) \to 0$ implies $x(k) \to 0$.

### Exercise 4.3: i-IOSS and convergence

Prove Proposition 4.2, which states that if system

$$x^+ = f(x,w) \qquad y = g(x)$$

is i-IOSS, and $w_1(k) \to w_2(k)$ and $y_1(k) \to y_2(k)$ as $k \to \infty$, then

$$x(k;z_1,\mathbf{w}_1) \to x(k;z_2,\mathbf{w}_2) \quad \text{as } k \to \infty \quad \text{for all } z_1, z_2$$

### Exercise 4.4: Observability and detectability of linear time-invariant systems and OSS

Consider the linear time-invariant system

$$x^+ = Ax \qquad y = Cx$$

(a) Show that if the system is observable, then the system is OSS.

(b) Show that the system is detectable if and only if the system is OSS.

### Exercise 4.5: Observability and detectability of linear time-invariant system and IOSS

Consider the linear time-invariant system with input

$$x^+ = Ax + Gw \qquad y = Cx$$

(a) Show that if the system is observable, then the system is IOSS.

(b) Show that the system is detectable if and only if the system is IOSS.

### Exercise 4.6: Max or sum?

Given $\gamma_1, \gamma_2 \in \mathcal{K}$, show there exists $a, b > 0$ such that

$$a(\gamma_1(x) + \gamma_2(y)) \le \max(\gamma_1(x), \gamma_2(y)) \le b(\gamma_1(x) + \gamma_2(y))$$

for all $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$. Therefore it is equivalent if ISS or OSS is defined in terms of inequalities using the max or the sum.

### Exercise 4.7: Linear systems and incremental stability

Show that for a linear time-invariant system, i-ISS (i-OSS, i-IOSS) is equivalent to ISS (OSS, IOSS).

### Exercise 4.8: Nonlinear observability and Lipschitz continuity implies i-OSS

Consider the following definition of observability for nonlinear systems in which $f$ and $h$ are Lipschitz continuous. A system

$$x^+ = f(x) \qquad y = h(x)$$

is observable if there exists $N_o \in \mathbb{I}_{\ge 1}$ and $\mathcal{K}$-function $\gamma$ such that

$$\sum_{k=0}^{N_o-1} |y(k;x_1) - y(k;x_2)| \ge \gamma(|x_1 - x_2|) \tag{4.62}$$

holds for all $x_1, x_2 \in \mathbb{R}^n$. This definition was used by Rao et al. (2003) in showing stability of nonlinear MHE to initial condition error under zero state and measurement disturbances.

(a) Show that this form of nonlinear observability implies i-OSS.

(b) Show that i-OSS does not imply this form of nonlinear observability and, therefore, i-OSS is a weaker assumption.

The i-OSS concept generalizes the linear system concept of detectability to nonlinear systems.

### Exercise 4.9: Robust GAS implies GAS in estimation

Show that robust GAS of an estimator implies GAS for the estimator.

### Exercise 4.10: Relationships between observability, FSO, MHE detectability and i-IOSS

Show that for the nonlinear system $x^+ = f(x,w)$, $y = h(x)$ with Lipschitz continuous $f$ and $h$, the following relationships hold between observability, FSO, MHE detectability, and i-IOSS (detectability).

$$\begin{aligned}&\text{observable} \implies \text{FSO} \implies \text{MHE detectable} \implies \text{i-IOSS}\\ &\text{observable} \not\Leftarrow \text{FSO} \not\Leftarrow \text{MHE detectable} \not\Leftarrow \text{i-IOSS}\end{aligned}$$

### Exercise 4.11: Observability, FSO, and detectability of linear systems

Consider the linear time-invariant system

$$x^+ = Ax \qquad y = Cx$$

and its observability canonical form. What conditions must the system satisfy to be

(a) observable?

(b) final-state observable (FSO)?

(c) detectable?

### Exercise 4.12: Dynamic programming recursion for Kalman predictor

In the Kalman predictor, we use forward DP to solve at stage $k$

$$\min_{x,w} \ell(x,w) + V_k^-(x) \qquad \text{s.t. } z = Ax + w$$

in which $x$ is the state at the current stage and $z$ is the state at the next stage. The stage cost and arrival cost are given by

$$\ell(x,w) = (1/2)(\,|y(k) - Cx|^2_{R^{-1}} + w'Q^{-1}w) \qquad V_k^-(x) = (1/2)\,|x - \hat{x}^-(k)|^2_{(P^-(k))^{-1}}$$

and we wish to find the value function $V^0(z)$, which we denote $V^-_{k+1}(z)$ in the Kalman predictor estimation problem.

(a) Combine the two $x$ terms to obtain

$$\min_{x,w} \frac{1}{2}\left(w'Q^{-1}w + (x - \hat{x}(k))'P(k)^{-1}(x - \hat{x}(k))\right) \qquad \text{s.t. } z = Ax + w$$

and, using the third part of Example 1.1, show

$$\begin{aligned}P(k) &= P^-(k) - P^-(k)C'(CP^-(k)C' + R)^{-1}CP^-(k)\\ L(k) &= P^-(k)C'(CP^-(k)C' + R)^{-1}C'R^{-1}\\ \hat{x}(k) &= \hat{x}^-(k) + L(k)(y(k) - C\hat{x}^-(k))\end{aligned}$$

(b) Add the $w$ term and use the inverse form in Exercise 1.18 to show the optimal cost is given by

$$\begin{aligned}V^0(z) &= (1/2)(z - A\hat{x}^-(k+1))'(P^-(k+1))^{-1}(z - A\hat{x}^-(k+1))\\ \hat{x}^-(k+1) &= A\hat{x}(k)\\ P^-(k+1) &= AP(k)A' + Q\end{aligned}$$

Substitute the results for $\hat{x}(k)$ and $P(k)$ above and show

$$
\begin{aligned}
V_{k+1}^{-}(z) &= (1/2)(z-\hat{x}^{-}(k+1))'(P^{-}(k+1))^{-1}(z-\hat{x}(k+1)) \\
P^{-}(k+1) &= Q + AP^{-}(k)A' - AP^{-}(k)C'(CP^{-}(k)C'+R)^{-1}CP^{-}(k)A \\
\hat{x}^{-}(k+1) &= A\hat{x}^{-}(k) + \widetilde{L}(k)(y(k)-C\hat{x}^{-}(k)) \\
\widetilde{L}(k) &= AP^{-}(k)C'(CP^{-}(k)C'+R)^{-1}
\end{aligned}
$$

(c) Compare and contrast this form of the estimation problem to the one given in Exercise 1.29 that describes the Kalman filter.

### Exercise 4.13: Duality, cost to go, and covariance

Using the duality variables of Table 4.2, translate Theorem 4.10 into the version that is relevant to the state estimation problem.

### Exercise 4.14: Estimator convergence for $(A, G)$ not stabilizable

What happens to the stability of the optimal estimator if we violate the condition

$$(A, G) \text{ stabilizable}$$

(a) Is the steady-state Kalman filter a stable estimator? Is the full information estimator a stable estimator? Are these two answers contradictory? Work out the results for the case $A = 1, G = 0, C = 1, P^{-}(0) = 1, Q = 1, R = 1$.
Hint: you may want to consult de Souza, Gevers, and Goodwin (1986).

(b) Can this phenomenon happen in the LQ regulator? Provide the interpretation of the time-varying regulator that corresponds to the time-varying filter given above. Does this make sense as a regulation problem?

### Exercise 4.15: Exponential stability of the Kalman predictor

Establish that the Kalman predictor defined in Section 4.2.1 is a globally exponentially stable estimator. What is the corresponding linear quadratic regulator?

### Exercise 4.16: The resampling theorem

Generalize the proof of Theorem 4.35 to cover any number of samples.
Hint: you may find the multinomial expansion formula useful

$$(x_1 + x_2 + \cdots + x_s)^k = \sum_{r_1=0}^{k} \sum_{r_2=0}^{k} \cdots \sum_{r_s=0}^{k} a(r_1, r_2, \ldots, r_s) x_1^{r_1} x_2^{r_2} \cdots x_s^{r_s}$$

in which the coefficients in the expansion formula are given by Feller (1968, p.37)

$$a(r_1, r_2, \ldots, r_s) = \begin{cases} \dfrac{k!}{r_1! r_2! \cdots r_s!} & r_1 + r_2 + \cdots + r_s = k \\ 0 & r_1 + r_2 + \cdots + r_s \neq k \end{cases} \tag{4.63}$$

### Exercise 4.17: Pruning while resampling

Sometimes it is convenient in a simulation to reduce the number of samples when resampling a density. In many discrete processes, for example, the number of possible states that may be reached in the simulation increases with time. To keep the number of samples constant, we may wish to remove samples at each time through the resampling process. Consider a modification of Theorem 4.35 in which the number of resamples is $\tilde{s}$, which does not have to be equal to $s$.

**Theorem 4.41** (Resampling and pruning)**.** *Consider a sampled density $p(x)$ with $s$ samples at $x = x_i$ and associated weights $w_i$*

$$p(x) = \sum_{i=1}^{s} w_i \delta(x - x_i) \qquad w_i \geq 0 \qquad \sum_{i=1}^{s} w_i = 1$$

*Consider the resampling procedure that gives a resampled density with $\tilde{s} > 0$ samples*

$$\tilde{p}(x) = \sum_{i=1}^{\tilde{s}} \tilde{w}_i \delta(x - \tilde{x}_i)$$

*in which the $\tilde{x}_i$ are chosen according to resample probability $p_r$*

$$p_r(\tilde{x}_i) = \begin{cases} w_j, & \tilde{x}_i = x_j \\ 0, & \tilde{x}_i \neq x_j \end{cases}$$

*and with uniform weights $\tilde{w}_i = 1/\tilde{s}$. Consider a function $f(\cdot)$ defined on a set $X$ containing the points $x_i$.*

*Under this resampling procedure, the expectation over resampling of any integral of the resampled density is equal to that same integral of the original density*

$$\mathcal{E}_r \left( \int f(x) \tilde{p}(x) dx \right) = \int f(x) p(x) dx \qquad \text{all } f$$

(a) Is the proposed theorem correct? If so, prove it. If not, provide a counterexample.

(b) What do you suppose happens in a simulation if we perform aggressive pruning by always choosing $\tilde{s} = 1$?

### Exercise 4.18: Properties of the importance function

It is stated in the chapter that $p(\mathbf{x}(k+1)|\mathbf{y}(k+1))$ does not satisfy the second importance function property listed in (4.58)

$$q(\mathbf{x}(k+1)|\mathbf{y}(k+1)) = q(x(k+1)|x(k), y(k+1))\, q(\mathbf{x}(k)|\mathbf{y}(k)) \tag{4.64}$$

Derive a similar property that $p(\mathbf{x}(k+1)|\mathbf{y}(k+1))$ *does* satisfy. What has been altered in (4.64)? Why do you think this change has been made?

### Exercise 4.19: A single sample of joint density

Consider again Example 4.31 in which we have $s_x$ and $s_y$ samples of the marginals of independent random variables $\xi$ and $\eta$, respectively

$$\xi \sim \{x_i, w_{xi}\} \qquad w_{xi} = 1/s_x, \quad i = 1, \ldots, s_x$$
$$\eta \sim \{y_j, w_{yj}\} \qquad w_{yj} = 1/s_y, \quad j = 1, \ldots, s_y$$

and wish to sample the joint density $p_{\xi,\eta}(x,y) = p_\xi(x)p_\eta(y)$. Show that selecting any single sample is a valid sample of the joint density

$$\{(x_1, y_1), w\}, \quad w = 1$$

### Exercise 4.20: Kolmogorov-Smirnov limit theorem for sampling error

Consider again $s$ mutually independent samples taken from cumulative distribution $P(x)$ to produce the sampled cumulative distribution $P_s(x;s)$ as discussed in Section 4.7.2. Define sampling error as in the chapter

$$D_s = \sup_x |P_s(x;s) - P(x)|$$

(a) Reproduce the results of Example 4.30. Plot the actual and limiting distributions for $D_s$ for $s = 10, 100, 1000$ when sampling a normal distribution with unit variance. Your result should resemble Figure 4.14

(b) Now compute the actual and limiting probability *densities* of the sampling error $p(D_s)$ rather than the distribution $\Pr(D_s)$. Give a formula for $l(z) = dL(z)/dz$. Plot $p(D_s)$ for $s = 10, 100, 1000$ samples for sampling the normal distribution with unit variance.

### Exercise 4.21: Sampled density from a weighted importance function

Given a weighted sample of an importance function $q(x)$

$$q_s(x) = \sum_{i=1}^{s} w_i^- \delta(x - x_i) \qquad \sum_i w_i^- = 1$$

(a) Show that the sampled density

$$\overline{p}_s(x) = \sum_{i=1}^{s} w_i \delta(x - x_i) \qquad w_i = w_i^- \frac{p(x_i)}{q(x_i)}$$

converges to $p(x)$ as sample size increases.

(b) Show that the sampled density is unbiased for all samples sizes.

### Exercise 4.22: Sampled density from a weighted importance function when unable to evaluate the density

Given a weighted sample of an importance function $q(x)$

$$q_s(x) = \sum_{i=1}^{s} w_i^- \delta(x - x_i) \qquad \sum_i w_i^- = 1$$

and a density of the following form

$$p(x) = \frac{h(x)}{\int h(x)dx}$$

in which $p(x)$ cannot be conveniently evaluated but $h(x)$ can be evaluated.

(a) Show that the sampled density

$$\overline{p}_s(x) = \sum_{i=1}^{s} \overline{w}_i \delta(x - x_i) \qquad w_i = w_i^- \frac{h(x_i)}{q(x_i)} \qquad \overline{w}_i = \frac{w_i}{\sum_j w_j}$$

converges to $p(x)$ as sample size increases.

(b) Show that the sampled density is biased for all finite sample sizes.

### Exercise 4.23: Unbiased particle filter with importance sampling

Show that an unbiased particle filter using importance sampling is given by

$$\overline{p}_s(x(k)|\mathbf{y}(k)) = \{x_i(k), \widetilde{w}_i(k)\}$$

$$\widetilde{w}_i(k+1) = \widetilde{w}_i(k) \frac{p(y(k+1)|x_i(k+1))\, p(x_i(k+1)|x_i(k))}{p(y(k+1)|\mathbf{y}(k))\, q(x_i(k+1)|x_i(k), y(k+1))}$$

in which $x_i(k)$ are samples of the importance function $q(x(k)|x_i(k-1), y(k))$. Note that normalization of $\widetilde{w}_i$ is not required in this form of a particle filter, but evaluation of $p(y(k+1)|\mathbf{y}(k))$ is required.

### Exercise 4.24: Variance of the unbiased particle filter with importance sampling

Show that the variance of the weights of the unbiased particle filter given in Exercise 4.23 increases with time.

### Exercise 4.25: Optimal importance function for a linear system

The optimal importance function is given in (4.60), repeated here

$$q(x_i(k)|x_i(k-1), y(k)) = p(x_i(k)|x_i(k-1), y(k))$$

For the linear time-invariant model, this conditional density is the following normal density (Doucet et al., 2000)

$$p(x_i(k)|x_i(k-1), y(k)) \sim N(\overline{x}(k), \overline{P})$$

$$\overline{x}(k) = \overline{P}Q^{-1}(Ax_i(k-1) + Bu(k-1)) + \overline{P}C'R^{-1}y(k)$$

$$\overline{P} = \left(Q^{-1} + C'R^{-1}C\right)^{-1}$$

Establish this result by first considering the linear transformation between $(x_i(k), y(k))$ and $x_i(k-1), w(k), v(k)$, and then using the formulas for taking conditional densities of normals.

### Exercise 4.26: Equivalance of detectability and IOSS for continuous-time, linear, time-invariant system

Consider the continuous-time, linear, time-invariant system with input

$$\dot{x} = Ax + Bu \qquad y = Cx$$

Show that the system is detectable if and only if the system is IOSS.

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# 5
# Output Model Predictive Control

## 5.1 Introduction

In Chapter 2 we show how model predictive control (MPC) may be employed to control a *deterministic* system, that is, a system in which there are no uncertainties and the state is known. In Chapter 3 we show how to control an *uncertain* system in which uncertainties are present but the state is known. Here we address the problem of MPC of an uncertain system in which the state is *not* fully known. We assume that there are outputs available that may be used to estimate the state as shown in Chapter 4. These outputs are used by the model predictive controller to generate control actions; hence the name *output MPC*.

Output feedback control is, in general, more complex than state feedback control since knowledge of the state provides considerable information. If the state is known, optimal control is, in general, a time-varying function of the current state *even if* the system is uncertain as, for example, when it is subject to an additive disturbance. In this case, the state must include the state of the disturbance.

Generally, however, the state is not known; instead, a noisy measurement $y(t)$ of the state is available at each time $t$. Since the state $x$ is not known, it is replaced by a hyperstate $p$ that summarizes all prior information (previous inputs and outputs and the prior distribution of the initial state) and that has the "state" property: future values of $p$ can be determined from the current value of $p$ and current and future inputs and outputs. Usually $p(t)$ is the conditional density of $x(t)$ given the prior density $p(0)$ of $x(0)$ and the current available "information" $I(t) := \{y(0), y(1), \ldots, y(t-1), u(0), u(1), \ldots, u(t-1)\}$. If the current hyperstate is known, future hyperstates have to be predicted since future noisy measurements of the state are not known. So

the hyperstate satisfies an uncertain difference equation of the form

$$p^+ = \phi(p, u, \psi) \tag{5.1}$$

where $\{\psi(t)\}$ is a sequence of random variables; the problem of controlling a system with unknown state $x$ is transformed into the problem of controlling an uncertain system with known state $p$. For example, if the underlying system is described by

$$x^+ = Ax + Bu + w$$
$$y = Cx + v$$

where $\{w(t)\}$ and $\{v(t)\}$ are sequences of zero mean normal independent random variables with variances $\Sigma_w$ and $\Sigma_v$, respectively, and if the prior density $p(0)$ of $x(0)$ is normal with density $n(\bar{x}_0, \Sigma_0)$ then, as is well known, $p(t)$ is the normal density $n(\hat{x}(t), \Sigma(t))$ so that the hyperstate $p(t)$ is finitely parameterized by $(\hat{x}(t), \Sigma(t))$; hence the evolution equation for $p(t)$ is defined by the evolution equation for $(\hat{x}, \Sigma)$, that is by:

$$\hat{x}(t+1) = A\hat{x}(t) + Bu + L(t)\psi(t) \tag{5.2}$$
$$\Sigma(t+1) = \Phi(\Sigma(t)) \tag{5.3}$$

in which

$$\Phi(\Sigma) := A\Sigma A' - A\Sigma C'(C'\Sigma C + \Sigma_v)^{-1}C\Sigma A' + \Sigma_w$$
$$\psi(t) := y(t) - C\hat{x}(t) = C\tilde{x}(t) + v(t)$$
$$\tilde{x}(t) := x(t) - \hat{x}(t)$$

The initial conditions for (5.2) and (5.3) are

$$\hat{x}(0) = \bar{x}_0 \qquad \Sigma(0) = \Sigma_0$$

These are, of course, the celebrated Kalman filter equations derived in Chapter 1. The random variables $\tilde{x}$ and $\psi$ have the following densities: $\tilde{x}(t) \sim n(0, \Sigma(t))$ and $\psi(t) \sim n(0, \Sigma_v + C'\Sigma(t)C)$. The finite dimensional equations (5.2) and (5.3) replace the difference equation (5.1) for the hyperstate $p$ that is a conditional density and, therefore, infinite dimensional in general. The sequence $\{\psi(t)\}$ is known as the *innovation* sequence; $\psi(t)$ is the "new" information contained in $y(t)$.

Output control, in general, requires control of the hyperstate $p$ which may be computed and is, therefore, known, but which satisfies

a complex evolution equation $p^+ = \phi(p, u, \psi)$ where $\psi$ is a random disturbance. Controlling $p$ is a problem of the same type as that considered in Chapter 3, but considerably more complex since the function $p(\cdot)$ is infinite dimensional. Because of the complexity of the evolution equation for $p$, the separation principle is often invoked; assuming that the state $x$ is known, a stabilizing controller $u = \kappa(x)$ and an observer or filter yielding an estimate $\hat{x}$ of the state are separately designed; the control $u = \kappa(\hat{x})$ is then applied to the plant. Indeed, this form of control is actually optimal for the linear quadratic Gaussian (LQG) optimal control problem considered briefly above but is not necessarily stabilizing when the system is nonlinear and constrained. We propose a variant of this procedure, modified to cope with state and control constraints. The state estimate $\hat{x}$ satisfies an uncertain difference equation with an additive disturbance of the same type as that considered in Chapter 3. Hence we employ tube MPC, similar to that employed in Chapter 3, but modified to ensure that state estimation error, not considered in Chapter 3, does not result in transgression of the control and state constraints. An advantage of the method presented here is that its online complexity is comparable to that of conventional MPC.

As in Chapter 3, a caveat is necessary. Because of the inherent complexity of output MPC, different compromises between simplicity and efficiency are possible; for this reason, output MPC remains an active research area and alternative methods, available or yet to be developed, may be preferred.

## 5.2 A Method for Output MPC

Suppose the system to be controlled is described by

$$x^+ = f(x, u, w) \tag{5.4}$$

$$y = h(x, v) \tag{5.5}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$; the disturbance $w$ lies in $\mathbb{R}^n$, and the measurement noise $v$ lies in $\mathbb{R}^p$ A prime requirement for simplification is replacement of the infinite dimensional hyperstate $p$, which is $n(\hat{x}, \Sigma)$ in the linear Gaussian case, by something considerably simpler. The hyperstate $p$, being a conditional density, may be regarded as a continuum of nested confidence regions, each of which is a subset of $\mathbb{R}^n$. Our initial simplification is the replacement, if this is possible, of this continuum of confidence regions by a single region of the form $\{\hat{x}\} \oplus \mathbb{\Sigma} \subseteq \mathbb{R}^n$, where $\hat{x}$ is the "center" of the confidence region and $\mathbb{\Sigma}$

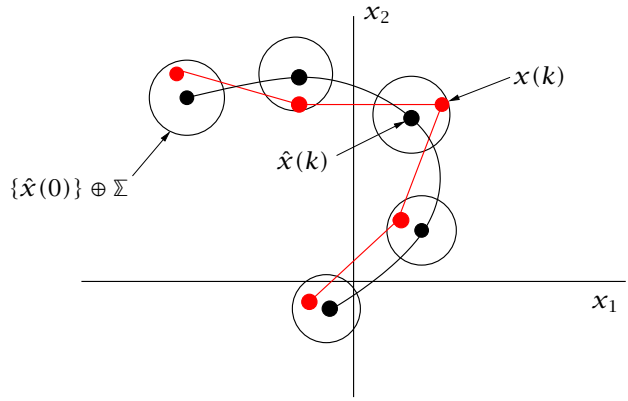


**Figure 5.1:** State estimator tube.

*now* denotes a subset of $\mathbb{R}^n$ rather than a variance. If the problem is stochastic, $\{\hat{x}\} \oplus \mathbb{\Sigma}$ may be a $\beta$ confidence region for $p$, i.e., a region satisfying $\Pr\{x \in \{\hat{x}\} \oplus \mathbb{\Sigma} | I\} = \beta$. When all disturbances are bounded, the usual assumption in robust MPC, $\mathbb{\Sigma}$ is chosen to ensure that all possible values of $x$ lie in the set $\{\hat{x}\} \oplus \mathbb{\Sigma}$. The finite dimensional variable $(\hat{x}, \mathbb{\Sigma})$ replaces the infinite dimensional object $p$.[1] In the linear time-invariant case, the state estimator $(\hat{x}, \mathbb{\Sigma})$ evolves, as shown in the sequel, according to

$$\hat{x}^+ = \phi(\hat{x}, u, \psi) \tag{5.6}$$

$$\mathbb{\Sigma}^+ = \Phi(\mathbb{\Sigma}) \tag{5.7}$$

in which $\psi$ is a random variable in the stochastic case and a bounded disturbance taking values in $\mathbb{\Psi}$ when $w$ and $v$ are bounded. In the latter case, $x \in \{\hat{x}\} \oplus \mathbb{\Sigma}$ implies $x^+ \in \{\hat{x}^+\} \oplus \mathbb{\Sigma}^+$ for all $\psi \in \mathbb{\Psi}$.

More generally, let $X \subseteq \mathbb{R}^n$ denote the set of states consistent with the current information $I$; although $X = \{\hat{x}\} \oplus \mathbb{\Sigma}$ in the linear case, the evolution of $X$, in the nonlinear case, is more complex than (5.6) and (5.7). The hope remains that $X$ has an outer approximation of the form $\{\hat{x}\} \oplus \mathbb{\Sigma}$ where $\hat{x}$ may be obtained by one of the methods described in Chapter 4; however $\mathbb{\Sigma}$ may no longer be independent of the observation sequence.

As illustrated in Figure 5.1, the evolution equations generate a *tube*, which is the set sequence $\{\{\hat{x}(t)\} \oplus \mathbb{\Sigma}(t)\}$; at time $t$ the center of the

[1] The object $\{\hat{x}\} \oplus \mathbb{\Sigma}$ may be regarded (Moitié, Quincampoix, and Veliov, 2002) as the "state" for the output MPC problem.

tube is $\hat{x}(t)$ and the "cross section" is $\mathbb{\Sigma}(t)$. When the disturbances are bounded, which is the only case we consider in the sequel, all possible realizations of the state trajectory $\{x(t)\}$ lie in the set $\{\{\hat{x}(t)\} \oplus \mathbb{\Sigma}(t)\}$ for all $t$; the dashed line is a sample trajectory of $x(t)$.

From (5.6), the estimator trajectory $\{\hat{x}(t)\}$ is influenced both by the control that is applied and by the disturbance sequence $\{\psi(t)\}$. If the trajectory were influenced only by the control, we could choose the control to satisfy both the control constraints and to cause the estimator tube to lie in a region such that the state constraints are satisfied by all possible realizations of the state trajectory. Hence the output MPC problem would reduce to a conventional MPC problem with modified constraints in which the state is $\hat{x}$, rather than $x$. The new state constraint is $\hat{x} \in \hat{\mathbb{X}}$ where $\hat{\mathbb{X}}$ is chosen to ensure that $\hat{x} \in \hat{\mathbb{X}}$ implies $x \in \mathbb{X}$ and, therefore, satisfies $\hat{\mathbb{X}} \subseteq \mathbb{X} \ominus \mathbb{\Sigma}$ if $\mathbb{\Sigma}$ does not vary with time $t$.

But the estimator state $\{\hat{x}(t)\}$ is influenced by the disturbance $\psi$ (see (5.6)), so it cannot be precisely controlled. The problem of controlling the system described by (5.6) is the same type of problem studied in Chapter 3, where the system was described by $x^+ = f(x, u, w)$ with the estimator state $\hat{x}$, which is accessible, replacing the actual state $x$. Hence we may use the techniques presented in Chapter 3 to choose a control that forces $\hat{x}$ to lie in another tube $\{\{z(t)\} \oplus \mathbb{S}(t)\}$ where the set sequence $\{\mathbb{S}(t)\}$ that defines the cross section of the tube is precomputed, and $\{z(t)\}$ that defines the center of the tube is the state trajectory of the nominal (deterministic) system defined by

$$z^+ = \phi(z, u, 0) \tag{5.8}$$

which is the nominal version of (5.6). Equations (5.8) is obtained by replacing $\psi$ by 0 in the original equations. Thus we get two tubes, one embedded in the other. At time $t$ the estimator state $\hat{x}(t)$ lies in the set $\{z(t)\} \oplus \mathbb{S}(t)$, and $x(t)$ lies in the set $\{\hat{x}(t)\} \oplus \mathbb{\Sigma}(t)$, so that for all $t$

$$x(t) \in \{z(t)\} \oplus \mathbb{\Gamma}(t) \qquad \mathbb{\Gamma}(t) := \mathbb{\Sigma}(t) \oplus \mathbb{S}(t)$$

The tubes $\{\{z(t)\} \oplus \mathbb{S}(t)\}$, in which the trajectory $\{\hat{x}(t)\}$ lies, and $\{\{z(t)\} \oplus \mathbb{\Gamma}(t)\}$, in which the state trajectory $\{x(t)\}$ lies, are shown in Figure 5.2. The tube $\{\{z(t)\} \oplus \mathbb{S}(t)\}$ is embedded in the larger tube $\{\{z(t)\} \oplus \mathbb{\Gamma}(t)\}$.

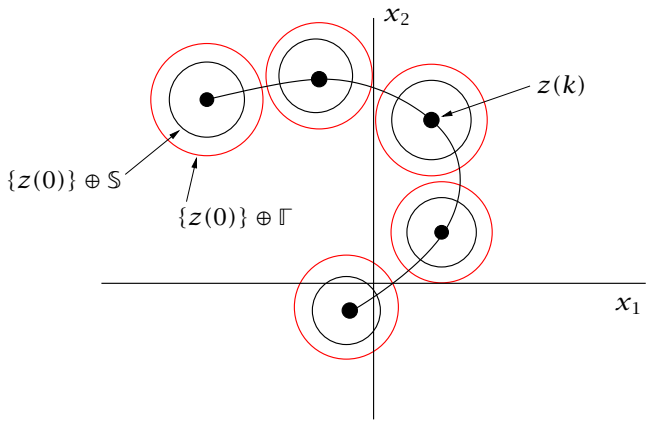


**Figure 5.2:** State tube.

## 5.3 Linear Constrained Systems: Time-Invariant Case

### 5.3.1 Introduction

We consider the following uncertain linear time-invariant system

$$\begin{aligned} x^+ &= Ax + Bu + w \\ y &= Cx + v \end{aligned} \tag{5.9}$$

in which $x \in \mathbb{R}^n$ is the current state, $u \in \mathbb{R}^m$ is the current control action, $x^+$ is the successor state, $w \in \mathbb{R}^n$ is an unknown state disturbance, $y \in \mathbb{R}^p$ is the current measured output, $v \in \mathbb{R}^p$ is an unknown output disturbance, the pair $(A, B)$ is assumed to be controllable, and the pair $(A, C)$ observable. The state and additive disturbances $w$ and $v$ are known only to the extent that they lie, respectively, in the $C$-sets[2] $\mathbb{W} \subseteq \mathbb{R}^n$ and $\mathbb{N} \subseteq \mathbb{R}^p$. Let $\phi(i; x(0), \mathbf{u}, \mathbf{w})$ denote the solution of (5.9) at time $i$ if the initial state at time 0 is $x(0)$, and the control and disturbance sequences are, respectively, $\mathbf{u} := \{u(0), u(1), \ldots\}$ and $\mathbf{w} := \{w(0), w(1), \ldots\}$. The system (5.9) is subject to the following set of hard state and control constraints

$$x \in \mathbb{X} \qquad u \in \mathbb{U} \tag{5.10}$$

where $\mathbb{X} \subseteq \mathbb{R}^n$ and $\mathbb{U} \subseteq \mathbb{R}^m$ are polyhedral and polytopic sets respectively; both sets contain the origin as an interior point.

[2] Recall, a $C$-set is a convex, compact set containing the origin.

### 5.3.2 State Estimator

To estimate the state a simple Luenberger observer is employed

$$\begin{aligned} \hat{x}^+ &= A\hat{x} + Bu + L(y - \hat{y}) \\ \hat{y} &= C\hat{x} \end{aligned} \tag{5.11}$$

where $\hat{x} \in \mathbb{R}^n$ is the current observer state (state estimate), $u \in \mathbb{R}^m$ is the current control action, $\hat{x}^+$ is the successor state of the observer system, $\hat{y} \in \mathbb{R}^p$ is the current observer output, and $L \in \mathbb{R}^{n \times p}$. The output injection matrix $L$ is chosen to satisfy $\rho(A_L) < 1$ where $A_L := A - LC$.

The estimated state $\hat{x}$ therefore satisfies the following uncertain difference equation

$$\hat{x}^+ = A\hat{x} + Bu + L(C\tilde{x} + v)$$

where the state estimation error $\tilde{x}$ is defined by $\tilde{x} := x - \hat{x}$ so that $x = \hat{x} + \tilde{x}$. Since $x^+ = Ax + Bu + w$, the state estimation error $\tilde{x}$ satisfies

$$\tilde{x}^+ = A_L\tilde{x} + \tilde{w} \qquad \tilde{w} := w - Lv \tag{5.12}$$

Because $w$ and $v$ are bounded, so is $\tilde{w}$; in fact, $\tilde{w}$ takes values in the $C$-set $\tilde{\mathbb{W}}$ defined by

$$\tilde{\mathbb{W}} := \mathbb{W} \oplus (-L\mathbb{N})$$

We recall the following standard definitions (Blanchini, 1999):

**Definition 5.1** (Positive invariance; robust positive invariance)**.** A set $\Omega \subseteq \mathbb{R}^n$ is *positive invariant* for the system $x^+ = f(x)$ and the constraint set $\mathbb{X}$ if $\Omega \subseteq \mathbb{X}$ and $f(x) \in \Omega,\ \forall x \in \Omega$.
A set $\Omega \subseteq \mathbb{R}^n$ is *robust positive invariant* for the system $x^+ = f(x, w)$ and the constraint set $(\mathbb{X}, \mathbb{W})$ if $\Omega \subseteq \mathbb{X}$ and $f(x, w) \in \Omega,\ \forall w \in \mathbb{W}$, $\forall x \in \Omega$.

Since $\rho(A_L) < 1$ and $\tilde{\mathbb{W}}$ is compact, there exists, as shown in Kolmanovsky and Gilbert (1998), Theorem 4.1, a robust positive invariant set $\mathbb{\Sigma} \subseteq \mathbb{R}^n$, satisfying

$$A_L\mathbb{\Sigma} \oplus \tilde{\mathbb{W}} = \mathbb{\Sigma} \tag{5.13}$$

Hence, for all $\tilde{x} \in \mathbb{\Sigma}$, $\tilde{x}^+ = A_L\tilde{x} + \tilde{w} \in \mathbb{\Sigma}$ for all $\tilde{w} \in \tilde{\mathbb{W}}$; the term *robust* in the description of $\mathbb{\Sigma}$ refers to this property. In fact, $\mathbb{\Sigma}$ is the *minimal* robust, positive invariant set for $\tilde{x}^+ = A_L\tilde{x} + \tilde{w}$, $\tilde{w} \in \tilde{\mathbb{W}}$, i.e., a set that is a subset of all robust positive invariant sets. There exist techniques

(Raković, Kerrigan, Kouramas, and Mayne, 2005) for obtaining, for every $\epsilon > 0$, a polytopic, nonminimal, robust, positive invariant set $\mathbb{Z}^0$ that satisfies $d_H(\mathbb{Z}, \mathbb{Z}^0) \leq \epsilon$ where $d_H(\cdot, \cdot)$ is the Hausdorff metric. An immediate consequence of (5.13) is:

**Proposition 5.2** (Proximity of state and state estimate)**.** *If the initial system and observer states, $x(0)$ and $\hat{x}(0)$ respectively, satisfy $\{x(0)\} \in \{\hat{x}(0)\} \oplus \mathbb{Z}$, then $x(i) \in \{\hat{x}(i)\} \oplus \mathbb{Z}$ for all $i \in \mathbb{I}_{\geq 0}$, and all admissible disturbance sequences* $\mathbf{w}$ *and* $\boldsymbol{\nu}$*.*

The assumption that $\widetilde{x}(i) \in \mathbb{Z}$ for all $i$ is a *steady-state* assumption; if $\widetilde{x}(0) \in \mathbb{Z}$, then $\widetilde{x}(i) \in \mathbb{Z}$ for all $i$. If, on the other hand, $\widetilde{x}(0) \in \mathbb{Z}(0)$ where $\mathbb{Z}(0) \supseteq \mathbb{Z}$, then it is possible to show that $\widetilde{x}(i) \in \mathbb{Z}(i)$ for all $i \in \mathbb{I}_{\geq 0}$ where $\mathbb{Z}(i) \to \mathbb{Z}$ in the Hausdorff metric as $i \to \infty$; the sequence $\{\mathbb{Z}(i)\}$ satisfies $\mathbb{Z}(0) \supseteq \mathbb{Z}(1) \supseteq \mathbb{Z}(2) \supseteq \cdots \supseteq \mathbb{Z}$. Hence, it is reasonable to assume that if the estimator has been running for a "long" time, it is in steady state.

Hence we have obtained a state estimator, with "state" $(\hat{x}, \mathbb{Z})$ satisfying

$$\begin{aligned} \hat{x}^+ &= A\hat{x} + Bu + L(y - \hat{y}) \\ \mathbb{Z}^+ &= \mathbb{Z} \end{aligned} \tag{5.14}$$

and $x(i) \in \hat{x}(i) \oplus \mathbb{Z}$ for all $i \in \mathbb{I}_{\geq 0}$, thus meeting the requirements specified in Section 5.2. Knowing this, our remaining task is to control $\hat{x}(i)$ so that the resultant closed-loop system is stable and satisfies all constraints.

### 5.3.3 Controlling $\hat{x}$

Since $\widetilde{x}(i) \in \mathbb{Z}$ for all $i$, we seek a method for controlling the observer state $\hat{x}(i)$ in such a way that $x(i) = \hat{x}(i) + \widetilde{x}(i)$ satisfies the state constraint $x(i) \in \mathbb{X}$ for all $i$. The state constraint $x(i) \in \mathbb{X}$ will be satisfied if we control the estimator state to satisfy $\hat{x}(i) \in \mathbb{X} \ominus \mathbb{Z}$ for all $i$. The estimator state satisfies (5.14) which can be written in the form

$$\hat{x}^+ = A\hat{x} + Bu + \delta \tag{5.15}$$

where the disturbance $\delta$ is defined by

$$\delta := L(y - \hat{y}) = L(C\widetilde{x} + \nu)$$

and, therefore, always lies in the $C$-set $\mathbb{\Delta}$ defined by

$$\mathbb{\Delta} := L(C\mathbb{Z} \oplus \mathbb{N})$$

The problem of controlling $\hat{x}$ is, therefore, the same as that of controlling an uncertain system with known state. This problem was extensively discussed in Chapter 3. We can therefore use the approach of Chapter 3 here with $\hat{x}$ replacing $x$, $\delta$ replacing $w$, $\mathbb{X} \ominus \mathbb{Z}$ replacing $\mathbb{X}$ and $\mathbb{A}$ replacing $\mathbb{W}$.

To control (5.15) we use, as in Chapter 3, a combination of open-loop and feedback control, i.e., we choose the control $u$ as follows

$$u = v + Ke \qquad e := \hat{x} - z \tag{5.16}$$

where $z$ is the state of a nominal (deterministic) system that we shall shortly specify; $v$ is the feedforward component of the control $u$, and $Ke$ is the feedback component. The matrix $K$ is chosen to satisfy $\rho(A_K) < 1$ where $A_K := A + BK$. The feedforward component $v$ of the control $u$ generates, as we show subsequently, a trajectory $\{z(i)\}$, which is the center of the tube in which the state estimator trajectory $\{\hat{x}(i)\}$ lies. The feedback component $Ke$ attempts to steer the trajectory $\{\hat{x}(i)\}$ of the state estimate toward the center of the tube and thereby controls the cross section of the tube. The controller is *dynamic* since it incorporates the nominal dynamic system.

With this control, $\hat{x}$ satisfies the following difference equation

$$\hat{x}^+ = A\hat{x} + Bv + BKe + \delta \qquad \delta \in \mathbb{A} \tag{5.17}$$

The nominal (deterministic) system describing the evolution of $z$ is obtained by neglecting the disturbances $BKe$ and $\delta$ in (5.17) yielding

$$z^+ = Az + Bv$$

The deviation $e = \hat{x} - z$ between the state $\hat{x}$ of the estimator and the state $z$ of the nominal system satisfies

$$e^+ = A_K e + \delta \qquad A_K := A + BK \tag{5.18}$$

The feedforward component $v$ of the control $u$ generates the trajectory $\{z(i)\}$, which is the center of the tube in which the state estimator trajectory $\{\hat{x}(i)\}$ lies. Because $\mathbb{A}$ is a $C$-set and $\rho(A_K) < 1$, there exists a robust positive invariant $C$-set $\mathbb{S}$ satisfying

$$A_K \mathbb{S} \oplus \mathbb{A} = \mathbb{S}$$

An immediate consequence is the following.

**Proposition 5.3** (Proximity of state estimate and nominal state)**.** *If the initial states of the estimator and nominal system, $\hat{x}(0)$ and $z(0)$ respectively, satisfy $\hat{x}(0) \in \{z(0)\} \oplus \mathbb{S}$, then $\hat{x}(i) \in \{z(i)\} \oplus \mathbb{S}$ and $u(i) \in \{v(i)\} \oplus K\mathbb{S}$ for all $i \in \mathbb{I}_{\geq 0}$, and all admissible disturbance sequences $\mathbf{w}$ and $\boldsymbol{\nu}$.*

It follows from Proposition 5.3 that the state estimator trajectory $\hat{\mathbf{x}}$ remains in the tube $\hat{\mathbf{X}}(z(0), \mathbf{v}) := \{\{z(i)\} \oplus \mathbb{S} \mid i \in \mathbb{I}_{\geq 0}\}$ and the control trajectory $\mathbf{v}$ remains in the tube $\hat{\mathbf{V}}(\mathbf{v}) := \{\{v(i)\} \oplus K\mathbb{S} \mid i \in \mathbb{I}_{\geq 0}\}$ provided that $e(0) \in \mathbb{S}$. Hence, from Propositions 5.2 and 5.3, the state trajectory $\mathbf{x}$ lies in the tube $\mathbf{X}(z(0), \mathbf{v}) := \{\{z(i)\} \oplus \mathbb{\Gamma} \mid i \in \mathbb{I}_{\geq 0}\}$ where $\mathbb{\Gamma} := \mathbb{S} \oplus \mathbb{Z}$ provided that $\tilde{x}(0) = x(0) - \hat{x}(0) \in \mathbb{Z}$ *and* $e(0) \in \mathbb{S}$. This information may be used to construct a robust output feedback model predictive controller using the procedures outlined in Chapter 3 for robust state feedback MPC of systems; the major difference is that we now control the estimator state $\hat{x}$ and use the fact that the actual state $x$ lies in $\{\hat{x}\} \oplus \mathbb{Z}$.

### 5.3.4 Output MPC

Model predictive controllers can now be constructed as described in Chapter 3, which dealt with robust control when the state was known. There is an obvious difference in that we are now concerned with controlling $\hat{x}$ whereas, in Chapter 3, our concern was control of $x$. We describe here the appropriate modification of the simple model predictive controller presented in Section 3.4.2. We adopt the same procedure of defining a nominal optimal control problem with tighter constraints than in the original problem. The solution to this problem defines the center of a tube in which solutions to the original system lie, and the tighter constraints in the nominal problem ensure that the original constraints are satisfied by the actual system.

The nominal system is described by

$$z^+ = Az + Bv \tag{5.19}$$

The nominal optimal control problem is the minimization of the cost function $\bar{V}_N(z, \mathbf{v})$ where

$$\bar{V}_N(z, \mathbf{v}) := \sum_{k=0}^{N-1} \ell(z(k), v(k)) + V_f(z(N)) \tag{5.20}$$

subject to satisfaction by the state and control sequences of (5.19) and the *tighter* constraints

$$z(i) \in \mathbb{Z} \subseteq \mathbb{X} \ominus \mathbb{\Gamma} \qquad \mathbb{\Gamma} := \mathbb{S} \oplus \mathbb{\Sigma} \tag{5.21}$$

$$v(i) \in \mathbb{V} \subseteq \mathbb{U} \ominus K\mathbb{S} \tag{5.22}$$

as well as a terminal constraint $z(N) \in \mathbb{Z}_f \subseteq \mathbb{Z}$. Notice that $\mathbb{\Gamma}$ appears in (5.21) whereas $\mathbb{S}$, the set in which $e = \hat{x} - z$ lies, appears in (5.22); this differs from the case studied in Chapter 3 where the same set appears in both equations. The sets $\mathbb{W}$ and $\mathbb{N}$ are assumed to be sufficiently small to ensure the following condition.

**Assumption 5.4** (Constraint bounds). $\mathbb{\Gamma} = \mathbb{S} \oplus \mathbb{\Sigma} \subseteq \mathbb{X}$ and $K\mathbb{S} \subseteq \mathbb{U}$.

If Assumption 5.4 holds, the sets on the right-hand side of (5.21) and (5.22) are not empty; it can be seen from their definitions that the sets $\mathbb{\Sigma}$ and $\mathbb{S}$ tend to the set $\{0\}$ as $\mathbb{W}$ and $\mathbb{N}$ tend to the set $\{0\}$ in the sense that $d_H(\mathbb{W}, \{0\}) \to 0$ and $d_H(\mathbb{N}, \{0\}) \to 0$.

It follows from Propositions 5.2 and 5.3, if Assumption 5.4 holds, that satisfaction of the constraints (5.21) and (5.22) by the nominal system ensures satisfaction of the constraints (5.10) by the original system. The nominal optimal control problem is, therefore,

$$\mathbb{P}_N(z): \quad \bar{V}_N^0(z) = \min_{\mathbf{v}} \{\bar{V}_N(z, \mathbf{v}) \mid \mathbf{v} \in \mathcal{V}_N(z)\}$$

where the constraint set $\mathcal{V}_N(z)$ is defined by

$$\mathcal{V}_N(z) := \{\mathbf{v} \mid v(k) \in \mathbb{V} \text{ and } \bar{\phi}(k; z, \mathbf{v}) \in \mathbb{Z} \ \forall k \in \{0, 1, \ldots, N-1\}, \\ \bar{\phi}(N; z, \mathbf{v}) \in \mathbb{Z}_f\} \tag{5.23}$$

In (5.23), $\mathbb{Z}_f \subseteq \mathbb{Z}$ is the terminal constraint set, and $\bar{\phi}(k; z, \mathbf{v})$ denotes the solution of $z^+ = Az + Bv$ at time $k$ if the initial state at time 0 is $z$ and the control sequence is $\mathbf{v} = \{v(0), v(1), \ldots, v(N-1)\}$. Let $\mathbf{v}^0(z)$ denote the minimizing control sequence; the stage cost $\ell(\cdot)$ is chosen to ensure uniqueness of $\mathbf{v}^0(z)$. The implicit MPC control law for the nominal system is $\bar{\kappa}_N(\cdot)$ defined by

$$\bar{\kappa}_N(z) := v^0(0; z)$$

where $v^0(0; z)$ is the first element in the sequence $\mathbf{v}^0(z)$. The domain of $\bar{V}_N^0(\cdot)$ and $\mathbf{v}^0(\cdot)$, and, hence, of $\bar{\kappa}_N(\cdot)$, is $\mathcal{Z}_N$ defined by

$$\mathcal{Z}_N := \{z \in \mathbb{Z} \mid \mathcal{V}_N(z) \neq \emptyset\} \tag{5.24}$$

$\mathcal{Z}_N$ is the set of initial states $z$ that can be steered to $\mathbb{Z}_f$ by an admissible control $\mathbf{v}$ that satisfies the state and control constraints, (5.21) and (5.22), and the terminal constraint. From (5.16), the implicit control law for the state estimator $\hat{x}^+ = A\hat{x} + Bu + \delta$ is $\kappa_N(\cdot)$ defined by

$$\kappa_N(\hat{x}, z) := \bar{\kappa}_N(z) + K(\hat{x} - z)$$

The controlled composite system with state $(\hat{x}, z)$ satisfies

$$\hat{x}^+ = A\hat{x} + B\kappa_N(\hat{x}, z) + \delta \tag{5.25}$$

$$z^+ = Az + B\bar{\kappa}_N(z) \tag{5.26}$$

with initial state $(\hat{x}(0), z(0))$ satisfying $\hat{x}(0) \in \{z(0)\} \oplus \mathbb{S}$, $z(0) \in \mathcal{Z}_N$. These constraints are satisfied if $z(0) = \hat{x}(0) \in \mathcal{Z}_N$. The control algorithm may be formally stated as follows:

**Robust control algorithm (linear constrained systems).**

**Initialization:** At time 0, set $i = 0$, set $\hat{x} = \hat{x}(0)$ and set $z = \hat{x}$.

**Step 1:** At time $i$, solve the nominal optimal control problem $\bar{\mathbb{P}}_N(z)$ to obtain the current nominal control action $v = \bar{\kappa}_N(z)$ and the control $u = v + K(\hat{x} - z)$.

**Step 2:** If $\hat{x} \notin \{z\} \oplus \mathbb{S}$ or $u \notin \{v\} \oplus K\mathbb{S}$, set $z = \hat{x}$ and re-solve $\bar{\mathbb{P}}_N(z)$ to obtain $v = \bar{\kappa}_N(z)$ and $u = v$.

**Step 3:** Apply the control $u$ to the system being controlled.

**Step 4:** (a) Compute the successor state estimate $\hat{x}^+ = A\hat{x} + Bu + L(y - C\hat{x})$. (b) Compute the successor state $z^+ = f(z, v)$ of the nominal system.

**Step 5:** Set $(\hat{x}, z) = (\hat{x}^+, z^+)$, set $i = i + 1$ and go to Step 1.

In normal operation, Step 2 is not activated; Propositions 5.2 and 5.3 ensure that the constraints are satisfied. In the event of an unanticipated event, Step 2 is activated, the controller is reinitialized and normal operation resumed. If Step 2 is activated, $v = \bar{\kappa}_N(\hat{x})$ and $u = v$. Hence *nominal* MPC would ensue if Step 2 were activated at each sample.

If the terminal cost $V_f(\cdot)$ and terminal constraint set $\mathbb{Z}_f$ satisfy the stability Assumptions 2.12 and 2.13 of Chapter 2, and if Assump-

tion 5.4 is satisfied, the value function $\bar{V}_N^0(\cdot)$ satisfies

$$\begin{aligned} \bar{V}_N^0(z) &\geq \ell(z, \bar{\kappa}_N(z)) && \forall z \in \mathcal{Z}_N \\ \Delta\bar{V}_N^0(z) &\leq -\ell(z, \bar{\kappa}_N(z)) && \forall z \in \mathcal{Z}_N \\ \bar{V}_N^0(z) &\leq V_f(z) && \forall z \in \mathbb{Z}_f \end{aligned}$$

in which $\Delta\bar{V}_N^0(z) := \bar{V}_N^0(f(z, \bar{\kappa}_N(z))) - \bar{V}_N^0(z)$.

As shown in Section 3.4.3, if, in addition to Assumption 5.4, (i) the stability Assumptions 2.12 and 2.13 are satisfied, (ii) $\ell(z, v) = (1/2)(|z|_Q^2 + |v|_R^2)$ where $Q$ and $R$ are positive definite, (iii) $V_f(z) = (1/2)\,|Z|_{P_f}^2$ where $P_f$ is positive definite, and (iv) $\mathcal{Z}_N$ is a $C$-set, then there exist positive constants $c_1$ and $c_2$ such that

$$\begin{aligned} \bar{V}_N^0(z) &\geq c_1|z|^2 && \forall z \in \mathcal{Z}_N \\ \Delta\bar{V}_N^0(z) &\leq -c_1|z|^2 && \forall z \in \mathcal{Z}_N \\ \bar{V}_N^0(z) &\leq c_2|z|^2 && \forall z \in \mathcal{Z}_N \end{aligned}$$

It follows from Chapter 2 that the origin is exponentially stable for the nominal system $z^+ = Az + B\bar{\kappa}_N(z)$ with a region of attraction $\mathcal{Z}_N$ so that there exists a $c > 0$ and a $\gamma \in (0, 1)$ such that

$$|z(i)| \leq c|z(0)|\gamma^i$$

for all $z(0) \in \mathcal{Z}_N$, all $i \in \mathbb{I}_{\geq 0}$. Also $z(i) \in \mathcal{Z}_N$ for all $i \in \mathbb{I}_{\geq 0}$ if $z(0) \in \mathcal{Z}_N$ so that problem $\mathbb{P}_N(z(i))$ is always feasible. Because the state $\hat{x}(i)$ of the state estimator always lies in $\{z(i)\} \oplus \mathbb{S}$, and the state $x(i)$ of the system being controlled always lies in $\{z(i)\} \oplus \mathbb{T}$, it follows that $\hat{x}(i)$ converges robustly and exponentially fast to $\mathbb{S}$, and $x(i)$ converges robustly and exponentially fast to $\mathbb{T}$. We are now in a position to establish exponential stability of $\mathcal{A} := \mathbb{S} \times \{0\}$ with a region of attraction $(\mathcal{Z}_N \oplus \mathbb{S}) \times \mathcal{Z}_N$ for the composite system (5.25) and (5.26).

**Proposition 5.5** (Exponential stability of output MPC)**.** *The set* $\mathcal{A} := \mathbb{S} \times \{0\}$ *is exponentially stable with a region of attraction* $(\mathcal{Z}_N \oplus \mathbb{S}) \times \mathcal{Z}_N$ *for the composite system* (5.25) *and* (5.26)*.*

*Proof.* Let $\phi := (\hat{x}, z)$ denote the state of the composite system. Then $|\phi|_{\mathcal{A}}$ is defined by

$$|\phi|_{\mathcal{A}} = |\hat{x}|_{\mathbb{S}} + |z|$$

where $|\hat{x}|_{\mathbb{S}} := d(\hat{x}, \mathbb{S})$. But $\hat{x} \in \{z\} \oplus \mathbb{S}$ implies

$$|\hat{x}|_{\mathbb{S}} = d(\hat{x}, \mathbb{S}) = d(z + e, \mathbb{S}) \leq d(z + e, e) = |z|$$

since $e \in \mathbb{S}$. Hence $|\phi|_{\mathcal{A}} \leq 2|z|$ so that

$$|\phi(i)|_{\mathcal{A}} \leq 2|z(i)| \leq 2c|z(0)|\gamma^i \leq 2c|\phi(0)|\gamma^i$$

for all $\phi(0) \in (\mathcal{Z}_N \oplus \mathbb{S}) \times \mathcal{Z}_N$. Since, for all $z(0) \in \mathcal{Z}_N$, $z(i) \in \mathbb{Z}$ and $v(i) \in \mathbb{V}$, it follows that $\hat{x}(i) \in \{z(i)\} \oplus \mathbb{S}$, $x(i) \in \mathbb{X}$, and $u(i) \in \mathbb{U}$ for all $i \in \mathbb{I}_{\geq 0}$. Thus $\mathcal{A} := \mathbb{S} \times \{0\}$ is exponentially stable with a region of attraction $(\mathcal{Z}_N \oplus \mathbb{S}) \times \mathcal{Z}_N$ for the composite system (5.25) and (5.26). ■

It follows from Proposition 5.5 that $x(i)$, which lies in the set $\{z(i)\} \oplus \mathbb{\Gamma}$, $\mathbb{\Gamma} := \mathbb{S} \oplus \mathbb{\Sigma}$, converges to the set $\mathbb{\Gamma}$. In fact $x(i)$ converges to a set that is, in general, smaller than $\mathbb{\Gamma}$ since $\mathbb{\Gamma}$ is a conservative bound on $\tilde{x}(i) + e(i)$. We determine this smaller set as follows. Let $\phi := (\tilde{x}, e)$ and let $\psi := (w, v)$; $\phi$ is the state of the two error systems and $\psi$ is a bounded disturbance lying in a $C$-set $\mathbb{\Psi} := \mathbb{W} \times \mathbb{N}$. Then, from (5.12) and (5.18) the state $\phi$ evolves according to

$$\phi^+ = \tilde{A}\phi + \tilde{B}\psi \tag{5.27}$$

where

$$\tilde{A} := \begin{bmatrix} A_L & 0 \\ LC & A_K \end{bmatrix} \qquad \tilde{B} := \begin{bmatrix} I & -L \\ 0 & L \end{bmatrix}$$

Because $\rho(A_L) < 1$ and $\rho(A_K) < 1$, it follows that $\rho(\tilde{A}) < 1$. Since $\rho(\tilde{A}) < 1$ and $\mathbb{\Psi}$ is compact, there exists a robust positive invariant set $\mathbb{\Phi} \subseteq \mathbb{R}^n \times \mathbb{R}^n$ for (5.27) satisfying

$$\tilde{A}\mathbb{\Phi} \oplus \tilde{B}\mathbb{\Psi} = \mathbb{\Phi}$$

Hence $\phi(i) \in \mathbb{\Phi}$ for all $i \in \mathbb{I}_{\geq 0}$ if $\phi(0) \in \mathbb{\Phi}$. Since $x(i) = z(i) + e(i) + \tilde{x}(i)$, it follows that $x(i) \in \{z(i)\} \oplus H\mathbb{\Phi}$, $H := \begin{bmatrix} I_n & I_n \end{bmatrix}$, for all $i \in \mathbb{I}_{\geq 0}$ provided that $x(0)$, $\hat{x}(0)$ and $z(0)$ satisfy $(\tilde{x}(0), e(0)) \in \mathbb{\Phi}$ where $\tilde{x}(0) = x(0) - \hat{x}(0)$ and $e(0) = \hat{x}(0) - z(0)$. If these initial conditions are satisfied, $x(i)$ converges robustly and exponentially fast to the set $H\mathbb{\Phi}$.

The remaining robust controllers presented in Section 3.4 of Chapter 3 may be similarly modified to obtain a robust output model predictive controller.

## 5.4 Linear Constrained Systems: Time-Varying Case

### 5.4.1 Introduction

In the previous section we considered the case when the state estimator was time-invariant in the sense that the state estimation error $\tilde{x}(i)$ lies

in a constant set $\mathbb{Z}$ for all $i$. The state estimator, in this case, is analogous to the steady-state Kalman filter for which the state estimation error has constant variance. In this section we consider the case where the initial state estimation error $\tilde{x}(0)$ lies in a set $\mathbb{Z}(0)$, which is larger than the time-invariant set $\mathbb{Z}$ considered in Section 5.3. We show subsequently that in the time-varying case, the estimation error lies in a set $\mathbb{Z}(i)$ at time $i$ where $\mathbb{Z}(i)$ converges to $\mathbb{Z}$ as $i$ tends to infinity. Because the set $\mathbb{Z}(i)$ in which the state estimation error lies is now time varying, the nominal optimal control problem has time-varying constraints and thus requires a different approach. Although this section shows that extension of the tube-based controller to the time-varying case is theoretically relatively simple, implementation is considerably more complex; readers whose main interest is in controllers that may be implemented simply should omit this section. To deal with the time-varying case, we extend slightly standard definitions of positive invariance and robust positive invariance.

**Definition 5.6** (Positive invariance; time-varying case)**.** A sequence $\{\mathbb{Z}(i)\}$ of sets is positive invariant for the time-varying system $x^+ = f(x,i)$, $i^+ = i+1$ if, for all $i \in \mathbb{I}_{\geq 0}$, all $x \in \mathbb{Z}(i)$, $f(x,i) \in \mathbb{Z}(i+1)$.

**Definition 5.7** (Robust positive invariance; time-varying case)**.** A sequence $\{\mathbb{Z}(i)\}$ of sets is robust positive invariant for the time-varying system $x^+ = f(x,w,i)$, $i^+ = i+1$ where the disturbance $w$ lies in the set $\mathbb{W}$ if, for all $i \in \mathbb{I}_{\geq 0}$, all $x \in \mathbb{Z}(i)$, $f(x,w,i) \in \mathbb{Z}(i+1)$ for all $w \in \mathbb{W}$.

We assume, as before, that $(A,B,C)$ is stabilizable and detectable.

### 5.4.2 State Estimator

The state estimator is defined as in (5.11) in Section 5.3.2. The state estimate $\hat{x}$ satisfies the difference equation

$$\hat{x}^+ = A\hat{x} + Bu + \delta \qquad \delta := L(y - C\hat{x})$$

and the state estimation error $\tilde{x}$ satisfies

$$\tilde{x}^+ = A_L\tilde{x} + \tilde{w} \qquad \tilde{w} := w - Lv$$

### 5.4.3 Controlling $x$ and $\hat{x}$

As before, we use MPC to control the state estimator and the system $x^+ = Ax + Bu$ by controlling the nominal system

$$z^+ = Az + Bv$$

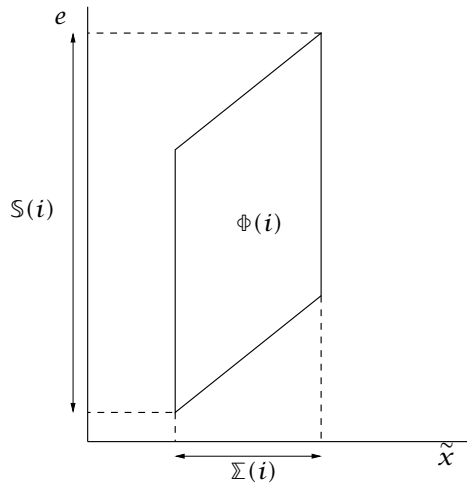


**Figure 5.3:** The sets $\Phi(i)$, $\mathbb{X}(i)$ and $\mathbb{S}(i)$.

and setting $u = v + Ke$, $e := \hat{x} - z$. With this control, the composite system whose state is $\phi := (\tilde{x}, e)$ satisfies

$$\begin{aligned} \tilde{x}^+ &= A_L\tilde{x} + w - Lv \qquad w \in \mathbb{W} \\ e^+ &= A_K e + LC\tilde{x} + Lv \qquad v \in \mathbb{N} \end{aligned}$$

where $A_K := A+BK$. The difference equations for the composite system may be written in the more compact form

$$\phi^+ = \tilde{A}\phi + \tilde{B}\psi$$

where the composite state $\phi := (\tilde{x}, e)$ lies in $\mathbb{R}^n \times \mathbb{R}^n$ and the bounded disturbance $\psi := (w, v)$ lies in the constant compact set $\Psi := \mathbb{W} \times \mathbb{N}$; the state matrix $\tilde{A}$ and the disturbance matrix $\tilde{B}$ are defined by

$$\tilde{A} = \begin{bmatrix} A_L & 0 \\ LC & A_K \end{bmatrix} \qquad \tilde{B} = \begin{bmatrix} I & -L \\ 0 & L \end{bmatrix}$$

We assume that $K$ and $L$ are such that $\rho(A_K) < 1$ and $\rho(A_L) < 1$; hence $\rho(\tilde{A}) < 1$. Consider the set sequence $\{\Phi(i)\}$ defined by

$$\Phi(i+1) = \tilde{A}\Phi(i) \oplus \tilde{B}\Psi$$

with initial condition $\Phi(0) = \mathbb{X}(0) \times \mathbb{S}(0)$ where $\tilde{x}(0)$ lies in $\mathbb{X}(0)$, the initial state uncertainty set, and $e(0)$ lies in $\mathbb{S}(0)$. A comprehensive

analysis of this coupled set of equations is provided in Raković (2007). It follows that the sequence $\{\Phi(i)\}$ is robust positive invariant for $\phi^+ = \widetilde{A}\phi + \widetilde{B}\psi$, $\psi \in \Psi$; if $\phi(0) = (\tilde{x}(0), e(0)) \in \Phi(0)$; then $\phi(i) = (\tilde{x}(i), e(i)) \in \Phi(i)$, $\tilde{x}(i) \in \mathbb{X}(i) := \begin{bmatrix} I_n & 0 \end{bmatrix} \Phi(i)$ and $e(i) \in \mathbb{S}(i) := \begin{bmatrix} 0 & I_n \end{bmatrix} \Phi(i)$ for all $i \in \mathbb{I}_{\geq 0}$. Since $x(i) = z(i) + e(i) + \tilde{x}(i)$ and $u(i) = v(i) + Ke(i)$, it follows that $x(i) \in \{z(i)\} \oplus \mathbb{\Gamma}(i)$, $\mathbb{\Gamma}_i := \begin{bmatrix} I_n & I_n \end{bmatrix} \Phi(i)$, and $u(i) \in \{v(i)\} \oplus K\mathbb{S}(i)$ for all $i \in \mathbb{I}_{\geq 0}$. See Figure 5.3. The following result provides further properties of the sequence $\{\Phi(i)\}$ that we will require in the sequel.

**Proposition 5.8** (Properties of composite system)**.** *If $\Phi(0)$ is compact and $0 \in \Phi(0)$, then $0 \in \Phi(i)$ for all $i \in \mathbb{I}_{\geq 0}$ and the sequence $\{\Phi(i)\}$ converges, in the Hausdorff metric to $\Phi$, the compact, minimal robust positive invariant set for $\phi^+ = \widetilde{A}\phi + \widetilde{B}\psi$, $\psi \in \Psi$. Moreover, $0 \in \Phi$, $\Phi$ satisfies $\Phi = \widetilde{A}\Phi \oplus \widetilde{B}\Psi$, and there exist $c > 0$ and $\lambda \in (0,1)$ such that $d_H(\Phi(i), \Phi) \leq c d_H(\Phi(0), \Phi)\lambda^i$ for all $i \in \mathbb{I}_{\geq 0}$. If, in addition, $\Phi(0)$ is a robust positive invariant set for the system $\phi^+ = \widetilde{A}\phi + \widetilde{B}\psi$, $\psi \in \Psi$, then, for each $i \in \mathbb{I}_{\geq 0}$, $\Phi(i)$ is robust positive invariant for $\phi^+ = \widetilde{A}\phi + \widetilde{B}\psi$, $\psi \in \Psi$, and $\{\Phi(i)\}$ is a monotonically nonincreasing sequence satisfying $0 \in \Phi(i)$ and $\Phi(i+1) \subseteq \Phi(i)$ for all $i \in \mathbb{I}_{\geq 0}$.*

*Proof.* It follows from the definition of the sequence $\{\Phi(i)\}$ that

$$\Phi(i) = \widetilde{A}^i \Phi(0) \oplus \mathcal{A}(i) \qquad \mathcal{A}(i) := \sum_{j=0}^{i-1} \widetilde{A}^j \widetilde{B} \Psi$$

where $\rho(\widetilde{A}) < 1$. The family of compact sets in $\mathbb{R}^n$ endowed with the Hausdorff metric is a complete space so any Cauchy sequence has a limit in this space. As shown in Kolmanovsky and Gilbert (1998), $\{\mathcal{A}(i)\}$ is a Cauchy sequence which, therefore, converges in the Hausdorff metric to the compact set $\Phi$ that satisfies $\Phi = \widetilde{A}\Phi \oplus \widetilde{B}\Psi$ and is the minimal robust positive invariant set for $\phi^+ = \widetilde{A}\phi + \widetilde{B}\psi$, $\psi \in \Psi$. Because $\Phi(0)$ is compact, and contains the origin, the set $\widetilde{A}^i \Phi(0)$ converges to $\{0\}$. Hence $\Phi(i)$ converges in the Hausdorff metric to $\Phi$. Clearly $0 \in \mathcal{A}(i)$ for all $i \in \mathbb{I}_{\geq 0}$. Because $0 \in \Phi(0)$, it follows that $0 \in \Phi(i)$ for all $i \in \mathbb{I}_{\geq 0}$; because $\Phi$ is closed, $0 \in \Phi$. By hypothesis $\Phi(0)$ is robust positive invariant for $\phi^+ = \widetilde{A}\phi + \widetilde{B}\psi$, $\psi \in \Psi(0)$ so that

$$\Phi(1) = \widetilde{A}\Phi(0) + \widetilde{B}\Psi \subseteq \Phi(0)$$

Let $i \in \mathbb{I}_{\geq 0}$ be arbitrary and assume that $\Phi(i)$ is robust positive invariant for $\phi^+ = \widetilde{A}\phi + \widetilde{B}\psi$, $\psi \in \Psi$. Then

$$\Phi(i+1) = \widetilde{A}\Phi(i) \oplus \widetilde{B}\Psi \subseteq \Phi(i)$$

so that $\Phi(i)$ is robust positive invariant for $\phi^+ = \widetilde{A}\phi + \widetilde{B}\psi$, $\psi \in \Psi$ and $\Phi(i+1) \subseteq \Phi(i)$. By induction, $\Phi(i)$ is robust positive invariant for $\phi^+ = \widetilde{A}\phi + \widetilde{B}\psi$, $\psi \in \Psi$ and $\Phi(i+1) \subseteq \Phi(i)$ for all $i \in \mathbb{I}_{\geq 0}$. Hence $\{\Phi(i)\}$ is a monotonically nonincreasing sequence. The proof that $d_H(\Phi(i), \Phi) \leq cd_H(\Phi(0), \Phi)\gamma^i$ is left as Exercise 5.8. ■

It follows that $\mathbb{\Sigma}(i) \to \mathbb{\Sigma}$ and $\mathbb{S}(i) \to \mathbb{S}$ as $i \to \infty$ where $\mathbb{\Sigma}$ and $\mathbb{S}$ satisfy

$$\mathbb{\Sigma} = A_L\mathbb{\Sigma} \oplus (\mathbb{W} \oplus (-L\mathbb{N}))$$
$$\mathbb{S} = A_K\mathbb{S} \oplus L(C\mathbb{\Sigma} \oplus \mathbb{N})$$

and are the minimal robust positive invariant sets for, respectively, $\widetilde{x}^+ = A_L\widetilde{x} + \widetilde{w}$, $\widetilde{w} \in (\mathbb{W} \oplus (-L\mathbb{N}))$ and $e^+ = A_K e + \delta$, $\delta \in L(C\mathbb{\Sigma} \oplus \mathbb{N})$; the sequences $\{\mathbb{\Sigma}(i)\}$ and $\{\mathbb{S}(i)\}$ are nonincreasing and converge in the Hausdorff metric to $\mathbb{\Sigma} = \begin{bmatrix} I_n & 0 \end{bmatrix} \Phi$ and $\mathbb{S} = \begin{bmatrix} 0 & I_n \end{bmatrix} \Phi$, respectively.

### 5.4.4 Control of the Nominal System

Since $x(i) \in \{z(i)\} \oplus \mathbb{\Gamma}(i)$ and $u(i) \in \{v(i)\} \oplus K\mathbb{S}(i)$ for all $i$, we can use MPC to control the sequences $\{z(i)\}$ and $\{v(i)\}$ so that $x(i) \in \mathbb{X}$ and $u(i) \in \mathbb{U}$ for all $i$. The constraints on $x$ and $u$ are satisfied if $z$ and $v$ are required to satisfy the tighter time-varying constraints

$$z(i) \in \mathbb{Z}_i := \mathbb{X} \ominus \mathbb{\Gamma}(i) \qquad v(i) \in \mathbb{V}_i := \mathbb{U} \ominus K\mathbb{S}(i)$$

for all $i$; $\mathbb{Z}_i$ and $\mathbb{V}_i$ may be replaced by outer approximating sets. For this to be possible, we assume

**Assumption 5.9** (Constraint bounds; time-varying case)**.** $\mathbb{\Gamma}(0) \subset \mathbb{X}$ and $K\mathbb{S}(0) \subset \mathbb{U}$.

Since both $\{\mathbb{\Gamma}(i)\}$ and $\{\mathbb{S}(i)\}$ are nonincreasing sequences, $\{\mathbb{Z}_i\}$ and $\{\mathbb{S}_i\}$ are nondecreasing sequences so that satisfaction of Assumption 5.9 ensures that $\mathbb{Z}_i$ and $\mathbb{V}_i$ are not empty for all $i \in \mathbb{I}_{\geq 0}$. The constraints are time varying, so the nominal MPC problem at time $k$, state $z$ is $\mathbb{P}_N(z, k)$ defined by

$$\mathbb{P}_N(z, k): \quad \bar{V}_N^0(z, k) = \min_{\mathbf{v}} \{\bar{V}_N(z, \mathbf{v}) \mid \mathbf{v} \in \mathcal{V}_N(z, k)\}$$

where the cost function $\bar{V}_N(\cdot)$ is defined by

$$\bar{V}_N(z, \mathbf{v}) := \sum_{k=0}^{N-1} \ell(z(k), v(k)) + V_f(z(N))$$

and the constraint set $\mathcal{V}_N(z,k)$ by

$$\mathcal{V}_N(z,k) := \{\mathbf{v} \mid v(i) \in \mathbb{V}_{k+i},\ z(i) \in \mathbb{Z}_{k+i},\ \forall i \in \{0,1,\ldots,N\}, \\ z(N) \in \mathbb{Z}_f\} \quad (5.28)$$

where, for all $i$, $z(i) := \bar{\phi}(i;z,\mathbf{v})$, the solution of $z^+ = Ax + Bv$ at time $i$ if the initial state at time 0 is $z$ and the nominal control sequence is $\mathbf{v} = \{v(0), v(1), \ldots, v(N-1)\}$. In (5.28), $\mathbb{Z}_f \subseteq \mathbb{Z}_N$ is the terminal constraint set and $z(i)$ is the predicted state at time $k+i$ which is why $z(i)$ is required to lie in the set $\mathbb{Z}_{k+i}$ and $v(i)$ to lie in $\mathbb{V}_{k+i}$; clearly $\mathbb{Z}_f \subseteq \mathbb{Z}_i$ for all $i \geq N$ so there is no need to make the terminal constraint set time varying. Let $\mathbf{v}^0(z,k) = \{v^0(0;z,k), v^0(1;z,k), \ldots, v^0(N;z,k)\}$ denote the minimizing control sequence; the stage cost $\ell(\cdot)$ is chosen to ensure uniqueness of $\mathbf{v}^0(z,k)$. The implicit MPC control law for the nominal system is $\bar{\kappa}_N(\cdot)$ defined by

$$\bar{\kappa}_N(z,k) := v^0(0;z,k)$$

where $v^0(0;z,k)$ is the first element in the sequence $\mathbf{v}^0(z,k)$. The domain of $\bar{V}_N^0(\cdot,k)$ and $\mathbf{v}(\cdot,k)$ and, hence, of $\bar{\kappa}_N(\cdot,k)$, is $\mathcal{Z}_N(k)$ defined by

$$\mathcal{Z}_N(k) := \{z \in \mathbb{Z}_k \mid \mathcal{V}_N(z,k) \neq \varnothing\}$$

$\mathcal{Z}_N(k)$ is the set of states $z$ at time $k$ that can be robustly steered to $\mathbb{Z}_f$ in $N$ steps by an admissible control $\mathbf{v}$. Because the constraints become weaker with time, the domain $\mathcal{Z}_N(k+1)$ of $\bar{V}_N^0(\cdot,k+1)$ is larger than the domain $\mathcal{Z}_N(k)$ of $\bar{V}_N^0(\cdot,k)$ for all $k > 0$; the sequence $\{\mathcal{Z}_N(k)\}$ is monotonically nondecreasing.

If the terminal cost $V_f(\cdot)$ and terminal constraint set $\mathbb{Z}_f$ satisfy the stability Assumptions 2.12 and 2.13 of Chapter 2, and if Assumption 5.9 is satisfied, the value function $\bar{V}_N^0(\cdot)$ satisfies, for all $k \in \mathbb{I}_{\geq 0}$

$$\begin{aligned} \bar{V}_N^0(z,k) &\geq \ell(z, \bar{\kappa}_N(z,k)) && \forall z \in \mathcal{Z}_N(k) \\ \Delta\bar{V}_N^0(z,k) &\leq -\ell(z, \bar{\kappa}_N(z,k)) && \forall z \in \mathcal{Z}_N(k) \\ \bar{V}_N^0(z,k) &\leq V_f(z) && \forall z \in \mathbb{Z}_f \end{aligned}$$

where $\Delta\bar{V}_N^0(z,k) := \bar{V}_N^0(f(z,\kappa_N(z)),k+1) - \bar{V}_N^0(z,k)$.

If, in addition, we assume that $\ell(z,v) = (1/2)(|z|_Q^2 + |v|_R^2)$ where $Q$ and $R$ are positive definite and $V_f(z) = (1/2)|z|_{P_f}^2$ where $P_f$ is positive definite, and if $\mathcal{Z}_N(0)$ is a $C$-set, then, as shown in Section 3.4.2, there

exist positive constants $c_1$ and $c_2$ such that

$$\begin{aligned} \bar{V}_N^0(z,k) &\geq \quad c_1|z|^2 \\ \Delta\bar{V}_N^0(z,k) &\leq -c_1|z|^2 \\ \bar{V}_N^0(z) &\leq \quad c_2|z|^2 \end{aligned}$$

for all $z \in \mathcal{Z}_N(k)$, all $k \in \mathbb{I}_{\geq 0}$. It follows from Chapter 2 that the origin is uniformly (in time $k$) exponentially stable for the nominal system $z^+ = Az + B\bar{\kappa}_N(z,k)$ with a region of attraction $\mathcal{Z}_N(0)$, and that $z(k) \in \mathcal{Z}_N(0) \subseteq \mathcal{Z}_N(k)$ for all $k \in \mathbb{I}_{\geq 0}$ if $z(0) \in \mathcal{Z}_N(0)$ so that problem $\mathbb{P}_N(z(k),k)$ is always feasible; here $z(k)$ is the solution of $z^+ = Az + B\bar{\kappa}_N(z,k)$ at time $k$ if the initial state is $z(0)$. There exists a $c > 0$ and a $\lambda \in (0,1)$ such that $|z(k)| \leq c|z(0)|\lambda^k$ for all $k \in \mathbb{I}_{\geq 0}$, all $z(0) \in \mathcal{Z}_N(0)$.

### 5.4.5 Control of the State Estimator

The implicit control law for the state estimator is $\kappa_N(\cdot)$ defined by

$$\kappa_N(\hat{x},z,k) := \bar{\kappa}_N(z,k) + K(\hat{x} - z)$$

Hence, the composite system with state $(\hat{x},z)$ satisfies

$$\hat{x}^+ = A\hat{x} + B\kappa_N(\hat{x},z,k) + \delta(k) \tag{5.29}$$

$$z^+ = Az + B\bar{\kappa}_N(z,k) \tag{5.30}$$

$$k^+ = k + 1 \tag{5.31}$$

with initial state $(\hat{x}(0),z(0))$ satisfying $\hat{x}(0) \in \{z(0)\} \oplus \mathbb{S}(0)$, $z(0) \in \mathcal{Z}_N(0)$; these constraints are satisfied if $z(0) = \hat{x}(0) \in \mathcal{Z}_N(0)$.

Also, from Proposition 5.8, the sequences $\{\Phi(k)\}$, $\{\mathbb{\Gamma}(k)\}$ and $\{\mathbb{S}(k)\}$ converge exponentially fast to $\Phi$, $\mathbb{\Gamma}$ and $\mathbb{S}$, respectively. We have the following result for robust time-varying output MPC:

**Proposition 5.10** (Exponential convergence of output MPC: time-varying case)**.** *There exists a $c > 0$ and a $\gamma \in (0,1)$ such that $|z(k)| \leq c|z(0)|\gamma^k$ and $d(x(k),\mathbb{\Gamma}) \leq c(|z(0)| + 1)\gamma^k$ for all $k \in \mathbb{I}_{\geq 0}$, all $x(0)$, $\hat{x}(0)$, $z(0)$ such that $(x(0) - \hat{x}(0), \hat{x}(0) - z(0)) \in \Phi(0)$, $z(0) \in \mathcal{Z}_N(0)$.*

*Proof.* If $z(0) \in \mathcal{Z}_N(0)$, we have $x(k) \in \{z(k)\} \oplus \mathbb{\Gamma}(k)$ for all $k \in \mathbb{I}_{\geq 0}$. From Proposition 5.8, there exists a $c > 0$ and a $\gamma \in (0,1)$ such that

$d_H(\mathbb{\Gamma}(k), \Gamma) \leq c\gamma^k$ and $|z(k)| \leq c\,|z(0)|\,\gamma^k$, $z(0) \in \mathcal{Z}_N(0)$, for all $k \in \mathbb{I}_{\geq 0}$. Hence

$$\begin{aligned} d(x(k), \Gamma) &\leq d_H(\{z(k)\} \oplus \mathbb{\Gamma}(k), \Gamma) \\ &\leq |z(k)| + d_H(\mathbb{\Gamma}(k), \Gamma) \\ &\leq c(|z(0)| + 1)\gamma^k \end{aligned} \tag{5.32}$$

for all $k \in \mathbb{I}_{\geq 0}$, all $(x(0), \hat{x}(0))$ such that $\phi(0) = (x(0) - \hat{x}(0), \hat{x}(0) - z(0)) \in \Phi(0)$ ■

Similarly it can be shown that there exist a possibly different $c > 0$ and $\gamma \in (0, 1)$ such that

$$d(\hat{x}(k), \mathbb{S}) \leq c(|z(0)| + 1)\gamma^k$$

for all $k \in \mathbb{I}_{\geq 0}$. This result is not as strong as the corresponding result in Proposition 5.5 where exponential stability of $\mathbb{S} \times \{0\}$ with a region of attraction $(\mathcal{Z}_N \oplus \mathbb{S}) \times \mathcal{Z}_N$ is established for the composite system (5.25) and (5.26); the time-varying nature of the problem appears to preclude a stronger result.

## 5.5 Offset-Free MPC

We are now in a position to give a more realistic solution to the problem of offset-free MPC, briefly introduced in Chapter 2 in a deterministic context. Suppose the system to be controlled is described by

$$\begin{aligned} x^+ &= Ax + B_d d + Bu + w_x \\ y &= Cx + C_d d + v \\ r &= Hy \qquad \tilde{r} = r - \bar{r} \end{aligned}$$

where $w_x$ and $v$ are unknown bounded disturbances taking values, respectively, in the compact sets $\mathbb{W}_x$ and $\mathbb{N}$ containing the origin in their interiors. We assume $d$ is constant, or almost constant, but unknown, and models an additive disturbance; $y = Cx + C_d d$ is the output of the system being controlled, $r$ is the controlled variable and $\bar{r}$ is its setpoint. The variable $\tilde{r}$ is the tracking error that we wish to minimize. We assume, for purposes of determining a control, that $d$ satisfies

$$d^+ = d + w_d$$

where $w_d$ is a bounded disturbance taking values in the compact set $\mathbb{W}_d$; in practice $d$ is bounded although this is not implied by our model.

| Set | Definition | Membership |
|---|---|---|
| $\mathbb{X}$ | state constraint set | $x \in \mathbb{X}$ |
| $\mathbb{U}$ | input constraint set | $u \in \mathbb{U}$ |
| $\mathbb{W}_x$ | state disturbance set | $w_x \in \mathbb{W}_x$ |
| $\mathbb{W}_d$ | integrating disturbance set | $w_d \in \mathbb{W}_d$ |
| $\mathbb{W}$ | total state disturbance set, $\mathbb{W}_x \times \mathbb{W}_d$ | $w \in \mathbb{W}$ |
| $\mathbb{N}$ | measurement error set | $v \in \mathbb{N}$ |
| $\widetilde{\mathbb{W}}$ | estimate error disturbance set, $\mathbb{W} \oplus (-L\mathbb{N})$ | $\tilde{w} \in \widetilde{\mathbb{W}}$ |
| $\Phi$ | total estimate error disturbance set, $\Phi = \widetilde{A}_L \Phi \oplus \widetilde{\mathbb{W}}$ | $\phi \in \Phi$ |
| $\mathbb{E}_x$ | state estimate error disturbance set, $\begin{bmatrix} I_n & 0 \end{bmatrix} \Phi$ | $\tilde{x} \in \mathbb{E}_x$ |
| $\mathbb{E}_d$ | integrating disturbance estimate error set, $\begin{bmatrix} 0 & I_p \end{bmatrix} \Phi$ | $\tilde{d} \in \mathbb{E}_d$ |
| $\mathbb{A}$ | innovation set, $L(\widetilde{C}\Phi \oplus \mathbb{N})$ | $L\tilde{y} \in \mathbb{A}$ |
| $\mathbb{A}_x$ | set containing state component of innovation, $L_x(\widetilde{C}\Phi \oplus \mathbb{N})$ | $L_x\tilde{y} \in \mathbb{A}_x$ |
| $\mathbb{A}_d$ | set containing integrating disturbance component of innovation, $L_d(\widetilde{C}\Phi \oplus \mathbb{N})$ | $L_d\tilde{y} \in \mathbb{A}_d$ |
| $\mathbb{S}$ | nominal state tracking error invariance set, $A_K\mathbb{S} \oplus \mathbb{A}_x = \mathbb{S}$ | $e \in \mathbb{S}$ <br> $\hat{x} \in \{z\} + \mathbb{S}$ |
| $\mathbb{\Gamma}$ | state tracking error invariance set, $\mathbb{S} + \mathbb{E}_x$ | $x \in \{z\} + \mathbb{\Gamma}$ |
| $\mathbb{V}$ | nominal input constraint set, $\mathbb{V} = \mathbb{U} \ominus K\mathbb{S}$ | $v \in \mathbb{V}$ |
| $\mathbb{Z}$ | nominal state constraint set, $\mathbb{Z} = \mathbb{X} \ominus \mathbb{\Gamma}$ | $z \in \mathbb{Z}$ |

**Table 5.1:** Summary of the sets and variables used in output MPC.

We assume that $x \in \mathbb{R}^n$, $d \in \mathbb{R}^p$ ,$u \in \mathbb{R}^m$, $y \in \mathbb{R}^r$, and $e \in \mathbb{R}^q$, $q \leq r$ and that the system to be controlled is subject to the usual state and control constraints

$$x \in \mathbb{X} \qquad u \in \mathbb{U}$$

where $\mathbb{X}$ is polyhedral and $\mathbb{U}$ is polytopic.

### 5.5.1 Estimation

Given the numerous sets that are required to specify the output feedback case we are about to develop, Table 5.1 may serve as a reference for the sets defined in the chapter and the variables that are members of these sets.

Since both $x$ and $d$ are unknown, it is necessary to estimate them. For estimation purposes, it is convenient to work with the composite system whose state is $\phi := (x, d)$. This system may be described more compactly by

$$\phi^+ = \tilde{A}\phi + \tilde{B}u + w$$
$$y = \tilde{C}\phi + v$$

in which

$$\tilde{A} := \begin{bmatrix} A & B_d \\ 0 & I \end{bmatrix} \qquad \tilde{B} := \begin{bmatrix} B \\ 0 \end{bmatrix} \qquad \tilde{C} := \begin{bmatrix} C & C_d \end{bmatrix}$$

and $w := (w_x, w_d)$ takes values in $\mathbb{W} = \mathbb{W}_x \times \mathbb{W}_d$. A necessary and sufficient condition for the detectability of $(\tilde{A}, \tilde{C})$ is given in Lemma 1.8 in Chapter 1; a sufficient condition is detectability of $(A, C)$ coupled with invertibility of $C_d$. If $(\tilde{A}, \tilde{C})$ is detectable, the state may be estimated using the time-invariant observer or filter described by

$$\hat{\phi}^+ = \tilde{A}\hat{\phi} + \tilde{B}u + \delta \qquad \delta := L(y - \tilde{C}\hat{\phi})$$

in which $L$ is such that $\rho(\tilde{A}_L) < 1$ where $\tilde{A}_L := \tilde{A} - L\tilde{C}$. Clearly $\delta = L\tilde{y}$ where $\tilde{y} = \tilde{C}\tilde{\phi} + v$. The estimation error $\tilde{\phi} := \phi - \hat{\phi}$ satisfies

$$\tilde{\phi}^+ = \tilde{A}\tilde{\phi} + w - L(\tilde{C}\tilde{\phi} + v)$$

or, in simpler form

$$\tilde{\phi}^+ = \tilde{A}_L\tilde{\phi} + \tilde{w} \qquad \tilde{w} := w - Lv$$

Clearly $\tilde{w} = w - Lv$ takes values in the compact set $\tilde{\mathbb{W}}$ defined by

$$\tilde{\mathbb{W}} := \mathbb{W} \oplus (-L\mathbb{N})$$

If $w$ and $v$ are zero, $\tilde{\phi}$ decays to zero exponentially fast. Since $\rho(\tilde{A}_L) < 1$ and $\tilde{\mathbb{W}}$ is compact, there exists a robust positive invariant set $\Phi$ for $\tilde{\phi}^+ = \tilde{A}_L\tilde{\phi} + \tilde{w}$, $\tilde{w} \in \tilde{\mathbb{W}}$ satisfying

$$\Phi = \tilde{A}_L\Phi \oplus \tilde{\mathbb{W}}$$

Hence $\tilde{\phi}(i) \in \Phi$ for all $i \in \mathbb{I}_{\geq 0}$ if $\tilde{\phi}(0) \in \Phi$. Since $\tilde{\phi} = (\tilde{x}, \tilde{d}) \in \mathbb{R}^n \times \mathbb{R}^p$ where $\tilde{x} := x - \hat{x}$ and $\tilde{d} := d - \hat{d}$, we define the sets $\mathbb{X}_x$ and $\mathbb{X}_d$ as follows

$$\mathbb{X}_x := \begin{bmatrix} I_n & 0 \end{bmatrix} \Phi \qquad \mathbb{X}_d := \begin{bmatrix} 0 & I_p \end{bmatrix} \Phi$$

It follows that $\tilde{x}(i) \in \mathbb{X}_x$ and $\tilde{d}(i) \in \mathbb{X}_d$ for all $i \in \mathbb{I}_{\geq 0}$ if $\tilde{\phi}(0) = (\tilde{x}(0), \tilde{d}(0)) \in \Phi$. That $\tilde{\phi}(0) \in \Phi$ is a steady-state assumption.

### 5.5.2 Control

The estimation problem has a solution similar to previous solutions. The control problem is more difficult. As before, we control the estimator state, making allowance for state estimation error. The estimator state $\hat{\phi}$ satisfies the difference equation

$$\hat{\phi}^+ = \widetilde{A}\hat{\phi} + \widetilde{B}u + \delta$$

where the disturbance $\delta$ is defined by

$$\delta := L\widetilde{y} = L(\widetilde{C}\widetilde{\phi} + \nu)$$

The disturbance $\delta = (\delta_x, \delta_d)$ lies in the $C-$set $\mathbb{\Delta}$ defined by

$$\mathbb{\Delta} := L(\widetilde{C}\mathbb{\Phi} \oplus \mathbb{N})$$

where the set $\mathbb{\Phi}$ is defined in Section 5.5.1. The system $\hat{\phi}^+ = \widetilde{A}\hat{\phi}+\widetilde{B}u+\delta$ is not stabilizable, however, so we examine the subsystems with states $\hat{x}$ and $\hat{d}$

$$\begin{aligned}\hat{x}^+ &= A\hat{x} + B_d\hat{d} + Bu + \delta_x\\ \hat{d}^+ &= \hat{d} + \delta_d\end{aligned}$$

where the disturbances $\delta_x$ and $\delta_d$ are components of $\delta$ ($\delta = (\delta_x, \delta_d)$) and are defined by

$$\delta_x := L_x\widetilde{y} = L_x(\widetilde{C}\widetilde{\phi} + \nu) \qquad \delta_d := L_d\widetilde{y} = L_d(\widetilde{C}\widetilde{\phi} + \nu)$$

The matrices $L_x$ and $L_d$ are the corresponding components of $L$. The disturbance $\delta_x$ and $\delta_d$ lie in the $C-$sets $\mathbb{\Delta}_x$ and $\mathbb{\Delta}_d$ defined by

$$\mathbb{\Delta}_x := \begin{bmatrix} I_n & 0 \end{bmatrix}\mathbb{\Delta} = L_x[\widetilde{C}\mathbb{\Phi} \oplus \mathbb{N}] \qquad \mathbb{\Delta}_d := \begin{bmatrix} 0 & I_p \end{bmatrix}\mathbb{\Delta} = L_d[\widetilde{C}\mathbb{\Phi} \oplus \mathbb{N}]$$

We assume that $(A, B)$ is a stabilizable pair so the tube methodology may be employed to control $\hat{x}$. The system $\hat{d}^+ = \hat{d} + \delta_d$ is uncontrollable. The central trajectory is therefore described by

$$\begin{aligned}z^+ &= Az + B_d\hat{d} + Bv\\ \hat{d}^+ &= \hat{d}\end{aligned}$$

We obtain $v = \bar{\kappa}_N(z, \hat{d}, \bar{r})$ by solving a nominal optimal control problem defined later and set $u = v + Ke$, $e := \hat{x} - z$ where $K$ is chosen so that

$\rho(A_K) < 1$, $A_K := A + BK$; this is possible since $(A, B)$ is assumed to be stabilizable. It follows that $e := \hat{x} - z$ satisfies the difference equation

$$e^+ = A_K e + \delta_x \qquad \delta_x \in \mathbb{\Delta}_x$$

Because $\mathbb{\Delta}_x$ is compact and $\rho(A_K) < 1$, there exists a robust positive invariant set $\mathbb{S}$ for $e^+ = A_K e + \delta_x$, $\delta_x \in \mathbb{\Delta}_x$ satisfying

$$A_K \mathbb{S} \oplus \mathbb{\Delta}_x = \mathbb{S}$$

Hence $e(i) \in \mathbb{S}$ for all $i \in \mathbb{I}_{\geq 0}$ if $e(0) \in \mathbb{S}$. So, as in Proposition 5.3, the states and controls of the estimator and nominal system satisfy $\hat{x}(i) \in \{z(i)\} \oplus \mathbb{S}$ and $u(i) \in \{v(i)\} \oplus K\mathbb{S}$ for all $i \in \mathbb{I}_{\geq 0}$ if the initial states $\hat{x}(0)$ and $z(0)$ satisfy $\hat{x}(0) \in \{z(0)\} \oplus \mathbb{S}$. Using the fact established previously that $\tilde{\mathbf{x}}(i) \in \mathbb{\Sigma}_x$ for all $i$, we can also conclude that $x(i) = z(i) + e(i) + \tilde{\mathbf{x}}(i) \in \{z(i)\} \oplus \mathbb{\Gamma}$ and that $u(i) = v(i) + Ke(i) \in \{v(i)\} + K\mathbb{S}$ for all $i$ where $\mathbb{\Gamma} := \mathbb{S} \oplus \mathbb{\Sigma}_x$ provided, of course, that $\phi(0) \in \{\hat{\phi}(0)\} \oplus \mathbb{\Phi}$ and $x(0) \in \{\hat{x}(0)\} \oplus \mathbb{S}$. These conditions are equivalent to $\tilde{\boldsymbol{\phi}}(0) \in \mathbb{\Phi}$ and $e(0) \in \mathbb{S}$ where, for all $i$, $e(i) := \hat{x}(i) - z(i)$. Hence $x(i)$ lies in $\mathbb{X}$ and $u(i)$ lies in $\mathbb{U}$ if $z(i) \in \mathbb{Z} := \mathbb{X} \ominus \mathbb{\Gamma}$ and $v(i) \in \mathbb{V} := \mathbb{U} \ominus K\mathbb{S}$.

Thus $\hat{x}(i)$ and $x(i)$ evolve in known neighborhoods of the central state $z(i)$ that we can control. Although we know that the uncontrollable state $d(i)$ lies, for all $i$, in the set $\{\hat{d}(i)\} \oplus i\mathbb{\Sigma}_d$, the evolution of $\hat{d}(i)$ is an uncontrollable random walk and is, therefore, unbounded; if the initial value of $\hat{d}$ at time 0 is $\hat{d}_0$, then $\hat{d}(i)$ lies in the set $\{\hat{d}_0\} \oplus \mathbb{W}_d$ that increases without bound as $i$ increases. This behavior is a defect in our model for the disturbance $d$; the model is useful for estimation purposes, but is unrealistic in permitting unbounded values for $d$. Hence we assume in the sequel that $d$ evolves in a compact $C-$set $X_d$. We can modify the observer to ensure that $\hat{d}$ lies in $X_d$ but find it simpler to observe that, if $d$ lies in $X_d$, $\hat{d}$ must lie in $X_d \oplus \mathbb{\Sigma}_d$.

**Target Calculation.** We are now in a position to specify the optimal control problem whose solution yields $v = \bar{\kappa}_N(z, k)$ and, hence, $u = v + K(\hat{x} - z)$. Our first task is to determine the target state $\bar{z}$ and associated control $\bar{v}$; we require our estimate of the tracking error $\tilde{r} = r - \bar{r}$ to be zero. Since our estimate of the measurement noise $\nu$ is 0 and since our best estimate of $d$ when the target state is reached is $\hat{d}$, we require

$$\hat{r} - \bar{r} = H(C\bar{z} + C_d\hat{d}) - \bar{r} = 0$$

We also require the target state to be an equilibrium state satisfying, therefore, $\bar{z} = A\bar{z} + B_d\hat{d} + B\bar{v}$ for some control $\bar{v}$. Given $(\hat{d}, \bar{r})$, the

target equilibrium pair $(\bar{z}, \bar{v})(\hat{d}, \bar{r})$ is computed as follows

$$(\bar{z}, \bar{v})(\hat{d}, \bar{r}) = \arg\min_{z,v}\{L(z, v) \mid z = Az+B_d\hat{d}+Bv,\ H(Cz+C_d\hat{d}) = \bar{r},\ z \in \mathbb{Z},\ v \in \mathbb{V}\}$$

where $L(\cdot)$ is an appropriate cost function; e.g. $L(v) = (1/2)|v|_R^2$. The equality constraints in this optimization problem can be satisfied if the matrix $\left[\begin{smallmatrix} I-A & -B \\ HC & 0 \end{smallmatrix}\right]$ has full rank. As the notation indicates, the target equilibrium pair $(\bar{z}, \bar{v})(\hat{d}, \bar{r})$ is not constant but varies with the estimate of the disturbance state $d$.

**MPC algorithm.** The control objective is to steer the central state $z$ to the target state $\bar{z}(\hat{d}, \bar{r})$ while satisfying the state and control constraints $x \in \mathbb{X}$ and $u \in \mathbb{U}$. It is desirable that $z(i)$ converges to $\bar{z}(\hat{d}, \bar{r})$ if $\hat{d}$ remains constant in which case $x(i)$ converges to the set $\{\bar{z}(\hat{d}, \bar{r})\} \oplus \mathbb{\Gamma}$. To achieve this objective, we define the deterministic optimal control problem

$$\bar{\mathbb{P}}_N(z, \hat{d}, \bar{r}): \quad V_N^0(z, \hat{d}, \bar{r}) := \min_{\mathbf{v}}\{V_N(z, \hat{d}, \bar{r}, \mathbf{v}) \mid \mathbf{v} \in \mathcal{V}_N(z, \hat{d}, \bar{r})\}$$

in which the cost $V_N(\cdot)$ and the constraint set $\mathcal{V}_N(z, \hat{d}, \bar{r})$ are defined by

$$V_N(z, \hat{d}, \bar{r}, \mathbf{v}) := \sum_{i=0}^{N-1} \ell(z(i) - \bar{z}(\hat{d}, \bar{r}), v(i) - \bar{v}(\hat{d}, \bar{r})) + V_f(z(N), \bar{z}(\hat{d}, \bar{r}))$$

$$\mathcal{V}_N(z, \hat{d}, \bar{r}) := \{\mathbf{v} \mid z(i) \in \mathbb{Z}, v(i) \in \mathbb{V}\ \forall i \in \mathbb{I}_{0:N-1}, z(N) \in \mathbb{Z}_f(\bar{z}(\hat{d}, \bar{r}))\}$$

where, for each $i$, $z(i) = \bar{\phi}(i; z, \hat{d}, \mathbf{v})$, the solution of $z^+ = Az+B_d\hat{d}+Bv$ when the initial state is $z$, the control sequence is $\mathbf{v}$, and the disturbance $\hat{d}$ is constant. The terminal cost is zero when the terminal state is equal to the target state and the target state lies in the center of the terminal constraint set. The solution to $\bar{\mathbb{P}}_N(z, \hat{d}, \bar{r})$ is

$$\mathbf{v}^0(z, \hat{d}, \bar{r}) = \{v^0(0; z, \hat{d}, \bar{r}), v^0(1; z, \hat{d}, \bar{r}), \ldots, v^0(N-1; z, \hat{d}, \bar{r})\}$$

and the implicit model control law $\bar{\kappa}_N(\cdot)$ is defined by

$$\bar{\kappa}_N(z, \hat{d}, \bar{r}) := v^0(0; z, \hat{d}, \bar{r})$$

where $v^0(0; z, \hat{d}, \bar{r})$ is the first element in the sequence $\mathbf{v}^0(z, \hat{d}, \bar{r})$. The control $u$ applied to the plant and the observer is $u = \kappa_N(\hat{x}, z, \hat{d}, \bar{r})$ where $\kappa_N(\cdot)$ is defined by

$$\kappa_N(\hat{x}, z, \hat{d}, \bar{r}) := \bar{\kappa}_N(z, \hat{d}, \bar{r}) + K(\hat{x} - z)$$

Although the optimal control problem $\bar{\mathbb{P}}_N(z, \hat{d}, \bar{r})$ is deterministic, $\hat{d}$ is random, so that the sequence $\{z(i)\}$ is random. The control algorithm may now be formally stated:

**Robust control algorithm (offset-free MPC).**

**Initialization:** At time 0, set $i = 0$, set $\hat{\phi} = \hat{\phi}(0)$ in which $\hat{\phi} = (\hat{x}, \hat{d})$ and set $z = \hat{x}$.

**Step 1 (Compute control):** At time $i$, solve the "nominal" optimal control problem $\bar{\mathbb{P}}_N(z, \hat{d}, \bar{r})$ to obtain the current "nominal" control action $v = \bar{\kappa}_N(z, \hat{d}, \bar{r})$ and the control action $u = v + K(\hat{x} - z)$.

**Step 2 (Check):** If $\bar{\mathbb{P}}_N(z, \hat{d}, \bar{r})$ is infeasible, adopt safety/recovery procedure.

**Step 3 (Apply control):** Apply the control $u$ to the system being controlled.

**Step 4 (Update):** (a) Compute the successor state estimate $\hat{\phi}^+ = \tilde{A}\hat{x} + \tilde{B}u + L(y - \tilde{C}\hat{\phi})$. (b) Compute the successor state $z^+ = Az + B_d\hat{d} + Bv$ of the nominal system.

**Step 5:** Set $(\hat{\phi}, z) = (\hat{\phi}^+, z^+)$, set $i = i + 1$, and go to Step 1.

In normal operation, Step 2 is not activated; Propositions 5.2 and 5.3 ensure that the constraints $\hat{x} \in \{z\} \oplus \mathbb{S}$ and $u \in \{v\} \oplus K\mathbb{S}$ are satisfied. If an unanticipated event occurs and Step 2 is activated, the controller can be reinitialized by setting $v = \bar{\kappa}_N(\hat{x}, \hat{d}, \bar{r})$, setting $u = v$ and relaxing constraints if necessary.

### 5.5.3 Stability Analysis

We give here an informal discussion of the stability properties of the controller because offset-free MPC of constrained uncertain systems remains an area of current research. The controller described above is motivated by the following consideration: nominal MPC is able to handle "slow" uncertainties such as the drift of a target point if the value function $V_N^0(\cdot)$ is Lipschitz continuous."Fast" uncertainties, however, are better handled by the tube controller that generates, using MPC, a suitable central trajectory that uses a "fast" ancillary controller to steer trajectories of the uncertain system toward the central trajectory. As shown above, the controller ensures that $x(i) \in \{z(i)\} \oplus \mathbb{\Gamma}$ for all $i$; its

success therefore depends on the ability of the controlled nominal system $z^+ = Az + B_d\hat{d} + B\bar{\kappa}_N(z, \hat{d}, \bar{r})$, $v = \bar{\kappa}_N(z, \hat{d}, \bar{r})$, to track the target $\bar{z}(\hat{d}, \bar{r})$, which varies as $\hat{d}$ evolves.

Assuming that the standard stability assumptions are satisfied for the nominal optimal control problem $\bar{\mathbb{P}}_N(z, \hat{d}, \bar{r})$ defined above, we have

$$\begin{aligned} V_N^0(z, \hat{d}, \bar{r}) &\geq c_1 |z - \bar{z}(\hat{d}, \bar{r})|^2 \\ \Delta V_N^0(z, \hat{d}, \bar{r}) &\leq -c_1 |z - \bar{z}(\hat{d}, \bar{r})|^2 \\ V_N^0(z, \hat{d}, \bar{r}) &\leq c_2 |z - \bar{z}(\hat{d}, \bar{r})|^2 \end{aligned}$$

for all $z \in \mathcal{Z}_N(\hat{d}, \bar{r})$ where, since $(\hat{d}, \bar{r})$ is constant,

$$\Delta V_N^0(z, \hat{d}, \bar{r}) := V_N^0(Az + B_d\hat{d} + B\bar{\kappa}_N(z, \hat{d}, \bar{r}), \hat{d}, \bar{r}) - V_N^0(z, \hat{d}, \bar{r})$$

and, for each $(\hat{d}, \bar{r})$, $\mathcal{Z}_N(\hat{d}, \bar{r}) = \{z \mid \mathcal{V}_N(z, \hat{d}, \bar{r}) \neq \emptyset\}$ is the domain of $V_N^0(\cdot, \hat{d}, \bar{r})$.

**Constant $\hat{d}$.** If $\hat{d}$ remains constant, $\bar{z}(\hat{d}, \bar{r})$ is exponentially stable for $z^+ = Az + B_d\hat{d} + B\bar{\kappa}_N(z, \hat{d}, \bar{r})$ with a region of attraction $\mathcal{Z}_N(\hat{d}, \bar{r})$. It can be shown, as in the proof of Proposition 5.5, that the set $\mathcal{A}(\hat{d}, \bar{r}) := (\{\bar{z}(\hat{d}, \bar{r})\} \oplus \mathbb{S}) \times \{\bar{z}(\hat{x}, \bar{r})\}$ is exponentially stable for the composite system $\hat{x}^+ = A\hat{x} + B_d\hat{d} + B\kappa_N(\hat{x}, z, \hat{d}, \bar{r}) + \delta_x$, $z^+ = Az + B_d\hat{d} + B\bar{\kappa}_N(z, \hat{d})$, $\delta_x \in \mathbb{\Delta}_x$, with a region of attraction $(\mathcal{Z}_N(\hat{d}, \bar{r}) \oplus \mathbb{S}) \times \mathcal{Z}_N(\hat{d}, \bar{r})$. Hence $x(i) \in \{z(i)\} \oplus \mathbb{\Gamma}$ tends to the set $\{\bar{z}(\hat{d}, \bar{r})\} \oplus \mathbb{\Gamma}$ exponentially fast. If the external disturbance $w$ is zero, $\mathbb{W} = \{0\}$. If, in addition, $\mathbb{N} = \{0\}$, then $\mathbb{\Delta} = \{0\}$ and $\mathbb{S} = \{0\}$ and $x(i) \to \bar{z}(\hat{d}, \bar{r})$ exponentially fast so that the tracking error $\tilde{r}(i) \to 0$ as $i \to \infty$.

**Slowly varying $\hat{d}$.** If $\hat{d}$ is varying, the inequality for $\Delta V_N^0(\cdot)$ must be replaced by

$$\begin{aligned} \Delta V_N^0(z, \hat{d}, \bar{r}, w_d) = V_N^0(Az + B_d\hat{d} + B\bar{\kappa}_N(z, \hat{d}, \bar{r}), \hat{d} + \delta_d, \bar{r}) - V_N^0(z, \hat{d}, \bar{r}) \\ \leq -c_1 \left| z - \bar{z}(\hat{d}, \bar{r}) \right|^2 + \bar{k} |\delta_d| \end{aligned}$$

where $\bar{k}$ is a Lipschitz constant for $V_N^0(\cdot)$. Employing the approach adopted in Section 3.2.4, it can be shown, if $\mathbb{\Delta}_d$ is sufficiently small, that there exist two sublevel sets $S_b(\hat{d}, \bar{r}) := \{z \mid V_N^0(z, \hat{d}, \bar{r}) \leq b\}$ and $S_c(\hat{d}, \bar{r}) := \{z \mid V_N^0(z, \hat{d}, \bar{r}) \leq c\}$, $c > b$ such that $z(0) \in S_c(\hat{d}(0), \bar{r})$ implies the existence of a finite time $i_0$ such that $z(i) \in S_b(\hat{d}(i), \bar{r})$ for all $i \geq i_0$. The center of each set is the target state $\bar{z}(\hat{d}(i), \bar{r})$ so that, if

$\mathbb{A}_d$ is small and recursive feasibility is maintained, $z(i)$ remains close to the target state; $x(i) \in \{z(i)\} \oplus \mathbb{\Gamma}$ also remains close if, in addition, $\mathbb{W}_x$ and $\mathbb{N}$ are small. Recursive feasibility is ensured if there are no state or terminal constraints in the nominal optimal control problem.

## 5.6 Nonlinear Constrained Systems

### 5.6.1 Introduction

For simplicity, we consider here the following uncertain, discrete time, nonlinear system

$$x^+ = f(x,u) + w \qquad y = h(x) + v \tag{5.33}$$

where $x \in \mathbb{R}^n$ is the current state, $u \in \mathbb{R}^m$ is the current control action, $x^+$ is the successor state, $w \in \mathbb{R}^n$ is an unknown state disturbance, $y \in \mathbb{R}^p$ is the current measured output, and $v \in \mathbb{R}^p$ is an unknown output disturbance. The state and additive disturbances $w$ and $v$ are known only to the extent that they lie, respectively, in the $C$ sets $\mathbb{W} \subseteq \mathbb{R}^n$ and $\mathbb{N} \subseteq \mathbb{R}^p$. Let $\phi(i; x(0), \mathbf{u}, \mathbf{w})$ denote the solution of (5.9) at time $i$ if the initial state at time 0 is $x(0)$, and the control and disturbance sequences are, respectively, $\mathbf{u} := \{u(0), u(1), \ldots\}$ and $\mathbf{w} := \{w(0), w(1), \ldots\}$. The system (5.33) is subject to the following set of hard state and control constraints

$$(x,u) \in \mathbb{X} \times \mathbb{U}$$

in which $\mathbb{X} \subseteq \mathbb{R}^n$ and $\mathbb{U} \subseteq \mathbb{R}^m$ are polyhedral and polytopic sets, respectively, with each set containing the origin in its interior. Output MPC of nonlinear systems remains an active area of research; the proposals to follow are speculative.

### 5.6.2 State Estimator

Several state estimators for nonlinear systems are described in Chapter 4. For each $t$, let $\mathcal{I}(t)$ denote the information available to the state estimator at time $t$: for a full information estimator

$$\mathcal{I}(t) := \{(y(j), u(j)) \mid j \in \mathbb{I}_{-\infty:t}\}$$

whereas for a moving horizon estimator

$$\mathcal{I}(t) := \{(y(j), u(j)) \mid j \in \mathbb{I}_{t-T:t}\}$$

where $T$ is the horizon. For each $t, j$ let $\hat{x}(t|j)$ denote the estimate of $x(t)$ give data $\mathcal{I}(j)$; for simplicity, we use $\hat{x}(t)$ to denote $\hat{x}(t|t-1)$. We make the strong assumption that we have available an estimator satisfying the following difference equation

$$\hat{x}(t+1) = f(\hat{x}(t), u(t)) + \delta$$

where $\delta \in \mathbb{\Delta}$ and $\mathbb{\Delta}$ is a compact subset of $\mathbb{R}^n$. Since

$$\hat{x}(t+1) = f(\hat{x}(t|t), u(t)) + w(t) = f(\hat{x}(t), u(t)) + \delta(t)$$

where

$$\delta(t) := [f(\hat{x}(t|t), u(t)) - f(\hat{x}(t), u(t))] + w(t)$$

the form of the evolution equation for $\hat{x}(t)$ is acceptable; the assumption that $\mathbb{\Delta}$ is constant is conservative. However controlling a random system with a time-varying bound on the disturbance would be considerably more complicated.

Our second assumption is the the state estimation error $\tilde{x}(t) := x(t) - \hat{x}(t)$ lies in a compact set $\mathbb{\Sigma}_x$. This is also a conservative assumption, made for simplicity.

Before proceeding to propose a tube-based controller, we examine briefly nominal MPC.

### 5.6.3 Nominal MPC

In nominal output MPC, the control $u$ is determined by solving an optimal control problem $\bar{\mathbb{P}}_N(\hat{x})$ for the nominal deterministic system defined by

$$z^+ = f(z, u) \qquad z(0) = \hat{x}$$

where $\hat{x}$ is the current estimate of the state $x$. This yields the implicit control law $\bar{\kappa}_N(\cdot)$ so the control $u$ applied to the system $x^+ = f(x, u) + w$ when the current state estimate is $\hat{x}$ is

$$u = \bar{\kappa}_N(\hat{x})$$

Because the evolution of the state $x$ differs from the evolution of the state estimate $\hat{x}$, the control $u = \bar{\kappa}_N(\hat{x})$ is not necessarily stabilizing. If the ingredients $V_f(\cdot)$ and $\mathbb{Z}_f$ of the optimal control problem $\bar{\mathbb{P}}_N(\hat{x})$ are chosen appropriately, and $\ell(\cdot)$ is quadratic and positive definite, the value function $\bar{V}_N^0(\cdot)$ satisfies the usual inequalities:

$$c_1|z|^2 \le \bar{V}_N^0(z) \le c_2|z|^2$$
$$\bar{V}_N^0(z^+) \le \bar{V}_N^0(z) - c_1|z|^2$$

where $z^+ = f(z, \bar{\kappa}_N(z))$. These inequalities are sufficient to establish the exponential stability of the origin for the nominal system $z^+ = f(z, \bar{\kappa}_N(z))$ with a region of attraction $\mathcal{Z}_N$ which is the domain of the value function if bounded, or an appropriate level set of the value function otherwise.

### 5.6.4 Tube-Based Output MPC

We apply the methodology of Chapter 3, Section 3.6 to the control of the uncertain system $\hat{x}^+ = f(\hat{x}, u) + \delta$, $\delta \in \mathbb{\Delta}$, making allowance for the fact that $x(i)$ lies in $\{\hat{x}(i)\} \oplus \mathbb{\Sigma}_x$ for all $i$. This method of control permits, in principle, larger disturbances.

We assume, therefore, that we have an implicit, stabilizing, control law $v = \bar{\kappa}_N(z)$ for the nominal system $z^+ = f(z, v)$. This control law is chosen to satisfy the tightened constraints

$$z \in \mathbb{Z} \qquad v \in \mathbb{V}$$

We discuss the choice of $\mathbb{Z}$ and $\mathbb{V}$ later. The control law is obtained by solving the nominal control problem $\bar{\mathbb{P}}_N(z)$ whose solution also yields the "central" state and control trajectories $\{\mathbf{z}^*(i;z)\}$ and $\{\mathbf{u}^*(i;z)\}$; these trajectories are the solutions of

$$z^+ = f(z, \bar{\kappa}_N(z)), \qquad v = \bar{\kappa}_N(z)$$

with initial state $z(0) = z$.

The second ingredient of the tube-based controller is the ancillary controller that attempts to steer the trajectories of the uncertain system $\hat{x}^+ = f(\hat{x}, u) + \delta$ toward the central path defined above. This determines $u$ by solving the ancillary problem $\mathbb{P}_N(\hat{x}, z)$ defined by

$$\bar{V}_N^0(\hat{x}, z) = \min_{\mathbf{u}} \{V_N(\hat{x}, z, \mathbf{u}) \mid \mathbf{u} \in \mathcal{U}_N(\hat{x}, z)\}$$

in which the cost function $\bar{V}_N^0(\cdot)$ is defined, as in Chapter 3, by

$$V_N(\hat{x}, z, \mathbf{u}) := \sum_{i=0}^{N-1} \ell(\hat{x}(i) - z^*(i;z), u(i) - v^*(i;z))$$

where $\hat{x}(i) := \bar{\phi}(i; \hat{x}, \mathbf{u})$, the solution at time $i$ of the *nominal* system $z^+ = f(z, u)$ with initial state $\hat{x}$ and control sequence $\mathbf{u}$; $z^*(i;z) := \bar{\phi}(i; z, \bar{\kappa}_N(\cdot))$, the solution at time $i$ of the controlled nominal system

$z^+ = f(z, \bar{\kappa}_N(z))$ MPC with initial state $z$, and $v^*(i;z) = \bar{\kappa}_N(z^*(i;z))$. The constraint set $\mathcal{U}_N(\hat{x}, z)$ is defined by

$$\mathcal{U}_N(\hat{x}, z) := \{\mathbf{u} \in \mathbb{R}^{Nm} \mid \bar{\phi}(N; \hat{x}, \mathbf{u}) = z^*(N; z)\}$$

The terminal equality constraint is chosen for simplicity. Because of the terminal equality constraint, there is no terminal cost. The terminal constraint $\hat{x}(N) = z^*(N;z)$ induces the implicit constraint $\mathbf{u} \in \mathcal{U}_N(\hat{x}, z)$ on the control sequence $\mathbf{u}$. For each $z \in \mathcal{Z}_N$, the domain of the value function $\bar{V}_N^0(\cdot, z)$, and of the minimizer $\mathbf{u}^0(\cdot, z)$, is the set $\hat{X}_N(z)$ defined by

$$\hat{X}_N(z) := \{\hat{x} \mid \mathcal{U}_N(\hat{x}, z) \neq \emptyset\}$$

The minimizing control sequence is

$$\mathbf{u}^0(\hat{x}, z) = \{u^0(0; \hat{x}, z), u^0(1; \hat{x}, z), \ldots, u^0(N-1; \hat{x}, z)\}$$

and the control applied to the estimator system (when the estimator state is $\hat{x}$ and the state of the nominal system is $z$) is $u^0(0; \hat{x}, z)$, the first element in this sequence. The corresponding optimal state sequence is

$$\hat{\mathbf{x}}^0(\hat{x}, z) = \{\hat{x}^0(0; \hat{x}, z), \hat{x}^0(1; \hat{x}, z), \ldots, \hat{x}^0(N; \hat{x}, z)\}$$

The implicit ancillary control law is, therefore, $\kappa_N(\cdot)$ defined by

$$\kappa_N(\hat{x}, z) := u^0(0; \hat{x}, z)$$

The controlled composite system satisfies

$$\begin{aligned} \hat{x}^+ &= f(\hat{x}, \kappa_N(\hat{x}, z)) + \delta \\ z^+ &= f(z, \bar{\kappa}_N(z)) \end{aligned}$$

For each $c > 0$, each $z \in \mathcal{Z}_N$, let $S_c(z) := \{\hat{x} \mid \bar{V}_N^0(\hat{x}, z) \leq c\}$. With appropriate assumptions, there exists a $c \in (0, \infty)$ such that if $\hat{x}(0) \in S_c(z(0))$, then $\hat{x}(i) \in S_c(z^0(i; z(0)))$ for all $i \in \mathbb{I}_{\geq 0}$ and all admissible disturbance and measurement noise sequences, $\mathbf{w}$ and $\boldsymbol{\nu}$. In other words, $c$ is such that $S_c(\cdot)$ is $\hat{x}$-robust positive invariant for the controlled composite system. It follows from the discussion previously that the solutions $\hat{x}(i)$ and $z(i)$ of the controlled composite system satisfy

$$\begin{aligned} &z(i) \to 0 \text{ as } i \to \infty \\ &\hat{x}(i) \in S_c(z(i)) \quad \forall i \in \mathbb{I}_{\geq 0} \\ &x(i) \in S_c(z(i)) \oplus \mathbb{Z}_x \quad \forall i \in \mathbb{I}_{\geq 0} \end{aligned}$$

provided that $z(0) \in \mathcal{Z}_N$ and $\hat{x}(0) \in S_c(z(0))$. Thus, the constraint sets $\mathbb{Z}$ and $\mathbb{V}$ required for the nominal optimal control problem should satisfy

$$\begin{aligned}
&\mathbb{Z}_f \subseteq \mathbb{Z} \\
&S_c(z) \oplus \mathbb{E}_x \subseteq \mathbb{X} \quad \forall z \in \mathcal{Z}_N \\
&\kappa_N(\hat{x}, z) \in \mathbb{U} \ \forall \hat{x} \in S_c(z), \ \forall z \in \mathcal{Z}_N
\end{aligned}$$

If these conditions are satisfied, the solutions $\hat{x}(i)$ and $z(i)$ of the controlled composite system and the associated control $u(i) = \kappa_N(\hat{x}(i), z(i))$ satisfy

$$\begin{aligned}
&z(i) \to 0 \text{ as } i \to \infty \\
&x(i) \in S_c(z(i)) \oplus \mathbb{E}_x \subseteq \mathbb{X} \quad \forall i \in \mathbb{I}_{\geq 0} \\
&u(i) \in \mathbb{U} \quad \forall i \in \mathbb{I}_{\geq 0}
\end{aligned}$$

Compared with the corresponding conditions in Chapter 3, we see that the state constraint set $\mathbb{Z}$ now has to satisfy a stronger requirement than the condition $S_c(z) \subseteq \mathbb{X}$ for all $z \in \mathbb{Z}$ precisely because of the state estimation error, which is bounded by $\mathbb{E}_x$. Hence the state constraint set $\mathbb{Z}$ required here is smaller than that required in Chapter 3.

### 5.6.5 Choosing $\mathbb{Z}$ and $\mathbb{V}$

Because the sets $\mathbb{E}_x$ and $S_d(z)$ cannot be easily computed, a pragmatic approach is required for choosing $\mathbb{Z}$ and $\mathbb{V}$. One simple, if conservative, possibility is to set $\mathbb{Z} = \alpha\mathbb{X}$ and $\mathbb{V} = \beta\mathbb{U}$ where $\alpha$ and $\beta$ lie in $(0, 1)$. The tuning parameters $\alpha$ and $\beta$ may be adjusted using data obtained by Monte Carlo simulation or from operation. If constraints are violated in the simulation, or in operation, $\alpha$ and $\beta$ may be reduced; if the constraints are too conservative, $\alpha$ and $\beta$ may be increased.

## 5.7 Notes

The problem of output feedback control has been extensively discussed in the general control literature. It is well known that, for linear systems, a stabilizing state feedback controller and an observer may be separately designed and combined to give a stabilizing output feedback controller (the separation principle). For nonlinear systems, Teel and Praly (1994) show that global stabilizability and complete uniform

observability are sufficient to guarantee semiglobal stabilizability using a dynamic observer and provide useful references to related work on this topic.

Although output MPC in which nominal MPC is combined with a separately designed observer is widely used in industry since the state is seldom available, it has received relatively little attention in the literature because of the inherent difficulty in establishing asymptotic stability. An extra complexity in MPC is the presence of hard constraints. A useful survey, more comprehensive than these notes, is provided in Findeisen, Imsland, Allgöwer, and Foss (2003). Thus Michalska and Mayne (1995) show for deterministic systems that, for any subset of the region of attraction of the full state feedback system, there exists a sampling time and convergence rate for the observer such that the subset also lies in the region of attraction of the output feedback system. A more sophisticated analysis in Imsland, Findeisen, Allgöwer, and Foss (2003) using continuous time MPC shows that the region of attraction and rate of convergence of the output feedback system can approach that of the state feedback system as observer gain increases.

We consider systems with input disturbances and noisy state measurement; we employ the "tube" methodology that has its roots in the work of Bertsekas and Rhodes (1971), and Glover and Schweppe (1971) on constrained discrete time systems subject to bounded disturbances. Reachability of a "target set" and a "target tube" are considered in these papers. These concepts were substantially developed in the context of continuous time systems in Khurzhanski and Valyi (1997); Aubin (1991); Kurzhanski and Filippova (1993). The theory for discrete time systems is considerably simpler; a modern tube-based theory for optimal control of discrete time uncertain systems with imperfect state measurement appears in Moitié et al. (2002). As in this chapter, they regard a set $X$ of states $x$ that are consistent with past measurements as the "state" of the optimal control problem. The set $X$ satisfies an uncertain "full information" difference equation of the form $X^+ = f^*(X, u, \mathbb{W}, v)$ so the output feedback optimal control problem reduces to robust control of an uncertain system with known state $X$. The optimal control problem remains difficult because the state $X$, a subset of $\mathbb{R}^n$, is difficult to obtain numerically and determination of a control law as a function of $(X, t)$ prohibitive. In Mayne, Raković, Findeisen, and Allgöwer (2006, 2009) the output feedback problem is simplified considerably by replacing $X(t)$ by a simple outer approximation $\{\hat{x}(t)\} \oplus \mathbb{Z}_x$ in the time-invariant case and by $\{\hat{x}(t)\} \oplus \mathbb{Z}_x(t)$ in the

time-varying case. The set $\mathbb{\Sigma}_X$, or the sequence $\{\mathbb{\Sigma}_X(t)\}$, may be precomputed so the difficult evolution equation for $X$ is replaced by a simple evolution equation for $\hat{x}$; in the linear case, the Luenberger observer or Kalman filter describes the evolution of $\hat{x}$. The output feedback control problem reduces to control of an uncertain system with known state $\hat{x}$.

## 5.8 Exercises

### Exercise 5.1: Hausdorff distance between a set and a subset

Show that $d_H(\mathbb{A},\mathbb{B}) = \max_{a\in\mathbb{A}} d(a,\mathbb{B})$ if $\mathbb{A}$ and $\mathbb{B}$ are two compact subsets of $\mathbb{R}^n$ satisfying $\mathbb{B}\subseteq\mathbb{A}$.

### Exercise 5.2: Hausdorff distance between sets $\mathbb{A}\oplus\mathbb{B}$ and $\mathbb{B}$

Show that $d_H(\mathbb{A}\oplus\mathbb{B},\mathbb{A}) \leq |\mathbb{B}|$ if $\mathbb{A}$ and $\mathbb{B}$ are two compact subsets of $\mathbb{R}^n$ satisfying $0\in\mathbb{B}$ in which $|\mathbb{B}| := \max_b\{|b| \mid b\in\mathbb{B}\}$.

### Exercise 5.3: Hausdorff distance between sets $\{z\}\oplus\mathbb{B}$ and $\mathbb{A}$

Show that $d_H(\{z\}\oplus\mathbb{B},\mathbb{A}) \leq |z| + d_H(\mathbb{A},\mathbb{B})$ if $\mathbb{A}$ and $\mathbb{B}$ are two compact sets in $\mathbb{R}^n$.

### Exercise 5.4: Hausdorff distance between sets $\{z\}\oplus\mathbb{A}$ and $\mathbb{A}$

Show that $d_H(\{z\}\oplus\mathbb{A},\mathbb{A}) = |z|$ if $z$ is a point and $\mathbb{A}$ is a compact set in $\mathbb{R}^n$.

### Exercise 5.5: Hausdorff distance between sets $\mathbb{A}\oplus\mathbb{C}$ and $\mathbb{B}\oplus\mathbb{C}$

Show that $d_H(\mathbb{A}\oplus\mathbb{C},\mathbb{B}\oplus\mathbb{C}) = d_H(\mathbb{A},\mathbb{B})$ if $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ are compact subsets of $\mathbb{R}^n$ satisfying $\mathbb{B}\subseteq\mathbb{A}$.

### Exercise 5.6: Hausdorff distance between sets $F\mathbb{A}$ and $F\mathbb{B}$

Let $\mathbb{A}$ and $\mathbb{B}$ be two compact sets in $\mathbb{R}^n$ satisfying $\mathbb{A}\subseteq\mathbb{B}$, and let $F\in\mathbb{R}^{n\times n}$. Show that $d_H(F\mathbb{A},F\mathbb{B}) \leq |F| d_H(\mathbb{A},\mathbb{B})$ in which $|F|$ is the induced norm of $F$ satisfying $|Fx| \leq |F|\,|x|$ and $|x| := d(x,0)$.

### Exercise 5.7: Linear combination of sets; $\lambda_1\mathbb{W}\oplus\lambda_2\mathbb{W} = (\lambda_1+\lambda_2)\mathbb{W}$

If $\mathbb{W}$ is a convex set, show that $\lambda_1\mathbb{W}\oplus\lambda_2\mathbb{W} = (\lambda_1+\lambda_2)\mathbb{W}$ for any $\lambda_1,\ \lambda_2\in\mathbb{R}_{\geq 0}$. Hence show $\mathbb{W}\oplus\lambda\mathbb{W}\oplus\lambda^2\mathbb{W}\oplus\ \cdots = (1-\lambda)^{-1}\mathbb{W}$ if $\lambda\in[0,1)$.

### Exercise 5.8: Hausdorff distance between the sets $\Phi(i)$ and $\Phi$

Show that there exist $c>0$ and $\gamma\in(0,1)$ such that

$$d_H(\Phi(i),\Phi) \leq c\, d_H(\Phi(0),\Phi)\gamma^i$$

in which

$$\Phi(i) = \widetilde{A}\Phi(i-1) + \widetilde{B}\Psi$$
$$\Phi = \widetilde{A}\Phi + \widetilde{B}\Psi$$

and $\widetilde{A}$ is a stable matrix ($\rho(\widetilde{A}) < 1$).

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# 6
# Distributed Model Predictive Control

## 6.1 Introduction and Preliminary Results

In many large-scale control applications, it becomes convenient to break the large plantwide problem into a set of smaller and simpler subproblems in which the local inputs are used to regulate the local outputs. The overall plantwide control is then accomplished by the composite behavior of the interacting, local controllers. There are many ways to design the local controllers, some of which produce guaranteed properties of the overall plantwide system. We consider four control approaches in this chapter: decentralized, noncooperative, cooperative, and centralized control. The first three methods require the local controllers to optimize over only their local inputs. Their computational requirements are identical. The communication overhead is different, however. Decentralized control requires no communication between subsystems. Noncooperative and cooperative control require the input sequences and the current states or state estimates for all the other local subsystems. Centralized control solves the large, complex plantwide optimization over all the inputs. Communication is not a relevant property for centralized control because all information is available in the single plantwide controller. We use centralized control in this chapter to provide a benchmark of comparison for the distributed controllers.

We have established the basic properties of centralized MPC, both with and without state estimation, in Chapters 2, 3, and 5. In this chapter, we analyze some basic properties of the three distributed approaches: decentralized, noncooperative, and cooperative MPC. We show that the conditions required for closed-loop stability of decentralized control and noncooperative control are often violated for models of chemical plants under reasonable decompositions into subsystems. For ensuring closed-loop stability of a wide class of plantwide models

and decomposition choices, cooperative control emerges as the most attractive option for distributed MPC. We then establish the closed-loop properties of cooperative MPC for unconstrained and constrained linear systems with and without state estimation. We also discuss current challenges facing this method, such as input constraints that are coupled between subsystems.

In our development of distributed MPC, we require some basic results on two topics: how to organize and solve the linear algebra of linear MPC, and how to ensure stability when using suboptimal MPC. We cover these two topics in the next sections, and then turn to the distributed MPC approaches.

### 6.1.1 Least Squares Solution

In comparing various forms of linear distributed MPC it proves convenient to see the MPC quadratic program for the sequence of states and inputs as a single large linear algebra problem. To develop this linear algebra problem, we consider first the *unconstrained* LQ problem of Chapter 1, which we solved efficiently with dynamic programming (DP) in Section 1.3.3

$$V(x(0),\mathbf{u}) = \frac{1}{2}\sum_{k=0}^{N-1}\left(x(k)'Qx(k)+u(k)'Ru(k)\right)+(1/2)x(N)'P_f x(N)$$

subject to

$$x^+ = Ax+Bu$$

In this section, we first take the direct but brute-force approach to finding the optimal control law. We write the model solution as

$$\begin{bmatrix} x(1)\\ x(2)\\ \vdots\\ x(N)\end{bmatrix} = \underbrace{\begin{bmatrix} A\\ A^2\\ \vdots\\ A^N\end{bmatrix}}_{\mathcal{A}} x(0) + \underbrace{\begin{bmatrix} B & 0 & \cdots & 0\\ AB & B & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ A^{N-1}B & A^{N-2}B & \cdots & B\end{bmatrix}}_{\mathcal{B}} \begin{bmatrix} u(0)\\ u(1)\\ \vdots\\ u(N-1)\end{bmatrix} \tag{6.1}$$

or using the input and state sequences

$$\mathbf{x} = \mathcal{A}x(0)+\mathcal{B}\mathbf{u}$$

The objective function can be expressed as

$$V(x(0),\mathbf{u}) = (1/2)\left(x'(0)Qx(0)+\mathbf{x}'\mathcal{Q}\mathbf{x}+\mathbf{u}'\mathcal{R}\mathbf{u}\right)$$

in which

$$\begin{aligned}\mathcal{Q} &= \text{diag}\left(\begin{bmatrix} Q & Q & \dots & P_f \end{bmatrix}\right) \in \mathbb{R}^{Nn\times Nn} \\ \mathcal{R} &= \text{diag}\left(\begin{bmatrix} R & R & \dots & R \end{bmatrix}\right) \in \mathbb{R}^{Nm\times Nm}\end{aligned} \tag{6.2}$$

**Eliminating the state sequence.** Substituting the model into the objective function and *eliminating* the state sequence gives a quadratic function of $\mathbf{u}$

$$V(x(0),\mathbf{u}) = (1/2)x'(0)(Q + \mathcal{A}'\mathcal{Q}\mathcal{A})x(0) + \mathbf{u}'(\mathcal{B}'\mathcal{Q}\mathcal{A})x(0) + (1/2)\mathbf{u}'(\mathcal{B}'\mathcal{Q}\mathcal{B} + \mathcal{R})\mathbf{u} \tag{6.3}$$

and the optimal solution for the entire set of inputs is obtained in one shot

$$\mathbf{u}^0(x(0)) = -(\mathcal{B}'\mathcal{Q}\mathcal{B} + \mathcal{R})^{-1}\mathcal{B}'\mathcal{Q}\mathcal{A}\, x(0)$$

and the optimal cost is

$$V^0(x(0)) = \left(\frac{1}{2}\right)x'(0)\left(Q + \mathcal{A}'\mathcal{Q}\mathcal{A} - \mathcal{A}'\mathcal{Q}\mathcal{B}(\mathcal{B}'\mathcal{Q}\mathcal{B} + \mathcal{R})^{-1}\mathcal{B}'\mathcal{Q}\mathcal{A}\right)x(0)$$

If used explicitly, this procedure for computing $\mathbf{u}^0$ would be inefficient because $\mathcal{B}'\mathcal{Q}\mathcal{B} + \mathcal{R}$ is an $(mN \times mN)$ matrix. Notice that in the DP formulation one has to invert instead an $(m\times m)$ matrix $N$ times, which is computationally less expensive.[1] Notice also that unlike DP, the least squares approach provides *all* input moves as a function of the *initial* state, $x(0)$. The gain for the control law comes from the first input move in the sequence

$$K(0) = -\begin{bmatrix} I_m & 0 & \cdots & 0 \end{bmatrix}(\mathcal{B}'\mathcal{Q}\mathcal{B} + \mathcal{R})^{-1}\mathcal{B}'\mathcal{Q}\mathcal{A}$$

It is not immediately clear that the $K(0)$ and $V^0$ given above from the least squares approach are equivalent to the result from the Riccati iteration, (1.11)–(1.15) of Chapter 1, but since we have solved the same optimization problem, the two results are the same.[2]

**Retaining the state sequence.** In this section we set up the least squares problem again, but with an eye toward improving its efficiency. Retaining the state sequence and adjoining the model equations as

[1] Would you prefer to invert by hand 100 $(1 \times 1)$ matrices or a single $(100 \times 100)$ dense matrix?

[2] Establishing this result directly is an exercise in using the partitioned matrix inversion formula. The next section provides another way to show they are equivalent.

equality constraints is a central idea in optimal control and is described in standard texts (Bryson and Ho, 1975, p. 44). We apply this standard approach here. Wright (1997) provides a discussion of this problem in the linear model MPC context and the extensions required for the quadratic programming problem when there are inequality constraints on the states and inputs. Including the state with the input in the sequence of unknowns, we define the enlarged vector $\mathbf{z}$ to be

$$\mathbf{z} = \begin{bmatrix} u(0) \\ x(1) \\ u(1) \\ x(2) \\ \vdots \\ u(N-1) \\ x(N) \end{bmatrix}$$

The objective function is

$$\min_{\mathbf{u}} (1/2)(x'(0)Qx(0) + \mathbf{z}'H\mathbf{z})$$

in which

$$H = \operatorname{diag}\left(\begin{bmatrix} R & Q & R & Q & \cdots & R & P_f \end{bmatrix}\right)$$

The constraints are

$$D\mathbf{z} = d$$

in which

$$D = -\begin{bmatrix} B & -I & & & & & \\ & A & B & -I & & & \\ & & & \ddots & & & \\ & & & & A & B & -I \end{bmatrix} \qquad d = \begin{bmatrix} A \\ 0 \\ \vdots \\ 0 \end{bmatrix} x(0)$$

We now substitute these results into (1.58) and obtain the linear algebra problem

$$\begin{bmatrix} R & & & & & & & B' & & & \\ & Q & & & & & & -I & A' & & \\ & & R & & & & & & B' & & \\ & & & Q & & & & & -I & & \\ & & & & \ddots & & & & & \ddots & \\ & & & & & R & & & & & B' \\ & & & & & & P_f & & & & -I \\ B & -I & & & & & & & & & \\ & A & B & -I & & & & & & & \\ & & & & \ddots & & & & & & \\ & & & & & B & -I & & & & \end{bmatrix} \begin{bmatrix} u(0) \\ x(1) \\ u(1) \\ x(2) \\ \vdots \\ u(N-1) \\ x(N) \\ \lambda(1) \\ \lambda(2) \\ \vdots \\ \lambda(N) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ -A \\ 0 \\ \vdots \\ 0 \end{bmatrix} x(0)$$

| Method | FLOPs |
|---|---|
| dynamic programming (DP) | $Nm^3$ |
| dense least squares | $N^3m^3$ |
| banded least squares | $N(2n+m)(3n+m)^2$ |

**Table 6.1:** Computational cost of solving finite horizon LQR problem.

This equation is rather cumbersome, but if we reorder the unknown vector to put the Lagrange multiplier together with the state and input from the same time index, and reorder the equations, we obtain the following banded matrix problem

$$\begin{bmatrix} R & B' & & & & & & & & & & \\ B & & -I & & & & & & & & & \\ & -I & Q & & & & & & & & & \\ & & & R & & & & & & & & \\ & & & \ddots & \ddots & & & & & & & \\ & & & & & R & B' & & & & & \\ & & & & A & B & & -I & & & & \\ & & & & & & -I & Q & & A' & & \\ & & & & & & & & R & B' & & \\ & & & & & & & A & B & & -I & \\ & & & & & & & & & -I & P_f \end{bmatrix} \begin{bmatrix} u(0) \\ \lambda(1) \\ x(1) \\ u(1) \\ \vdots \\ u(N-2) \\ \lambda(N-1) \\ x(N-1) \\ u(N-1) \\ \lambda(N) \\ x(N) \end{bmatrix} = \begin{bmatrix} 0 \\ -A \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} x(0) \quad (6.4)$$

The banded structure allows a more efficient solution procedure. The floating operation (FLOP) count for the factorization of a banded matrix is $O(LM^2)$ in which $L$ is the dimension of the matrix and $M$ is the bandwidth. This compares to the regular FLOP count of $O(L^3)$ for the factorization of a regular dense matrix. The bandwidth of the matrix in (6.4) is $3n+m$ and the dimension of the matrix is $N(2n+m)$. Therefore the FLOP count for solving this equation is $O(N(2n+m)(3n+m)^2)$. Notice that this approach reduces the $N^3$ dependence of the previous MPC solution method. That is the computational advantage provided by these adjoint methods for treating the model constraints. Table 6.1 summarizes the computational cost of the three approaches. As shown in the table, DP is highly efficient. When we add input and state inequality constraints to the control problem and the state dimension is large, however, we cannot conveniently apply DP. The dense least squares computational cost is high if we wish to compute a large number of moves in the horizon. Note the cost of dense least squares scales with the third power of horizon length $N$. As we have discussed in Chap-

ter 2, considerations of control theory favor large $N$. Another factor increasing the computational cost is the trend in industrial MPC implementations to larger multivariable control problems with more states and inputs, i.e., larger $m$ and $n$. Therefore, the adjoint approach using banded least squares method becomes important for industrial applications in which the problems are large and a solid theoretical foundation for the control method is desirable.

We might obtain more efficiency than the banded structure if we view (6.4) as a block tridiagonal matrix and use the method provided by Golub and Van Loan (1996, p. 174). The final fine tuning of the solution method for this class of problems is a topic of current research, but the important point is that, whatever final procedure is selected, the computational cost will be linear in $N$ as in DP instead of cubic in $N$ as in dense least squares.

Furthermore, if we wish to see the connection to the DP solution, we can proceed as follows. Substitute $\Pi(N) = P_f$ as in (1.12) of Chapter 1 and consider the last three-equation block of the matrix appearing in (6.4)

$$\begin{bmatrix} & R & B' & \\ A & B & & -I \\ & & -I & \Pi(N) \end{bmatrix} \begin{bmatrix} x(N-1) \\ u(N-1) \\ \lambda(N) \\ x(N) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We can eliminate this small set of equations and solve for $u(N-1)$, $\lambda(N)$, $x(N)$ in terms of $x(N-1)$, resulting in

$$\begin{bmatrix} u(N-1) \\ \lambda(N) \\ x(N) \end{bmatrix} = \begin{bmatrix} -(B'\Pi(N)B+R)^{-1}B'\Pi(N)A \\ \Pi(N)(I-B(B'\Pi(N)B+R)^{-1}B'\Pi(N))A \\ (I-B(B'\Pi(N)B+R)^{-1}B'\Pi(N))A \end{bmatrix} x(N-1)$$

Notice that in terms of the Riccati matrix, we also have the relationship

$$A'\lambda(N) = \Pi(N-1)x(N-1) - Qx(N-1)$$

We then proceed to the next to last block of three equations

$$\begin{bmatrix} & R & B' & & \\ A & B & & -I & \\ & & -I & Q & A' \end{bmatrix} \begin{bmatrix} x(N-2) \\ u(N-2) \\ \lambda(N-1) \\ x(N-1) \\ u(N-1) \\ \lambda(N) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Note that the last equation gives

$$\lambda(N-1) = Qx(N-1) + A'\lambda(N) = \Pi(N-1)x(N-1)$$

Using this relationship and continuing on to solve for $x(N-1)$, $\lambda(N-1)$, $u(N-2)$ in terms of $x(N-2)$ gives

$$\begin{bmatrix} u(N-2) \\ \lambda(N-1) \\ x(N-1) \end{bmatrix} = \begin{bmatrix} -(B'\Pi(N-1)B+R)^{-1}B'\Pi(N-1)A \\ \Pi(N-1)(I-B(B'\Pi(N-1)B+R)^{-1}B'\Pi(N-1))A \\ (I-B(B'\Pi(N-1)B+R)^{-1}B'\Pi(N-1))A \end{bmatrix} x(N-2)$$

Continuing on through each previous block of three equations produces the Riccati iteration and feedback gains of (1.11)–(1.14). The other unknowns, the multipliers, are simply

$$\lambda(k) = \Pi(k)x(k) \qquad k = 1, 2, \ldots, N$$

so the cost to go at each stage is simply $x(k)'\lambda(k)$, and we see the nice connection between the Lagrange multipliers and the cost of the LQR control problem.

### 6.1.2 Stability of Suboptimal MPC

When using distributed MPC, it may be necessary or convenient to implement the control without solving the complete optimization. We then have a form of suboptimal MPC, which was first considered in Chapter 2, Section 2.8. Before adding the complexity of the distributed version, we wish to further develop a few features of suboptimal MPC in the centralized, single-player setting. These same features arise in the distributed, many-player setting as we discuss subsequently.

We consider a specific variation of suboptimal MPC in which a starting guess is available from the control trajectory at the previous time and we take a fixed number of steps of an optimization algorithm. The exact nature of the optimization method is not essential, but we do restrict the method so that each iteration is feasible and decreases the value of the cost function. To initialize the suboptimal controller, we are given an initial state $x(0) = x_0$, and we generate an initial control sequence $\mathbf{u}(0) = \mathbf{h}(x_0)$. We consider input constraints $u(i) \in \mathbb{U} \subseteq \mathbb{R}^m$, $i \in \mathbb{I}_{0:N-1}$, which we also write as $\mathbf{u} \in \mathbb{U}^N \subseteq \mathbb{R}^N$. As in Chapter 2 we denote the set of feasible states as $\mathcal{X}_N$. These are the states for which the initial control sequence $\mathbf{h}(x_0)$ is well defined. The suboptimal MPC algorithm is as follows.

**Suboptimal MPC algorithm.**

**Data:** Integer $N_{\text{iter}}$.

**Initialize:** Set current state $x = x_0$, current control sequence, $\mathbf{u} = \mathbf{h}(x_0)$.

**Step 1 (State evolution):** Apply control $u = u(0)$ to the system. Obtain state at next sample, $x^+$. For the nominal system

$$x^+ = f(x, u(0))$$

**Step 2 (Warm start):** Denote the warm start for the next sample time as $\tilde{\mathbf{u}}^+$. We use

$$\tilde{\mathbf{u}}^+ = \{u(1), u(2), \ldots, u(N-1), 0\}$$

in which $x(N) = \phi(N; x, \mathbf{u})$. The warm start $\tilde{\mathbf{u}}^+$ therefore is a function of $(x, \mathbf{u})$. This warm start is a simplified version of the one considered in Chapter 2, in which the final control move in the warm start was determined by the control law $\kappa_f(x)$. In distributed MPC it is simpler to use zero for the final control move in the warm start.

**Step 3 (Iteration of an optimization method):** The controller performs $N_{\text{iter}}$ iterations of a feasible path optimization algorithm to obtain an improved control sequence using initial state $x^+$. The final input sequence $\mathbf{u}^+$ is a function of the state initial condition and the warm start $(x^+, \tilde{\mathbf{u}})$. Noting that $x^+$ and $\tilde{\mathbf{u}}$ are both functions of $(x, \mathbf{u})$, the input sequence $\mathbf{u}^+$ can also be expressed as function of only $(x, \mathbf{u})$

$$\mathbf{u}^+ = g(x, \mathbf{u})$$

**Step 4 (Next time step):** Update state and input sequence: $x \leftarrow x^+$, $\mathbf{u} \leftarrow \mathbf{u}^+$. Go to Step 1.

We establish later in the chapter that the system cost function $V(x, \mathbf{u})$ satisfies the following properties for the form of suboptimal MPC generated by distributed MPC. There exist constants $a, b, c > 0$ such that

$$a\,|(x, \mathbf{u})|^2 \le V(x, \mathbf{u}) \le b\,|(x, \mathbf{u})|^2$$
$$V(x^+, \mathbf{u}^+) - V(x, \mathbf{u}) \le -c\,|(x, u(0))|^2$$

These properties are similar to those required for a valid Lyapunov function. The difference is that the cost decrease here does not depend on the size of $\mathbf{u}$, but only $x$ and the first element of $\mathbf{u}$, $u(0)$. This cost decrease is sufficient to establish that $x(k)$ and $u(k)$ converge to zero, but allows the possibility that $\mathbf{u}(k)$ is large even though $x(k)$ is small. That fact prevents us from establishing the solution $x(k) = 0$ for all $k$ is Lyapunov stable. We can establish that the solution $x(k) = 0$ for all $k$ is Lyapunov stable at $k = 0$ only. We cannot establish uniform Lyapunov stability nor Lyapunov stability for any $k > 0$. The problem is not that our proof technique is deficient. There is no reason to *expect* that the solution $x(k) = 0$ for all $k$ is Lyapunov stable for suboptimal MPC. The lack of Lyapunov stability of $x(k) = 0$ for all $k$ is a subtle issue and warrants some discussion. To make these matters more precise, consider the following standard definitions of Lyapunov stability at time $k$ and uniform Lyapunov stability (Vidyasagar, 1993, p. 136).

**Definition 6.1** (Lyapunov stability)**.** The zero solution $x(k) = 0$ for all $k$ is stable (in the sense of Lyapunov) at $k = k_0$ if for any $\varepsilon > 0$ there exists a $\delta(k_0, \varepsilon) > 0$ such that

$$|x(k_0)| < \delta \implies |x(k)| < \varepsilon \quad \forall k \geq k_0 \tag{6.5}$$

Lyapunov stability is defined at a time $k_0$. Uniform stability is the concept that guarantees that the zero solution is not losing stability with time. For a uniformly stable zero solution, $\delta$ in Definition 6.1 is *not* a function of $k_0$, so that (6.5) holds for all $k_0$.

**Definition 6.2** (Uniform Lyapunov stability)**.** The zero solution $x(k) = 0$ for all $k$ is uniformly stable (in the sense of Lyapunov) if for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that

$$|x(k_0)| < \delta \implies |x(k)| < \varepsilon \quad \forall k \geq k_0 \quad \forall k_0$$

Exercise 6.6 gives an example of a linear system for which $x(k)$ converges exponentially to zero with increasing $k$ for all $x(0)$, but the zero solution $x(k) = 0$ for all $k$ is Lyapunov stable only at $k = 0$. It is not uniformly Lyapunov stable nor Lyapunov stable for any $k > 0$. Without further restrictions, suboptimal MPC admits this same type of behavior.

To ensure uniform Lyapunov stability, we add requirements to suboptimal MPC beyond obtaining only a cost decrease. Here we impose the constraint

$$|\mathbf{u}| \leq d\,|x| \qquad x \in r\mathcal{B}$$

in which $d, r > 0$. This type of constraint was first introduced in (2.43) of Chapter 2. In that arrangement of suboptimal MPC it was simplest to switch to local controller $u = \kappa_f(x)$ when the state entered $\mathbb{X}_f$ to automatically enforce this constraint. In this chapter we instead include the constraint explicitly in the distributed MPC optimization problem and do not switch to a local controller. Both alternatives provide (uniform) Lyapunov stability of the solution $x(k) = 0$ for all $k$. The following lemma summarizes the conditions we use later in the chapter for establishing exponential stability of distributed MPC. A similar lemma establishing asymptotic stability of suboptimal MPC was given by Scokaert, Mayne, and Rawlings (1999) (Theorem 1).

**Definition 6.3** (Exponential stability)**.** Let $\mathbb{X}$ be positive invariant for $x^+ = f(x)$. Then the origin is exponentially stable for $x^+ = f(x)$ with a region of attraction $\mathbb{X}$ if there exists $c > 0$ and $\gamma < 1$ such that

$$\left|\phi(i;x)\right| \leq c\,|x|\,\gamma^i$$

for all $i \geq 0, x \in \mathbb{X}$.

Consider next the suboptimal MPC controller. Let the system satisfy $(x^+, \mathbf{u}^+) = (f(x,\mathbf{u}), g(x,\mathbf{u}))$ with initial sequence $\mathbf{u}(0) = \mathbf{h}(x(0))$. The controller constraints are $x(i) \in \mathbb{X} \subseteq \mathbb{R}^n$ for all $i \in \mathbb{I}_{0:N}$ and $u(i) \in \mathbb{U} \subseteq \mathbb{R}^m$ for all $i \in \mathbb{I}_{0:N-1}$. Let $\mathcal{X}_N$ denote the set of states for which the MPC controller is feasible. The suboptimal MPC system satisfies the following. Given $r > 0$, there exist $a, b, c > 0$ such that

$$\begin{aligned}
a\,|(x,\mathbf{u})|^2 \leq V(x,\mathbf{u}) \leq b\,|(x,\mathbf{u})|^2 \qquad & x \in \mathcal{X}_N \quad \mathbf{u} \in \mathbb{U}^N \\
V(x^+,\mathbf{u}^+) - V(x,\mathbf{u}) \leq -c\,|(x,u(0))|^2 \qquad & x \in \mathcal{X}_N \quad \mathbf{u} \in \mathbb{U}^N \\
|\mathbf{u}| \leq d\,|x| \qquad & x \in r\mathcal{B}
\end{aligned}$$

**Lemma 6.4** (Exponential stability of suboptimal MPC)**.** *The origin is exponentially stable for the closed-loop system under suboptimal MPC with region of attraction $\mathcal{X}_N$ if either of the following assumptions holds*

*(a) $\mathbb{U}$ is compact. In this case, $\mathcal{X}_N$ may be unbounded.*

*(b) $\mathcal{X}_N$ is compact. In this case $\mathbb{U}$ may be unbounded.*

Exercises 6.7 and 6.8 explore what to conclude about exponential stability when both $\mathbb{U}$ and $\mathcal{X}_N$ are unbounded.

*Proof.* First we show that the origin of the extended state $(x,\mathbf{u})$ is exponentially stable for $x(0) \in \mathcal{X}_N$.

(a) For the case $\mathbb{U}$ compact, we use the same argument used to prove Proposition 2.18 of Chapter 2. We have $|\mathbf{u}| \leq d\,|x|\,, x \in r\mathcal{B}$. Consider the optimization

$$\max_{\mathbf{u}\in\mathbb{U}^N} |\mathbf{u}| = s > 0$$

The solution exists by the Weierstrass theorem since $\mathbb{U}$ is compact, which implies $\mathbb{U}^N$ is compact. Then we have $|\mathbf{u}| \leq (s/r)\,|x|$ for $x \in \mathcal{X}_N \setminus r\mathcal{B}$, so we have $|\mathbf{u}| \leq d'\,|x|$ for $x \in \mathcal{X}_N$ in which $d' = \max(d, s/r)$.

(b) For the case $\mathcal{X}_N$ compact, consider the optimization

$$\max_{x\in\mathcal{X}_N} V(x, \mathbf{h}(x)) = \bar{V} > 0$$

The solution exists because $\mathcal{X}_N$ is compact and $\mathbf{h}(\cdot)$ and $V(\cdot)$ are continuous. Define the compact set $\bar{\mathbb{U}}$ by

$$\bar{\mathbb{U}} = \{\mathbf{u} \mid V(x, \mathbf{u}) \leq \bar{V}, \quad x \in \mathcal{X}_N\}$$

The set is bounded because $V(x, \mathbf{u}) \geq a\,|(x, \mathbf{u})|^2 \geq a\,|\mathbf{u}|^2$. The set is closed because $V$ is continuous. The significance of this set is that for all $k \geq 0$ and all $x \in \mathcal{X}_N$, $\mathbf{u}(k) \in \bar{\mathbb{U}}$. Therefore we have established that $\mathcal{X}_N$ compact implies $\mathbf{u}(k)$ evolves in a compact set as in the previous case when $\mathbb{U}$ is assumed compact. Using the same argument as in that case, we have established that there exists $d' > 0$ such that $|\mathbf{u}| \leq d'\,|x|$ for all $x \in \mathcal{X}_N$.

For the two cases, we therefore have established for all $x \in \mathcal{X}_N$, $\mathbf{u} \in \mathbb{U}^N$ (case (a)) or $\mathbf{u} \in \bar{\mathbb{U}}$ (case (b))

$$|(x, \mathbf{u})| \leq |x| + |\mathbf{u}| \leq |x| + d'\,|x| \leq (1 + d')\,|x|$$

which gives $|x| \geq c'\,|(x, \mathbf{u})|$ with $c' = 1/(1 + d') > 0$. Hence, there exists $a_3 = c(c')^2$ such that $V(x^+, \mathbf{u}^+) - V(x, \mathbf{u}) \leq -a_3\,|(x, \mathbf{u})|^2$ for all $x \in \mathcal{X}_N$. Therefore the extended state $(x, \mathbf{u})$ satisfies the standard conditions of an exponential stability Lyapunov function (see Theorem B.14 in Appendix B) with $a_1 = a, a_2 = b, a_3 = c(c')^2, \sigma = 2$ for $(x, \mathbf{u}) \in \mathcal{X}_N \times \mathbb{U}^N$ (case (a)) or $\mathcal{X}_N \times \bar{\mathbb{U}}$ (case (b)). Therefore for all $x(0) \in \mathcal{X}_N$, $k \geq 0$,

$$|(x(k), \mathbf{u}(k))| \leq \alpha\,|(x(0), \mathbf{u}(0))|\,\gamma^k$$

in which $\alpha > 0$ and $0 < \gamma < 1$.

Finally we remove the input sequence and establish that the origin for the state (rather than the extended state) is exponentially stable for the closed-loop system. We have for all $x(0) \in \mathcal{X}_N$ and $k \geq 0$

$$\begin{aligned} |x(k)| &\leq |(x(k), \mathbf{u}(k))| \leq \alpha\, |(x(0), \mathbf{u}(0))|\, \gamma^k \\ &\leq \alpha(|x(0)| + |\mathbf{u}(0)|)\gamma^k \leq \alpha(1 + d')\, |x(0)|\, \gamma^k \\ &\leq \alpha'\, |x(0)|\, \gamma^k \end{aligned}$$

in which $\alpha' = \alpha(1 + d') > 0$, and we have established exponential stability of the origin on the feasible set $\mathcal{X}_N$. ■

We also consider later in the chapter the effects of state estimation error on the closed-loop properties of distributed MPC. For analyzing stability under perturbations, the following lemma is useful. Here $e$ plays the role of estimation error.

**Lemma 6.5** (Exponential stability with mixed powers of norm)**.** *Consider a dynamic system*

$$(x^+, e^+) = f(x, e)$$

*with a zero steady-state solution, $f(0,0) = (0,0)$. Assume there exists a function $V : \mathbb{R}^{n+m} \to \mathbb{R}_{\geq 0}$ that satisfies the following for all $(x, e) \in \mathbb{R}^n \times \mathbb{R}^m$*

$$a(|x|^\sigma + |e|^\gamma) \leq V((x, e)) \leq b(|x|^\sigma + |e|^\gamma) \tag{6.6}$$

$$V(f(x, e)) - V((x, e)) \leq -c(|x|^\sigma + |e|^\gamma) \tag{6.7}$$

*with constants $a, b, c, \sigma, \gamma > 0$. Then the zero steady-state solution is globally exponentially stable for $(x^+, e^+) = f(x, e)$.*

The proof of this lemma is discussed in Exercise 6.9. We also require a converse theorem for exponential stability.

**Lemma 6.6** (Converse theorem for exponential stability)**.** *If the zero steady-state solution of $x^+ = f(x)$ is globally exponentially stable, then there exists Lipschitz continuous $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ that satisfies the following: there exist constants $a, b, c, \sigma > 0$, such that for all $x \in \mathbb{R}^n$*

$$\begin{aligned} a\, |x|^\sigma \leq V(x) &\leq b\, |x|^\sigma \\ V(f(x)) - V(x) &\leq -c\, |x|^\sigma \end{aligned}$$

*Moreover, any $\sigma > 0$ is valid, and the constant $c$ can be chosen as large as one wishes.*

The proof of this lemma is discussed in Exercise B.3.

## 6.2 Unconstrained Two-Player Game

To introduce clearly the concepts and notation required to analyze distributed MPC, we start with a two-player game. We then generalize to an $M$-player game in the next section.

Let $(A_{11}, B_{11}, C_{11})$ be a minimal state space realization of the $(u_1, y_1)$ input-output pair. Similarly, let $(A_{12}, B_{12}, C_{12})$ be a minimal state space realization of the $(u_2, y_1)$ input-output pair. The dimensions are $u_1 \in \mathbb{R}^{m_1}$, $y_1 \in \mathbb{R}^{p_1}$, $x_{11} \in \mathbb{R}^{n_{11}}$, $x_{12} \in \mathbb{R}^{n_{12}}$ with $n_1 = n_{11} + n_{12}$. Output $y_1$ can then be represented as the following, possibly nonminimal, state space model

$$\begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix}^+ = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{12} \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} + \begin{bmatrix} B_{11} \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ B_{12} \end{bmatrix} u_2$$
$$y_1 = \begin{bmatrix} C_{11} & C_{12} \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix}$$

Proceeding in an analogous fashion with output $y_2$ and inputs $u_1$ and $u_2$, we model $y_2$ with the following state space model

$$\begin{bmatrix} x_{22} \\ x_{21} \end{bmatrix}^+ = \begin{bmatrix} A_{22} & 0 \\ 0 & A_{21} \end{bmatrix} \begin{bmatrix} x_{22} \\ x_{21} \end{bmatrix} + \begin{bmatrix} B_{22} \\ 0 \end{bmatrix} u_2 + \begin{bmatrix} 0 \\ B_{21} \end{bmatrix} u_1$$
$$y_2 = \begin{bmatrix} C_{22} & C_{21} \end{bmatrix} \begin{bmatrix} x_{22} \\ x_{21} \end{bmatrix}$$

We next define player one's local cost functions

$$V_1(x_1(0), \mathbf{u}_1, \mathbf{u}_2) = \sum_{k=0}^{N-1} \ell_1(x_1(k), u_1(k)) + V_{1f}(x_1(N))$$

in which

$$x_1 = \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix}$$

Note that the first local objective is affected by the second player's inputs through the model evolution of $x_1$, i.e., through the $x_{12}$ states. We choose the stage cost to account for the first player's inputs and outputs

$$\ell_1(x_1, u_1) = (1/2)(y_1' \overline{Q}_1 y_1 + u_1' R_1 u_1)$$
$$\ell_1(x_1, u_1) = (1/2)(x_1' Q_1 x_1 + u_1' R_1 u_1)$$

in which

$$Q_1 = C_1' \overline{Q}_1 C_1 \qquad C_1 = \begin{bmatrix} C_{11} & C_{12} \end{bmatrix}$$

Motivated by the warm start to be described later, for stable systems, we choose the terminal penalty to be the infinite horizon cost to go under zero control

$$V_{1f}(x_1(N)) = (1/2) x_1'(N) P_{1f} x_1(N)$$

We choose $P_{1f}$ as the solution to the following Lyapunov equation assuming $A_1$ is stable

$$A_1' P_{1f} A_1 - P_{1f} = -Q_1 \tag{6.8}$$

We proceed analogously to define player two's local objective function and penalties

$$V_2(x_2(0), \mathbf{u}_1, \mathbf{u}_2) = \sum_{k=0}^{N-1} \ell_2(x_2(k), u_2(k)) + V_{2f}(x_2(N))$$

In centralized control and the cooperative game, the two players share a common objective, which can be considered to be the overall plant objective

$$V(x_1(0), x_2(0), \mathbf{u}_1, \mathbf{u}_2) = \rho_1 V_1(x_1(0), \mathbf{u}_1, \mathbf{u}_2) + \rho_2 V_2(x_2(0), \mathbf{u}_2, \mathbf{u}_1)$$

in which the parameters $\rho_1, \rho_2$ are used to specify the relative weights of the two subsystems in the overall plant objective. Their values are restricted so $\rho_1, \rho_2 > 0$, $\rho_1 + \rho_2 = 1$ so that both local objectives must have some nonzero effect on the overall plant objective.

### 6.2.1 Centralized Control

Centralized control requires the solution of the systemwide control problem. It can be stated as

$$\begin{aligned} \min_{\mathbf{u}_1, \mathbf{u}_2} & V(x_1(0), x_2(0), \mathbf{u}_1, \mathbf{u}_2) \\ \text{s.t. } x_1^+ &= A_1 x_1 + \overline{B}_{11} u_1 + \overline{B}_{12} u_2 \\ x_2^+ &= A_2 x_2 + \overline{B}_{22} u_2 + \overline{B}_{21} u_1 \end{aligned}$$

in which

$$A_1 = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{12} \end{bmatrix} \qquad A_2 = \begin{bmatrix} A_{22} & 0 \\ 0 & A_{21} \end{bmatrix}$$

$$\overline{B}_{11} = \begin{bmatrix} B_{11} \\ 0 \end{bmatrix} \qquad \overline{B}_{12} = \begin{bmatrix} 0 \\ B_{12} \end{bmatrix} \qquad \overline{B}_{21} = \begin{bmatrix} 0 \\ B_{21} \end{bmatrix} \qquad \overline{B}_{22} = \begin{bmatrix} B_{22} \\ 0 \end{bmatrix}$$

This optimal control problem is more complex than all of the distributed cases to follow because the decision variables include both $\mathbf{u}_1$ and $\mathbf{u}_2$. Because the performance is optimal, centralized control is a natural benchmark against which to compare the distributed cases: cooperative, noncooperative, and decentralized MPC. The plantwide stage cost and terminal cost can be expressed as quadratic functions of the subsystem states and inputs

$$\ell(x,u) = (1/2)(x'Qx + u'Ru)$$
$$V_f(x) = (1/2)x'P_f x$$

in which

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad Q = \begin{bmatrix} \rho_1 Q_1 & 0 \\ 0 & \rho_2 Q_2 \end{bmatrix}$$
$$R = \begin{bmatrix} \rho_1 R_1 & 0 \\ 0 & \rho_2 R_2 \end{bmatrix} \qquad P_f = \begin{bmatrix} \rho_1 P_{1f} & 0 \\ 0 & \rho_2 P_{2f} \end{bmatrix} \tag{6.9}$$

and we have the standard MPC problem considered in Chapters 1 and 2

$$\min_{\mathbf{u}} V(x(0), \mathbf{u})$$
$$\text{s.t. } x^+ = Ax + Bu \tag{6.10}$$

in which

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \qquad B = \begin{bmatrix} \overline{B}_{11} & \overline{B}_{12} \\ \overline{B}_{21} & \overline{B}_{22} \end{bmatrix} \tag{6.11}$$

Given the terminal penalty in (6.8), stability of the closed-loop centralized system is guaranteed for all choices of system models and tuning parameters subject to the usual stabilizability assumption on the system model.

### 6.2.2 Decentralized Control

Centralized and decentralized control define the two extremes in distributing the decision making in a large-scale system. Centralized control has full information and optimizes the full control problem over all decision variables. Decentralized control, on the other hand, optimizes only the local objectives and has no information about the actions of the other subsystems. Player one's objective function is

$$V_1(x_1(0), \mathbf{u}_1) = \sum_{k=0}^{N-1} \ell_1(x_1(k), u_1(k)) + V_{1f}(x_1(N))$$

We then have player one's decentralized control problem

$$\min_{\mathbf{u}_1} V_1(x_1(0), \mathbf{u}_1)$$
$$\text{s.t. } x_1^+ = A_1 x_1 + \overline{B}_{11} u_1$$

We know the optimal solution for this kind of LQ problem is a linear feedback law

$$u_1^0 = K_1 x_1(0)$$

Notice that in decentralized control, player one's model does not account for the inputs of player two, and already contains model error. In the decentralized problem, player one requires no information about player two. The communication overhead for decentralized control is therefore minimal, which is an implementation advantage, but the resulting performance may be quite poor for systems with reasonably strong coupling. We compute an optimal $K_1$ for system one $(A_1, \overline{B}_{11}, Q_1, R_1)$ and optimal $K_2$ for system 2. The closed-loop system evolution is then

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^+ = \begin{bmatrix} A_1 + \overline{B}_{11} K_1 & \overline{B}_{12} K_2 \\ \overline{B}_{21} K_1 & A_2 + \overline{B}_{22} K_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

and we know only that $A_{11} + \overline{B}_{11} K_1$ and $A_{22} + \overline{B}_{22} K_2$ are stable matrices. Obviously the stability of the closed-loop, decentralized system is fragile and depends in a sensitive way on the sizes of the interaction terms $\overline{B}_{12}$ and $\overline{B}_{21}$ and feedback gains $K_1$, $K_2$.

### 6.2.3 Noncooperative Game

In the noncooperative game, player one optimizes $V_1(x_1(0), \mathbf{u}_1, \mathbf{u}_2)$ over $\mathbf{u}_1$ and player two optimizes $V_2(x_2(0), \mathbf{u}_1, \mathbf{u}_2)$ over $\mathbf{u}_2$. From player one's perspective, player two's planned inputs $\mathbf{u}_2$ are known disturbances affecting player one's output through the dynamic model. Part of player one's optimal control problem is therefore to compensate for player two's inputs with his optimal $\mathbf{u}_1$ sequence in order to optimize his local objective $V_1$. Similarly, player two considers player one's inputs as a known disturbance and solves an optimal control problem that removes their effect in his local objective $V_2$. Because this game is noncooperative ($V_1 \neq V_2$), the struggle between players one and two can produce an outcome that is bad for both of them as we show subsequently. Notice that unlike decentralized control, there is no model error in the noncooperative game. Player one knows exactly the effect

of the actions of player two and vice versa. Any poor nominal performance is caused by the noncooperative game, not model error.

Summarizing the noncooperative control problem statement, player one's model is

$$x_1^+ = A_1 x_1 + \overline{B}_{11} u_1 + \overline{B}_{12} u_2$$

and player one's objective function is

$$V_1(x_1(0), \mathbf{u}_1, \mathbf{u}_2) = \sum_{k=0}^{N-1} \ell_1(x_1(k), u_1(k)) + V_{1f}(x_1(N))$$

Note that $V_1$ here depends on $\mathbf{u}_2$ because the state trajectory $x_1(k), k \geq 1$ depends on $\mathbf{u}_2$ as shown in player one's dynamic model. We then have player one's noncooperative control problem

$$\min_{\mathbf{u}_1} V_1(x_1(0), \mathbf{u}_1, \mathbf{u}_2)$$
$$\text{s.t. } x_1^+ = A_1 x_1 + \overline{B}_{11} u_1 + \overline{B}_{12} u_2$$

**Solution to player one's optimal control problem.** We now solve player one's optimal control problem. Proceeding as in Section 6.1.1 we define

$$\mathbf{z} = \begin{bmatrix} u_1(0) \\ x_1(1) \\ \vdots \\ u_1(N-1) \\ x_1(N) \end{bmatrix} \qquad H = \text{diag}\left(\begin{bmatrix} R_1 & Q_1 & \cdots & R_1 & P_{1f} \end{bmatrix}\right)$$

and can express player one's optimal control problem as

$$\min_{\mathbf{z}} (1/2)(\mathbf{z}' H \mathbf{z} + x_1(0)' Q_1 x_1(0))$$
$$\text{s.t. } D\mathbf{z} = d$$

in which

$$D = -\begin{bmatrix} \overline{B}_{11} & -I & & & & & \\ & A_1 & \overline{B}_{11} & -I & & & \\ & & & \ddots & & & \\ & & & & A_1 & \overline{B}_{11} & -I \end{bmatrix}$$

$$d = \begin{bmatrix} A_1 x_1(0) + \overline{B}_{12} u_2(0) \\ \overline{B}_{12} u_2(1) \\ \vdots \\ \overline{B}_{12} u_2(N-1) \end{bmatrix}$$

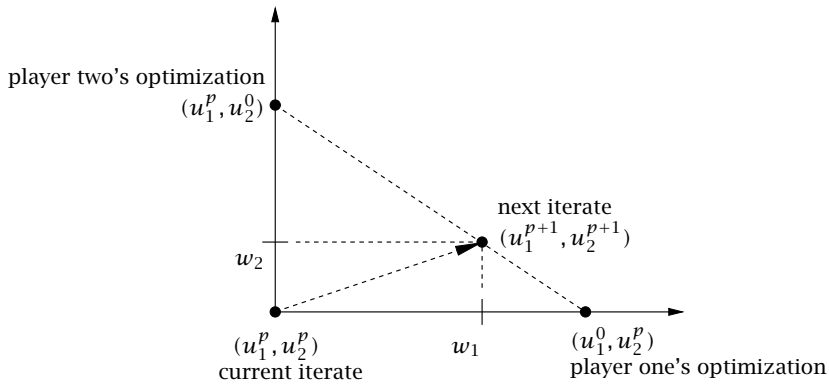


**Figure 6.1:** Convex step from $(u_1^p, u_2^p)$ to $(u_1^{p+1}, u_2^{p+1})$; the parameters $w_1$, $w_2$ with $w_1 + w_2 = 1$ determine location of next iterate on line joining the two players' optimizations: $(u_1^0, u_2^p)$ and $(u_1^p, u_2^0)$.

We then apply (1.58) to obtain

$$\begin{bmatrix} H & -D' \\ -D & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} 0 \\ -\widetilde{A}_1 \end{bmatrix} x_1(0) + \begin{bmatrix} 0 \\ -\widetilde{B}_{12} \end{bmatrix} \mathbf{u}_2 \tag{6.12}$$

in which we have defined

$$\boldsymbol{\lambda} = \begin{bmatrix} \lambda(1) \\ \lambda(2) \\ \vdots \\ \lambda(N) \end{bmatrix} \qquad \widetilde{A}_1 = \begin{bmatrix} A_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \widetilde{B}_{12} = \begin{bmatrix} \overline{B}_{12} & & & \\ & \overline{B}_{12} & & \\ & & \ddots & \\ & & & \overline{B}_{12} \end{bmatrix}$$

Solving this equation and picking out the rows of $\mathbf{z}$ corresponding to the elements of $\mathbf{u}_1$ gives

$$\mathbf{u}_1^0 = K_1 x_1(0) + L_1 \mathbf{u}_2$$

and we see player one's optimal decision depends linearly on his initial state, but also on *player two's decision.* This is the key difference between decentralized control and noncooperative control. In noncooperative control, player two's decisions are communicated to player one and player one accounts for them in optimizing the local objective.

**Convex step.** Let $p \in \mathbb{I}_{\geq 0}$ denote the integer-valued iteration in the optimization problem. Looking ahead to the $M$-player game, we do not take the full step, but a convex combination of the the current optimal solution, $\mathbf{u}_1^0$, and the current iterate, $\mathbf{u}_1^p$

$$\mathbf{u}_1^{p+1} = w_1\mathbf{u}_1^0 + (1-w_1)\mathbf{u}_1^p \qquad 0 < w_1 < 1$$

This iteration is displayed in Figure 6.1. Notice we have chosen a distributed optimization of the Gauss-Jacobi type (see Bertsekas and Tsitsiklis, 1997, pp.219–223).

We place restrictions on the systems under consideration before analyzing stability of the controller.

**Assumption 6.7** (Unconstrained two-player game)**.**

(a) All subsystems, $A_{ij}, i = 1,2, j = 1,2$, are stable.

(b) The controller penalties $Q_1, Q_2, R_1, R_2$ are positive definite.

The assumption of stable models is purely for convenience of exposition. We treat unstable, stabilizable systems in Section 6.3.

**Convergence of the players' iteration.** To understand the convergence of the two players' iterations, we express both players' moves as follows

$$\begin{aligned}\mathbf{u}_1^{p+1} &= w_1\mathbf{u}_1^0 + (1-w_1)\mathbf{u}_1^p \\ \mathbf{u}_2^{p+1} &= w_2\mathbf{u}_2^0 + (1-w_2)\mathbf{u}_2^p \\ 1 &= w_1 + w_2 \qquad 0 < w_1, w_2 < 1\end{aligned}$$

or

$$\begin{bmatrix}\mathbf{u}_1\\ \mathbf{u}_2\end{bmatrix}^{p+1} = \begin{bmatrix}w_1 I & 0\\ 0 & w_2 I\end{bmatrix}\begin{bmatrix}\mathbf{u}_1^0\\ \mathbf{u}_2^0\end{bmatrix} + \begin{bmatrix}(1-w_1)I & 0\\ 0 & (1-w_2)I\end{bmatrix}\begin{bmatrix}\mathbf{u}_1\\ \mathbf{u}_2\end{bmatrix}^p$$

The optimal control for each player is

$$\begin{bmatrix}\mathbf{u}_1^0\\ \mathbf{u}_2^0\end{bmatrix} = \begin{bmatrix}K_1 & 0\\ 0 & K_2\end{bmatrix}\begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix} + \begin{bmatrix}0 & L_1\\ L_2 & 0\end{bmatrix}\begin{bmatrix}\mathbf{u}_1\\ \mathbf{u}_2\end{bmatrix}^p$$

Substituting the optimal control into the iteration gives

$$\begin{bmatrix}\mathbf{u}_1\\ \mathbf{u}_2\end{bmatrix}^{p+1} = \underbrace{\begin{bmatrix}w_1K_1 & 0\\ 0 & w_2K_2\end{bmatrix}}_{\overline{K}}\begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix} + \underbrace{\begin{bmatrix}(1-w_1)I & w_1L_1\\ w_2L_2 & (1-w_2)I\end{bmatrix}}_{L}\begin{bmatrix}\mathbf{u}_1\\ \mathbf{u}_2\end{bmatrix}^p$$

Finally writing this equation in the plantwide notation, we express the iteration as

$$\mathbf{u}^{p+1} = \overline{K}x(0) + L\mathbf{u}^{p}$$

The convergence of the two players' control iteration is governed by the eigenvalues of $L$. If $L$ is stable, the control sequence converges to

$$\mathbf{u}^{\infty} = (I - L)^{-1}\overline{K}x(0) \qquad |\lambda| < 1 \text{ for } \lambda \in \text{eig}(L)$$

in which

$$(I - L)^{-1}\overline{K} = \begin{bmatrix} w_1 I & -w_1 L_1 \\ -w_2 L_2 & w_2 I \end{bmatrix}^{-1} \begin{bmatrix} w_1 K_1 & 0 \\ 0 & w_2 K_2 \end{bmatrix}$$

$$(I - L)^{-1}\overline{K} = \begin{bmatrix} I & -L_1 \\ -L_2 & I \end{bmatrix}^{-1} \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}$$

Note that the weights $w_1$, $w_2$ do not appear in the converged input sequence. The $\mathbf{u}_1^{\infty}$, $\mathbf{u}_2^{\infty}$ pair have the equilibrium property that neither player can improve his position given the other player's current decision. This point is called a Nash equilibrium (Başar and Olsder, 1999, p. 4). Notice that the distributed MPC game does not have a Nash equilibrium if the eigenvalues of $L$ are on or outside the unit circle. If the controllers have sufficient time during the control system's sample time to iterate to convergence, then the effect of the initial control sequence is removed by using the converged control sequence. If the iteration has to be stopped before convergence, the solution is

$$\mathbf{u}^{p+1} = L^{p}\mathbf{u}^{[0]} + \sum_{j=0}^{p-1} L^{j}\overline{K}x(0) \qquad 0 \le p$$

in which $\mathbf{u}^{[0]}$ is the $p = 0$ (initial) input sequence. We use the brackets with $p = 0$ to distinguish this initial input sequence from an optimal input sequence.

**Stability of the closed-loop system.** We assume the Nash equilibrium is stable and there is sufficient computation time to iterate to convergence.

We require a matrix of zeros and ones to select the first move from the input sequence for injection into the plant. For the first player, the

required matrix is

$$u_1(0) = E_1 \mathbf{u}_1$$
$$E_1 = \begin{bmatrix} I_{m_1} & 0_{m_1} & \dots & 0_{m_1} \end{bmatrix} \qquad m_1 \times m_1 N \text{ matrix}$$

The closed-loop system is then

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^+ = \underbrace{\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} \overline{B}_{11} & \overline{B}_{12} \\ \overline{B}_{21} & \overline{B}_{22} \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} \begin{bmatrix} I & -L_1 \\ -L_2 & I \end{bmatrix}^{-1} \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}}_{K} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Using the plantwide notation for this equation and defining the feedback gain $K$ gives

$$x^+ = (A + BK)x$$

The stability of the closed loop with converged, noncooperative control is therefore determined by the eigenvalues of $(A + BK)$.

We next present three simple examples to show that (i) the Nash equilibrium may not be stable ($L$ is unstable), (ii) the Nash equilibrium may be stable but the closed loop is unstable ($L$ is stable, $A + BK$ is unstable), and (iii) the Nash equilibrium may be stable and the closed loop is stable ($L$ is stable, $A + BK$ is stable). Which situation arises depends in a nonobvious way on all of the problem data: $A_1$, $A_2$, $\overline{B}_{11}$, $\overline{B}_{12}$, $\overline{B}_{21}$, $\overline{B}_{22}$, $Q_1$, $Q_2$, $P_{1f}$, $P_{2f}$, $R_1$, $R_2$, $w_1$, $w_2$, $N$. One has to examine the eigenvalues of $L$ and $A + BK$ for each application of interest to know how the noncooperative distributed MPC is going to perform. Even for a fixed dynamic model, when changing tuning parameters such as $Q$, $P_f$, $R$, $w$, one has to examine eigenvalues of $L$ and $A + BK$ to know the effect on the closed-loop system. This is the main drawback of the noncooperative game. In many control system design methods, such as all forms of MPC presented in Chapter 2, closed-loop properties such as exponential stability are guaranteed for the *nominal* system for all choices of performance tuning parameters. Noncooperative distributed MPC does not have this feature and a stability analysis is required. We show in the next section that cooperative MPC does not suffer from this drawback, at the cost of slightly more information exchange.

**Example 6.8: Nash equilibrium is unstable**

Consider the following transfer function matrix for a simple two-input two-output system

$$\begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \begin{bmatrix} u_1(s) \\ u_2(s) \end{bmatrix}$$

in which

$$G(s) = \begin{bmatrix} \dfrac{1}{s^2 + 2(0.2)s + 1} & \dfrac{0.5}{0.225s + 1} \\ \dfrac{-0.5}{(0.5s+1)(0.25s+1)} & \dfrac{1.5}{0.75s^2 + 2(0.8)(0.75)s + 1} \end{bmatrix}$$

Obtain discrete time models $(A_{ij}, B_{ij}, C_{ij})$ for each of the four transfer functions $G_{ij}(s)$ using a sample time of $T = 0.2$ and zero-order holds on the inputs. Set the control cost function parameters to be

$$\overline{Q}_1 = \overline{Q}_2 = 1 \qquad \overline{P}_{1f} = \overline{P}_{2f} = 0 \qquad R_1 = R_2 = 0.01$$
$$N = 30 \qquad w_1 = w_2 = 0.5$$

Compute the eigenvalues of the $L$ matrix for this system using noncooperative MPC. Show the Nash equilibrium is unstable and the closed-loop system is therefore unstable. Discuss why this system is problematic for noncooperative control.

**Solution**

For this problem $L$ is a $60 \times 60$ matrix ($N(m_1 + m_2)$). The magnitudes of the largest eigenvalues are

$$|\operatorname{eig}(L)| = \begin{bmatrix} 1.11 & 1.11 & 1.03 & 1.03 & 0.914 & 0.914 & \cdots \end{bmatrix}$$

The noncooperative iteration does not converge. The steady-state gains for this system are

$$G(0) = \begin{bmatrix} 1 & 0.5 \\ -0.5 & 1.5 \end{bmatrix}$$

and we see that the diagonal elements are reasonably large compared to the nondiagonal elements. So the *steady-state* coupling between the two systems is relatively weak. The dynamic coupling is unfavorable, however. The response of $y_1$ to $u_2$ is more than four times faster than the response of $y_1$ to $u_1$. The faster input is the disturbance and the slower input is used for control. Likewise the response of $y_2$ to $u_1$ is

three times faster than the response of $y_2$ to $u_2$. Also in the second loop, the faster input is the disturbance and the slower input is used for control. These pairings are unfavorable dynamically, and that fact is revealed in the instability of $L$ and lack of a Nash equilibrium for the noncooperative dynamic regulation problem. □

**Example 6.9: Nash equilibrium is stable but closed loop is unstable**

Switch the outputs for the previous example and compute the eigenvalues of $L$ and $(A+BK)$ for the noncooperative distributed MPC regulator for the system

$$G(s) = \begin{bmatrix} \dfrac{-0.5}{(0.5s+1)(0.25s+1)} & \dfrac{1.5}{0.75s^2+2(0.8)(0.75)s+1} \\ \dfrac{1}{s^2+2(0.2)s+1} & \dfrac{0.5}{0.225s+1} \end{bmatrix}$$

Show in this case that the Nash equilibrium is stable, but the noncooperative regulator destabilizes the system. Discuss why this system is problematic for noncooperative control.

**Solution**

For this case the largest magnitude eigenvalues of $L$ are

$$|\operatorname{eig}(L)| = \begin{bmatrix} 0.63 & 0.63 & 0.62 & 0.62 & 0.59 & 0.59 & \cdots \end{bmatrix}$$

and we see the Nash equilibrium for the noncooperative game is stable. So we have removed the first source of closed-loop instability by switching the input-output pairings of the two subsystems. There are seven states in the complete system model, and the magnitudes of the eigenvalues of the closed-loop regulator $(A+BK)$ are

$$|\operatorname{eig}(A+BK)| = \begin{bmatrix} 1.03 & 1.03 & 0.37 & 0.37 & 0.77 & 0.77 & 0.04 \end{bmatrix}$$

which also gives an unstable closed-loop system. We see the distributed noncooperative regulator has destabilized a stable open-loop system. The problem with this pairing is the steady-state gains are now

$$G(0) = \begin{bmatrix} -0.5 & 1.5 \\ 1 & 0.5 \end{bmatrix}$$

If one computes any steady-state interaction measure, such as the relative gain array (RGA), we see the new pairings are poor from a steady-state interaction perspective

$$\text{RGA} = \begin{bmatrix} 0.14 & 0.86 \\ 0.86 & 0.14 \end{bmatrix}$$

Neither pairing of the inputs and outputs is closed-loop stable with noncooperative distributed MPC.

Decentralized control with this pairing is discussed in Exercise 6.10. □

**Example 6.10: Nash equilibrium is stable and the closed loop is stable**

Next consider the system

$$G(s) = \begin{bmatrix} \dfrac{1}{s^2 + 2(0.2)s + 1} & \dfrac{0.5}{0.9s + 1} \\ \dfrac{-0.5}{(2s+1)(s+1)} & \dfrac{1.5}{0.75s^2 + 2(0.8)(0.75)s + 1} \end{bmatrix}$$

Compute the eigenvalues of $L$ and $A + BK$ for this system. What do you conclude about noncooperative distributed MPC for this system?

**Solution**

This system is not difficult to handle with distributed control. The gains are the same as in the original pairing in Example 6.8, and the steady-state coupling between the two subsystems is reasonably weak. Unlike Example 6.8, however, the responses of $y_1$ to $u_2$ and $y_2$ to $u_1$ have been slowed so they are not faster than the responses of $y_1$ to $u_1$ and $y_2$ to $u_2$, respectively. Computing the eigenvalues of $L$ and $A + BK$ for noncooperative control gives

$$|\operatorname{eig}(L)| = \begin{bmatrix} 0.61 & 0.61 & 0.59 & 0.59 & 0.56 & 0.56 & 0.53 & 0.53 \cdots \end{bmatrix}$$

$$|\operatorname{eig}(A + BK)| = \begin{bmatrix} 0.88 & 0.88 & 0.74 & 0.67 & 0.67 & 0.53 & 0.53 \end{bmatrix}$$

The Nash equilibrium is stable since $L$ is stable, and the closed loop is stable since both $L$ and $A + BK$ are stable. □

These examples reveal the simple fact that communicating the actions of the other controllers does not guarantee acceptable closed-loop behavior. If the coupling of the subsystems is weak enough, both dynamically and in steady state, then the closed loop is stable. In this

sense, noncooperative MPC has few advantages over completely decentralized control, which has this same basic property.

We next show how to obtain much better closed-loop properties while maintaining the small size of the distributed control problems.

### 6.2.4 Cooperative Game

In the cooperative game, the two players share a common objective, which can be considered to be the overall plant objective

$$V(x_1(0), x_2(0), \mathbf{u}_1, \mathbf{u}_2) = \rho_1 V_1(x_1(0), \mathbf{u}_1, \mathbf{u}_2) + \rho_2 V_2(x_2(0), \mathbf{u}_2, \mathbf{u}_1)$$

in which the parameters $\rho_1$, $\rho_2$ are used to specify the relative weights of the two subsystems in the overall plant objective. In the cooperative problem, each player keeps track of *how his input affects the other player's output* as well as his own output. We can implement this cooperative game in several ways. The implementation leading to the simplest notation is to combine $x_1$ and $x_2$ into a single model

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^+ = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \overline{B}_{11} \\ \overline{B}_{21} \end{bmatrix} u_1 + \begin{bmatrix} \overline{B}_{12} \\ \overline{B}_{22} \end{bmatrix} u_2$$

and then express player one's stage cost as

$$\ell_1(x_1, x_2, u_1) = \frac{1}{2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' \begin{bmatrix} \rho_1 Q_1 & 0 \\ 0 & \rho_2 Q_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \frac{1}{2} u_1' (\rho_1 R_1) u_1 + \text{const.}$$

$$V_{1f}(x_1, x_2) = \frac{1}{2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' \begin{bmatrix} \rho_1 P_{1f} & 0 \\ 0 & \rho_2 P_{2f} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Notice that $u_2$ does not appear because the contribution of $u_2$ to the stage cost cannot be affected by player one, and can therefore be neglected. The cost function is then expressed as

$$V(x_1(0), x_2(0), \mathbf{u}_1, \mathbf{u}_2) = \sum_{k=0}^{N-1} \ell_1(x_1(k), x_2(k), u_1(k)) + V_{1f}(x_1(N), x_2(N))$$

Player one's optimal control problem is

$$\min_{\mathbf{u}_1} V(x_1(0), x_2(0), \mathbf{u}_1, \mathbf{u}_2)$$

$$\text{s.t. } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^+ = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \overline{B}_{11} \\ \overline{B}_{21} \end{bmatrix} u_1 + \begin{bmatrix} \overline{B}_{12} \\ \overline{B}_{22} \end{bmatrix} u_2$$

Note that this form is identical to the noncooperative form presented previously if we redefine the terms (noncooperative $\longrightarrow$ cooperative)

$$x_1 \to \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad A_1 \to \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \quad \overline{B}_{11} \to \begin{bmatrix} \overline{B}_{11} \\ \overline{B}_{21} \end{bmatrix} \quad \overline{B}_{12} \to \begin{bmatrix} \overline{B}_{12} \\ \overline{B}_{22} \end{bmatrix}$$

$$Q_1 \to \begin{bmatrix} \rho_1 Q_1 & 0 \\ 0 & \rho_2 Q_2 \end{bmatrix} \qquad R_1 \to \rho_1 R_1 \quad P_{1f} \to \begin{bmatrix} \rho_1 P_{1f} & 0 \\ 0 & \rho_2 P_{2f} \end{bmatrix}$$

Any computational program written to solve either the cooperative or noncooperative optimal control problem can be used to solve the other.

**Eliminating states $\mathbf{x}_2$.** An alternative implementation is to remove states $x_2(k), k \geq 1$ from player one's optimal control problem by substituting the dynamic model of system two. This implementation reduces the size of the dynamic model because only states $x_1$ are retained. This reduction in model size may be important in applications with many players. The removal of states $x_2(k), k \geq 1$ also introduces linear terms into player one's objective function. We start by using the dynamic model for $x_2$ to obtain

$$\begin{bmatrix} x_2(1) \\ x_2(2) \\ \vdots \\ x_2(N) \end{bmatrix} = \begin{bmatrix} A_2 \\ A_2^2 \\ \vdots \\ A_2^N \end{bmatrix} x_2(0) + \begin{bmatrix} \overline{B}_{21} & & & \\ A_2\overline{B}_{21} & \overline{B}_{21} & & \\ \vdots & \vdots & \ddots & \\ A_2^{N-1}\overline{B}_{21} & A_2^{N-2}\overline{B}_{21} & \dots & \overline{B}_{21} \end{bmatrix} \begin{bmatrix} u_1(0) \\ u_1(1) \\ \vdots \\ u_1(N-1) \end{bmatrix} +$$

$$\begin{bmatrix} \overline{B}_{22} & & & \\ A_2\overline{B}_{22} & \overline{B}_{22} & & \\ \vdots & \vdots & \ddots & \\ A_2^{N-1}\overline{B}_{22} & A_2^{N-2}\overline{B}_{22} & \dots & \overline{B}_{22} \end{bmatrix} \begin{bmatrix} u_2(0) \\ u_2(1) \\ \vdots \\ u_2(N-1) \end{bmatrix}$$

Using more compact notation, we have

$$\mathbf{x}_2 = \mathcal{A}_2 x_2(0) + \mathcal{B}_{21}\mathbf{u}_1 + \mathcal{B}_{22}\mathbf{u}_2$$

We can use this relation to replace the cost contribution of $\mathbf{x}_2$ with linear and quadratic terms in $\mathbf{u}_1$ as follows

$$\sum_{k=0}^{N-1} x_2(k)'Q_2x_2(k) + x_2(N)'P_{2f}x_2(N) =$$

$$\mathbf{u}_1' \left[\mathcal{B}_{21}'\mathcal{Q}_2\mathcal{B}_{21}\right]\mathbf{u}_1 + 2\left[x_2(0)'\mathcal{A}_2' + \mathbf{u}_2'\mathcal{B}_{22}'\right]\mathcal{Q}_2\mathcal{B}_{21}\,\mathbf{u}_1 + \text{constant}$$

in which

$$\mathcal{Q}_2 = \text{diag}\left(\begin{bmatrix} Q_2 & Q_2 & \dots & P_{2f} \end{bmatrix}\right) \qquad Nn_2 \times Nn_2 \text{ matrix}$$

and the constant term contains products of $x_2(0)$ and $\mathbf{u}_2$, which are constant with respect to player one's decision variables and can therefore be neglected.

Next we insert the new terms created by eliminating $\mathbf{x}_2$ into the cost function. Assembling the cost function gives

$$\min_{\mathbf{z}} (1/2)\mathbf{z}'\widetilde{H}\mathbf{z} + h'\mathbf{z}$$
$$\text{s.t. } D\mathbf{z} = d$$

and (1.58) again gives the necessary and sufficient conditions for the optimal solution

$$\begin{bmatrix} \widetilde{H} & -D' \\ -D & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} 0 \\ -\widetilde{A}_1 \end{bmatrix} x_1(0) + \begin{bmatrix} -\widetilde{A}_2 \\ 0 \end{bmatrix} x_2(0) + \begin{bmatrix} -\widetilde{B}_{22} \\ -\widetilde{B}_{12} \end{bmatrix} \mathbf{u}_2 \quad (6.13)$$

in which

$$\widetilde{H} = H + E'\mathcal{B}_{21}'\mathcal{Q}_2\mathcal{B}_{21}E \qquad \widetilde{B}_{22} = E'\mathcal{B}_{21}'\mathcal{Q}_2\mathcal{B}_{22} \qquad \widetilde{A}_2 = E'\mathcal{B}_{21}'\mathcal{Q}_2\mathcal{A}_2$$
$$E = I_N \otimes \begin{bmatrix} I_{m_1} & 0_{m_1,n_1} \end{bmatrix}$$

See also Exercise 6.13 for details on constructing the padding matrix $E$. Comparing the cooperative and noncooperative dynamic games, (6.13) and (6.12), we see the cooperative game has made three changes: (i) the quadratic penalty $H$ has been modified, (ii) the effect of $x_2(0)$ has been included with the term $\widetilde{A}_2$, and (iii) the influence of $\mathbf{u}_2$ has been modified with the term $\widetilde{B}_{22}$. Notice that the size of the vector $\mathbf{z}$ has not changed, and we have accomplished the goal of keeping player one's dynamic model in the cooperative game the same size as his dynamic model in the noncooperative game.

Regardless of the implementation choice, the cooperative optimal control problem is no more complex than the noncooperative game considered previously. The extra information required by player one in the cooperative game is $x_2(0)$. Player one requires $\mathbf{u}_2$ in both the cooperative and noncooperative games. Only in decentralized control does player one not require player two's input sequence $\mathbf{u}_2$. The other extra required information, $A_2, B_{21}, Q_2, R_2, P_{2f}$, are fixed parameters and making their values available to player one is a minor communication overhead.

Proceeding as before, we solve this equation for $\mathbf{z}^0$ and pick out the rows corresponding to the elements of $\mathbf{u}_1^0$ giving

$$\mathbf{u}_1^0(x(0), \mathbf{u}_2) = \begin{bmatrix} K_{11} & K_{12} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + L_1\mathbf{u}_2$$

Combining the optimal control laws for each player gives

$$\begin{bmatrix}\mathbf{u}_1^0\\ \mathbf{u}_2^0\end{bmatrix} = \begin{bmatrix}K_{11} & K_{12}\\ K_{21} & K_{22}\end{bmatrix}\begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix} + \begin{bmatrix}0 & L_1\\ L_2 & 0\end{bmatrix}\begin{bmatrix}\mathbf{u}_1\\ \mathbf{u}_2\end{bmatrix}^p$$

in which the gain matrix multiplying the state is a full matrix for the cooperative game. Substituting the optimal control into the iteration gives

$$\begin{bmatrix}\mathbf{u}_1\\ \mathbf{u}_2\end{bmatrix}^{p+1} = \underbrace{\begin{bmatrix}w_1K_{11} & w_1K_{12}\\ w_2K_{21} & w_2K_{22}\end{bmatrix}}_{\overline{K}}\begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix} + \underbrace{\begin{bmatrix}(1-w_1)I & w_1L_1\\ w_2L_2 & (1-w_2)I\end{bmatrix}}_{L}\begin{bmatrix}\mathbf{u}_1\\ \mathbf{u}_2\end{bmatrix}^p$$

Finally writing this equation in the plantwide notation, we express the iteration as

$$\mathbf{u}^{p+1} = \overline{K}x(0) + L\mathbf{u}^p$$

**Exponential stability of the closed-loop system.** In the case of cooperative control, we consider the closed-loop system with a finite number of iterations, $p$. With finite iterations, distributed MPC becomes a form of *suboptimal* MPC as discussed in Sections 6.1.2 and 2.8. To analyze the behavior of the cooperative controller with a finite number of iterations, we require the cost decrease achieved by a single iteration, which we derive next. First we write the complete system evolution as in (6.10)

$$x^+ = Ax + Bu$$

in which $A$ and $B$ are defined in (6.11). We can then use (6.3) to express the overall cost function

$$V(x(0),\mathbf{u}) = (1/2)x'(0)(Q + \mathcal{A}'\mathcal{Q}\mathcal{A})x(0) + \mathbf{u}'(\mathcal{B}'\mathcal{Q}\mathcal{A})x(0) + (1/2)\mathbf{u}'H_{\mathbf{u}}\mathbf{u}$$

in which $\mathcal{A}$ and $\mathcal{B}$ are given in (6.1), the cost penalties $\mathcal{Q}$ and $\mathcal{R}$ are given in (6.2) and (6.9), and

$$H_{\mathbf{u}} = \mathcal{B}'\mathcal{Q}\mathcal{B} + \mathcal{R}$$

The overall cost is a positive definite quadratic function in $\mathbf{u}$ because $R_1$ and $R_2$ are positive definite, and therefore so are $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}$.

The iteration in the two players' moves satisfies

$$\begin{aligned}(\mathbf{u}_1^{p+1},\mathbf{u}_2^{p+1}) &= \left((w_1\mathbf{u}_1^0 + (1-w_1)\mathbf{u}_1^p),(w_2\mathbf{u}_2^0 + (1-w_2)\mathbf{u}_2^p)\right)\\ &= (w_1\mathbf{u}_1^0,(1-w_2)\mathbf{u}_2^p) + ((1-w_1)\mathbf{u}_1^p, w_2\mathbf{u}_2^0)\\ (\mathbf{u}_1^{p+1},\mathbf{u}_2^{p+1}) &= w_1(\mathbf{u}_1^0,\mathbf{u}_2^p) + w_2(\mathbf{u}_1^p,\mathbf{u}_2^0) \qquad (6.14)\end{aligned}$$

Exercise 6.18 analyzes the cost decrease for a convex step with a positive definite quadratic function and shows

$$V(x(0),\mathbf{u}_1^{p+1},\mathbf{u}_2^{p+1}) = V(x(0),\mathbf{u}_1^{p},\mathbf{u}_2^{p}) - \frac{1}{2}\left[\mathbf{u}^p - \mathbf{u}^0(x(0))\right]' P \left[\mathbf{u}^p - \mathbf{u}^0(x(0))\right] \quad (6.15)$$

in which $P > 0$ is given by

$$P = H_{\mathbf{u}} D^{-1} \widetilde{H} D^{-1} H_{\mathbf{u}} \qquad \widetilde{H} = D - N$$

$$D = \begin{bmatrix} w_1^{-1} H_{\mathbf{u},11} & 0 \\ 0 & w_2^{-1} H_{\mathbf{u},22} \end{bmatrix} \qquad N = \begin{bmatrix} -w_1^{-1} w_2 H_{\mathbf{u},11} & H_{\mathbf{u},12} \\ H_{\mathbf{u},21} & -w_1 w_2^{-1} H_{\mathbf{u},22} \end{bmatrix}$$

and $H_{\mathbf{u}}$ is partitioned for the two players' input sequences. Notice that the cost decrease achieved in a single iteration is quadratic in the distance from the optimum. An important conclusion is that *each iteration in the cooperative game reduces the systemwide cost.* This cost reduction is the key property that gives cooperative MPC its excellent convergence properties, as we show next.

The two players' warm starts at the next sample are given by

$$\widetilde{\mathbf{u}}_1^+ = \{u_1(1), u_1(2), \ldots, u_1(N-1), 0\}$$
$$\widetilde{\mathbf{u}}_2^+ = \{u_2(1), u_2(2), \ldots, u_2(N-1), 0\}$$

We define the following linear time-invariant functions $g_1^p$ and $g_2^p$ as the outcome of applying the control iteration procedure $p$ times

$$\mathbf{u}_1^p = g_1^p(x_1, x_2, \mathbf{u}_1, \mathbf{u}_2)$$
$$\mathbf{u}_2^p = g_2^p(x_1, x_2, \mathbf{u}_1, \mathbf{u}_2)$$

in which $p \geq 0$ is an integer, $x_1$ and $x_2$ are the states, and $\mathbf{u}_1, \mathbf{u}_2$ are the input sequences from the previous sample, used to generate the warm start for the iteration. Here we consider $p$ to be constant with time, but Exercise 6.20 considers the case in which the controller iterations may vary with sample time. The system evolution is then given by

$$x_1^+ = A_1 x_1 + \overline{B}_{11} u_1 + \overline{B}_{12} u_2 \qquad x_2^+ = A_2 x_2 + \overline{B}_{21} u_1 + \overline{B}_{22} u_2$$
$$\mathbf{u}_1^+ = g_1^p(x_1, x_2, \mathbf{u}_1, \mathbf{u}_2) \qquad \mathbf{u}_2^+ = g_2^p(x_1, x_2, \mathbf{u}_1, \mathbf{u}_2) \quad (6.16)$$

By the construction of the warm start, $\widetilde{\mathbf{u}}_1^+, \widetilde{\mathbf{u}}_2^+$, we have

$$V(x_1^+, x_2^+, \widetilde{\mathbf{u}}_1^+, \widetilde{\mathbf{u}}_2^+) = V(x_1, x_2, \mathbf{u}_1, \mathbf{u}_2) - \rho_1 \ell_1(x_1, u_1) - \rho_2 \ell_2(x_2, u_2)$$
$$+ (1/2)\rho_1 x_1(N)' \left[A_1' P_{1f} A_1 - P_{1f} + Q_1\right] x_1(N)$$
$$+ (1/2)\rho_2 x_2(N)' \left[A_2' P_{2f} A_2 - P_{2f} + Q_2\right] x_2(N)$$

From our choice of terminal penalty satisfying (6.8), the last two terms are zero giving

$$V(x_1^+, x_2^+, \tilde{\mathbf{u}}_1^+, \tilde{\mathbf{u}}_2^+) = V(x_1, x_2, \mathbf{u}_1, \mathbf{u}_2) \\ - \rho_1 \ell_1(x_1, u_1) - \rho_2 \ell_2(x_2, u_2) \quad (6.17)$$

**No optimization, $p = 0$.** If we do no further optimization, then we have $\mathbf{u}_1^+ = \tilde{\mathbf{u}}_1^+$, $\mathbf{u}_2^+ = \tilde{\mathbf{u}}_2^+$, and the equality

$$V(x_1^+, x_2^+, \mathbf{u}_1^+, \mathbf{u}_2^+) = V(x_1, x_2, \mathbf{u}_1, \mathbf{u}_2) - \rho_1 \ell_1(x_1, u_1) - \rho_2 \ell_2(x_2, u_2)$$

The input sequences add a zero at each sample until $\mathbf{u}_1 = \mathbf{u}_2 = 0$ at time $k = N$. The system decays exponentially under zero control and the closed loop is exponentially stable.

**Further optimization, $p \geq 1$.** We next consider the case in which optimization is performed. Equation 6.15 then gives

$$V(x_1^+, x_2^+, \mathbf{u}_1^+, \mathbf{u}_2^+) \leq V(x_1^+, x_2^+, \tilde{\mathbf{u}}_1^+, \tilde{\mathbf{u}}_2^+) - \\ \left[\tilde{\mathbf{u}}^+ - \mathbf{u}^0(x^+)\right]' P \left[\tilde{\mathbf{u}}^+ - \mathbf{u}^0(x^+)\right] \qquad p \geq 1$$

with equality holding for $p = 1$. Using this result in (6.17) gives

$$V(x_1^+, x_2^+, \mathbf{u}_1^+, \mathbf{u}_2^+) \leq V(x_1, x_2, \mathbf{u}_1, \mathbf{u}_2) - \rho_1 \ell_1(x_1, u_1) - \rho_2 \ell_2(x_2, u_2) \\ - \left[\tilde{\mathbf{u}}^+ - \mathbf{u}^0(x^+)\right]' P \left[\tilde{\mathbf{u}}^+ - \mathbf{u}^0(x^+)\right]$$

Since $V$ is bounded below by zero and $\ell_1$ and $\ell_2$ are positive functions, we conclude the time sequence $V(x_1(k), x_2(k), \mathbf{u}_1(k), \mathbf{u}_2(k))$ converges. and therefore $x_1(k)$, $x_2(k)$, $u_1(k)$, and $u_2(k)$ converge to zero. Moreover, since $P > 0$, the last term implies that $\tilde{\mathbf{u}}^+$ converges to $\mathbf{u}^0(x^+)$, which converges to zero because $x^+$ converges to zero. Therefore, the entire input sequence $\mathbf{u}$ converges to zero. Because the total system evolution is a linear time-invariant system, the convergence is exponential. Even though we are considering here a form of *suboptimal* MPC, we do not require an additional inequality constraint on $\mathbf{u}$ because the problem considered here is *unconstrained* and the iterations satisfy (6.15).

### 6.2.5 Tracking Nonzero Setpoints

For tracking nonzero setpoints, we compute steady-state targets as discussed in Section 1.5. The steady-state input-output model is given by

$$y_s = G u_s \qquad G = C(I - A)^{-1} B$$

in which $G$ is the steady-state gain of the system. The two subsystems are denoted

$$\begin{bmatrix} y_{1s} \\ y_{2s} \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} u_{1s} \\ u_{2s} \end{bmatrix}$$

For simplicity, we assume that the targets are chosen to be the measurements ($H = I$). Further, we assume that both local systems are square, and that the local targets can be reached exactly with the local inputs. This assumption means that $G_{11}$ and $G_{22}$ are square matrices of full rank. We remove all of these assumptions when we treat the constrained two-player game in the next section. If there is model error, integrating disturbance models are required as discussed in Chapter 1. We discuss these later.

The target problem also can be solved with any of the four approaches discussed so far. We consider each.

**Centralized case.** The centralized problem gives in one shot both inputs required to meet both output setpoints

$$\begin{aligned} u_s &= G^{-1} y_{\mathrm{sp}} \\ y_s &= y_{\mathrm{sp}} \end{aligned}$$

**Decentralized case.** The decentralized problem considers only the diagonal terms and computes the following steady inputs

$$u_s = \begin{bmatrix} G_{11}^{-1} & \\ & G_{22}^{-1} \end{bmatrix} y_{\mathrm{sp}}$$

Notice these inputs produce offset in both output setpoints

$$y_s = \begin{bmatrix} I & G_{12}G_{22}^{-1} \\ G_{21}G_{11}^{-1} & I \end{bmatrix} y_{\mathrm{sp}}$$

**Noncooperative case.** In the noncooperative game, each player attempts to remove offset in only its outputs. Player one solves the following problem

$$\begin{aligned} \min_{u_1} (y_1 - y_{1sp})' \overline{Q}_1 (y_1 - y_{1sp}) \\ \text{s.t. } y_1 = G_{11}u_1 + G_{12}u_2 \end{aligned}$$

Because the target can be reached exactly, the optimal solution is to find $u_1$ such that $y_1 = y_{1sp}$, which gives

$$u_{1s}^0 = G_{11}^{-1} \left( y_{1sp} - G_{12} u_2^p \right)$$

Player two solves the analogous problem. If we iterate on the two players' solutions, we obtain

$$\begin{bmatrix} u_{1s} \\ u_{2s} \end{bmatrix}^{p+1} = \underbrace{\begin{bmatrix} w_1 G_{11}^{-1} & \\ & w_2 G_{22}^{-1} \end{bmatrix}}_{\overline{K}_s} \begin{bmatrix} y_{1sp} \\ y_{2sp} \end{bmatrix} + \underbrace{\begin{bmatrix} w_2 I & -w_1 G_{11}^{-1} G_{12} \\ -w_2 G_{22}^{-1} G_{21} & w_1 I \end{bmatrix}}_{L_s} \begin{bmatrix} u_{1s} \\ u_{2s} \end{bmatrix}^{p}$$

This iteration can be summarized by

$$u_s^{p+1} = \overline{K}_s y_{\text{sp}} + L_s u_s^p$$

If $L_s$ is stable, this iteration converges to

$$u_s^{\infty} = (I - L_s)^{-1} \overline{K}_s y_{\text{sp}}$$
$$u_s^{\infty} = G^{-1} y_{\text{sp}}$$

and we have no offset. We already have seen that we cannot expect the dynamic noncooperative iteration to converge. The next several examples explore the issue of whether we can expect at least the steady-state iteration to be stable.

**Cooperative case.** In the cooperative case, both players work on minimizing the offset in both outputs. Player one solves

$$\min_{u_1} (1/2) \begin{bmatrix} y_1 - y_{1sp} \\ y_2 - y_{2sp} \end{bmatrix}' \begin{bmatrix} \rho_1 \overline{Q}_1 & \\ & \rho_2 \overline{Q}_2 \end{bmatrix} \begin{bmatrix} y_1 - y_{1sp} \\ y_2 - y_{2sp} \end{bmatrix}$$
$$\text{s.t.} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} G_{11} \\ G_{21} \end{bmatrix} u_1 + \begin{bmatrix} G_{12} \\ G_{22} \end{bmatrix} u_2$$

We can write this in the general form

$$\min_{r_s} (1/2) r_s' H r_s + h' r_s$$
$$\text{s.t. } D r_s = d$$

in which

$$r_s = \begin{bmatrix} y_{1s} \\ y_{2s} \\ u_{1s} \end{bmatrix} \qquad H = \begin{bmatrix} \rho_1 \overline{Q}_1 & & \\ & \rho_2 \overline{Q}_2 & \\ & & 0 \end{bmatrix} \qquad h = \begin{bmatrix} -Q y_{\text{sp}} \\ 0 \end{bmatrix}$$
$$D = \begin{bmatrix} I & -G_1 \end{bmatrix} \qquad d = G_2 u_2 \qquad G_1 = \begin{bmatrix} G_{11} \\ G_{12} \end{bmatrix} \qquad G_2 = \begin{bmatrix} G_{12} \\ G_{22} \end{bmatrix}$$

We can then solve the linear algebra problem

$$\begin{bmatrix} H & -D' \\ -D & 0 \end{bmatrix} \begin{bmatrix} r_s \\ \lambda_s \end{bmatrix} = - \begin{bmatrix} h \\ d \end{bmatrix}$$

and identify the linear gains between the optimal $u_{1s}$ and the setpoint $y_{\text{sp}}$ and player two's input $u_{2s}$

$$u_{1s}^0 = K_{1s} y_{\text{sp}} + L_{1s} u_{2s}^p$$

Combining the optimal control laws for each player gives

$$\begin{bmatrix} u_{1s}^0 \\ u_{2s}^0 \end{bmatrix} = \begin{bmatrix} K_{1s} \\ K_{2s} \end{bmatrix} y_{\text{sp}} + \begin{bmatrix} 0 & L_{1s} \\ L_{2s} & 0 \end{bmatrix} \begin{bmatrix} u_{1s} \\ u_{2s} \end{bmatrix}^p$$

Substituting the optimal control into the iteration gives

$$\begin{bmatrix} u_{1s} \\ u_{2s} \end{bmatrix}^{p+1} = \underbrace{\begin{bmatrix} w_1 K_{1s} \\ w_2 K_{2s} \end{bmatrix}}_{\overline{K}_s} y_{\text{sp}} + \underbrace{\begin{bmatrix} (1-w_1)I & w_1 L_{1s} \\ w_2 L_{2s} & (1-w_2)I \end{bmatrix}}_{L_s} \begin{bmatrix} u_{1s} \\ u_{2s} \end{bmatrix}^p$$

Finally writing this equation in the plantwide notation, we express the iteration as

$$u_s^{p+1} = \overline{K}_s y_{\text{sp}} + L_s u_s^p$$

As we did with the cooperative regulation problem, we can analyze the optimization problem to show that this iteration is always stable and converges to the centralized target. Next we explore the use of these approaches in some illustrative examples.

**Example 6.11: Stability and offset in the distributed target calculation**

Consider the following two-input, two-output system with steady-state gain matrix and setpoint

$$\begin{bmatrix} y_{1s} \\ y_{2s} \end{bmatrix} = \begin{bmatrix} -0.5 & 1.0 \\ 2.0 & 1.0 \end{bmatrix} \begin{bmatrix} u_{1s} \\ u_{2s} \end{bmatrix} \qquad \begin{bmatrix} y_{1sp} \\ y_{2sp} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(a) Show the first 10 iterations of the noncooperative and cooperative steady-state cases starting with the decentralized solution as the initial guess.

Describe the differences. Compute the eigenvalues of $L$ for the cooperative and noncooperative cases. Discuss the relationship between these eigenvalues and the result of the iteration calculations.

Mark also the solution to the centralized and decentralized cases on your plots.

(b) Switch the pairings and repeat the previous part. Explain your results.

**Solution**

(a) The first 10 iterations of the noncooperative steady-state calculation are shown in Figure 6.2. Notice the iteration is unstable and the steady-state target does not converge. The cooperative case is shown in Figure 6.3. This case is stable and the iterations converge to the centralized target and achieve zero offset. The magnitudes of the eigenvalues of $L_s$ for the noncooperative (nc) and cooperative (co) cases are given by

$$|\text{eig}(L_{snc})| = \{1.12, 1.12\} \qquad |\text{eig}(L_{sco})| = \{0.757, 0.243\}$$

Stability of the iteration is determined by the magnitudes of the eigenvalues of $L_s$.

(b) Reversing the pairings leads to the following gain matrix in which we have reversed the labels of the outputs for the two systems

$$\begin{bmatrix} y_{1s} \\ y_{2s} \end{bmatrix} = \begin{bmatrix} 2.0 & 1.0 \\ -0.5 & 1.0 \end{bmatrix} \begin{bmatrix} u_{1s} \\ u_{2s} \end{bmatrix}$$

The first 10 iterations of the noncooperative and cooperative games are shown in Figures 6.4 and 6.5. For this pairing, the noncooperative case also converges to the centralized target. The eigenvalues are given by

$$|\text{eig}(L_{snc})| = \{0.559, 0.559\} \qquad |\text{eig}(L_{sco})| = \{0.757, 0.243\}$$

The eigenvalues of the cooperative case are unaffected by the reversal of pairings. □

Given the stability analysis of the simple unconstrained two-player game, we remove from further consideration two options we have been

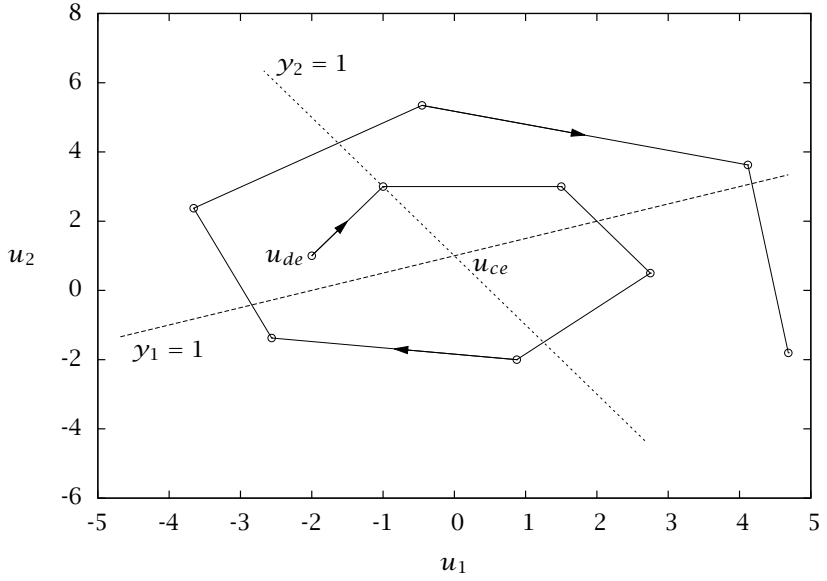


**Figure 6.2:** Ten iterations of the noncooperative steady-state calculation, $u^{[0]} = u_{de}$; iterations are unstable, $u^p \to \infty$.

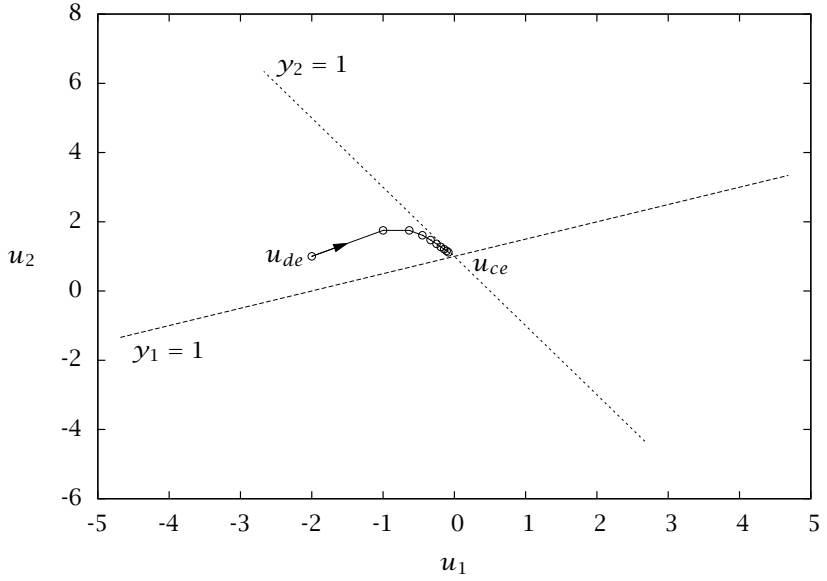


**Figure 6.3:** Ten iterations of the cooperative steady-state calculation, $u^{[0]} = u_{de}$; iterations are stable, $u^p \to u_{ce}$.

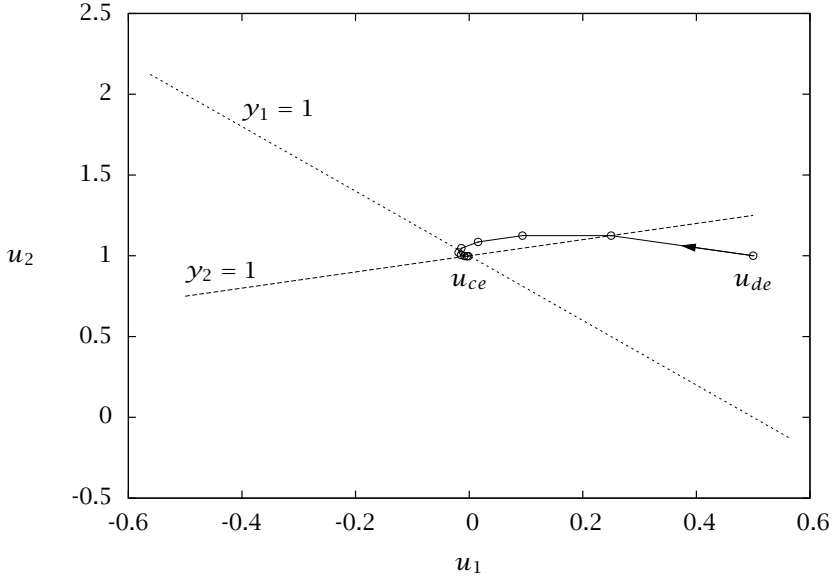


**Figure 6.4:** Ten iterations of the noncooperative steady-state calculation, $u^{[0]} = u_{de}$; iterations are now stable with reversed pairing.

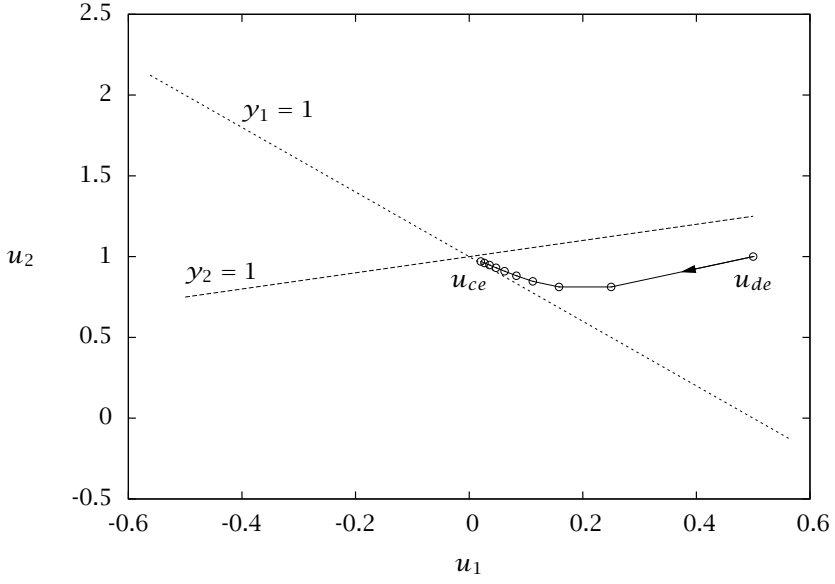


**Figure 6.5:** Ten iterations of the cooperative steady-state calculation, $u^{[0]} = u_{de}$; iterations remain stable with reversed pairing.

discussing to this point: noncooperative control and decentralized control. We next further develop the theory of cooperative MPC and compare its performance to centralized MPC in more general and challenging situations.

### 6.2.6 State Estimation

Given output measurements, we can express the state estimation problem also in distributed form. Player one uses local measurements of $y_1$ and knowledge of both inputs $u_1$ and $u_2$ to estimate state $x_1$

$$\hat{x}_1^+ = A_1\hat{x}_1 + \overline{B}_{11}u_1 + \overline{B}_{12}u_2 + L_1(y_1 - C_1\hat{x}_1)$$

Defining estimate error to be $e_1 = x_1 - \hat{x}_1$ gives

$$e_1^+ = (A_1 - L_1C_1)e_1$$

Because all the subsystems are stable, we know $L_1$ exists so that $A_1 - L_1C_1$ is stable and player one's local estimator is stable. The estimate error for the two subsystems is then given by

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}^+ = \begin{bmatrix} A_{L1} & \\ & A_{L2} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \tag{6.18}$$

in which $A_{Li} = A_i - L_iC_i$.

**Closed-Loop Stability.** The dynamics of the estimator are given by

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}^+ = \begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} \overline{B}_{11} & \overline{B}_{12} \\ \overline{B}_{21} & \overline{B}_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} L_1C_1 & \\ & L_2C_2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

In the control law we use the state estimate in place of the state, which is unmeasured and unknown. We consider two cases.

**Converged controller.** In this case the distributed control law converges to the centralized controller, and we have

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}$$

The closed-loop system evolves according to

$$\begin{bmatrix}\hat{x}_1\\ \hat{x}_2\end{bmatrix}^+ = \left\{\begin{bmatrix}A_1 & \\ & A_2\end{bmatrix} + \begin{bmatrix}\overline{B}_{11} & \overline{B}_{12}\\ \overline{B}_{21} & \overline{B}_{22}\end{bmatrix}\begin{bmatrix}K_{11} & K_{12}\\ K_{21} & K_{22}\end{bmatrix}\right\}\begin{bmatrix}\hat{x}_1\\ \hat{x}_2\end{bmatrix} + \begin{bmatrix}L_1C_1 & \\ & L_2C_2\end{bmatrix}\begin{bmatrix}e_1\\ e_2\end{bmatrix}$$

The $A+BK$ term is stable because this term is the same as in the stabilizing centralized controller. The perturbation is exponentially decaying because the distributed estimators are stable. Therefore $\hat{x}$ goes to zero exponentially, which, along with $e$ going to zero exponentially, implies $x$ goes to zero exponentially.

**Finite iterations.** Here we use the state plus input sequence description given in (6.16), which, as we have already noted, is a linear time-invariant system. With estimate error, the system equation is

$$\begin{bmatrix}\hat{x}_1^+\\ \hat{x}_2^+\\ \mathbf{u}_1^+\\ \mathbf{u}_2^+\end{bmatrix} = \begin{bmatrix}A_1\hat{x}_1 + \overline{B}_{11}u_1 + \overline{B}_{12}u_2\\ A_2\hat{x}_2 + \overline{B}_{21}u_1 + \overline{B}_{22}u_2\\ g_1^p(\hat{x}_1,\hat{x}_2,\mathbf{u}_1,\mathbf{u}_2)\\ g_2^p(\hat{x}_1,\hat{x}_2,\mathbf{u}_1,\mathbf{u}_2)\end{bmatrix} + \begin{bmatrix}L_1C_1e_1\\ L_2C_2e_2\\ 0\\ 0\end{bmatrix}$$

Because there is again only one-way coupling between the estimate error evolution, (6.18), and the system evolution given above, the composite system is exponentially stable.

## 6.3 Constrained Two-Player Game

Now that we have introduced most of the notation and the fundamental ideas, we consider more general cases. Because we are interested in establishing stability properties of the controlled systems, we focus exclusively on *cooperative distributed MPC* from this point forward. In this section we consider convex input constraints on the two players. We assume output constraints have been softened with exact soft constraints and added to the objective function, so do not consider output constraints explicitly. The input constraints break into two significant categories: coupled and uncoupled constraints. We treat each of these in turn.

We also allow unstable systems and replace Assumption 6.7 with the following more general restrictions on the systems and controller parameters.

**Assumption 6.12** (Constrained two-player game)**.**

(a) The systems $(\underline{A}_i, \underline{B}_i)$, $i = 1, 2$ are stabilizable, in which $\underline{A}_i = \text{diag}(A_{1i}, A_{2i})$ and $\underline{B}_i = \begin{bmatrix} B_{1i} \\ B_{2i} \end{bmatrix}$.

(b) The systems $(A_i, C_i)$, $i = 1, 2$ are detectable.

(c) The input penalties $R_1, R_2$ are positive definite, and the state penalties $Q_1, Q_2$ are semidefinite.

(d) The systems $(A_1, Q_1)$ and $(A_2, Q_2)$ are detectable.

(e) The horizon is chosen sufficiently long to zero the unstable modes, $N \geq \max_{i \in \mathbb{I}_{1:2}} \underline{n}_i^u$, in which $\underline{n}_i^u$ is the number of unstable modes of $\underline{A}_i$, i.e., number of $\lambda \in \text{eig}(\underline{A}_i)$ such that $|\lambda| \geq 1$.

Assumption (b) implies that we have $L_i$ such that $(A_i - L_i C_i)$, $i = 1, 2$ is stable. Note that the stabilizable and detectable conditions of Assumption 6.12 are automatically satisfied if we obtain the state space models from a minimal realization of the input/output models for $(u_i, y_j)$, $i, j = 1, 2$.

**Unstable modes.** To handle unstable systems, we add constraints to zero the unstable modes at the end of the horizon. To set up this constraint, consider the real Schur decomposition of $A_{ij}$ for $i, j \in \mathbb{I}_{1:2}$

$$A_{ij} = \begin{bmatrix} S_{ij}^s & S_{ij}^u \end{bmatrix} \begin{bmatrix} A_{ij}^s & - \\ & A_{ij}^u \end{bmatrix} \begin{bmatrix} S_{ij}^{s\prime} \\ S_{ij}^{u\prime} \end{bmatrix} \tag{6.19}$$

in which $A_{ij}^s$ is upper triangular and stable, and $A_{ij}^u$ is upper triangular with all unstable eigenvalues.[3] Given the Schur decomposition (6.19), we define the matrices

$$\begin{aligned} S_i^s &= \text{diag}(S_{i1}^s, S_{i2}^s) & A_i^s &= \text{diag}(A_{i1}^s, A_{i2}^s) \quad i \in \mathbb{I}_{1:2} \\ S_i^u &= \text{diag}(S_{i1}^u, S_{i2}^u) & A_i^u &= \text{diag}(A_{i1}^u, A_{i2}^u) \quad i \in \mathbb{I}_{1:2} \end{aligned}$$

These matrices satisfy the Schur decompositions

$$A_i = \begin{bmatrix} S_i^s & S_i^u \end{bmatrix} \begin{bmatrix} A_i^s & - \\ & A_i^u \end{bmatrix} \begin{bmatrix} S_i^{s\prime} \\ S_i^{u\prime} \end{bmatrix} \quad i \in \mathbb{I}_{1:2}$$

We further define the matrices $\Sigma_1, \Sigma_2$ as the solutions to the Lyapunov equations

$$A_1^{s\prime} \Sigma_1 A_1^s - \Sigma_1 = -S_1^{s\prime} Q_1 S_1^s \qquad A_2^{s\prime} \Sigma_2 A_2^s - \Sigma_2 = -S_2^{s\prime} Q_2 S_2^s \tag{6.20}$$

[3] If $A_{ij}$ is stable, then there is no $A_{ij}^u$ and $S_{ij}^u$.

We then choose the terminal penalty for each subsystem to be the cost to go under zero control

$$P_{1f} = S_1^s \Sigma_1 S_1^{s\prime} \qquad P_{2f} = S_2^s \Sigma_2 S_2^{s\prime}$$

### 6.3.1 Uncoupled Input Constraints

We consider convex input constraints of the following form

$$Hu(k) \le h \qquad k = 0, 1, \ldots, N$$

Defining convex set $\mathbb{U}$

$$\mathbb{U} = \{u | Hu \le h\}$$

we express the input constraints as

$$u(k) \in \mathbb{U} \qquad k = 0, 1, \ldots, N$$

We drop the time index and indicate the constraints are applied over the entire input sequence using the notation $\mathbf{u} \in \mathbb{U}$. In the uncoupled constraint case, the two players' inputs must satisfy

$$\mathbf{u}_1 \in \mathbb{U}_1 \qquad \mathbf{u}_2 \in \mathbb{U}_2$$

in which $\mathbb{U}_1$ and $\mathbb{U}_2$ are convex subsets of $\mathbb{R}^{m_1}$ and $\mathbb{R}^{m_2}$, respectively. The constraints are termed *uncoupled* because there is no interaction or coupling of the inputs in the constraint relation. Player one then solves the following constrained optimization

$$\begin{aligned}
&\min_{\mathbf{u}_1} V(x_1(0), x_2(0), \mathbf{u}_1, \mathbf{u}_2) \\
\text{s.t. } &\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^+ = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \overline{B}_{11} \\ \overline{B}_{21} \end{bmatrix} u_1 + \begin{bmatrix} \overline{B}_{12} \\ \overline{B}_{22} \end{bmatrix} u_2 \\
&\mathbf{u}_1 \in \mathbb{U}_1 \\
&S_{j1}^{u\prime} x_{j1}(N) = 0 \quad j \in \mathbb{I}_{1:2} \\
&|\mathbf{u}_1| \le d_1(|x_{11}(0)| + |x_{21}(0)|) \quad x_{11}(0), x_{21}(0) \in r\mathcal{B}
\end{aligned}$$

in which we include the system's hard input constraints, the stability constraint on the unstable modes, and the Lyapunov stability constraints. Exercise 6.22 discusses how to write the constraint $|\mathbf{u}_1| \le d_1 |x_1(0)|$ as a set of linear inequalities on $\mathbf{u}_1$. Similarly, player two

solves

$$\min_{\mathbf{u}_2} V(x_1(0), x_2(0), \mathbf{u}_1, \mathbf{u}_2)$$

$$\text{s.t.} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^+ = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \overline{B}_{11} \\ \overline{B}_{21} \end{bmatrix} u_1 + \begin{bmatrix} \overline{B}_{12} \\ \overline{B}_{22} \end{bmatrix} u_2$$

$$\mathbf{u}_2 \in \mathbb{U}_2$$

$$S_{j2}^{u\,\prime} x_{j2}(N) = 0 \quad j \in \mathbb{I}_{1:2}$$

$$|\mathbf{u}_2| \le d_2(|x_{21}(0)| + |x_{22}(0)|) \quad x_{12}(0), x_{22}(0) \in r\mathcal{B}$$

We denote the solutions to these problems as

$$\mathbf{u}_1^0(x_1(0), x_2(0), \mathbf{u}_2) \qquad \mathbf{u}_2^0(x_1(0), x_2(0), \mathbf{u}_1)$$

The feasible set $\mathcal{X}_N$ for the unstable system is the set of states for which the unstable modes can be brought to zero in $N$ moves while satisfying the input constraints.

Given an initial iterate, $(\mathbf{u}_1^p, \mathbf{u}_2^p)$, the next iterate is defined to be

$$(\mathbf{u}_1, \mathbf{u}_2)^{p+1} = w_1(\mathbf{u}_1^0(x_1(0), x_2(0), \mathbf{u}_2^p), \mathbf{u}_2^p) + w_2(\mathbf{u}_1^p, \mathbf{u}_2^0(x_1(0), x_2(0), \mathbf{u}_1^p))$$

To reduce the notational burden we denote this as

$$(\mathbf{u}_1, \mathbf{u}_2)^{p+1} = w_1(\mathbf{u}_1^0, \mathbf{u}_2^p) + w_2(\mathbf{u}_1^p, \mathbf{u}_2^0)$$

and the functional dependencies of $\mathbf{u}_1^0$ and $\mathbf{u}_2^0$ should be kept in mind.

This procedure provides three important properties, which we establish next.

1. The iterates are feasible: $(\mathbf{u}_1, \mathbf{u}_2)^p \in (\mathbb{U}_1, \mathbb{U}_2)$ implies $(\mathbf{u}_1, \mathbf{u}_2)^{p+1} \in (\mathbb{U}_1, \mathbb{U}_2)$. This follows from convexity of $\mathbb{U}_1$, $\mathbb{U}_2$ and the convex combination of the feasible points $(\mathbf{u}_1^p, \mathbf{u}_2^p)$ and $(\mathbf{u}_1^0, \mathbf{u}_2^0)$ to make $(\mathbf{u}_1, \mathbf{u}_2)^{p+1}$.

2. The cost decreases on iteration: $V(x_1(0), x_2(0), (\mathbf{u}_1, \mathbf{u}_2)^{p+1}) \le V(x_1(0), x_2(0), (\mathbf{u}_1, \mathbf{u}_2)^p)$ for all $x_1(0)$, $x_2(0)$, and for all feasible $(\mathbf{u}_1, \mathbf{u}_2)^p \in (\mathbb{U}_1, \mathbb{U}_2)$. The systemwide cost satisfies the following inequalities

$$\begin{aligned} V(x(0), \mathbf{u}_1^{p+1}, \mathbf{u}_2^{p+1}) &= V\left(x(0), \left(w_1(\mathbf{u}_1^0, \mathbf{u}_2^p) + w_2(\mathbf{u}_1^p, \mathbf{u}_2^0)\right)\right) \\ &\le w_1 V(x(0), (\mathbf{u}_1^0, \mathbf{u}_2^p)) + w_2 V(x(0), (\mathbf{u}_1^p, \mathbf{u}_2^0)) \\ &\le w_1 V(x(0), (\mathbf{u}_1^p, \mathbf{u}_2^p)) + w_2 V(x(0), (\mathbf{u}_1^p, \mathbf{u}_2^p)) \\ &= V(x(0), \mathbf{u}_1^p, \mathbf{u}_2^p) \end{aligned}$$

The first equality follows from (6.14). The next inequality follows from convexity of $V$. The next follows from optimality of $\mathbf{u}_1^0$ and $\mathbf{u}_2^0$, and the last follows from $w_1 + w_2 = 1$. Because the cost is bounded below, the cost iteration converges.

3. The converged solution of the cooperative problem is equal to the optimal solution of the centralized problem. Establishing this property is discussed in Exercise 6.26.

**Exponential stability of the closed-loop system.** We next consider the closed-loop system. The two players' warm starts at the next sample are as defined previously

$$\begin{aligned}\widetilde{\mathbf{u}}_1^+ &= \{u_1(1), u_1(2), \ldots, u_1(N-1), 0\}\\ \widetilde{\mathbf{u}}_2^+ &= \{u_2(1), u_2(2), \ldots, u_2(N-1), 0\}\end{aligned}$$

We define again the functions $g_1^p$, $g_2^p$ as the outcome of applying the control iteration procedure $p$ times

$$\begin{aligned}\mathbf{u}_1^p &= g_1^p(x_1, x_2, \mathbf{u}_1, \mathbf{u}_2)\\ \mathbf{u}_2^p &= g_2^p(x_1, x_2, \mathbf{u}_1, \mathbf{u}_2)\end{aligned}$$

The important difference between the previous unconstrained and this constrained case is that the functions $g_1^p$, $g_2^p$ are nonlinear due to the input constraints. The system evolution is then given by

$$\begin{aligned} x_1^+ &= A_1x_1 + \overline{B}_{11}u_1 + \overline{B}_{12}u_2 \qquad & x_2^+ &= A_2x_2 + \overline{B}_{21}u_1 + \overline{B}_{22}u_2\\ \mathbf{u}_1^+ &= g_1^p(x_1, x_2, \mathbf{u}_1, \mathbf{u}_2) & \mathbf{u}_2^+ &= g_2^p(x_1, x_2, \mathbf{u}_1, \mathbf{u}_2)\end{aligned}$$

We have the following cost using the warm start at the next sample

$$\begin{aligned} V(x_1^+, x_2^+, \widetilde{\mathbf{u}}_1^+, \widetilde{\mathbf{u}}_2^+) = V(x_1, x_2, \mathbf{u}_1, \mathbf{u}_2) - \rho_1\ell_1(x_1, u_1) - \rho_2\ell_2(x_2, u_2)\\ +(1/2)\rho_1 x_1(N)' \left[A_1'P_{1f}A_1 - P_{1f} + Q_1\right] x_1(N)\\ +(1/2)\rho_2 x_2(N)' \left[A_2'P_{2f}A_2 - P_{2f} + Q_2\right] x_2(N)\end{aligned}$$

Using the Schur decomposition (6.19) and the constraints $S_{ji}^{u\prime} x_{ji}(N) = 0$ for $i, j \in \mathbb{I}_{1:2}$, the last two terms can be written as

$$\begin{aligned}(1/2)\rho_1 x_1(N)' S_1^s \left[A_1^{s\prime}\Sigma_1 A_1^s - \Sigma_1 + S_1^{s\prime} Q_1 S_1^s\right] S_1^{s\prime} x_1(N)\\ +(1/2)\rho_2 x_2(N)' S_2^s \left[A_2^{s\prime}\Sigma_2 A_2^s - \Sigma_2 + S_2^{s\prime} Q_2 S_2^s\right] S_2^{s\prime} x_2(N)\end{aligned}$$

These terms are zero because of (6.20). Using this result and applying the iteration for the controllers gives

$$V(x_1^+, x_2^+, \mathbf{u}_1^+, \mathbf{u}_2^+) \leq V(x_1, x_2, \mathbf{u}_1, \mathbf{u}_2) - \rho_1 \ell_1(x_1, u_1) - \rho_2 \ell_2(x_2, u_2)$$

The Lyapunov stability constraints give (see also Exercise 6.28)

$$|(\mathbf{u}_1, \mathbf{u}_2)| \leq 2 \max(d_1, d_2) \, |(x_1, x_2)| \qquad (x_1, x_2) \in r\mathcal{B}$$

Given the cost decrease and this constraint on the size of the input sequence, we satisfy the conditions of Lemma 6.4, and conclude the solution $x(k) = 0$ for all $k$ is exponentially stable on all of $\mathcal{X}_N$ if either $\mathcal{X}_N$ is compact or $\mathbb{U}$ is compact.

### 6.3.2 Coupled Input Constraints

By contrast, in the coupled constraint case, the constraints are of the form

$$H_1 \mathbf{u}_1 + H_2 \mathbf{u}_2 \leq h \quad \text{or} \quad (\mathbf{u}_1, \mathbf{u}_2) \in \mathbb{U} \tag{6.21}$$

These constraints represent the players sharing some common resource. An example would be different subsystems in a chemical plant drawing steam or some other utility from a single plantwide generation plant. The total utility used by the different subsystems to meet their control objectives is constrained by the generation capacity.

The players solve the same optimization problems as in the uncoupled constraint case, with the exception that both players' input constraints are given by (6.21). This modified game provides only two of the three properties established for the uncoupled constraint case. These are

1. The iterates are feasible: $(\mathbf{u}_1, \mathbf{u}_2)^p \in \mathbb{U}$ implies $(\mathbf{u}_1, \mathbf{u}_2)^{p+1} \in \mathbb{U}$. This follows from convexity of $\mathbb{U}$ and the convex combination of the feasible points $(\mathbf{u}_1^p, \mathbf{u}_2^p)$ and $(\mathbf{u}_1^0, \mathbf{u}_2^0)$ to make $(\mathbf{u}_1, \mathbf{u}_2)^{p+1}$.

2. The cost decreases on iteration: $V(x_1(0), x_2(0), (\mathbf{u}_1, \mathbf{u}_2)^{p+1}) \leq V(x_1(0), x_2(0), (\mathbf{u}_1, \mathbf{u}_2)^p)$ for all $x_1(0)$, $x_2(0)$, and for all feasible $(\mathbf{u}_1, \mathbf{u}_2)^p \in \mathbb{U}$. The systemwide cost satisfies the same inequalities established for the uncoupled constraint case giving

$$V(x(0), \mathbf{u}_1^{p+1}, \mathbf{u}_2^{p+1}) \leq V(x(0), \mathbf{u}_1^p, \mathbf{u}_2^p)$$

   Because the cost is bounded below, the cost iteration converges.

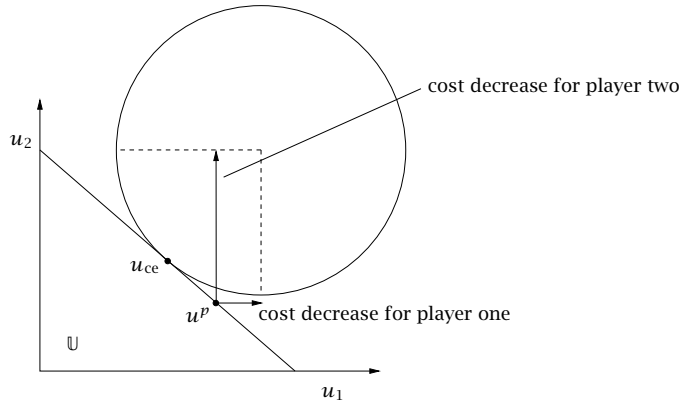


**Figure 6.6:** Cooperative control stuck on the boundary of $\mathbb{U}$ under coupled constraints; $u^{p+1} = u^p \neq u_{ce}$.

The converged solution of the cooperative problem is *not* equal to the optimal solution of the centralized problem, however. We have lost property 3 of the uncoupled case. To see how the convergence property is lost, consider Figure 6.6. Region $\mathbb{U}$ is indicated by the triangle and its interior. Consider point $u^p$ on the boundary of $\mathbb{U}$. Neither player one nor player two can improve upon the current point $u^p$ so the iteration has converged. But the converged point is clearly not the optimal point, $u_{ce}$.

Because of property 2, the nominal stability properties for the coupled and uncoupled cases are identical. The differences arise when the performance of cooperative control is compared to the benchmark of centralized control. Improving the performance of cooperative control in the case of coupled constraints is therefore a topic of current research. Current approaches include adding another player to the game, whose sole objective is to parcel out the coupled resource to the other players in a way that achieves optimality on iteration. This approach also makes sense from an engineering perspective because it is commonplace to design a dedicated control system for managing a shared resource such as steam or power among many plant units. The design of this single unit's control system is a reasonably narrow and well defined task compared to the design of a centralized controller for the entire plant.

### 6.3.3 Exponential Stability with Estimate Error

Consider next the constrained system evolution with estimate error

$$\begin{bmatrix} \hat{x}^+ \\ \mathbf{u}^+ \\ e^+ \end{bmatrix} = \begin{bmatrix} A\hat{x} + \overline{B}_1 u_1 + \overline{B}_2 u_2 + Le \\ g^p(\hat{x}, \mathbf{u}) \\ A_L e \end{bmatrix} \tag{6.22}$$

The estimate error is globally exponentially stable so we know from Lemma 6.6 that there exists a Lipschitz continuous Lyapunov function $J(\cdot)$ such that for all $e \in \mathbb{R}^n$

$$\begin{aligned} \overline{a}\,|e| \le J(e) \le \overline{b}\,|e| \\ J(e^+) - J(e) \le -\overline{c}\,|e| \end{aligned}$$

in which $\overline{b} > 0$, $\overline{a} > 0$, and we can choose constant $\overline{c} > 0$ as large as desired. In the subsequent development, we require this Lyapunov function to be based on the first power of the norm rather than the usual square of the norm to align with Lipschitz continuity of the Lyapunov function. From the stability of the solution $x(k) = 0$ for all $k$ for the *nominal* system, the cost function $V(\hat{x}, \mathbf{u})$ satisfies for all $\hat{x} \in \mathcal{X}_N$, $\mathbf{u} \in \mathbb{U}^N$

$$\begin{aligned} \widetilde{a}\,|(\hat{x}, \mathbf{u})|^2 \le V(\hat{x}, \mathbf{u}) \le \widetilde{b}\,|(\hat{x}, \mathbf{u})|^2 \\ V(A\hat{x} + \overline{B}_1 u_1 + \overline{B}_2 u_2, \mathbf{u}^+) - V(\hat{x}, \mathbf{u}) \le -\widetilde{c}\,|\hat{x}|^2 \\ |\mathbf{u}| \le d\,|\hat{x}| \qquad \hat{x} \in \widetilde{r}\mathcal{B} \end{aligned}$$

in which $\widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{r} > 0$. We propose $W(\hat{x}, \mathbf{u}, e) = V(\hat{x}, \mathbf{u}) + J(e)$ as a Lyapunov function candidate for the perturbed system. We next derive the required properties of $W(\cdot)$ to establish exponential stability of the solution $(x(k), e(k)) = 0$. From the definition of $W(\cdot)$ we have for all $(\hat{x}, \mathbf{u}, e) \in \mathcal{X}_N \times \mathbb{U}^N \times \mathbb{R}^n$

$$\begin{aligned} \widetilde{a}\,|(\hat{x}, \mathbf{u})|^2 + \overline{a}\,|e| \le W(\hat{x}, \mathbf{u}, e) \le \widetilde{b}\,|(\hat{x}, \mathbf{u})|^2 + \overline{b}\,|e| \\ a(|(\hat{x}, \mathbf{u})|^2 + |e|) \le W(\hat{x}, \mathbf{u}, e) \le b(|(\hat{x}, \mathbf{u})|^2 + |e|) \end{aligned} \tag{6.23}$$

in which $a = \min(\widetilde{a}, \overline{a}) > 0$, $b = \max(\widetilde{b}, \overline{b})$. Next we compute the cost change

$$W(\hat{x}^+, \mathbf{u}^+, e^+) - W(\hat{x}, \mathbf{u}, e) = V(\hat{x}^+, \mathbf{u}^+) - V(\hat{x}, \mathbf{u}) + J(e^+) - J(e)$$

The Lyapunov function $V$ is quadratic in $(x, \mathbf{u})$ and therefore Lipschitz continuous on bounded sets. Therefore, for all $\hat{x}, u_1, u_2, \mathbf{u}^+, e$ in some

bounded set,

$$\left| V(A\hat{x} + \overline{B}_1 u_1 + \overline{B}_2 u_2 + Le, \mathbf{u}^+) - V(A\hat{x} + \overline{B}_1 u_1 + \overline{B}_2 u_2, \mathbf{u}^+) \right| \le L_V |Le|$$

in which $L_V$ is the Lipschitz constant for $V$ with respect to its first argument. Using the system evolution we have

$$V(\hat{x}^+, \mathbf{u}^+) \le V(A\hat{x} + \overline{B}_1 u_1 + \overline{B}_2 u_2, \mathbf{u}^+) + L'_V |e|$$

in which $L'_V = L_V |L|$. Subtracting $V(\hat{x}, \mathbf{u})$ from both sides gives

$$V(\hat{x}^+, \mathbf{u}^+) - V(\hat{x}, \mathbf{u}) \le -\widetilde{c} |\hat{x}|^2 + L'_V |e|$$

Substituting this result into the equation for the change in $W$ gives

$$\begin{aligned} W(\hat{x}^+, \mathbf{u}^+, e^+) - W(\hat{x}, \mathbf{u}, e) &\le -\widetilde{c} |\hat{x}|^2 + L'_V |e| - \overline{c} |e| \\ &\le -\widetilde{c} |\hat{x}|^2 - (\overline{c} - L'_V) |e| \\ W(\hat{x}^+, \mathbf{u}^+, e^+) - W(\hat{x}, \mathbf{u}, e) &\le -c(|\hat{x}|^2 + |e|) \end{aligned} \tag{6.24}$$

in which we choose $\overline{c} > L'_V$, which is possible because we may choose $\overline{c}$ as large as we wish, and $c = \min(\widetilde{c}, \overline{c} - L'_V) > 0$. Notice this step is what motivated using the first power of the norm in $J(\cdot)$. Lastly, we require the constraint

$$|\mathbf{u}| \le d |\hat{x}| \qquad \hat{x} \in \widetilde{r}\mathcal{B} \tag{6.25}$$

**Lemma 6.13** (Exponential stability of perturbed system)**.** *If either $\mathcal{X}_N$ or $\mathbb{U}$ is compact, the origin for the state plus estimate error $(x, e)$ is exponentially stable for system* (6.22) *under cooperative distributed MPC.*

The proof is based on the properties (6.23), (6.24) and (6.25) of function $W(\hat{x}, \mathbf{u}, e)$ and is similar to the proof of Lemma 6.4. The region of attraction is the set of states and initial estimate errors for which the unstable modes of the two subsystems can be brought to zero in $N$ moves while satisfying the respective input constraints. If both subsystems are stable, for example, the region of attraction is $(x, e) \in \mathcal{X}_N \times \mathbb{R}^n$.

### 6.3.4 Disturbance Models and Zero Offset

**Integrating disturbance model.** As discussed in Chapter 1, we model the disturbance with an integrator to remove steady offset. The augmented models for the local systems are

$$\begin{bmatrix} x_i \\ d_i \end{bmatrix}^+ = \begin{bmatrix} A_i & B_{di} \\ 0 & I \end{bmatrix} \begin{bmatrix} x_i \\ d_i \end{bmatrix} + \begin{bmatrix} \overline{B}_{i1} \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} \overline{B}_{i2} \\ 0 \end{bmatrix} u_2$$

$$y_i = \begin{bmatrix} C_i & C_{di} \end{bmatrix} \begin{bmatrix} x_i \\ d_i \end{bmatrix} \qquad i = 1, 2$$

We wish to estimate both $x_i$ and $d_i$ from measurements $y_i$. To ensure this goal is possible, we make the following restriction on the disturbance models

**Assumption 6.14** (Disturbance models)**.**

$$\operatorname{rank}\begin{bmatrix} I - A_i & -B_{di} \\ C_i & C_{di} \end{bmatrix} = n_i + p_i \qquad i = 1, 2$$

It is always possible to satisfy this assumption by proper choice of $B_{di}, C_{di}$. From Assumption 6.12 (b), $(A_i, C_i)$ is detectable, which implies that the first $n_i$ columns of the square $(n_i + p_i) \times (n_i + p_i)$ matrix in Assumption 6.14 are linearly independent. Therefore the columns of $\begin{bmatrix} -B_{di} \\ C_{di} \end{bmatrix}$ can be chosen so that the entire matrix has rank $n_i + p_i$. Assumption 6.14 is equivalent to detectability of the following augmented system.

**Lemma 6.15** (Detectability of distributed disturbance model)**.** *Consider the augmented systems*

$$\widetilde{A}_i = \begin{bmatrix} A_i & B_{di} \\ 0 & I \end{bmatrix} \qquad \widetilde{C}_i = \begin{bmatrix} C_i & C_{di} \end{bmatrix} \qquad i = 1, 2$$

*The augmented systems* $(\widetilde{A}_i, \widetilde{C}_i), i = 1, 2$ *are detectable if and only if Assumption 6.14 is satisfied.*

Proving this lemma is discussed in Exercise 6.29. The detectability assumption then establishes the existence of $\widetilde{L}_i$ such that $(\widetilde{A}_i - \widetilde{L}_i\widetilde{C}_i), i = 1, 2$ are stable and the local integrating disturbances can be estimated from the local measurements.

**Centralized target problem.** We can solve the target problem at the plantwide level or as a distributed target problem at the subunit controller level. Consider first the centralized target problem with the disturbance model discussed in Chapter 1, (1.46)

$$\min_{x_s, u_s} \frac{1}{2} \left| u_s - u_{\text{sp}} \right|^2_{R_s} + \frac{1}{2} \left| Cx_s + C_d\hat{d}(k) - y_{\text{sp}} \right|^2_{Q_s}$$

subject to:

$$\begin{bmatrix} I - A & -B \\ HC & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} B_d\hat{d}(k) \\ r_{\text{sp}} - HC_d\hat{d}(k) \end{bmatrix}$$
$$Eu_s \le e$$

in which we have removed the state inequality constraints to be consistent with the regulator problem. We denote the solution to this problem $(x_s(k), u_s(k))$. Notice first that the solution of the target problem depends only on the disturbance estimate, $\hat{d}(k)$, and not the solution of the control problem. So we can analyze the behavior of the target by considering only the exponential convergence of the estimator. We restrict the plant disturbance $d$ so that the target problem is feasible, and denote the solution to the target problem for the plant disturbance, $\hat{d}(k) = d$, as $(x_s^*, u_s^*)$. Because the estimator is exponentially stable, we know that $\hat{d}(k) \to d$ as $k \to \infty$. Because the target problem is a positive definite QP, we know the solution is Lipschitz continuous on bounded sets in the term $\hat{d}(k)$, which appears linearly in the objective function and the right-hand side of the equality constraint. Therefore, if we also restrict the initial disturbance estimate error so that the target problem remains feasible for all time, we know $(x_s(k), u_s(k)) \to (x_s^*, u_s^*)$ and the rate of convergence is exponential.

**Distributed target problem.** Consider next the cooperative approach, in which we assume the input inequality constraints are uncoupled. In the constrained case, we try to set things up so each player solves a local target problem

$$\min_{x_{1s},u_{1s}} \frac{1}{2}\begin{bmatrix} y_{1s}-y_{1sp} \\ y_{2s}-y_{2sp} \end{bmatrix}' \begin{bmatrix} Q_{1s} & \\ & Q_{2s} \end{bmatrix} \begin{bmatrix} y_{1s}-y_{1sp} \\ y_{2s}-y_{2sp} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} u_{1s}-u_{1sp} \\ u_{2s}-u_{2sp} \end{bmatrix}' \begin{bmatrix} R_{1s} & \\ & R_{2s} \end{bmatrix} \begin{bmatrix} u_{1s}-u_{1sp} \\ u_{2s}-u_{2sp} \end{bmatrix}$$

subject to

$$\begin{bmatrix} I-A_1 & & -\overline{B}_{11} & -\overline{B}_{12} \\ & I-A_2 & -\overline{B}_{21} & -\overline{B}_{22} \\ H_1C_1 & & & \\ & H_2C_2 & & \end{bmatrix} \begin{bmatrix} x_{1s} \\ x_{2s} \\ u_{1s} \\ u_{2s} \end{bmatrix} = \begin{bmatrix} B_{d1}\hat{d}_1(k) \\ B_{d2}\hat{d}_2(k) \\ r_{1sp} - H_1C_{d1}\hat{d}_1(k) \\ r_{2sp} - H_2C_{d2}\hat{d}_2(k) \end{bmatrix}$$

$$E_1 u_{1s} \le e_1$$

in which

$$y_{1s} = C_1 x_{1s} + C_{d1}\hat{d}_1(k) \qquad y_{2s} = C_2 x_{2s} + C_{d2}\hat{d}_2(k) \tag{6.27}$$

But here we run into several problems. First, the constraints to ensure zero offset in both players' controlled variables are not feasible with

only the $u_{1s}$ decision variables. We require also $u_{2s}$, which is not available to player one. We can consider deleting the zero offset condition for player two's controlled variables, the last equality constraint. But if we do that for both players, then the two players have *different and coupled* equality constraints. That is a path to instability as we have seen in the noncooperative target problem. To resolve this issue, we move the controlled variables to the objective function, and player one solves instead the following

$$\min_{x_{1s},u_{1s}} \frac{1}{2} \begin{bmatrix} H_1 y_{1s} - r_{1sp} \\ H_2 y_{2s} - r_{2sp} \end{bmatrix}' \begin{bmatrix} T_{1s} & \\ & T_{2s} \end{bmatrix} \begin{bmatrix} H_1 y_{1s} - r_{1sp} \\ H_2 y_{2s} - r_{2sp} \end{bmatrix}$$

subject to (6.27) and

$$\begin{bmatrix} I - A_1 & & -\overline{B}_{11} & -\overline{B}_{12} \\ & I - A_2 & -\overline{B}_{21} & -\overline{B}_{22} \end{bmatrix} \begin{bmatrix} x_{1s} \\ x_{2s} \\ u_{1s} \\ u_{2s} \end{bmatrix} = \begin{bmatrix} B_{d1}\hat{d}_1(k) \\ B_{d2}\hat{d}_2(k) \end{bmatrix}$$

$$E_1 u_{1s} \le e_1 \tag{6.28}$$

The equality constraints for the two players appear coupled when written in this form. Coupled constraints admit the potential for the optimization to become stuck on the boundary of the feasible region, and not achieve the centralized target solution after iteration to convergence. But Exercise 6.30 discusses how to show that the equality constraints are, in fact, uncoupled. Also, the distributed target problem as expressed here may not have a unique solution when there are more manipulated variables than controlled variables. In such cases, a regularization term using the input setpoint can be added to the objective function. The controlled variable penalty can be converted to a linear penalty with a large penalty weight to ensure exact satisfaction of the controlled variable setpoint.

If the input inequality constraints are coupled, however, then the distributed target problem may indeed become stuck on the boundary of the feasible region and not eliminate offset in the controlled variables. If the input inequality constraints are coupled, we recommend using the centralized approach to computing the steady-state target. As discussed above, the centralized target problem eliminates offset in the controlled variables as long as it remains feasible given the disturbance estimates.

**Zero offset.** Finally we establish the zero offset property. As described in Chapter 1, the regulator is posed in deviation variables

$$\tilde{\hat{x}}(k) = \hat{x}(k) - x_s(k) \qquad \tilde{u}(k) = u(k) - u_s(k) \qquad \tilde{\mathbf{u}} = \mathbf{u} - u_s(k)$$

in which the notation $\mathbf{u} - u_s(k)$ means to subtract $u_s(k)$ from each element of the $\mathbf{u}$ sequence. Player one then solves

$$\begin{aligned}
&\min_{\tilde{\mathbf{u}}_1} V(\tilde{x}_1(0), \tilde{x}_2(0), \tilde{\mathbf{u}}_1, \tilde{\mathbf{u}}_2) \\
\text{s.t. } &\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}^+ = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} \overline{B}_{11} \\ \overline{B}_{21} \end{bmatrix} \tilde{u}_1 + \begin{bmatrix} \overline{B}_{12} \\ \overline{B}_{22} \end{bmatrix} \tilde{u}_2 \\
&\tilde{\mathbf{u}}_1 \in \mathbb{U}_1 \ominus u_s(k) \\
&S_{1u}' \tilde{x}_1(N) = 0 \\
&|\tilde{\mathbf{u}}_1| \le d_1 |\tilde{x}_1(0)|
\end{aligned}$$

Notice that because the input constraint is shifted by the input target, we must retain feasibility of the regulation problem by restricting also the plant disturbance and its initial estimate error. If the two players' regulation problems remain feasible as the estimate error converges to zero, we have exponential stability of the zero solution from Lemma 6.13. Therefore we conclude

| | | |
|---|---|---|
| | $(\tilde{\hat{x}}(k), \tilde{u}(k)) \to (0,0)$ | Lemma 6.13 |
| $\Longrightarrow$ | $(\hat{x}(k), u(k)) \to (x_s(k), u_s(k))$ | definition of deviation variables |
| $\Longrightarrow$ | $(\hat{x}(k), u(k)) \to (x_s^*, u_s^*)$ | target problem convergence |
| $\Longrightarrow$ | $x(k) \to x_s^*$ | estimator stability |
| $\Longrightarrow$ | $r(k) \to r_{\text{sp}}$ | target equality constraint |

and we have *zero offset* in the plant controlled variable $r = Hy$. The rate of convergence of $r(k)$ to $r_{\text{sp}}$ is also exponential. As we saw here, this convergence depends on maintaining feasibility in both the target problem and the regulation problem at all times.

## 6.4 Constrained $M$-Player Game

We have set up the constrained two-player game so that the approach generalizes naturally to the $M$-player game. We do not have a lot of work left to do to address this general case. Recall $\mathbb{I}_{1:M}$ denotes the set

of integers $\{1, 2, \ldots, M\}$. We define the following systemwide variables

$$x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_M(0) \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_M \end{bmatrix} \qquad B_i = \begin{bmatrix} \overline{B}_{1i} \\ \overline{B}_{2i} \\ \vdots \\ \overline{B}_{Mi} \end{bmatrix} \quad \underline{B}_i = \begin{bmatrix} B_{1i} \\ B_{2i} \\ \vdots \\ B_{Mi} \end{bmatrix} \quad i \in \mathbb{I}_{1:M}$$

$$V(x(0), \mathbf{u}) = \sum_{j \in \mathbb{I}_{1:M}} \rho_j V_j(x_j(0), \mathbf{u})$$

Each player solves a similar optimization, so for $i \in \mathbb{I}_{1:M}$

$$\begin{aligned} &\min_{\mathbf{u}_i} V(x(0), \mathbf{u}) \\ &\text{s.t. } x^+ = Ax + \sum_{j \in \mathbb{I}_{1:M}} B_j u_j \\ &\mathbf{u}_i \in \mathbb{U}_i \\ &S_{ji}^{u\prime} x_{ji}(N) = 0 \quad j \in \mathbb{I}_{1:M} \\ &|\mathbf{u}_i| \leq d_i \sum_{j \in \mathbb{I}_{1:M}} \left| x_{ji}(0) \right| \qquad \text{if } x_{ji}(0) \in r\mathcal{B},\ j \in \mathbb{I}_{1:M} \end{aligned}$$

This optimization can be expressed as a quadratic program, whose constraints and linear cost term depend affinely on parameter $x$. The warm start for each player at the next sample is generated from purely local information

$$\widetilde{\mathbf{u}}_i^+ = \{u_i(1), u_i(2), \ldots, u_i(N-1), 0\} \quad i \in \mathbb{I}_{1:M}$$

The controller iteration is given by

$$\mathbf{u}^{p+1} = \sum_{j \in \mathbb{I}_{1:M}} w_j \left( \mathbf{u}_1^p, \ldots, \mathbf{u}_j^0, \ldots, \mathbf{u}_M^p \right)$$

in which $\mathbf{u}_i^0 = \mathbf{u}_i^0 \left( x(0), \mathbf{u}_{j \in \mathbb{I}_{1:M}, j \neq i}^p \right)$. The plantwide cost function then satisfies for any $p \geq 0$

$$\begin{aligned} V(x^+, \mathbf{u}^+) &\leq V(x, \mathbf{u}) - \sum_{j \in \mathbb{I}_{1:M}} \rho_j \ell_j(x_j, u_j) \\ |\mathbf{u}| &\leq d\,|x| \qquad x \in r\mathcal{B} \end{aligned}$$

For the $M$-player game, we generalize Assumption 6.12 of the two-player game to the following.

**Assumption 6.16** (Constrained $M$-player game)**.**

(a) The systems $(\underline{A}_i, \underline{B}_i)$, $i \in \mathbb{I}_{1:M}$ are stabilizable, in which $\underline{A}_i = \text{diag}(A_{1i}, A_{2i}, \cdots, A_M$

(b) The systems $(A_i, C_i)$, $i \in \mathbb{I}_{1:M}$ are detectable.

(c) The input penalties $R_i, i \in \mathbb{I}_{1:M}$ are positive definite, and $Q_i, i \in \mathbb{I}_{1:M}$ are semidefinite.

(d) The systems $(A_i, Q_i), i \in \mathbb{I}_{1:M}$ are detectable.

(e) The horizon is chosen sufficiently long to zero the unstable modes; $N \geq \max_{i \in \mathbb{I}_{1:M}}(\underline{n}_i^u)$, in which $\underline{n}_i^u$ is the number of unstable modes of $\underline{A}_i$.

(f) Zero offset. For achieving zero offset, we augment the models with integrating disturbances such that

$$\text{rank} \begin{bmatrix} I - A_i & -B_{di} \\ C_i & C_{di} \end{bmatrix} = n_i + p_i \qquad i \in \mathbb{I}_{1:M}$$

Applying Theorem 6.4 then establishes exponential stability of the solution $x(k) = 0$ for all $k$. The region of attraction is the set of states for which the unstable modes of each subsystem can be brought to zero in $N$ moves while satisfying the respective input constraints. These conclusions apply regardless of how many iterations of the players' optimizations are used in the control calculation. Although the closed-loop system is exponentially stable for both coupled and uncoupled constraints, the converged distributed controller is equal to the centralized controller only for the case of uncoupled constraints.

The exponential stability of the regulator implies that the states and inputs of the constrained $M$-player system converge to the steady-state target. The steady-state target can be calculated as a centralized or distributed problem. We assume the centralized target has a feasible, zero offset solution for the true plant disturbance. The initial state of the plant and the estimate error must be small enough that feasibility of the target is maintained under the nonzero estimate error.

## 6.5 Nonlinear Distributed MPC

In the nonlinear case, the usual model comes from physical principles and conservation laws of mass, energy and momentum. The state has a physical meaning and the measured outputs usually are a subset of

the state. We assume the model is of the form

$$\begin{aligned}\frac{dx_1}{dt} &= f_1(x_1, x_2, u_1, u_2)\\ y_1 &= C_1 x_1\\ \frac{dx_2}{dt} &= f_2(x_1, x_2, u_1, u_2)\\ y_2 &= C_2 x_2\end{aligned}$$

in which $C_1, C_2$ are matrices of zeros and ones selecting the part of the state that is measured in subsystems one and two. We generally cannot avoid state $x_2$ dependence in the differential equation for $x_1$. But often it is only a small subset of the entire state $x_2$ that appears in $f_1$, and vice versa. The reason in chemical process systems is that the two subsystems are generally coupled through a small set of process streams transferring mass and energy between the systems. These connecting streams isolate the coupling between the two systems and reduce the influence to a small part of the entire state required to describe each system.

Given these physical system models of the subsystems, the overall plant model is

$$\begin{aligned}\frac{dx}{dt} &= f(x, u)\\ y &= Cx\end{aligned}$$

in which

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \qquad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad C = \begin{bmatrix} C_1 & \\ & C_2 \end{bmatrix}$$

**Nonconvexity.** The basic difficulty in both the theory and application of nonlinear MPC is the nonconvexity in the control objective function caused by the nonlinear dynamic model. This difficulty applies even to centralized nonlinear MPC as discussed in Section 2.8, and motivates the development of suboptimal MPC. In the distributed case, nonconvexity causes even greater difficulties. As an illustration, consider the simple two-player, nonconvex game depicted in Figure 6.7. The cost function is

$$\begin{aligned}V(u_1, u_2) = e^{-2u_1} - 2e^{-u_1} + e^{-2u_2} - 2e^{-u_2}\\ + a\exp(-\beta((u_1 + 0.2)^2 + (u_2 + 0.2)^2))\end{aligned}$$

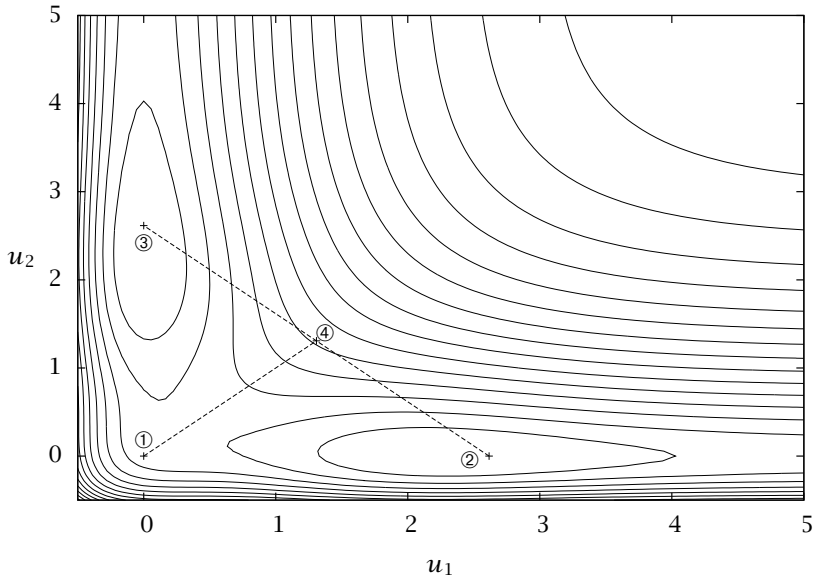


**Figure 6.7:** Cost contours for a two-player, nonconvex game; cost *increases* for the convex combination of the two players' optimal points.

in which $a = 1.1$ and $\beta = 0.4$. Each player optimizes the cooperative objective starting at ① and produces the points, $(u_1^0, u_2^p)$, denoted ② and $(u_1^p, u_2^0)$, denoted ③. Consider taking a convex combination of the two players' optimal points for the next iterate

$$(u_1^{p+1}, u_2^{p+1}) = w_1(u_1^0, u_2^p) + w_2(u_1^p, u_2^0) \qquad w_1 + w_2 = 1, \quad w_1, w_2 \geq 0$$

We see in Figure 6.7 that this iterate causes the objective function to *increase* rather than decrease for most values of $w_1, w_2$. For $w_1 = w_2 = 1/2$, we see clearly from the contours that $V$ at point ④ is greater than $V$ at point ①. The values of the four points are given in the following table

| Point | $u_1$ | $u_2$ | $V(u)$ |
|---|---|---|---|
| ① | 0 | 0 | −0.93 |
| ② | 2.62 | 0 | −1.10 |
| ③ | 0 | 2.62 | −1.10 |
| ④ | 1.31 | 1.31 | −0.76 |

The possibility of a cost increase leads to the possibility of closed-loop instability and precludes developing even a nominal control theory for this situation. In the centralized MPC problem, this nonconvexity issue can be addressed in the optimizer, which can move both inputs simultaneously and always avoid a cost increase. In the distributed case, the required information to avoid a cost increase is *by design* unavailable to the players.

One can of course consider adding another player to the game who has access to more systemwide information. This player takes the optimization results of the individual players and determines a search direction and step length that achieve a cost decrease for the overall system. This player is often known as a coordinator. The main challenge of this approach is that the design of the coordinator may not be significantly simpler than the design of the centralized controller. This issue remains a topic of current research.

## 6.6 Notes

At least three different fields have contributed substantially to the material presented in this chapter. We attempt here to point out briefly what each field has contributed, and indicate what literature the interested reader may wish to consult for further pursuing this and related subjects.

**Game theory.** Game theory emerged in the mid-1900s to analyze situations in which multiple players follow a common set of rules but have their own and different objectives that they try to optimize in competition with each other. Von Neumann and Morgenstern introduced the classic text on this subject, "Theory of Games and Economic Behavior," in 1944. A principle aim of game theory since its inception was to model and understand human *economic* behavior, especially as it arises in a capitalistic, free-market system. For that reason, much of the subsequent game theory literature was published in economics journals rather than systems theory journals. This field has contributed richly to the ideas and vocabulary used in this chapter to describe distributed control. For example, the game in which players have different objectives is termed *noncooperative*. The equilibrium of a noncooperative game is known as a *Nash equilibrium* (Nash, 1951). The Nash equilibrium is usually not Pareto optimal, which means that the outcomes for all players can be improved simultaneously from the Nash solution. A comprehensive overview of the game theory literature, especially the

parts relevant to control theory, is provided by Başar and Olsder (1999, Chapter 1), which is a highly recommended reference. Analyzing the equilibria of a noncooperative game is usually more complex than the cooperative game (optimal control problem). The closed-loop properties of a receding horizon implementation of any of these game theory solutions is not addressed in game theory. That topic is addressed by control theory.

**Distributed optimization.** The optimization community has extensively studied the issue of solving large-scale optimization problems using distributed optimization methods. The primary motivation in this field is to exploit parallel computing hardware and distributed data communication networks to solve large optimization problems faster. Bertsekas and Tsitsiklis provide an excellent and comprehensive overview of this field focusing on numerical algorithms for implementing the distributed approaches. The important questions that are addressed in designing a distributed optimization are: task allocation, communication, and synchronization (Bertsekas and Tsitsiklis, 1997, Chapter 1).

These basic concepts arise in distributed problems of all types, and therefore also in the distributed MPC problem, which provides good synergy between these fields. But one should also be aware of the structural distinctions between distributed optimization and distributed MPC. The primary obstacle to implementing centralized MPC for large-scale plants is not *computational* but *organizational*. The agents considered in distributed MPC are usually existing MPC systems already built for units or subsystems within an existing large-scale process. The plant management often is seeking to improve the plant performance by better coordinating the behavior of the different agents already in operation. Ignoring these structural constraints and treating the distributed MPC problem purely as a form of distributed optimization, ignores aspects of the design that are critical for successful industrial application (Rawlings and Stewart, 2008).

**Control theory.** Researchers have long studied the issue of how to distribute control tasks in a complex large-scale plant (Mesarović, Macko, and Takahara, 1970; Sandell Jr., Varaiya, Athans, and Safonov, 1978). The centralized controller and decentralized controller define two limiting design extremes. Centralized control accounts for all possible interactions, large and small, whereas decentralized control ignores them completely. In decentralized control the local agents have no

knowledge of each others' actions. It is well known that the nominal closed-loop system behavior under decentralized control can be arbitrarily poor (unstable) if the system interactions are not small. The following reviews provide general discussion of this and other performance issues involving decentralized control (Šiljak, 1991; Lunze, 1992; Larsson and Skogestad, 2000; Cui and Jacobsen, 2002).

The next level up in design complexity from decentralized control is noncooperative control. In this framework, the agents have interaction models and communicate at each iteration (Jia and Krogh, 2002; Motee and Sayyar-Rodsari, 2003; Dunbar and Murray, 2006). The advantage of noncooperative control over decentralized control is that the agents have accurate knowledge of the effects of all other agents on their local objectives. The basic issue to analyze and understand in this setup is the competition between the agents. Characterizing the noncooperative equilibrium is the subject of noncooperative game theory, and the impact of using that solution for feedback control is the subject of control theory. For example, Dunbar (2007) shows closed-loop stability for an extension of noncooperative MPC described in (Dunbar and Murray, 2006) that handles systems with interacting subsystem dynamics. The key assumptions are the existence of a stabilizing *decentralized* feedback law valid near the origin, and an inequality condition limiting the coupling between the agents.

Cooperative MPC was introduced by Venkat, Rawlings, and Wright (2007). They show that a receding horizon implementation of a cooperative game with any number of iterates of the local MPC controllers leads to closed-loop stability in the linear dynamics case. Venkat, Rawlings, and Wright (2006a,b) show that state estimation errors (output instead of state feedback) do not change the system closed-loop stability if the estimators are also asymptotically stable. Most of the theoretical results on cooperative MPC given in this chapter are presented in Venkat (2006) using an earlier, different notation. If implementable, this form of distributed MPC clearly has the best control properties. Although one can easily modify the agents' objective functions in a single large-scale process owned by a single company, this kind of modification may not be possible in other situations in which competing interests share critical infrastructure. The requirements of the many different classes of applications create exciting opportunities for continued research in this field.

## 6.7 Exercises

### Exercise 6.1: Three looks at solving the LQ problem

In the following exercise, you will write three codes to solve the LQR using Octave or MATLAB. The objective function is the LQR with mixed term

$$V = \frac{1}{2}\sum_{k=0}^{N-1}\left(x(k)'Qx(k) + u(k)'Ru(k) + 2x(k)'Mu(k)\right) + (1/2)x(N)'P_f x(N)$$

First, implement the method described in Section 6.1.1 in which you eliminate the state and solve the problem for the decision variable

$$\mathbf{u} = \{u(0), u(1), \ldots, u(N-1)\}$$

Second, implement the method described in Section 6.1.1 in which you do *not* eliminate the state and solve the problem for

$$\mathbf{z} = \{u(0), x(1), u(1), x(2), \ldots, u(N-1), x(N)\}$$

Third, use backward dynamic programming (DP) and the Riccati iteration to compute the closed-form solution for $u(k)$ and $x(k)$.

(a) Let

$$A = \begin{bmatrix} 4/3 & -2/3 \\ 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad C = \begin{bmatrix} -2/3 & 1 \end{bmatrix} \qquad x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$Q = C'C + 0.001I \qquad P_f = \Pi \qquad R = 0.001 \qquad M = 0$$

in which the terminal penalty, $P_f$ is set equal to $\Pi$, the steady-state cost to go. Compare the three solutions for $N = 5$. Plot $x(k)$, $u(k)$ versus time for the closed-loop system.

(b) Let $N = 50$ and repeat. Do any of the methods experience numerical problems generating an accurate solution? Plot the condition number of the matrix that is inverted in the first two methods versus $N$.

(c) Now consider the following unstable system

$$A = \begin{bmatrix} 27.8 & -82.6 & 34.6 \\ 25.6 & -76.8 & 32.4 \\ 40.6 & -122.0 & 51.9 \end{bmatrix} \qquad B = \begin{bmatrix} 0.527 & 0.548 \\ 0.613 & 0.530 \\ 1.06 & 0.828 \end{bmatrix} \qquad x(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Consider regulator tuning parameters and constraints

$$Q = I \qquad P_f = \Pi \qquad R = I \qquad M = 0$$

Repeat parts 6.1a and 6.1b for this system. Do you lose accuracy in any of the solution methods? What happens to the condition number of $H(N)$ and $S(N)$ as $N$ becomes large? Which methods are still accurate for this case? Can you explain what happened?

### Exercise 6.2: LQ as least squares

Consider the standard LQ problem

$$\min_{\mathbf{u}} V = \frac{1}{2}\sum_{k=0}^{N-1}\left(x(k)'Qx(k) + u(k)'Ru(k)\right) + (1/2)x(N)'P_f x(N)$$

subject to

$$x^+ = Ax + Bu$$

(a) Set up the dense Hessian least squares problem for the LQ problem with a horizon of three, $N = 3$. Eliminate the state equations and write out the objective function in terms of only the decision variables $u(0), u(1), u(2)$.

(b) What are the conditions for an optimum, i.e., what linear algebra problem do you solve to compute $u(0), u(1), u(2)$?

### Exercise 6.3: Lagrange multiplier method

Consider the general least squares problem

$$\min_{x} V(x) = \frac{1}{2}x'Hx + \text{const}$$

subject to

$$Dx = d$$

(a) What is the Lagrangian $L$ for this problem? What is the dimension of the Lagrange multiplier vector, $\lambda$?

(b) What are necessary and sufficient conditions for a solution to the optimization problem?

(c) Apply this approach to the LQ problem of Exercise 6.2 using the equality constraints to represent the model equations. What are $H, D, d$ for the LQ problem?

(d) Write out the linear algebra problem to be solved for the optimum.

(e) Contrast the two different linear algebra problems in these two approaches. Which do you want to use when $N$ is large and why?

### Exercise 6.4: Reparameterizing an unstable system

Consider again the LQR problem with cross term

$$\min_{\mathbf{u}} V = \frac{1}{2}\sum_{k=0}^{N-1}\left(x(k)'Qx(k) + u(k)'Ru(k) + 2x(k)'Mu(k)\right) + (1/2)x(N)'P_f x(N)$$

subject to

$$x^+ = Ax + Bu$$

and the three approaches of Exercise 6.1:

1. The method described in Section 6.1.1 in which you eliminate the state and solve the problem for the decision variable

$$\mathbf{u} = \{u(0), u(1), \ldots, u(N-1)\}$$

2. The method described in Section 6.1.1 in which you do *not* eliminate the state and solve the problem for

$$\mathbf{z} = \{u(0), x(1), u(1), x(2), \ldots, u(N-1), x(N)\}$$

3. The method of DP and the Riccati iteration to compute the closed-form solution for $u(k)$ and $x(k)$.

(a) You found that unstable $A$ causes numerical problems in the first method using large horizons. So let's consider a fourth method. Reparameterize the input in terms of a state feedback gain via

$$u(k) = Kx(k) + v(k)$$

in which $K$ is chosen so that $A + BK$ is a stable matrix. Consider the matrices in a transformed LQ problem

$$\min_{\mathbf{v}} V = \frac{1}{2}\sum_{k=0}^{N-1}\left(x(k)'\tilde{Q}x(k) + v(k)'\tilde{R}v(k) + 2x(k)'\tilde{M}v(k)\right) + (1/2)x(N)'\tilde{P}_f x(N)$$

subject to $x^+ = \tilde{A}x + \tilde{B}v$.

What are the matrices $\tilde{A}, \tilde{B}, \tilde{Q}, \tilde{P}_f, \tilde{R}, \tilde{M}$ such that the two problems give the same solution (state trajectory)?

(b) Solve the following problem using the first method and the fourth method and describe differences between the two solutions. Compare your results to the DP approach. Plot $x(k)$ and $u(k)$ versus $k$.

$$A = \begin{bmatrix} 27.8 & -82.6 & 34.6 \\ 25.6 & -76.8 & 32.4 \\ 40.6 & -122.0 & 51.9 \end{bmatrix} \qquad B = \begin{bmatrix} 0.527 & 0.548 \\ 0.613 & 0.530 \\ 1.06 & 0.828 \end{bmatrix} \qquad x(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Consider regulator tuning parameters and constraints

$$Q = P_f = I \qquad R = I \qquad M = 0 \qquad N = 50$$

### Exercise 6.5: Recursively summing quadratic functions

Consider generalizing Example 1.1 to an $N$-term sum. Let the $N$-term sum of quadratic functions be defined as

$$V(N, x) = \frac{1}{2}\sum_{i=1}^{N}(x - x(i))'X_i(x - x(i))$$

in which $x, x(i) \in \mathbb{R}^n$ are real $n$-vectors and $X_i \in \mathbb{R}^{n\times n}$ are positive definite matrices.

(a) Show that $V(N, x)$ can be found recursively

$$V(N, x) = (1/2)(x - v(N))'H(N)(x - v(N)) + \text{constant}$$

in which $v(i)$ and $H(i)$ satisfy the recursion

$$H(i+1) = H_i + X_{i+1} \qquad v(i+1) = H^{-1}(i+1)\left(H_i v_i + X_{i+1}x(i+1)\right)$$
$$H_1 = X_1 \qquad v_1 = x_1$$

Notice the recursively defined $v(m)$ and $H(m)$ provide the solutions and the Hessian matrices of the sequence of optimization problems

$$\min_x V(m, x) \qquad 1 \le m \le N$$

(b) Check your answer by solving the equivalent, but larger dimensional, constrained least squares problem (see Exercise 1.16)

$$\min_z (z - z_0)' \tilde{H} (z - z_0)$$

subject to

$$Dz = 0$$

in which $z, z_0 \in \mathbb{R}^{nN}$, $\tilde{H} \in \mathbb{R}^{nN \times nN}$ is a block diagonal matrix, $D \in \mathbb{R}^{n(N-1) \times nN}$

$$z_0 = \begin{bmatrix} x(1) \\ \vdots \\ x(N-1) \\ x(N) \end{bmatrix} \qquad \tilde{H} = \begin{bmatrix} X_1 & & & \\ & \ddots & & \\ & & X_{N-1} & \\ & & & X_N \end{bmatrix} \qquad D = \begin{bmatrix} I & -I & & \\ & \ddots & \ddots & \\ & & I & -I \end{bmatrix}$$

(c) Compare the size and number of matrix inverses required for the two approaches.

### Exercise 6.6: Why call the Lyapunov stability *nonuniform*?

Consider the following linear system

$$\begin{aligned} w^+ &= Aw \qquad & w(0) &= Hx(0) \\ x &= Cw \end{aligned}$$

with solution $w(k) = A^k w(0) = A^k H x(0)$, $x(k) = CA^k H x(0)$. Notice that $x(0)$ completely determines both $w(k)$ and $x(k)$, $k \geq 0$. Also note that zero is a solution, i.e., $x(k) = 0, k \geq 0$ satisfies the model.

(a) Consider the following case

$$A = \rho \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad H = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & -1 \end{bmatrix}$$

$$\rho = 0.925 \qquad \theta = \pi/4 \qquad x(0) = 1$$

Plot the solution $x(k)$. Does $x(k)$ converge to zero? Does $x(k)$ achieve zero exactly for finite $k > 0$?

(b) Is the zero solution $x(k) = 0$ Lyapunov stable? State your definition of Lyapunov stability, and prove your answer. Discuss how your answer is consistent with the special case considered above.

### Exercise 6.7: Exponential stability of suboptimal MPC with unbounded feasible set

Consider again Lemma 6.4 when both $\mathbb{U}$ and $\mathcal{X}_N$ are unbounded. Show that the suboptimal MPC controller is exponentially stable on the following sets.

(a) Any sublevel set of $V(x, \mathbf{h}(x))$

(b) Any compact subset of $\mathcal{X}_N$

### Exercise 6.8: A refinement to the warm start

Consider the following refinement to the warm start in the suboptimal MPC strategy. First add the requirement that the initialization strategy satisfies the following bound

$$\mathbf{h}(x) \le \bar{d}\,|x| \qquad x \in \mathcal{X}_N$$

in which $\bar{d} > 0$. Notice that all initializations considered in the chapter satisfy this requirement.

Then, at time $k$ and state $x$, in addition to the shifted input sequence from time $k-1$, $\widetilde{\mathbf{u}}$, evaluate the initialization sequence applied to the current state, $\mathbf{u} = \mathbf{h}(x)$. Select whichever of these two input sequence has lower cost as the warm start for time $k$. Notice also that this refinement makes the constraint

$$|\mathbf{u}| \le d\,|x| \qquad x \in r\mathcal{B}$$

redundant, and it can be removed from the MPC optimization.

Prove that this refined suboptimal strategy is exponentially stabilizing on the set $\mathcal{X}_N$. Notice that with this refinement, we do not have to assume that $\mathcal{X}_N$ is bounded or that $\mathbb{U}$ is bounded.

### Exercise 6.9: Exponential stability with mixed powers of the norm

Prove Lemma 6.5.

Hints: exponential convergence can be established as in standard exponential stability theorems. To establish Lyapunov stability, consider sublevel sets of the function $|x|^\sigma + |e|^\gamma$

$$L_\rho = \{(x,e)\,|\,|x|^\sigma + |e|^\gamma \le \rho\} \qquad \rho > 0$$

Choose the function $\overline{\rho}(R)$ to be the maximal $\rho$ value such that $L_{\overline{\rho}(R)} \subset B_R$. Similarly, choose $\overline{r}(\rho)$ to be the maximal $r$ value such that $B_{\overline{r}(\rho)} \subset L_\rho$. Use $\overline{\rho}$ and $\overline{r}$ to establish Lyapunov stability.

### Exercise 6.10: Decentralized control of Examples 6.8–6.10

Apply decentralized control to the systems in Examples 6.8–6.10. Which of these systems are closed-loop unstable with decentralized control? Compare this result to the result for noncooperative MPC.

### Exercise 6.11: Cooperative control of Examples 6.8–6.10

Apply cooperative MPC to the systems in Examples 6.8–6.10. Are any of these systems closed-loop unstable? Compare the closed-loop eigenvalues of converged cooperative control to centralized MPC, and discuss any differences.

### Exercise 6.12: Adding norms

Establish the following result used in the proof of Lemma 6.13. Given that $w \in \mathbb{R}^m$, $e \in \mathbb{R}^n$

$$\frac{1}{\sqrt{2}}(|w| + |e|) \le |(w,e)| \le |w| + |e| \qquad \forall w, e$$

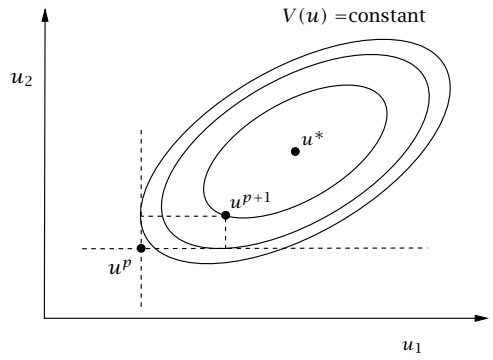


**Figure 6.8:** Optimizing a quadratic function in one set of variables at a time.

### Exercise 6.13: Padding matrices

Given a vector $\mathbf{z}$ and subvector $\mathbf{u}$

$$\mathbf{z} = \begin{bmatrix} u(0) \\ x(1) \\ u(1) \\ x(2) \\ \vdots \\ u(N-1) \\ x(N) \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(N-1) \end{bmatrix} \qquad x \in \mathbb{R}^n \qquad u \in \mathbb{R}^m$$

and quadratic function of $\mathbf{u}$

$$(1/2)\mathbf{u}'H\mathbf{u} + h'\mathbf{u}$$

Find the corresponding quadratic function of $\mathbf{z}$ so that

$$(1/2)\mathbf{z}'H_z\mathbf{z} + h_z'\mathbf{z} = (1/2)\mathbf{u}'H\mathbf{u} + h'\mathbf{u} \qquad \forall \mathbf{z}, \mathbf{u}$$

Hint: first find the padding matrix $E$ such that $\mathbf{u} = E\mathbf{z}$.

### Exercise 6.14: A matrix inverse

Compute the four partitioned elements in the two-player feedback gain $(I - L)^{-1}\overline{K}$

$$\mathbf{u}^\infty = (I - L)^{-1}\overline{K}x(0) \qquad |\operatorname{eig}(L)| < 1$$

in which

$$(I - L)^{-1}\overline{K} = \begin{bmatrix} I & -L_1 \\ -L_2 & I \end{bmatrix}^{-1} \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}$$

### Exercise 6.15: Optimizing one variable at a time

Consider the positive definite quadratic function partitioned into two sets of variables

$$V(u) = (1/2)u'Hu + c'u + d$$

$$V(u_1,u_2) = (1/2)\begin{bmatrix}u_1' & u_2'\end{bmatrix}\begin{bmatrix}H_{11} & H_{12}\\ H_{21} & H_{22}\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix} + \begin{bmatrix}c_1' & c_2'\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix} + d$$

in which $H > 0$. Imagine we wish to optimize this function by first optimizing over the $u_1$ variables holding $u_2$ fixed and then optimizing over the $u_2$ variables holding $u_1$ fixed as shown in Figure 6.8. Let's see if this procedure, while not necessarily efficient, is guaranteed to converge to the optimum.

(a) Given an initial point $(u_1^p, u_2^p)$, show that the next iteration is

$$\begin{aligned} u_1^{p+1} &= -H_{11}^{-1}\left(H_{12}u_2^p + c_1\right)\\ u_2^{p+1} &= -H_{22}^{-1}\left(H_{21}u_1^p + c_2\right) \end{aligned} \tag{6.29}$$

The procedure can be summarized as

$$u^{p+1} = Au^p + b \tag{6.30}$$

in which the iteration matrix $A$ and constant $b$ are given by

$$A = \begin{bmatrix}0 & -H_{11}^{-1}H_{12}\\ -H_{22}^{-1}H_{21} & 0\end{bmatrix} \qquad b = \begin{bmatrix}-H_{11}^{-1}c_1\\ -H_{22}^{-1}c_2\end{bmatrix} \tag{6.31}$$

(b) Establish that the optimization procedure converges by showing the iteration matrix is stable

$$|\operatorname{eig}(A)| < 1$$

(c) Given that the iteration converges, show that it produces the same solution as

$$u^* = -H^{-1}c$$

### Exercise 6.16: Monotonically decreasing cost

Consider again the iteration defined in Exercise 6.15.

(a) Prove that the cost function is monotonically decreasing when optimizing one variable at a time

$$V(u^{p+1}) < V(u^p) \qquad \forall\, u^p \neq -H^{-1}c$$

(b) Show that the following expression gives the size of the decrease

$$V(u^{p+1}) - V(u^p) = -(1/2)(u^p - u^*)'P(u^p - u^*)$$

in which

$$P = HD^{-1}\widetilde{H}D^{-1}H \qquad \widetilde{H} = D - N \qquad D = \begin{bmatrix}H_{11} & 0\\ 0 & H_{22}\end{bmatrix} \qquad N = \begin{bmatrix}0 & H_{12}\\ H_{21} & 0\end{bmatrix}$$

and $u^* = -H^{-1}c$ is the optimum.

Hint: to simplify the algebra, first change coordinates and move the origin of the coordinate system to $u^*$.

### Exercise 6.17: One variable at a time with convex step

Consider Exercise 6.15 but with the convex step for the iteration

$$\begin{bmatrix} u_1^{p+1} \\ u_2^{p+1} \end{bmatrix} = w_1 \begin{bmatrix} u_1^0(u_2^p) \\ u_2^p \end{bmatrix} + w_2 \begin{bmatrix} u_1^p \\ u_2^0(u_1^p) \end{bmatrix} \qquad 0 \le w_1, w_2 \quad w_1 + w_2 = 1$$

(a) Show that the iteration for the convex step is also of the form

$$u^{p+1} = Au^p + b$$

and the $A$ matrix and $b$ vector for this case are

$$A = \begin{bmatrix} w_2 I & -w_1 H_{11}^{-1} H_{12} \\ -w_2 H_{22}^{-1} H_{21} & w_1 I \end{bmatrix} \qquad b = \begin{bmatrix} -w_1 H_{11}^{-1} & \\ & -w_2 H_{22}^{-1} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

(b) Show that $A$ is stable.

(c) Show that this iteration also converges to $u^* = -H^{-1}c$.

### Exercise 6.18: Monotonically decreasing cost with convex step

Consider again the problem of optimizing one variable at a time with the convex step given in Exercise 6.17.

(a) Prove that the cost function is monotonically decreasing

$$V(u^{p+1}) < V(u^p) \qquad \forall u^p \neq -H^{-1}c$$

(b) Show that the following expression gives the size of the decrease

$$V(u^{p+1}) - V(u^p) = -(1/2)(u^p - u^*)' P (u^p - u^*)$$

in which

$$P = HD^{-1}\tilde{H}D^{-1}H \qquad \tilde{H} = D - N$$

$$D = \begin{bmatrix} w_1^{-1} H_{11} & 0 \\ 0 & w_2^{-1} H_{22} \end{bmatrix} \qquad N = \begin{bmatrix} -w_1^{-1} w_2 H_{11} & H_{12} \\ H_{21} & -w_1 w_2^{-1} H_{22} \end{bmatrix}$$

and $u^* = -H^{-1}c$ is the optimum.

Hint: to simplify the algebra, first change coordinates and move the origin of the coordinate system to $u^*$.

### Exercise 6.19: Splitting more than once

Consider the generalization of Exercise 6.15 in which we repeatedly decompose a problem into one-variable-at-a-time optimizations. For a three-variable problem we have the three optimizations

$$u_1^{p+1} = \arg\min_{u_1} V(u_1, u_2^p, u_3^p)$$

$$u_2^{p+1} = \arg\min_{u_2} V(u_1^p, u_2, u_3^p) \qquad u_3^{p+1} = \arg\min_{u_3} V(u_1^p, u_2^p, u_3)$$

Is it true that

$$V(u_1^{p+1}, u_2^{p+1}, u_3^{p+1}) \le V(u_1^p, u_2^p, u_3^p)$$

Hint: you may wish to consider the following example, $V(u) = (1/2)u'Hu + c'u$, in which

$$H = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \qquad c = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \qquad u^p = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

### Exercise 6.20: Time-varying controller iterations

We let $p_k \geq 0$ be a time-varying integer-valued index representing the iterations applied in the controller at time $k$.

$$\begin{aligned} x_1(k+1) &= A_1 x_1(k) + \overline{B}_{11} u_1(0;k) + \overline{B}_{12} u_2(0;k) \\ x_2(k+1) &= A_2 x_2(k) + \overline{B}_{21} u_1(0;k) + \overline{B}_{22} u_2(0;k) \\ \mathbf{u}_1(k+1) &= g_1^{p_k}(x_1(k), x_2(k), \mathbf{u}_1(k), \mathbf{u}_2(k)) \\ \mathbf{u}_2(k+1) &= g_2^{p_k}(x_1(k), x_2(k), \mathbf{u}_1(k), \mathbf{u}_2(k)) \end{aligned}$$

Notice the system evolution is time-varying even though the models are time invariant because we allow a time-varying sequence of controller iterations.

Show that cooperative MPC is exponentially stabilizing for any $p_k \geq 0$ sequence.

### Exercise 6.21: Stable interaction models

In some industrial applications it is preferable to partition the plant so that there are no unstable connections between subsystems. Any inputs $u_j$ that have unstable connections to outputs $y_i$ should be included in the $i$th subsystem inputs. Allowing an unstable connection between two subsystems may not be robust to faults and other kinds of system failures.[4] To implement this design idea in the two-player case, we replace Assumption 6.12 (b) with the following

**Assumption 6.12** (Constrained two-player game)**.**

(b') The interaction models $A_{ij}, i \neq j$ are stable.

Prove that Assumption 6.12 (b') implies Assumption 6.12 (b). It may be helpful to first prove the following lemma.

**Lemma 6.17** (Local detectability)**.** *Given partitioned system matrices*

$$\boldsymbol{A} = \begin{bmatrix} A & 0 \\ 0 & A_s \end{bmatrix} \qquad \boldsymbol{C} = \begin{bmatrix} C & C_s \end{bmatrix}$$

*in which $A_s$ is stable, the system $(\boldsymbol{A}, \boldsymbol{C})$ is detectable if and only if the system $(A, C)$ is detectable.*

Hint: use the Hautus lemma as the test for detectability.

Next show that this lemma and Assumption 6.12 (b') establishes the distributed detectability assumption 6.12 (b).

### Exercise 6.22: Norm constraints as linear inequalities

Consider the quadratic program (QP) in decision variable $u$ with parameter $x$

$$\begin{aligned} &\min_u (1/2) u' H u + x' D u \\ &\text{s.t. } Eu \leq Fx \end{aligned}$$

[4] We are not considering the common instability of base-level inventory management in this discussion. It is assumed that level control in storage tanks (integrators) is maintained at all times with simple, local level controllers. The internal unit flowrates dedicated for inventory management are not considered available inputs in the MPC problem.

in which $u \in \mathbb{R}^m$, $x \in \mathbb{R}^n$, and $H > 0$. The parameter $x$ appears linearly (affinely) in the cost function and constraints. Assume that we wish to add a norm constraint of the following form

$$|u|_\alpha \leq c\,|x|_\alpha \qquad \alpha = 2, \infty$$

(a) If we use the infinity norm, show that this problem can be posed as an equivalent QP with additional decision variables, and the cost function and constraints remain linear (affine) in parameter $x$. How many decision variables and constraints are added to the problem?

(b) If we use the two norm, show that this problem can be approximated by a QP whose solution does satisfy the constraints, but the solution may be suboptimal compared to the original problem.

### Exercise 6.23: Steady-state noncooperative game

Consider again the steady-state target problem for the system given in Example 6.11.

(a) Resolve the problem for the choice of convex step parameters $w_1 = 0.2$, $w_2 = 0.8$. Does the iteration for noncooperative control converge? Plot the iterations for the noncooperative and cooperative cases.

(b) Repeat for the convex step $w_1 = 0.8, w_2 = 0.2$. Are the results identical to the previous part? If not, discuss any differences.

(c) For what choices of $w_1, w_2$ does the target iteration converge using noncooperative control for the target calculation?

### Exercise 6.24: Optimality conditions for constrained optimization

Consider the convex quadratic optimization problem

$$\min_u V(u) \qquad \text{subject to} \quad u \in \mathbb{U}$$

in which $V$ is a convex quadratic function and $\mathbb{U}$ is a convex set. Show that $u^*$ is an optimal solution if and only if

$$\langle z - u^*, -\nabla\, V|_{u^*} \rangle \quad \leq 0 \qquad \forall z \in \mathbb{U} \tag{6.32}$$

Figure 6.9(a) depicts this condition for $u \in \mathbb{R}^2$. This condition motivates defining the normal cone (Rockafellar, 1970) to $\mathbb{U}$ at $u^*$ as follows

$$N(\mathbb{U}, u^*) = \{y \mid \langle z - u^*, y - u^* \rangle \leq 0 \quad \forall z \in \mathbb{U}\}$$

The optimality condition can be stated equivalently as $u^*$ is an optimal point if and only if the negative gradient is in the normal cone to $\mathbb{U}$ at $u^*$

$$-\nabla\, V|_{u^*} \in N(\mathbb{U}, u^*)$$

This condition and the normal cone are depicted in Figure 6.9(b).

### Exercise 6.25: Partitioned optimality conditions with constraints

Consider a partitioned version of the constrained optimization problem of Exercise 6.24 with uncoupled constraints

$$\min_{u_1, u_2} V(u_1, u_2) \qquad \text{subject to} \quad u_1 \in \mathbb{U}_1 \qquad u_2 \in \mathbb{U}_2$$

in which $V$ is a quadratic function and $\mathbb{U}_1$ and $\mathbb{U}_2$ are convex and nonempty.

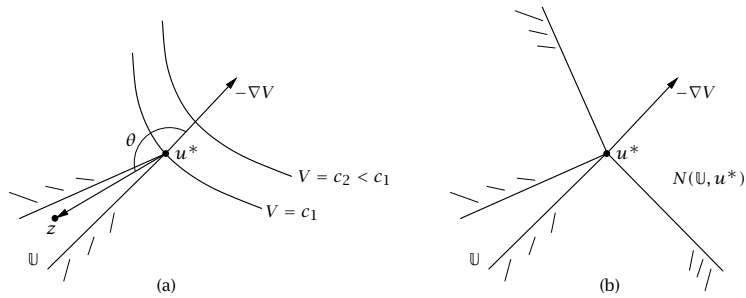


**Figure 6.9:** (a) Optimality of $u^*$ means the angle between $-\nabla V$ and any point $z$ in the feasible region must be greater than $90^\circ$ and less than $270^\circ$. (b) The same result restated: $u^*$ is optimal if and only if the negative gradient is in the normal cone to the feasible region at $u^*$, $-\nabla V|_{u^*} \in N(\mathbb{U}, u^*)$.

(a) Show that $(u_1^*, u_2^*)$ is an optimal solution if and only if

$$\begin{aligned} \langle z_1 - u_1^*, -\nabla_{u_1} V|_{(u_1^*, u_2^*)} \rangle &\leq 0 \qquad \forall z_1 \in \mathbb{U}_1 \\ \langle z_2 - u_2^*, -\nabla_{u_2} V|_{(u_1^*, u_2^*)} \rangle &\leq 0 \qquad \forall z_2 \in \mathbb{U}_2 \end{aligned} \tag{6.33}$$

(b) Extend the optimality conditions to cover the case

$$\min_{u_1,\ldots,u_M} V(u_1, \ldots, u_M) \qquad \text{subject to} \quad u_j \in \mathbb{U}_j \quad j = 1, \ldots, M$$

in which $V$ is a quadratic function and the $\mathbb{U}_j$ are convex and nonempty.

### Exercise 6.26: Constrained optimization of $M$ variables

Consider an optimization problem with $M$ variables and uncoupled constraints

$$\min_{u_1,u_2,\ldots,u_M} V(u_1, u_2, \ldots, u_M) \qquad \text{subject to} \quad u_l \in \mathbb{U}_j \quad j = 1, \ldots, M$$

in which $V$ is a strictly convex function. Assume that the feasible region is convex and nonempty and denote the unique optimal solution as $(u_1^*, u_2^*, \ldots, u_M^*)$ having cost $V^* = V(u_1^*, \ldots, u_M^*)$. Denote the $M$ one-variable-at-a-time optimization problems at iteration $k$

$$z_j^{p+1} = \arg\min_{u_j} V(u_1^p, \ldots, u_j, \ldots, u_M^p) \qquad \text{subject to } u_j \in \mathbb{U}_j$$

Then define the next iterate to be the following convex combination of the previous and new points

$$\begin{aligned} u_j^{p+1} &= \alpha_j^p z_j^{p+1} + (1 - \alpha_j^p) u_j^p \qquad j = 1, \ldots, M \\ \varepsilon \leq \alpha_j^p &< 1 \qquad 0 < \varepsilon \qquad j = 1, \ldots, M, \quad p \geq 1 \\ \sum_{j=1}^{M} \alpha_j^p &= 1, \qquad p \geq 1 \end{aligned}$$

Prove the following results.

(a) Starting with any feasible point, $(u_1^0, u_2^0, \ldots, u_M^0)$, the iterations $(u_1^p, u_2^p, \ldots, u_M^p)$ are feasible for $p \geq 1$.

(b) The objective function decreases monotonically from any feasible initial point

$$V(u_1^{p+1}, \ldots, u_M^{p+1}) \leq V(u_1^p, \ldots, u_M^p) \qquad \forall\, u_l^0 \in \mathbb{U}_j, j = 1, \ldots, M, \quad p \geq 1$$

(c) The cost sequence $V(u_1^p, u_2^p, \ldots, u_M^p)$ converges to the optimal cost $V^*$ from any feasible initial point.

(d) The sequence $(u_1^p, u_2^p, \ldots, u_M^p)$ converges to the optimal solution $(u_1^*, u_2^*, \ldots, u_M^*)$ from any feasible initial point.

### Exercise 6.27: The constrained two-variable special case

Consider the special case of Exercise 6.26 with $M = 2$

$$\min_{u_1, u_2} V(u_1, u_2) \qquad \text{subject to} \quad u_1 \in \mathbb{U}_1 \qquad u_2 \in \mathbb{U}_2$$

in which $V$ is a strictly positive quadratic function. Assume that the feasible region is convex and nonempty and denote the unique optimal solution as $(u_1^*, u_2^*)$ having cost $V^* = V(u_1^*, u_2^*)$. Consider the two one-variable-at-a-time optimization problems at iteration $k$

$$u_1^{p+1} = \arg\min_{u_1} V(u_1, u_2^p) \qquad u_2^{p+1} = \arg\min_{u_2} V(u_1^p, u_2)$$
$$\text{subject to } u_1 \in \mathbb{U}_1 \qquad \text{subject to } u_2 \in \mathbb{U}_2$$

We know from Exercise 6.15 that taking the full step in the unconstrained problem with $M = 2$ achieves a cost decrease. We know from Exercise 6.19 that taking the full step for an unconstrained problem with $M \geq 3$ does *not* provide a cost decrease in general. We know from Exercise 6.26 that taking a reduced step in the constrained problem for all $M$ achieves a cost decrease. That leaves open the case of a full step for a constrained problem with $M = 2$.

Does the full step in the constrained case for $M = 2$ guarantee a cost decrease? If so, prove it. If not, provide a counterexample.

### Exercise 6.28: Subsystem stability constraints

Show that the following uncoupled subsystem constraints imply an overall system constraint of the same type. The first is suitable for asymptotic stability and the second for exponential stability.

(a) Given $r_1, r_2 > 0$, and functions $\gamma_1$ and $\gamma_2$ of class $\mathcal{K}$, assume the following constraints are satisfied

$$|\mathbf{u}_1| \leq \gamma_1(|x_1|) \quad x_1 \in r_1\mathcal{B}$$
$$|\mathbf{u}_2| \leq \gamma_2(|x_2|) \quad x_2 \in r_2\mathcal{B}$$

Show that there exists $r > 0$ and function $\gamma$ of class $\mathcal{K}$ such that

$$|(\mathbf{u}_1, \mathbf{u}_2)| \leq \gamma(|(x_1, x_2)|) \qquad (x_1, x_2) \in r\mathcal{B} \tag{6.34}$$

(b) Given $r_1, r_2 > 0$, and constants $c_1, c_2, \sigma_1, \sigma_2 > 0$, assume the following constraints are satisfied

$$|\mathbf{u}_1| \leq c_1 |x_1|^{\sigma_1} \quad x_1 \in r_1 \mathcal{B}$$
$$|\mathbf{u}_2| \leq c_2 |x_2|^{\sigma_2} \quad x_2 \in r_2 \mathcal{B}$$

Show that there exists $r > 0$ and function $c, \sigma > 0$ such that

$$|(\mathbf{u}_1, \mathbf{u}_2)| \leq c \, |(x_1, x_2)|^{\sigma} \qquad (x_1, x_2) \in r\mathcal{B} \tag{6.35}$$

### Exercise 6.29: Distributed disturbance detectability

Prove Lemma 6.15.

Hint: use the Hautus lemma as the test for detectability.

### Exercise 6.30: Distributed target problem and uncoupled constraints

Player one's distributed target problem in the two-player game is given in (6.28)

$$\min_{x_{11s}, x_{21s}, u_{1s}} (1/2) \begin{bmatrix} H_1 y_{1s} - z_{1sp} \\ H_2 y_{2s} - z_{2sp} \end{bmatrix}' \begin{bmatrix} T_{1s} & \\ & T_{2s} \end{bmatrix} \begin{bmatrix} H_1 y_{1s} - z_{1sp} \\ H_2 y_{2s} - z_{2sp} \end{bmatrix}$$

subject to:

$$\begin{bmatrix} I - A_1 & & -\overline{B}_{11} & -\overline{B}_{12} \\ & I - A_2 & -\overline{B}_{21} & -\overline{B}_{22} \end{bmatrix} \begin{bmatrix} x_{1s} \\ x_{2s} \\ u_{1s} \\ u_{2s} \end{bmatrix} = \begin{bmatrix} B_{1d} \hat{d}_1(k) \\ B_{2d} \hat{d}_2(k) \end{bmatrix}$$

$$E_1 u_{1s} \leq e_1$$

Show that the constraints can be expressed so that the target problem constraints are uncoupled.

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# 7

# Explicit Control Laws for Constrained Linear Systems

## 7.1 Introduction

In preceding chapters we show how model predictive control (MPC) can be derived for a variety of control problems with constraints. It is interesting to recall the major motivation for MPC; solution of a *feedback* optimal control problem yielding a stabilizing control *law* is often prohibitively difficult. MPC sidesteps the problem of determining a stabilizing control *law* $\kappa(\cdot)$ by determining, instead, at each state $x$ encountered, a control *action* $u = \kappa(x)$ by solving a mathematical programming problem. This procedure, if repeated at *every* state $x$, yields an implicit control *law* $\kappa(\cdot)$ that solves the original feedback problem. In many cases, determining an explicit control law is impractical while solving a mathematical programming problem online for a given state is possible; this fact has led to the wide-scale adoption of MPC in the chemical process industry.

Some of the control problems for which MPC has been extensively used, however, have recently been shown to be amenable to analysis. One such problem is control of linear systems with polytopic constraints, for which determination of a stabilizing control law was thought in the past to be prohibitively difficult. It has been shown that it is possible, in principle, to determine a stabilizing control law for some of these control problems. Some authors have referred to these results as *explicit MPC* because they yield an explicit control law in contrast to MPC that yields a control action for each encountered state, thereby implicitly defining a control law. There are two objections to this terminology. First, determination of control laws for a wide variety of control problems has been the prime concern of control theory since its birth and certainly before the advent of MPC, an important tool

in this endeavor being dynamic programming (DP). These new results merely show that classical control-theoretic tools, such as DP, can be successfully applied to a wider range of problems than was previously thought possible. MPC is a useful method for implementing a control law that can, in principle, be determined using control-theoretic tools.

Second, some authors using this terminology have, perhaps inadvertently, implied that these results can be employed in place of conventional MPC. This is far from the truth, since only relatively simple problems, far simpler than those routinely solved in MPC applications, can be solved. That said, the results may be useful in applications where models with low state dimension, say six or less, are sufficiently accurate and where it is important that the control be rapidly computed. A previously determined control law generally yields the control action more rapidly than solving an optimal control problem. Potential applications include vehicle control.

In the next section we give a few simple examples of parametric programming. In subsequent sections we show how the solutions to parametric linear and quadratic programs may be obtained, and also show how these solutions may be used to solve optimal control problems when the system is linear, the cost quadratic or affine, and the constraints polyhedral.

## 7.2 Parametric Programming

A conventional optimization problem has the form $V^0 = \min_u\{V(u) \mid u \in \mathcal{U}\}$ where $u$ is the "decision" variable, $V(u)$ is the cost to be minimized, and $\mathcal{U}$ is the constraint set. The solution to a conventional optimization is a *point* or *set* in $\mathcal{U}$; the value $V^0$ of the problem satisfies $V^0 = V(u^0)$ where $u^0$ is the minimizer. A simple example of such a problem is $V^0 = \min_u\{a + bu + (1/2)cu^2 \mid u \in [-1, 1]\}$ where the solution is required for only *one* value of the parameters $a, b$ and $c$. The solution to this problem $u^0 = -b/c$ if $|b/c| \leq 1$, $u^0 = -1$ if $b/c \geq 1$ and $u^0 = 1$ if $b/c \leq -1$. This may be written more compactly as $u^0 = -\text{sat}(b/c)$ where $\text{sat}(\cdot)$ is the saturation function. The corresponding value is $V^0 = a - b^2/2c$ if $|b/c| \leq 1$, $V^0 = a - b + c^2/2$ if $b/c \geq 1$ and $V^0 = a + b + c^2/2$ if $b/c \leq -1$.

A parametric programming problem $\mathbb{P}(x)$ on the other hand, takes the form $V^0(x) = \min_u\{V(x, u) \mid u \in \mathcal{U}(x)\}$ where $x$ is a *parameter* so that the optimization problem, and its solution, depend on the value of the parameter. Hence, the solution to a parametric program-

ming problem $\mathbb{P}(x)$ is not a point or set but a *function* $x \mapsto u^0(x)$ that may be set valued; similarly the value of the problem is a function $x \mapsto V^0(x)$. At each $x$, the minimizer $u^0(x)$ may be a point or a set. Optimal control problems often take this form, with $x$ being the state, and $u$, in open-loop discrete time optimal control, being a control sequence; $u^0(x)$, the optimal control sequence, is a function of the initial state. In state feedback optimal control, necessary when uncertainty is present, DP is employed yielding a sequence of parametric optimization problems in each of which $x$ is the state and $u$ a control action; see Chapter 2. The programming problem in the first paragraph of this section may be regarded as a parametric programming problem with the parameter $x := (a, b, c)$, $V(x, u) := (x_1 + x_2u + (1/2)x_3u^2/2)$ and $\mathcal{U}(x) := [-1, 1]$; $\mathcal{U}(x)$, in this example, does not depend on $x$. The solution to this problem yields the functions $u^0(\cdot)$ and $V^0(\cdot)$ defined by $u^0(x) = -\text{sat}(x_2/x_3)$ and $V^0(x) = V(x, u^0(x)) = x_1 + x_2u^0(x) + (x_3/2)(u^0(x))^2$.

Because the minimizer and value of a parametric programming problem are *functions* rather than points or sets, we would not, in general, expect to be able to compute a solution. Surprisingly, parametric programs are relatively easily solved when the cost function $V(\cdot)$ is affine ($V(x, u) = a + b'x + c'u$) or quadratic ($V(x, u) = (1/2)x'Qx + x'Su + (1/2)u'Ru$) and $\mathcal{U}(x)$ is defined by a set of linear inequalities: $\mathcal{U}(x) = \{u \mid Mu \leq Nx + p\}$. The parametric constraint $u \in \mathcal{U}(x)$ may be conveniently expressed as $(x, u) \in \mathbb{Z}$ where $\mathbb{Z}$ is a subset of $(x, u)$-space which we will take to be $\mathbb{R}^n \times \mathbb{R}^m$; for each $x$

$$\mathcal{U}(x) = \{u \mid (x, u) \in \mathbb{Z}\}$$

We assume that $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. Let $\mathcal{X} \subset \mathbb{R}^n$ be defined by

$$\mathcal{X} := \{x \mid \exists u \text{ such that } (x, u) \in \mathbb{Z}\} = \{x \mid \mathcal{U}(x) \neq \emptyset\}$$

The set $\mathcal{X}$ is the domain of $V^0(\cdot)$ and $u^0(\cdot)$ and is thus the set of points $x$ for which a feasible solution of $\mathbb{P}(x)$ exists; it is the projection of $\mathbb{Z}$ (which is a set in $(x, u)$-space) onto $x$-space. See Figure 7.1, which illustrates $\mathbb{Z}$ and $\mathcal{U}(x)$ for the case when $\mathcal{U}(x) = \{u \mid Mu \leq Nx + p\}$; the set $\mathbb{Z}$ is thus defined by $\mathbb{Z} := \{(x, u) \mid Mu \leq Nx + p\}$. In this case, both $\mathbb{Z}$ and $\mathcal{U}(x)$ are polyhedral.

Before proceeding to consider parametric linear and quadratic programming, some simple examples may help the reader to appreciate the underlying ideas. Consider first a very simple parametric linear program $\min_u\{V(x, u) \mid (x, u) \in \mathbb{Z}\}$ where $V(x, u) := x + u$ and

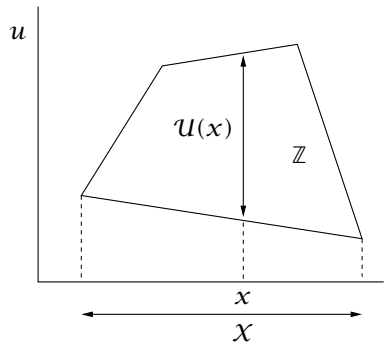


**Figure 7.1:** The sets $\mathbb{Z}$, $\mathcal{X}$ and $\mathcal{U}(x)$.

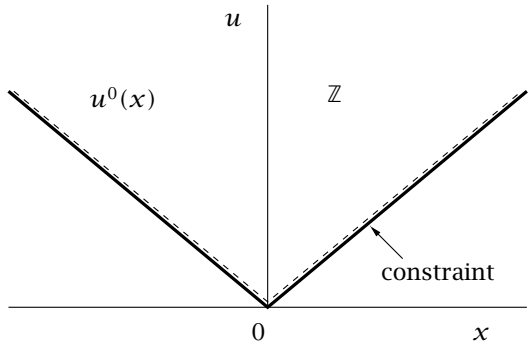


**Figure 7.2:** Parametric linear program.

$\mathbb{Z} := \{(x,u) \mid u + x \geq 0,\ u - x \geq 0\}$ so that $\mathcal{U}(x) = \{u \geq -x,\ u \geq x\}$. The problem is illustrated in Figure 7.2. The set $\mathbb{Z}$ is the region lying above the two solid lines $u = -x$ and $u = x$, and is convex. The gradient $\nabla_u V(x,u) = 1$ everywhere, so the solution, at each $x$, to the parametric program is the smallest $u$ in $\mathcal{U}(x)$, i.e., the smallest $u$ lying above the two lines $u = -x$ and $u = x$. Hence $u^0(x) = -x$ if $x \leq 0$ and $u^0(x) = x$ if $x \geq 0$, i.e., $u^0(x) = |x|$; the graph of $u^0(\cdot)$ is the dashed line in Figure 7.2. Both $u^0(\cdot)$ and $V^0(\cdot)$, where $V^0(x) = x + u^0(x)$, are *piecewise affine*, being affine in each of the two regions $X_1 := \{x \mid x \leq 0\}$ and $X_2 := \{x \mid x \geq 0\}$.

Next we consider an unconstrained parametric quadratic program $\min_u V(x,u)$ where $V(x,u) := (1/2)(x-u)^2 + u^2/2$. The problem is illustrated in Figure 7.3. For each $x \in \mathbb{R}$, $\nabla_u V(x,u) = -x + 2u$ and

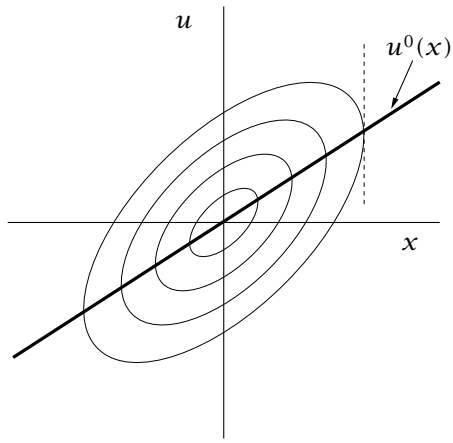


**Figure 7.3:** Unconstrained parametric quadratic program.

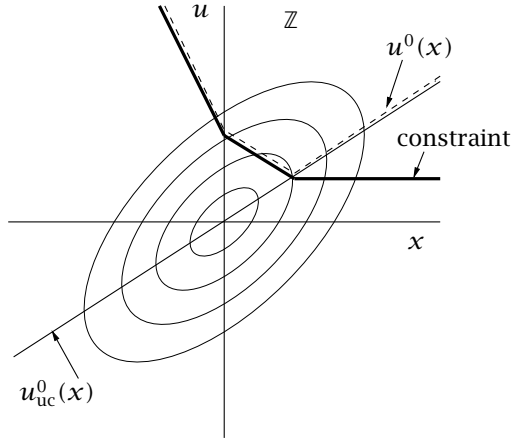


**Figure 7.4:** Parametric quadratic program.

$\nabla_{uu}V(x,u) = 2$ so that $u^0(x) = x/2$ and $V^0(x) = x^2/4$. Hence $u^0(\cdot)$ is linear and $V^0(\cdot)$ is quadratic in $\mathbb{R}$.

We now add the constraint set $\mathbb{Z} := \{(x,u) \mid u \geq 1,\ u + x/2 \geq 2,\ u + x \geq 2\}$; see Figure 7.4. The solution is defined on three regions, $X_1 := (-\infty, 0]$, $X_2 := [0,2]$, and $X_3 := [2,\infty)$. From the preceding example, the unconstrained minimum is achieved at $u^0_{\text{uc}}(x) = x/2$ shown by the solid straight line in Figure 7.4. Since $\nabla_u V(x,u) = -x + 2u$, $\nabla_u V(x,u) > 0$ for all $u > u^0_{\text{uc}}(x) = x/2$. Hence, in $X_1$, $u^0(x)$ lies

on the boundary of $\mathbb{Z}$ and satisfies $u^0(x) = 2 - x$. Similarly, in $X_2$, $u^0(x)$ lies on the boundary of $\mathbb{Z}$ and satisfies $u^0(x) = 2 - x/2$. Finally, in $X_3$, $u^0(x) = u^0_{\text{uc}}(x) = x/2$, the unconstrained minimizer, and lies in the interior of $\mathbb{Z}$ for $x > 1$. The third constraint $u \geq 2 - x$ is active in $X_1$, the second constraint $u \geq 2 - x/2$ is active in $X_2$, while no constraints are active in $X_3$. Hence the minimizer $u^0(\cdot)$ is piecewise affine, being affine in each of the regions $X_1$, $X_2$ and $X_3$. Since $V^0(x) = (1/2)|x - u^0(x)|^2 + u^0(x)^2/2$, the value function $V^0(\cdot)$ is piecewise quadratic, being quadratic in each of the regions $X_1$, $X_2$ and $X_3$.

We require, in the sequel, the following definitions:

**Definition 7.1** (Polytopic (polyhedral) partition)**.** A set $\mathcal{P} = \{\mathbb{Z}_i \mid i \in \mathcal{I}\}$, for some index set $\mathcal{I}$, is called a polytopic (polyhedral) partition of the polytopic (polyhedral) set $\mathbb{Z}$ if $\mathbb{Z} = \cup_{i\in\mathcal{I}}\mathbb{Z}_i$ and the sets $\mathbb{Z}_i$, $i \in \mathcal{I}$, are polytopes (polyhedrons) with nonempty interiors (relative to $\mathbb{Z}$)[1] that are nonintersecting: $\text{int}(\mathbb{Z}_i) \cap \text{int}(\mathbb{Z}_j) = \emptyset$ if $i \neq j$.

**Definition 7.2** (Piecewise affine function)**.** A function $f : \mathbb{Z} \to \mathbb{R}^m$ is said to be piecewise affine on a polytopic (polyhedral) partition $\mathcal{P} = \{\mathbb{Z}_i \mid i \in \mathcal{I}\}$ if it satisfies, for some $K_i$, $k_i$, $i \in \mathcal{I}$, $f(x) = K_i x + k_i$ for all $x \in \mathbb{Z}_i$, all $i \in \mathcal{I}$. Similarly, a function $f : \mathbb{Z} \to \mathbb{R}$ is said to be piecewise quadratic on a polytopic (polyhedral) partition $\mathcal{P} = \{\mathbb{Z}_i \mid i \in \mathcal{I}\}$ if it satisfies, for some $Q_i$, $r_i$, and $s_i$, $i \in \mathcal{I}$, $f(x) = (1/2)x'Q_i x + r_i' x + s_i$ for all $x \in \mathbb{Z}_i$, all $i \in \mathcal{I}$.

Note the piecewise affine and piecewise quadratic functions defined this way are not necessarily continuous and may, therefore, be set valued at the intersection of the defining polyhedrons. An example is the piecewise affine function $f(\cdot)$ defined by

$$\begin{aligned} f(x) &:= -x - 1 \quad x \in (-\infty, 0] \\ &:= x + 1 \quad x \in [0, \infty) \end{aligned}$$

This function is set valued at $x = 0$ where it has the value $f(0) = \{-1, 1\}$. We shall mainly be concerned with continuous piecewise affine and piecewise quadratic functions.

We now generalize the points illustrated by our example above and consider, in turn, parametric quadratic programming and parametric

[1] The interior of a set $S \subseteq \mathbb{Z}$ relative to the set $\mathbb{Z}$ is the set $\{z \in S \mid \varepsilon(z)\mathcal{B} \cap \text{aff}(\mathbb{Z}) \subseteq \mathbb{Z} \text{ for some } \varepsilon > 0\}$ where $\text{aff}(\mathbb{Z})$ is the intersection of all affine sets containing $\mathbb{Z}$.

linear programming and their application to optimal control problems. We deal with parametric quadratic programming first because it is more widely used and because, with reasonable assumptions, the minimizer is unique making the underlying ideas somewhat simpler to follow.

## 7.3 Parametric Quadratic Programming

### 7.3.1 Preliminaries

The parametric quadratic program $\mathbb{P}(x)$ is defined by

$$V^0(x) = \min_u \{V(x,u) | (x,u) \in \mathbb{Z}\}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. The cost function $V(\cdot)$ is defined by

$$V(x,u) := (1/2)x'Qx + u'Sx + (1/2)u'Ru + q'x + r'u + c$$

and the polyhedral constraint set $\mathbb{Z}$ is defined by

$$\mathbb{Z} := \{(x,u) \mid Mx \leq Nu + p\}$$

where $M \in \mathbb{R}^{r\times n}$, $N \in \mathbb{R}^{r\times m}$ and $p \in \mathbb{R}^r$; thus $\mathbb{Z}$ is defined by $r$ affine inequalities. Let $u^0(x)$ denote the solution of $\mathbb{P}(x)$ if it exists, i.e., if $x \in \mathcal{X}$, the domain of $V^0(\cdot)$; thus

$$u^0(x) := \arg\min_u \{V(x,u) \mid (x,u) \in \mathbb{Z}\}$$

The solution $u^0(x)$ is unique if $V(\cdot)$ is strictly convex in $u$; this is the case if $R$ is positive definite. Let the matrix $\mathcal{Q}$ be defined by

$$\mathcal{Q} := \begin{bmatrix} Q & S' \\ S & R \end{bmatrix}$$

For simplicity we assume, in the sequel:

**Assumption 7.3** (Strict convexity)**.** The matrix $\mathcal{Q}$ is positive definite.

Assumption 7.3 implies that both $R$ and $Q$ are positive definite. The cost function $V(\cdot)$ may be written in the form

$$V(x,u) = (1/2)(x,u)'\mathcal{Q}(x,u) + q'x + r'u + c$$

where, as usual, the vector $(x,u)$ is regarded as a column vector $(x',u')'$ in algebraic expressions. The parametric quadratic program may also be expressed as

$$V^0(x) := \min_u \{V(x,u) \mid u \in \mathcal{U}(x)\}$$

where the parametric constraint set $\mathcal{U}(x)$ is defined by

$$\mathcal{U}(x) := \{u \mid (x,u) \in \mathbb{Z}\} = \{u \in \mathbb{R}^m \mid Mu \leq Nx + p\}$$

For each $x$ the set $\mathcal{U}(x)$ is polyhedral. The domain $\mathcal{X}$ of $V^0(\cdot)$ and $u^0(\cdot)$ is defined by

$$\mathcal{X} := \{x \mid \exists u \in \mathbb{R}^m \text{ such that } (\mathrm{x},\mathrm{u}) \in \mathbb{Z}\} = \{x \mid \mathcal{U}(x) \neq \emptyset\}$$

For all $(x,u) \in \mathbb{Z}$, let the index set $I(x,u)$ specify the constraints that are *active* at $(x,u)$, i.e.,

$$I(x,u) := \{i \in \mathbb{I}_{1:r} \mid M_i u = N_i x + p_i\}$$

where $M_i$, $N_i$ and $p_i$ denote, respectively, the $i$th row of $M$, $N$ and $p$. Similarly, for any matrix or vector $A$ and any index set $I$, $A_I$ denotes the matrix or vector with rows $A_i$, $i \in I$. For any $x \in \mathcal{X}$, the indices set $I^0(x)$ specifies the constraints that are active at $(x, u^0(x))$, i.e.,

$$I^0(x) := I(x, u^0(x)) = \{i \in \mathbb{I}_{1:r} \mid M_i u^0(x) = N_i x + p_i\}$$

Since $u^0(x)$ is unique, $I^0(x)$ is well defined. Thus $u^0(x)$ satisfies the equation

$$M_x^0 u = N_x^0 x + p_x^0$$

where

$$M_x^0 := M_{I^0(x)},\ N_x^0 := N_{I^0(x)},\ p_x^0 := p_{I^0(x)} \tag{7.1}$$

### 7.3.2 Preview

We show in the sequel that $V^0(\cdot)$ is piecewise quadratic and $u^0(\cdot)$ piecewise affine on a polyhedral partition of $\mathcal{X}$, the domain of both these functions. To do this, we take an arbitrary point $x$ in $\mathcal{X}$, and show that $u^0(x)$ is the solution of an *equality* constrained quadratic program $\mathbb{P}(x) : \min_u \{V(x,u) \mid M_x^0 u = N_x^0 x + p_x^0\}$ in which the equality constraint is $M_x^0 u = N_x^0 x + p_x^0$. We then show that there is a polyhedral region $R_x^0 \subset \mathcal{X}$ in which $x$ lies and such that, for all $w \in R_x^0$, $u^0(w)$ is the solution of the equality constrained quadratic program $\mathbb{P}(w) : \min_u \{V(w,u) \mid M_x^0 u = N_x^0 w + p_x^0\}$ in which the equality constraints are the same as those for $\mathbb{P}(x)$. It follows that $u^0(\cdot)$ is affine and $V^0(\cdot)$ is quadratic in $R_x^0$. We then show that there are only a finite number of such polyhedral regions so that $u^0(\cdot)$ is piecewise affine, and $V^0(\cdot)$ piecewise quadratic, on a polyhedral partition of $\mathcal{X}$. To carry out this

program, we require a suitable characterization of optimality. We develop this in the next subsection. Some readers may prefer to jump to Proposition 7.8, which gives the optimality condition we employ in the sequel.

### 7.3.3 Optimality Condition for a Convex Program

Necessary and sufficient conditions for nonlinear optimization problems are developed in Section C.2 of Appendix C. Since we are concerned here with a relatively simple optimization problem where the cost is convex and the constraint set polyhedral, we give a self-contained exposition that uses the concept of a *polar cone*:

**Definition 7.4** (Polar cone)**.** The *polar cone* of a cone $C \subseteq \mathbb{R}^n$ is the cone $C^*$ defined by

$$C^* := \{g \in \mathbb{R}^n \mid \langle g, h\rangle \leq 0 \ \forall h \in C\}$$

We recall that a set $C \subseteq \mathbb{R}^n$ is a cone if $0 \in C$ and that $h \in C$ implies $\lambda h \in C$ for all $\lambda > 0$. A cone $C$ is said to be *generated* by $\{a_i \mid i \in I\}$ where $I$ is an index set if $C = \sum_{i\in I}\{\mu_i a_i \mid \mu_i \geq 0, i \in I\}$ in which case we write $C = \text{cone}\{a_i \mid i \in I\}$. We need the following result:

**Proposition 7.5** (Farkas's Lemma)**.** *Suppose $C$ is a polyhedral cone defined by*

$$C := \{h \mid Ah \leq 0\} = \{h \mid \langle a_i, h\rangle \leq 0 \mid i \in \mathbb{I}_{1:m}\}$$

*where, for each $i$, $a_i$ is the $i$th row of $A$. Then*

$$C^* = \text{cone}\{a_i \mid i \in \mathbb{I}_{1:m}\}$$

A proof of this result is given in Section C.2 of Appendix C; that $g \in \text{cone}\{a_i \mid i \in \mathbb{I}_{1:m}\}$ implies $\langle g, h\rangle \leq 0$ for all $h \in C$ is easily shown. An illustration of Proposition 7.5 is given in Figure 7.5.

Next we make use of a standard necessary and sufficient condition of optimality for optimization problems in which the cost is convex and differentiable and the constraint set is convex:

**Proposition 7.6** (Optimality conditions for convex set)**.** *Suppose, for each $x \in \mathcal{X}$, $u \mapsto V(x,u)$ is convex and differentiable and $\mathcal{U}(x)$ is convex. Then $u$ is optimal for $\min_u\{V(x,u) \mid u \in \mathcal{U}(x)\}$ if and only if*

$$u \in \mathcal{U}(x) \text{ and } \langle \nabla_u V(x,u), v - u\rangle \geq 0 \quad \forall v \in \mathcal{U}(x)$$

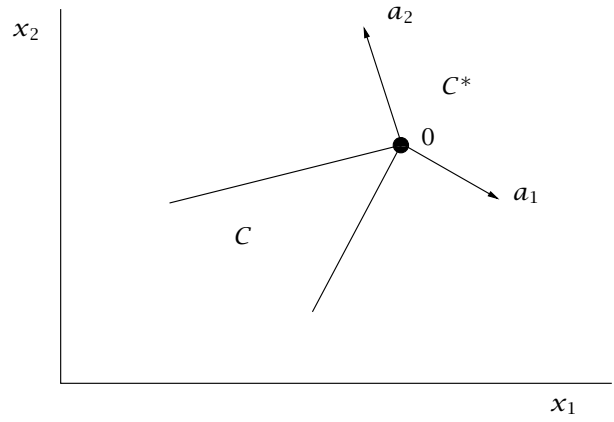


**Figure 7.5:** Polar cone.

*Proof.* This Proposition appears as Proposition C.9 in Appendix C where a proof is given. ■

In our case $\mathcal{U}(x)$, $x \in \mathcal{X}$, is polyhedral and is defined by

$$\mathcal{U}(x) := \{v \in \mathbb{R}^m \mid Mv \leq Nx + p\} \tag{7.2}$$

so $v \in \mathcal{U}(x)$ if and only if, for all $u \in \mathcal{U}(x)$, $v - u \in \mathcal{U}(x) - \{u\} := \{v - u \mid v \in \mathcal{U}(x)\}$. With $h := v - u$

$$\mathcal{U}(x)-\{u\} = \left\{ h \in \mathbb{R}^m \;\middle|\; \begin{array}{ll} M_i h \leq 0, & i \in I(x,u) \\ M_j h < N_j x + p_j - M_j u, & j \in \mathbb{I}_{1:r} \setminus I(x,u) \end{array} \right\}$$

since $M_i u = N_i x + p_i$ for all $i \in I(x,u)$. For each $z = (x,u) \in \mathbb{Z}$, let $C(x,u)$ denote the cone of first-order feasible variations of $u$; $C(x,u)$ is defined by

$$C(x,u) := \{h \in \mathbb{R}^m \mid M_i h \leq 0,\ i \in I(x,u)\}$$

Clearly

$$\mathcal{U}(x)-\{u\} = C(x,u) \cap \{h \in \mathbb{R}^m \mid M_i h < N_i x + p_i - M_i u,\ i \in \mathbb{I}_{1:r} \setminus I(x,u)\}$$

so that $\mathcal{U}(x) - \{u\} \subseteq C(x,u)$; for any $(x,u) \in \mathbb{Z}$, any $h \in C(x,u)$, there exists an $\alpha > 0$ such that $u + \alpha h \in \mathcal{U}(x)$. Proposition 7.6 may be expressed as: $u$ is optimal for $\min_u \{V(x,u) \mid u \in \mathcal{U}(x)\}$ if and only if

$$u \in \mathcal{U}(x) \text{ and } \langle \nabla_u V(x,u), h \rangle \geq 0 \quad \forall h \in \mathcal{U}(x) - \{u\}$$

We may now state a modified form of Proposition 7.6:

**Proposition 7.7** (Optimality conditions in terms of polar cone)**.** *Suppose for each $x \in \mathcal{X}$, $u \mapsto V(x,\cdot)$ is convex and differentiable, and $\mathcal{U}(x)$ is defined by* (7.2)*. Then $u$ is optimal for* $\min_u\{V(x,u) \mid u \in \mathcal{U}(x)\}$ *if and only if*

$$u \in \mathcal{U}(x) \text{ and } \langle \nabla_u V(x,u), h\rangle \geq 0 \quad \forall h \in C(x,u)$$

*Proof.* We show that the condition $\langle \nabla_u V(x,u), h\rangle \geq 0$ for all $h \in C(x,u)$ is equivalent to the condition $\langle \nabla_u V(x,u), h\rangle \geq 0$ for all $h \in \mathcal{U}(x) - \{u\}$ employed in Proposition 7.6. (i) Since $\mathcal{U}(x) - \{u\} \subseteq C(x,u)$, $\langle \nabla_u V(x,u), h\rangle \geq 0$ for all $h \in C(x,u)$ implies $\langle \nabla_u V(x,u), h\rangle \geq 0$ for all $h \in \mathcal{U}(x) - \{u\}$. (ii) $\langle \nabla_u V(x,u), h\rangle \geq 0$ for all $h \in \mathcal{U}(x) - \{u\}$ implies $\langle \nabla_u V(x,u), \alpha h\rangle \geq 0$ for all $h \in \mathcal{U}(x) - \{u\}$, all $\alpha > 0$. But, for any $h^* \in C(x,u)$, there exists an $\alpha \geq 1$ such that $h^* = \alpha h$ where $h := (1/\alpha)h^* \in \mathcal{U}(x) - \{u\}$. Hence $\langle \nabla_u V(x,u), h^*\rangle = \langle \nabla_u V(x,u), \alpha h\rangle \geq 0$ for all $h^* \in C(x,u)$. ∎

We now make use of Proposition 7.7 to obtain the optimality condition in the form we use in the sequel. For all $(x,u) \in \mathbb{Z}$, let $C^*(x,u)$ denote the polar cone to $C(x,u)$.

**Proposition 7.8** (Optimality conditions for linear inequalities)**.** *Suppose, for each $x \in \mathcal{X}$, $u \mapsto V(x,u)$ is convex and differentiable, and $\mathcal{U}(x)$ is defined by* (7.2)*. Then $u$ is optimal for* $\min_u\{V(x,u) \mid u \in \mathcal{U}(x)\}$ *if and only if*

$$u \in \mathcal{U}(x) \text{ and } -\nabla_u V(x,u) \in C^*(x,u) = \text{cone}\{M_i' \mid i \in I(x,u)\}$$

*Proof.* The desired result follows from a direct application of Proposition 7.5 to Proposition 7.7. ∎

Note that $C(x,u)$ and $C^*(x,u)$ are both cones so that each set contains the origin. In particular, $C^*(x,u)$ is generated by the gradients of the constraints active at $z = (x,u)$, and may be defined by a set of affine inequalities: for each $z \in \mathbb{Z}$, there exists a matrix $L_z$ such that

$$C^*(x,u) = C^*(z) = \{g \in \mathbb{R}^m \mid L_z g \leq 0\}$$

The importance of this result for us lies in the fact that the necessary and sufficient condition for optimality is satisfaction of two polyhedral constraints, $u \in \mathcal{U}(x)$ and $-\nabla_u V(x,u) \in C^*(x,u)$. Proposition 7.8 may also be obtained by direct application of Proposition C.12 of Appendix C; $C^*(x,u)$ may be recognized as $\mathcal{N}_{\mathcal{U}(x)}(u)$, the regular normal cone to the set $\mathcal{U}(x)$ at $u$.

### 7.3.4 Solution of the Parametric Quadratic Program

For the parametric programming problem $\mathbb{P}(x)$, the parametric cost function is

$$V(x,u) := (1/2)x'Qx + u'Sx + (1/2)u'Ru + q'x + r'u + c$$

and the parametric constraint set is

$$\mathcal{U}(x) := \{u \mid Mu \leq Nx + p\}$$

Hence, the cost gradient is

$$\nabla_u V(x,u) = Ru + Sx + r$$

where, because of Assumption 7.3, $R$ is positive definite. Hence, the necessary and sufficient condition for the optimality of $u$ for the parametric quadratic program $\mathbb{P}(x)$ is

$$\begin{aligned} &Mu \leq Nx + p \\ &-(Ru + Sx + r) \in C^*(x,u) \end{aligned}$$

where $C^*(x,u) = \text{cone}\{M_i' \mid i \in I(x,u)\}$, the cone generated by the gradients of the active constraints, is polyhedral. We cannot use this characterization of optimality directly to solve the parametric programming problem since $I(x,u)$ and, hence, $C^*(x,u)$, varies with $(x,u)$. Given any $x \in \mathcal{X}$, however, there exists the possibility of a region containing $x$ such that $I^0(x) \subseteq I^0(w)$ for all $w$ in this region. We make use of this observation as follows. It follows from the definition of $I^0(x)$ that the unique solution $u^0(x)$ of $\mathbb{P}(x)$ satisfies the equation

$$\begin{aligned} &M_i u = N_i x + p_i, \quad i \in I^0(x), \text{ i.e.,} \\ &M_x^0 u = N_x^0 x + p_x^0 \end{aligned}$$

where $M_x^0$, $N_x^0$ and $p_x^0$ are defined in (7.1). Hence $u^0(x)$ is the solution of the equality constrained problem

$$V^0(x) = \min_u \{V(x,u) \mid M_x^0 u = N_x^0 x + p_x^0\}$$

If the active constraint set remains constant near the point $x$ or, more precisely, if $I^0(x) \subseteq I^0(w)$ for all $w$ in some region in $\mathbb{R}^n$ containing $x$, then, for all $w$ in this region, $u^0(w)$ satisfies the equality constraint

$M_x^0 u = N_x^0 w + p_x^0$. This motivates us to consider the simple equality constrained problem $\mathbb{P}_x(w)$ defined by

$$V_x^0(w) = \min_u \{V(w,u) \mid M_x^0 u = N_x^0 w + p_x^0\}$$
$$u_x^0(w) = \arg\min_u \{V(w,u) \mid M_x^0 u = N_x^0 w + p_x^0\}$$

The subscript $x$ indicates that the equality constraints in $\mathbb{P}_x(w)$ depend on $x$. Problem $\mathbb{P}_x(w)$ is an optimization problem with a quadratic cost function and linear equality constraints and is, therefore, easily solved; see the exercises at the end of this chapter. Its solution is

$$V_x^0(w) = (1/2)w'Q_x w + r_x' w + s_x \tag{7.3}$$
$$u_x^0(w) = K_x w + k_x \tag{7.4}$$

for all $w$ such that $I^0(w) = I^0(x)$ where $Q_x \in \mathbb{R}^{n\times n}$, $r_x \in \mathbb{R}^n$, $s_x \in \mathbb{R}$, $K_x \in \mathbb{R}^{m\times n}$ and $k_x \in \mathbb{R}^m$ are easily determined. Clearly, $u_x^0(x) = u^0(x)$; but, is $u_x^0(w)$, the optimal solution to $\mathbb{P}_x(w)$, the optimal solution $u^0(w)$ to $\mathbb{P}(w)$ in some region containing $x$ and, if it is, what is the region? Our optimality condition answers this question. For all $x \in \mathcal{X}$, let the region $R_x^0$ be defined by

$$R_x^0 := \left\{ w \;\middle|\; \begin{matrix} u_x^0(w) \in \mathcal{U}(w) \\ -\nabla_u V(w, u_x^0(w)) \in C^*(x, u^0(x)) \end{matrix} \right\} \tag{7.5}$$

Because of the equality constraint $M_x^0 u = N_x^0 w + p_x^0$ in problem $\mathbb{P}_x(w)$, it follows that $I(w, u_x^0(w)) \supseteq I(x, u^0(x))$, and that $C(w, u_x^0(w)) \subseteq C(x, u^0(x))$ and $C^*(w, u_x^0(w)) \supseteq C^*(x, u^0(x))$ for all $w \in R_x^0$. Hence $w \in R_x^0$ implies $u_x^0(w) \in \mathcal{U}(w)$ and $-\nabla_u V(w, u_x^0(w)) \in C^*(w, u_x^0(w))$ for all $w \in R_x^0$ which, by Proposition 7.8, is a necessary and sufficient condition for $u_x^0(w)$ to be optimal for $\mathbb{P}(w)$. In fact, $I(w, u_x^0(w)) = I(x, u^0(x))$ so that $C^*(w, u_x^0(w)) = C^*(x, u^0(x))$ for all $w$ in the interior of $R_x^0$. The obvious conclusion of this discussion is

**Proposition 7.9** (Solution of $\mathbb{P}(w)$, $w \in R_x^0$)**.** *For any $x \in \mathcal{X}$, $u_x^0(w)$ is optimal for $\mathbb{P}(w)$ for all $w \in R_x^0$.*

The constraint $u_x^0(w) \in \mathcal{U}(w)$ may be expressed as

$$M(K_x w + k_x) \leq Nw + p$$

which is a linear inequality in $w$. Similarly, since $\nabla_u V(w,u) = Ru + Sw + r$ and since $C^*(x, u^0(x)) = \{g \mid L_x^0 g \leq 0\}$ where $L_x^0 = L_{(x,u^0(x))}$, the constraint $-\nabla_u V(x, u_x^0(w)) \in C(x, u^0(x))$ may be expressed as

$$-L_x^0(R(K_x w + k_x) + Sw + r) \leq 0$$

which is also an affine inequality in the variable $w$. Thus, for each $x$, there exists a matrix $F_x$ and vector $f_x$ such that

$$R_x^0 = \{w \mid F_x w \leq f_x\}$$

so that $R_x^0$ is polyhedral. Since $u_x^0(x) = u^0(x)$, it follows that $u_x^0(x) \in \mathcal{U}(x)$ and $-\nabla_u V(x, u_x^0(x)) \in C^*(x, u^0(x))$ so that $x \in R_x^0$.

Our next task is to bound the number of distinct regions $R_x^0$ that exist as we permit $x$ to range over $\mathcal{X}$. We note, from its definition, that $R_x^0$ is determined, through the constraint $M_x^0 u = N_x^0 w + p_x^0$ in $\mathbb{P}_x(w)$, through $u_x^0(\cdot)$ and through $C^*(x, u^0(x))$, by $I^0(x)$ so that $R_{x_1}^0 \neq R_{x_2}^0$ implies that $I^0(x_1) \neq I^0(x_2)$. Since the number of subsets of $\{1, 2, \ldots, p\}$ is finite, the number of distinct regions $R_x^0$ as $x$ ranges over $\mathcal{X}$ is finite. Because each $x \in \mathcal{X}$ lies in the set $R_x^0$, there exists a discrete set of points $X \subset \mathcal{X}$ such that $\mathcal{X} = \cup\{R_x^0 \mid x \in X\}$. We have proved:

**Proposition 7.10** (Piecewise quadratic (affine) cost (solution))**.**

*(a) There exists a set $X$ of a finite number of points in $\mathcal{X}$ such that $\mathcal{X} = \cup\{R_x^0 \mid x \in X\}$ and $\{R_x^0 \mid x \in X\}$ is a polyhedral partition of $\mathcal{X}$.*

*(b) The value function $V^0(\cdot)$ of the parametric piecewise quadratic program $\mathbb{P}$ is piecewise quadratic in $\mathcal{X}$, being quadratic and equal to $V_x^0(\cdot)$, defined in* (7.3) *in each polyhedron $R_x$, $x \in X$. Similarly, the minimizer $u^0(\cdot)$ is piecewise affine in $\mathcal{X}$, being affine and equal to $u_x^0(\cdot)$ defined in* (7.4) *in each polyhedron $R_x^0$, $x \in X$.*

**Example 7.11: Parametric QP**

Consider the example in Section 7.2. This may be expressed as

$$V^0(x) = \min_u V(x, u), \quad V(x, u) := \{(1/2)x^2 - ux + u^2 \mid Mu \leq Nx + p\}$$

where

$$M = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}, \quad N = \begin{bmatrix} 0 \\ 1/2 \\ 1 \end{bmatrix}, \quad p = \begin{bmatrix} -1 \\ -2 \\ -2 \end{bmatrix}$$

At $x = 1$, $u^0(x) = 3/2$ and $I^0(x) = \{2\}$. The equality constrained optimization problem $\mathbb{P}_x(w)$ is

$$V_x^0(w) = \min_u \{(1/2)w^2 - uw + u^2 \mid -u = (1/2)w - 2\}$$

so that $u^0(w) = 2 - w/2$. Hence

$$R_x^0 := \left\{ w \;\middle|\; \begin{matrix} Mu_x^0(w) \le Nw + p(w) \\ -\nabla_u V(w, u_x^0(w)) \in C^*(x, u^0(x)) \end{matrix} \right\}$$

Since $M_2 = -1$, $C^*(x) = \text{cone}\{M_i' \mid i \in I^0(x)\} = \text{cone}\{M_2'\} = \{h \in \mathbb{R} \mid h \le 0\}$; also

$$\nabla_u V(w, u_x^0(w)) = -w + 2u^0(w) = -w + 2(2 - w/2) = -2w + 4$$

so that $R_x^0$ is defined by the following inequalities:

$$\begin{aligned} (1/2)w - 2 &\le -1 && \text{or } w \le 2 \\ (1/2)w - 2 &\le (1/2)w - 2 && \text{or } w \in \mathbb{R} \\ (1/2)w - 2 &\le w - 2 && \text{or } w \ge 0 \\ 2w - 4 &\le 0 && \text{or } w \le 2 \end{aligned}$$

which reduces to $w \in [0, 2]$ so $R_x^0 = [0, 2]$ when $x = 1$; $[0, 2]$ is the set $X_2$ determined in Section 7.2. □

**Example 7.12: Explicit optimal control**

We return to the MPC problem presented in Example 2.5 of Chapter 2

$$\begin{aligned} V^0(x, \mathbf{u}) &= \min_{\mathbf{u}} \{V(x, \mathbf{u}) \mid \mathbf{u} \in \mathcal{U}\} \\ V(x, \mathbf{u}) &:= (3/2)x^2 + [2x, x]\mathbf{u} + (1/2)\mathbf{u}' H \mathbf{u} \\ H &:= \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \\ \mathcal{U} &:= \{\mathbf{u} \mid M\mathbf{u} \le p\} \end{aligned}$$

where

$$M := \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}, \qquad p := \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

It follows from the solution to Example 2.5 that

$$u^0(2) = \begin{bmatrix} -1 \\ -(1/2) \end{bmatrix}$$

and $I^0(x) = \{2\}$. The equality constrained optimization problem at $x = 2$ is

$$V_x^0(w) = \min_{\mathbf{u}} \{(3/2)w^2 + 2wu_1 + wu_2 + (1/2)\mathbf{u}' H \mathbf{u} \mid u_1 = -1\}$$

so that

$$u_x^0(w) = \begin{bmatrix} -1 \\ (1/2) - (1/2)w \end{bmatrix}$$

Hence $u_x^0(2) = [-1, -1/2]' = u^0(2)$ as expected. Since $M_x^0 = M_2 = [-1, 0]$, $C^*(x, u^0(x)) = \{g \in \mathbb{R}^2 \mid g_1 \le 0\}$. Also

$$\nabla_{\mathbf{u}} V(w, \mathbf{u}) = \begin{bmatrix} 2w + 3u_1 + u_2 \\ w + u_1 + 2u_2 \end{bmatrix}$$

so that

$$\nabla_{\mathbf{u}} V(w, \mathbf{u}_x^0(w)) = \begin{bmatrix} (3/2)w - (5/2) \\ 0 \end{bmatrix}$$

Hence $R_x^0$, $x = 2$ is the set of $w$ satisfying the following inequalities

$$\begin{aligned} (1/2) - (1/2)w &\le 1 \quad \text{or } w \ge -1 \\ (1/2) - (1/2)w &\ge -1 \text{ or } w \le 3 \\ -(3/2)w + (5/2) &\le 0 \quad \text{or } w \ge (5/3) \end{aligned}$$

which reduces to $w \in [5/3, 3]$; hence $R_x^0 = [5/3, 3]$ when $x = 2$ as shown in Example 2.5. □

### 7.3.5 Continuity of $V^0(\cdot)$ and $u^0(\cdot)$

Continuity of $V^0(\cdot)$ and $u^0(\cdot)$ follows from Theorem C.34 in Appendix C. We present here a simpler proof, however, based on the above analysis. We use the fact that the parametric quadratic problem is strictly convex, i.e., for each $x \in \mathcal{X}$, $u \mapsto V(x, u)$ is strictly convex and $\mathcal{U}(x)$ is convex, so that the minimizer $u^0(x)$ is unique as shown in Proposition C.8 of Appendix C.

Let $X = \{x_i \mid i \in \mathbb{I}_{1:I}\}$ denote the set defined in Proposition 7.10(a). For each $i \in \mathbb{I}_{i:I}$, let $R_i := R_{x_i}^0$, $V_i(\cdot) := V_{x_i}^0(\cdot)$ and $u_i(\cdot) := u_{x_i}^0(\cdot)$. From Proposition 7.10, $u^0(x) = u_i(x)$ for each $x \in R_i$, each $i \in \mathbb{I}_{1:I}$ so that $u^0(\cdot)$ is affine and hence continuous in the interior of each $R_i$, and also continuous at any point $x$ on the boundary of $\mathcal{X}$ such that $x$ lies in a single region $R_i$. Consider now a point $x$ lying in the intersection of several regions, $x \in \cap_{i \in J} R_i$, where $J$ is a subset of $\mathbb{I}_{1:I}$. Then, by Proposition 7.10, $u_i(x) = u^0(x)$ for all $x \in \cap_{i \in J} R_i$, all $i \in J$. Each $u_i(\cdot)$ is affine and, therefore, continuous, so that $u^0(\cdot)$ is continuous in $\cap_{i \in J} R_i$. Hence $u^0(\cdot)$ is continuous in $\mathcal{X}$. Because $V(\cdot)$ is continuous and $u^0(\cdot)$ is continuous in $\mathcal{X}$, the value function

$V^0(\cdot)$ defined by $V^0(x) = V(x, u^0(x))$ is also continuous in $\mathcal{X}$. Let $S$ denote any bounded subset of $\mathcal{X}$. Then, since $V^0(x) = V_i(x) = (1/2)x'Q_i x + r_i' x + s_i$ for all $x \in R_i$, all $i \in \mathbb{I}_{1:I}$ where $Q_i := Q_{x_i}$, $r_i := r_{x_i}$ and $s_i := s_{x_i}$, it follows that $V^0(\cdot)$ is Lipschitz continuous in each set $R_i \cap S$ and, hence, Lipschitz continuous in $\mathcal{X} \cap S$. We have proved the following.

**Proposition 7.13** (Continuity of cost and solution)**.** *The value function $V^0(\cdot)$ and the minimizer $u^0(\cdot)$ are continuous in $\mathcal{X}$. Moreover, the value function and the minimizer are Lipschitz continuous on bounded sets.*

## 7.4 Constrained Linear Quadratic Control

We now show how parametric quadratic programming may be used to solve the optimal receding horizon control problem when the system is linear, the constraints polyhedral, and the cost is quadratic. The system is described, as before, by

$$x^+ = Ax + Bu \tag{7.6}$$

and the constraints are, as before,

$$x \in \mathbb{X}, \quad u \in \mathbb{U} \tag{7.7}$$

where $\mathbb{X}$ is a polyhedron containing the origin in its interior and $\mathbb{U}$ is a polytope also containing the origin in its interior. There may be a terminal constraint of the form

$$x(N) \in \mathbb{X}_f \tag{7.8}$$

where $\mathbb{X}_f$ is a polyhedron containing the origin in its interior. The cost is

$$V_N(x, \mathbf{u}) = \left[ \sum_{i=0}^{N-1} \ell(x(i), u(i)) \right] + V_f(x(N)) \tag{7.9}$$

in which, for all $i$, $x(i) = \phi(i; x, \mathbf{u})$, the solution of (7.6) at time $i$ if the initial state at time 0 is $x$ and the control sequence is $\mathbf{u} := \{u(0), u(1), \ldots, u(N-1)\}$. The functions $\ell(\cdot)$ and $V_f(\cdot)$ are quadratic

$$\ell(x, u) := (1/2)x'Qx + (1/2)u'Ru, \quad V_f(x) := (1/2)x'Q_f x \tag{7.10}$$

The state and control constraints (7.7) induce, via the difference equation (7.6), an implicit constraint $(x, \mathbf{u}) \in \mathcal{Z}$ where

$$\mathcal{Z} := \{(x, \mathbf{u}) \mid x(i) \in \mathbb{X},\ u(i) \in \mathbb{U}, i \in \mathbb{I}_{0:N-1}, x(N) \in \mathbb{X}_f\} \tag{7.11}$$

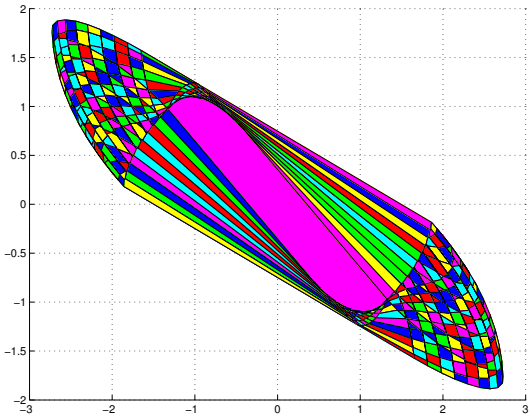


**Figure 7.6:** Regions $R_x$, $x \in X$ for a second-order example; from Mayne and Raković (2003)

where, for all $i$, $x(i) = \phi(i; x, \mathbf{u})$. It is easily seen that $\mathbb{Z}$ is polyhedral since, for each $i$, $x(i) = A^i x + M_i \mathbf{u}$ for some matrix $M_i$ in $\mathbb{R}^{n \times Nm}$; here $\mathbf{u}$ is regarded as the column vector $\begin{bmatrix} u(0)' & u(1)' & \cdots & u(N-1)' \end{bmatrix}'$. Clearly $x(i) = \phi(i; x, \mathbf{u})$ is linear in $(x, \mathbf{u})$. The constrained linear optimal control problem may now be defined by

$$V_N^0(x) = \min_{\mathbf{u}} \{V_N(x, \mathbf{u}) \mid (x, \mathbf{u}) \in \mathbb{Z}\}$$

Using the fact that for each $i$, $x(i) = A^i x + M_i \mathbf{u}$, it is possible to determine matrices $\mathbf{Q} \in \mathbb{R}^{n \times n}$, $\mathbf{R} \in \mathbb{R}^{Nm \times Nm}$, and $\mathbf{S} \in \mathbb{R}^{Nm \times n}$ such that

$$V_N(x, \mathbf{u}) = (1/2) x' \mathbf{Q} x' + (1/2) \mathbf{u}' \mathbf{R} \mathbf{u} + \mathbf{u}' \mathbf{S} x \tag{7.12}$$

Similarly, as shown above, there exist matrices $\mathbf{M}$, $\mathbf{N}$ and a vector $\mathbf{p}$ such that

$$\mathbb{Z} = \{(x, \mathbf{u}) \mid \mathbf{M}\mathbf{u} \leq \mathbf{N}x + \mathbf{p}\} \tag{7.13}$$

This is precisely the parametric problem studied in Section 7.3, so that the solution $\mathbf{u}^0(x)$ to $\mathbb{P}(x)$ is piecewise affine on a polytopic partition $\mathcal{P} = \{R_x \mid x \in X\}$ of $\mathcal{X}$ the projection of $\mathbb{Z} \subset \mathbb{R}^n \times \mathbb{R}^{Nm}$ onto $\mathbb{R}^n$, being affine in each of the constituent polytopes of $\mathcal{P}$. The receding horizon control law is $x \mapsto u^0(0; x)$, the first element of $\mathbf{u}^0(x)$. Two examples are shown in Figures 7.6 and 7.7.

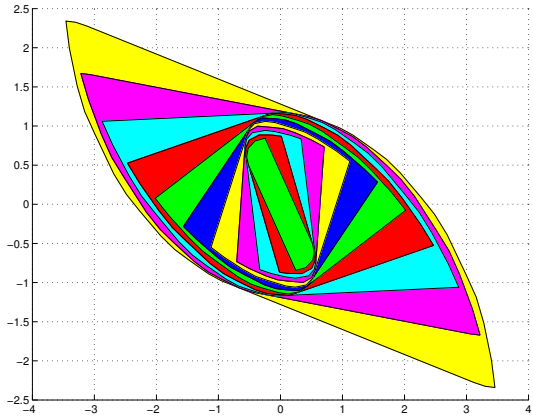


**Figure 7.7:** Regions $R_x,\ x \in X$ for a second-order example.

## 7.5 Parametric Piecewise Quadratic Programming

The parametric quadratic program $\mathbb{P}(x)$ is defined, as before, by

$$V^0(x) = \min_u \{V(x,u) \mid (x,u) \in \mathbb{Z}\} \tag{7.14}$$

where $x \in \mathcal{X} \subset \mathbb{R}^n$ and $u \in \mathbb{R}^m$, but now the cost function $V(\cdot)$ is assumed to be continuous, strictly convex, and piecewise quadratic on a polytopic partition $\mathcal{P} = \{\mathbb{Z}_i \mid i \in \mathcal{I}\}$ of the set $\mathbb{Z}$ so that

$$V(z) = V_i(z) = (1/2)z'\mathcal{Q}_i z + s_i' z + c_i$$

for all $z \in \mathbb{Z}_i$, all $i \in \mathcal{I}$ where $\mathcal{I}$ is an index set. In (7.14), the matrix $Q_i$ and the vector $s_i$ have the structure

$$\mathcal{Q}_i = \begin{bmatrix} Q_i & S_i' \\ S_i & R_i \end{bmatrix} \qquad s_i = \begin{bmatrix} q_i \\ r_i \end{bmatrix}$$

so that for all $i \in \mathcal{I}$,

$$V_i(x,u) = (1/2)x'Q_i x + u'S_i x + (1/2)u'R_i u + q_i' x + r_i' u + c$$

For each $x$, the function $u \mapsto V_i(x,u)$ is quadratic and depends on $x$. The constraint set $\mathbb{Z}$ is defined, as above, by

$$\mathbb{Z} := \{(x,u) \mid Mu \le Nx + p\}$$

Let $u^0(x)$ denote the solution of $\mathbb{P}(x)$, i.e.,

$$u^0(x) = \arg\min_u \{V(x,u) \mid (x,u) \in \mathbb{Z}\}$$

The solution $u^0(x)$ is unique if $V(\cdot)$ is strictly convex in $u$ at each $x$; this is the case if each $R_i$ is positive definite. The parametric piecewise quadratic program may also be expressed, as before, as

$$V^0(x) = \min_u \{V(x,u) \mid u \in \mathcal{U}(x)\}$$
$$u^0(x) = \arg\min_u \{V(x,u) \mid u \in \mathcal{U}(x)\}$$

where the parametric constraint set $\mathcal{U}(x)$ is defined by

$$\mathcal{U}(x) := \{u \mid (x,u) \in \mathbb{Z}\} = \{u \mid Mu \leq Nx + p\}$$

Let $\mathcal{X} \subset \mathbb{R}^n$ be defined by

$$\mathcal{X} := \{x \mid \exists u \text{ such that } (x,u) \in \mathbb{Z}\} = \{x \mid \mathcal{U}(x) \neq \emptyset\}$$

The set $\mathcal{X}$ is the domain of $V^0(\cdot)$ and of $u^0(\cdot)$ and is thus the set of points $x$ for which a feasible solution of $\mathbb{P}(x)$ exists; it is the projection of $\mathbb{Z}$, which is a set in $(x,u)$-space, onto $x$-space as shown in Figure 7.1. We make the following assumption in the sequel.

**Assumption 7.14** (Continuous, piecewise quadratic function)**.** The function $V(\cdot)$ is continuous, strictly convex, and piecewise quadratic on the polytopic partition $\mathcal{P} = \{\mathbb{Z}_i \mid i \in \mathcal{I} := \mathbb{I}_{1:q}\}$ of the polytope $\mathbb{Z}$ in $\mathbb{R}^n \times \mathbb{R}^m$; $V(x,u) = V_i(x,u)$ where $V_i(\cdot)$ is a positive definite quadratic function of $(x,u)$ for all $(x,u) \in \mathbb{Z}_i$, all $i \in \mathcal{I}$, and $q$ is the number of constituent polytopes in $\mathcal{P}$.

The assumption of continuity places restrictions on the quadratic functions $V_i(\cdot)$, $i \in \mathcal{I}$. For example, we must have $V_i(z) = V_j(z)$ for all $z \in \mathbb{Z}_i \cap \mathbb{Z}_j$. Assumption 7.14 implies that the piecewise quadratic programming problem $\mathbb{P}(x)$ satisfies the hypotheses of Theorem C.34 so that the value function $V^0(\cdot)$ is continuous. It follows from Assumption 7.14 and Theorem C.34 that $V^0(\cdot)$ is strictly convex and continuous and that the minimizer $u^0(\cdot)$ is continuous. Assumption 7.14 implies that $Q_i$ is positive definite for all $i \in \mathcal{I}$. For each $x$, let the set $\mathcal{U}(x)$ be defined by

$$\mathcal{U}(x) := \{u \mid (x,u) \in \mathbb{Z}\}$$

Thus $\mathcal{U}(x)$ is the set of admissible $u$ at $x$, and $\mathbb{P}(x)$ may be expressed in the form $V^0(x) = \min_u \{V(x,u) \mid u \in \mathcal{U}(x)\}$.

For each $i \in \mathcal{I}$, we define an "artificial" problem $\mathbb{P}_i(x)$ as follows

$$V_i^0(x) := \min_u \{V_i(x,u) \mid (x,u) \in \mathbb{Z}_i\}$$
$$u_i^0(x) := \arg\min_u \{V_i(x,u) \mid (x,u) \in \mathbb{Z}_i\}$$

The cost $V_i(x,u)$ in the above equations may be replaced by $V(x,u)$ since $V(x,u) = V_i(x,u)$ in $\mathbb{Z}_i$. The problem is artificial because it includes constraints (the boundaries of $\mathbb{Z}_i$) that are not necessarily constraints of the original problem. We introduce this problem because it helps us to understand the solution of the original problem. For each $i \in \mathbb{I}_{1:p}$, let the set $\mathcal{U}_i(x)$ be defined as follows

$$\mathcal{U}_i(x) := \{u \mid (x,u) \in \mathbb{Z}_i\}$$

Thus the set $\mathcal{U}_i(x)$ is the set of admissible $u$ at $x$, and problem $\mathbb{P}_i(x)$ may be expressed as $V_i^0(x) := \min_u \{V_i(x,u) \mid u \in \mathcal{U}_i(x)\}$; the set $\mathcal{U}_i(x)$ is polytopic. For each $i$, problem $\mathbb{P}_i(x)$ may be recognized as a standard parametric quadratic program discussed in Section 7.4. Because of the piecewise nature of $V(\cdot)$, we require another definition.

**Definition 7.15** (Active polytope (polyhedron))**.** A polytope (polyhedron) $\mathbb{Z}_i$ in a polytopic (polyhedral) partition $\mathcal{P} = \{\mathbb{Z}_i \mid i \in \mathcal{I}\}$ of a polytope (polyhedron) $\mathbb{Z}$ is said to be *active* at $z \in \mathbb{Z}$ if $z = (x,u) \in \mathbb{Z}_i$. The index set specifying the polytopes active at $z \in \mathbb{Z}$ is

$$S(z) := \{i \in \mathcal{I} \mid z \in \mathbb{Z}_i\}$$

A polytope $\mathbb{Z}_i$ in a polytopic partition $\mathcal{P} = \{\mathbb{Z}_i \mid i \in \mathcal{I}\}$ of a polytope $\mathbb{Z}$ is said to be *active* for problem $\mathbb{P}(x)$) if $(x, u^0(x)) \in \mathbb{Z}_i$. The index set specifying polytopes active at $(x, u^0(x))$ is $S^0(x)$ defined by

$$S^0(x) := S(x, u^0(x)) = \{i \in \mathcal{I} \mid (x, u^0(x)) \in \mathbb{Z}_i\}$$

Because we know how to solve the "artificial" problems $\mathbb{P}_i(x)$, $i \in \mathcal{I}$ that are parametric quadratic programs, it is natural to ask if we can recover the solution of the original problem $\mathbb{P}(x)$ from the solutions to these simpler problems. This question is answered by the following proposition.

**Proposition 7.16** (Solving $\mathbb{P}$ using $\mathbb{P}_i$)**.** *For any $x \in \mathcal{X}$, $u$ is optimal for $\mathbb{P}(x)$ if and only if $u$ is optimal for $\mathbb{P}_i(x)$ for all $i \in S(x,u)$.*

*Proof.* (i) Suppose $u$ is optimal for $\mathbb{P}(x)$ but, contrary to what we wish to prove, there exists an $i \in S(x,u) = S^0(x)$ such that $u$ is not optimal for $\mathbb{P}_i(x)$. Hence there exists a $v \in \mathbb{R}^m$ such that $(x,v) \in \mathbb{Z}_i$ and $V(x,v) = V_i(x,v) < V_i(x,u) = V(x,u) = V^0(x)$, a contradiction of the optimality of $u$ for $\mathbb{P}(x)$. (ii) Suppose $u$ is optimal for $\mathbb{P}_i(x)$ for all $i \in S(x,u)$ but, contrary to what we wish to prove, $u$ is not optimal for $\mathbb{P}(x)$. Hence $V^0(x) = V(x,u^0(x)) < V(x,u)$. If $u^0(x) \in \mathbb{Z}^{(x,u)} := \cup_{i \in S(x,u)} \mathbb{Z}_i$, we have a contradiction of the optimality of $u$ in $\mathbb{Z}^{(x,u)}$. Assume then that $u^0(x) \in \mathbb{Z}_j$, $j \notin S(x,u)$; for simplicity, assume further that $\mathbb{Z}_j$ is adjacent to $\mathbb{Z}^{(x,u)}$. Then, there exists a $\lambda \in (0,1]$ such that $u^\lambda := u + \lambda(u^0(x) - u) \in \mathbb{Z}^{(x,u)}$; if not, $j \in S(x,u)$, a contradiction. Since $V(\cdot)$ is strictly convex, $V(x,u^\lambda) < V(x,u)$, which contradicts the optimality of $u$ in $\mathbb{Z}^{(x,u)}$. The case when $\mathbb{Z}_j$ is not adjacent to $\mathbb{Z}^{(x,u)}$ may be treated similarly. ∎

To obtain a parametric solution, we proceed as before. We select a point $x \in \mathcal{X}$ and obtain the solution $u^0(x)$ to $\mathbb{P}(x)$ using a standard algorithm for convex programs. The solution $u^0(x)$ satisfies an equality constraint $E_x u = F_x x + g_x$, which we employ to define, for any $w \in \mathcal{X}$ near $x$ an easily solved equality constrained optimization problem $\mathbb{P}_x(w)$ that is derived from the problems $\mathbb{P}_i(x)$, $i \in S^0(x)$. Finally, we show that the solution to this simple problem is also a solution to the original problem $\mathbb{P}(w)$ at all $w$ in a polytope $R_x \subset \mathcal{X}$ in which $x$ lies.

For each $i \in \mathcal{I}$, $\mathbb{Z}_i$ is defined by

$$\mathbb{Z}_i := \{(x,u) \mid M^i u \le N^i x + p^i\}$$

Let $M^i_j$, $N^i_j$ and $q^i_j$ denote, respectively, the $j$th row of $M^i$, $N^i$ and $q^i$, and let $I_i(x,u)$ and $I^0_i(x)$, defined by

$$I_i(x,u) := \{j \mid M^i_j u = N^i_j x + p^i_j\}, \quad I^0_i(x) := I_i(x, u^0_i(x))$$

denote, respectively, the active constraint set at $(x,u) \in \mathbb{Z}_i$ and the active constraint set for $\mathbb{P}_i(x)$. Because we now use subscript $i$ to specify $\mathbb{Z}_i$, we change our notation slightly and now let $C_i(x,u)$ denote the cone of first-order feasible variations for $\mathbb{P}_i(x)$ at $u \in \mathcal{U}_i(x)$, i.e.,

$$C_i(x,u) := \{h \in \mathbb{R}^m \mid M^i_j h \le 0 \ \forall j \in I_i(x,u)\}$$

Similarly, we define the polar cone $C^*_i(x,u)$ of the cone $C_i(x,u)$ at

$h = 0$ by

$$C_i^*(x,u) := \{v \in \mathbb{R}^m \mid v'h \le 0 \ \forall h \in C_i(x,u)\}$$
$$= \left\{ \sum_{j \in I_i(x,u)} (M_j^i)'\lambda_j \;\middle|\; \lambda_j \ge 0, j \in I_i(x,u) \right\}$$

As shown in Proposition 7.7, a necessary and sufficient condition for the optimality of $u$ for problem $\mathbb{P}_i(x)$ is

$$-\nabla_u V_i(x,u) \in C_i^*(x,u), \quad u \in \mathcal{U}_i(x) \tag{7.15}$$

If $u$ lies in the interior of $\mathcal{U}_i(x)$ so that $I_i^0(x) = \emptyset$, condition (7.15) reduces to $\nabla_u V_i(x,u) = 0$. For any $x \in \mathcal{X}$, the solution $u^0(x)$ of the piecewise parametric program $\mathbb{P}(x)$ satisfies

$$M_j^i u = N_j^i x + p_j^i, \ \forall j \in I_i^0(x), \ \forall i \in S^0(x) \tag{7.16}$$

To simplify our notation, we rewrite the equality constraint (7.16) as

$$E_x u = F_x x + g_x$$

where the subscript $x$ denotes the fact that the constraints are precisely those constraints that are active for the problems $\mathbb{P}_i(x)$, $i \in S^0(x)$. The fact that $u^0(x)$ satisfies these constraints and is, therefore, the unique solution of the optimization problem

$$V^0(x) = \min_u \{V(x,u) \mid E_x u = F_x x + g_x\}$$

motivates us to define the equality constrained problem $\mathbb{P}_x(w)$ for $w \in \mathcal{X}$ near $x$ by

$$V_x^0(w) = \min_u \{V_x(w,u) \mid E_x u = F_x w + g_x\}$$

where $V_x(w,u) := V_i(w,u)$ for all $i \in S^0(x)$ and is, therefore, a positive definite quadratic function of $(x,u)$. The notation $V_x^0(w)$ denotes the fact that the parameter in the parametric problem $\mathbb{P}_x(w)$ is now $w$ but the data for the problem, namely $(E_x, F_x, g_x)$, is derived from the solution $u^0(x)$ of $\mathbb{P}(x)$ and is, therefore, $x$-dependent. Problem $\mathbb{P}_x(w)$ is a simple equality constrained problem in which the cost $V_x(\cdot)$ is quadratic and the constraints $E_x u = F_x w + g_x$ are linear. Let $V_x^0(w)$ denote the value of $\mathbb{P}_x(w)$ and $u_x^0(w)$ its solution. Then

$$V_x^0(w) = (1/2)w'Q_x w + r_x' w + s_x$$
$$u_x^0(w) = K_x w + k_x \tag{7.17}$$

where $Q_x, r_x, s_x, K_x$ and $k_x$ are easily determined. It is easily seen that $u_x^0(x) = u^0(x)$ so that $u_x^0(x)$ is optimal for $\mathbb{P}(x)$. Our hope is that $u_x^0(w)$ is optimal for $\mathbb{P}(w)$ for all $w$ in some neighborhood $R_x$ of $x$. We now show this is the case.

**Proposition 7.17** (Optimality of $u_x^0(w)$ in $R_x$)**.** *Let $x$ be an arbitrary point in $\mathcal{X}$. Then,*

*(a) $u^0(w) = u_x^0(w)$ and $V^0(w) = V_x^0(w)$ for all $w$ in the set $R_x$ defined by*

$$R_x := \left\{ w \in \mathbb{R}^n \;\middle|\; \begin{array}{c} u_x^0(w) \in \mathcal{U}_i(w) \;\; \forall i \in S^0(x) \\ -\nabla_u V_i(w, u_x^0(w)) \in C_i^*(x, u^0(x)) \;\; \forall i \in S^0(x) \end{array} \right\}$$

*(b) $R_x$ is a polytope*

*(c) $x \in R_x$*

*Proof.*

(a) Because of the equality constraint 7.16 it follows that $I_i(w, u_x(w)) \supseteq I_i(x, u^0(x))$ and that $S(w, u_x^0(w)) \supseteq S(x, u^0(x))$ for all $i \in S(x, u^0(x)) = S^0(x)$, all $w \in R_x$. Hence $C_i(w, u_x^0(w)) \subseteq C_i(x, u^0(x))$, which implies $C_i^*(w, u_x^0(w)) \supseteq C_i^*(x, u^0(x))$ for all $i \in S(x, u^0(x)) \subseteq S(w, u_x^0(w))$. It follows from the definition of $R_x$ that $u_x^0(w) \in \mathcal{U}_i(w)$ and that $-\nabla_u V_i(w, u_x^0(w)) \in C_i^*(w, u_x^0(w))$ for all $i \in S(w, u_x^0(w))$. Hence $u = u_x^0(w)$ satisfies necessary and sufficient for optimality for $\mathbb{P}_i(w)$ for all $i \in S(w, u)$, all $w \in R_x$ and, by Proposition 7.16, necessary and sufficient conditions of optimality for $\mathbb{P}(w)$ for all $w \in R_x$. Hence $u_x^0(w) = u^0(w)$ and $V_x^0(w) = V^0(w)$ for all $w \in R_x$.

(b) That $R_x$ is a polytope follows from the facts that the functions $w \mapsto u_x^0(w)$ and $w \mapsto \nabla_u V_i(w, u_x^0(w))$ are affine, the sets $\mathbb{Z}_i$ are polytopic and the sets $C_i^0(x, u^0(x))$ are polyhedral; hence $(w, u_x^0(w)) \in \mathbb{Z}_i$ is a polytopic constraint and $-\nabla_u V_i(w, u_x^0(w)) \in C_i^*(x, u^0(x))$ a polyhedral constraint on $w$.

(c) That $x \in R_x$ follows from Proposition 7.16 and the fact that $u_x^0(x) = u^0(x)$. ■

Reasoning as in the proof of Proposition 7.10, we obtain:

**Proposition 7.18** (Piecewise quadratic (affine) solution)**.** *There exists a finite set of points $X$ in $\mathcal{X}$ such that $\{R_x \mid x \in X\}$ is a polytopic partition of $\mathcal{X}$. The value function $V^0(\cdot)$ for $\mathbb{P}(x)$ is strictly convex and*

*piecewise quadratic and the minimizer $u^0(\cdot)$ is piecewise affine in $\mathcal{X}$ being equal, respectively, to the quadratic function $V_x^0(\cdot)$ and the affine function $u_x^0(\cdot)$ in each region $R_x$, $x \in \mathcal{X}$.*

## 7.6 DP Solution of the Constrained LQ Control Problem

A disadvantage in the procedure described in Section 7.4 for determining the piecewise affine receding horizon control law is the dimension $Nm$ of the decision variable $\mathbf{u}$. It seems natural to inquire whether or not DP, which replaces a multistage decision problem by a sequence of relatively simple single-stage problems, provides a simpler solution. We answer this question by showing how DP may be used to solve the constrained linear quadratic (LQ) problem discussed in Section 7.4. For all $j \in \mathbb{I}_{1:N}$, let $V_j^0(\cdot)$, the optimal value function at time-to-go $j$, be defined by

$$\begin{aligned} V_j^0(x) &:= \min_u \{V_j(x,u) \mid (x,\mathbf{u}) \in \mathbb{Z}_j\} \\ V_j(x,\mathbf{u}) &:= \sum_{i=0}^{j-1} \ell(x(i),u(i)) + V_f(x(j)) \\ \mathbb{Z}_j &:= \{(x,\mathbf{u}) \mid x(i) \in \mathbb{X}, u(i) \in \mathbb{U}, i \in \mathbb{I}_{0:j-1},\ x(j) \in \mathbb{X}_f\} \end{aligned}$$

with $x(i) := \phi(i;x,\mathbf{u})$; $V_j^0(\cdot)$ is the value function for $\mathbb{P}_j(x)$. As shown in Chapter 2, the constrained DP recursion is

$$V_{j+1}^0(x) = \min_u \{\ell(x,u) + V_j^0(f(x,u)) \mid u \in \mathbb{U}, f(x,u) \in \mathcal{X}_j\} \quad (7.18)$$

$$\mathcal{X}_{j+1} = \{x \in \mathbb{X} \mid \exists\ u \in \mathbb{U} \text{ such that } f(x,u) \in \mathcal{X}_j\} \quad (7.19)$$

where $f(x,u) := Ax + Bu$ with boundary condition

$$V_0^0(\cdot) = V_f(\cdot), \quad \mathcal{X}_0 = \mathbb{X}_f$$

The minimizer of (7.18) is $\kappa_{j+1}(x)$. In the equations, the subscript $j$ denotes time to go, so that current time $i = N - j$. For each $j$, $\mathcal{X}_j$ is the domain of the value function $V_j^0(\cdot)$ and of the control law $\kappa_j(\cdot)$, and is the set of states that can be steered to the terminal set $\mathbb{X}_f$ in $j$ steps or less by an admissible control that satisfies the state and control constraints. The time-invariant receding horizon control law for horizon $j$ is $\kappa_j(\cdot)$ whereas the optimal policy for problem $\mathbb{P}_j(x)$ is $\{\kappa_j(\cdot), \kappa_{j-1}(\cdot), \ldots, \kappa_1(\cdot)\}$. The data of the problem are identical to the data in Section 7.4.

We know from Section 7.4 that $V_j^0(\cdot)$ is continuous, strictly convex and piecewise quadratic, and that $\kappa_j(\cdot)$ is continuous and piecewise affine on a polytopic partition $\mathcal{P}_{\mathcal{X}_j}$ of $\mathcal{X}_j$. Hence the function $(x,u) \mapsto V(x,u) := \ell(x,u) + V_j^0(Ax+Bu)$ is continuous, strictly convex and piecewise quadratic on a polytopic partition $\mathcal{P}_{\mathbb{Z}_{j+1}}$ of the polytope $\mathbb{Z}_{j+1}$ defined by

$$\mathbb{Z}_{j+1} := \{(x,u) \mid x \in \mathbb{X}, u \in \mathbb{U}, Ax+Bu \in \mathcal{X}_j\}$$

The polytopic partition $\mathcal{P}_{\mathbb{Z}_{j+1}}$ of $\mathbb{Z}_{j+1}$ may be computed as follows: if $X$ is a constituent polytope of $\mathcal{X}_j$, then, from (7.19), the corresponding constituent polytope of $\mathcal{P}_{\mathbb{Z}_{j+1}}$ is the polytope $Z$ defined by

$$Z := \{z = (x,u) \mid x \in \mathbb{X}, u \in \mathbb{U}, Ax+Bu \in X\}$$

Thus $Z$ is defined by a set of linear inequalities; also $\ell(x,u)+V_j^0(f(x,u))$ is quadratic on $Z$. Thus the techniques of Section 7.5 can be employed for its solution, yielding the piecewise quadratic value function $V_{j+1}^0(\cdot)$, the piecewise affine control law $\kappa_{j+1}(\cdot)$, and the polytopic partition $\mathcal{P}_{\mathcal{X}_{j+1}}$ on which $V_{j+1}^0(\cdot)$ and $\kappa_{j+1}(\cdot)$ are defined. Each problem (7.18) is much simpler than the problem considered in Section 7.4 since $m$, the dimension of $u$, is much less than $Nm$, the dimension of $\mathbf{u}$. Thus, the DP solution is preferable to the direct method described in Section 7.4.

## 7.7 Parametric Linear Programming

### 7.7.1 Preliminaries

The parametric linear program $\mathbb{P}(x)$ is

$$V^0(x) = \min_u \{V(x,u) \mid (x,u) \in \mathbb{Z}\}$$

where $x \in \mathcal{X} \subset \mathbb{R}^n$ and $u \in \mathbb{R}^m$, the cost function $V(\cdot)$ is defined by

$$V(x,u) = q'x + r'u$$

and the constraint set $\mathbb{Z}$ is defined by

$$\mathbb{Z} := \{(x,u) \mid Mu \leq Nx + p\}$$

Let $u^0(x)$ denote the solution of $\mathbb{P}(x)$, i.e.,

$$u^0(x) = \arg\min_u \{V(x,u) \mid (x,u) \in \mathbb{Z}\}$$

The solution $u^0(x)$ may be set valued. The parametric linear program may also be expressed as

$$V^0(x) = \min_u \{V(x,u) \mid u \in \mathcal{U}(x)\}$$

where, as before, the parametric constraint set $\mathcal{U}(x)$ is defined by

$$\mathcal{U}(x) := \{u \mid (x,u) \in \mathbb{Z}\} = \{u \mid Mu \leq Nx + p\}$$

Also, as before, the domain of $V^0(\cdot)$ and $u^0(\cdot)$, i.e., the set of points $x$ for which a feasible solution of $\mathbb{P}(x)$ exists, is the set $\mathcal{X}$ defined by

$$\mathcal{X} := \{x \mid \exists u \text{ such that } (x,u) \in \mathbb{Z}\} = \{x \mid \mathcal{U}(x) \neq \emptyset\}$$

The set $\mathcal{X}$ is the projection of $\mathbb{Z}$ (which is a set in $(x,u)$-space) onto $x$-space; see Figure 7.1. We assume in the sequel that the problem is well posed, i.e., for each $x \in \mathcal{X}$, $V^0(x) > -\infty$. This excludes problems like $V^0(x) = \inf_u \{x + u \mid -x \leq 1, x \leq 1\}$ for which $V^0(x) = -\infty$ for all $x \in \mathcal{X} = [-1,1]$.

Let $\mathbb{I}_{1:p}$ denote, as usual, the index set $\{1,2,\ldots,p\}$. For all $(x,u) \in \mathbb{Z}$, let $I(x,u)$ denote the set of active constraints at $(x,u)$, i.e.,

$$I(x,u) := \{i \in \mathbb{I}_{1:p} \mid M_i u = N_i x + p_i\}$$

where $A_i$ denotes the $i$th row of any matrix (or vector) $A$. Similarly, for any matrix $A$ and any index set $I$, $A_I$ denotes the matrix with rows $A_i$, $i \in I$. If, for any $x \in \mathcal{X}$, $u^0(x)$ is unique, the set $I^0(x)$ of constraints active at $(x, u^0(x))$ is defined by

$$I^0(x) := I(x, u^0(x))$$

When $u^0(x)$ is unique, it is a vertex (a face of dimension zero) of the polyhedron $\mathcal{U}(x)$ and is the *unique* solution of

$$M_x^0 u = N_x^0 x + p_x^0$$

where

$$M_x^0 := M_{I^0(x)},\ N_x^0 := N_{I^0(x)},\ p_x^0 := p_{I^0(x)}$$

In this case, the matrix $M_x^0$ has rank $m$.

Any face $F$ of $\mathcal{U}(x)$ with dimension $d \in \{1,2,\ldots,m\}$ satisfies $M_i u = N_i x + p_i$ for all $i \in I_F$, all $u \in F$ for some index set $I_F \subseteq \mathbb{I}_{1:p}$. The matrix $M_{I_F}$ with rows $M_i$, $i \in I_F$, has rank $m - d$, and the face $F$ is defined by

$$F := \{u \mid M_i u = N_i x + p_i, i \in I_F\} \cap \mathcal{U}(x)$$

When $u^0(x)$ is not unique, it is a face of dimension $d \geq 1$ and the set $I^0(x)$ of active constraints is defined by

$$I^0(x) := \{i \mid M_i u = N_i x + p_i \ \forall u \in u^0(x)\} = \{i \mid i \in I(x,u) \ \forall u \in u^0(x)\}$$

The set $\{u \mid M_i u = N_i x + p_i,\ i \in I^0(x)\}$ is an affine hyperplane in which $u^0(x)$ lies. See Figure 7.8 where $u^0(x_1)$ is unique, a vertex of $\mathcal{U}(x_1)$, and $I^0(x_1) = \{2,3\}$. If, in Figure 7.8, $r = -e_1$, then $u^0(x_1) = F_2(x_1)$, a face of dimension 1; $u^0(x_1)$ is, therefore, set valued. Since $u \in \mathbb{R}^m$ where $m = 2$, $u^0(x_1)$ is a facet, i.e., a face of dimension $m-1 = 1$. Thus $u^0(x_1)$ is a set defined by $u^0(x_1) = \{u \mid M_1 u \leq N_1 x_1 + p_1,\ M_2 u = N_2 x_1 + p_2,\ M_3 u \leq N_3 x_1 + p_3\}$.

At each $z = (x,u) \in \mathbb{Z}$, i.e., for each $(x,u)$ such that $x \in \mathcal{X}$ and $u \in \mathcal{U}(x)$, the cone $C(z) = C(x,u)$ of first-order feasible variations is defined, as before, by

$$C(z) := \{h \in \mathbb{R}^m \mid M_i h \leq 0,\ i \in I(z)\} = \{h \in \mathbb{R}^m \mid M_{I(z)} h \leq 0\}$$

If $I(z) = I(x,u) = \emptyset$ (no constraints are active), $C(z) = \mathbb{R}^m$ (all variations are feasible).

Since $u \mapsto V(x,\cdot)$ is convex and differentiable, and $\mathcal{U}(x)$ is polyhedral for all $x$, the parametric linear program $\mathbb{P}(x)$ satisfies the assumptions of Proposition 7.8. Hence, repeating Proposition 7.8 for convenience, we have

**Proposition 7.19** (Optimality conditions for parametric linear program)**.** *A necessary and sufficient condition for $u$ to be a minimizer for the parametric linear program $\mathbb{P}(x)$ is*

$$u \in \mathcal{U}(x) \text{ and } -\nabla_u V(x,u) \in C^*(x,u)$$

*where $\nabla_u V(x,u) = r$ and $C^*(x,u)$ is the polar cone of $C(x,u)$.*

An important difference between this result and that for the parametric quadratic program is that $\nabla_u V(x,u) = r$ and, therefore, does not vary with $x$ or $u$. We now use this result to show that both $V^0(\cdot)$ and $u^0(\cdot)$ are piecewise affine. We consider the simple case when $u^0(x)$ is unique for all $x \in \mathcal{X}$.

### 7.7.2 Minimizer $u^0(x)$ Is Unique for all $x \in \mathcal{X}$

Before proceeding to obtain the solution to a parametric linear program when the minimizer $u^0(x)$ is unique for each $x \in \mathcal{X}$, we look first at

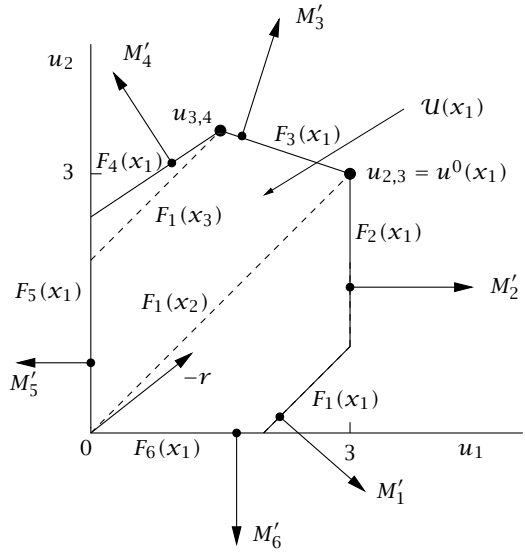


**Figure 7.8:** Solution to a parametric linear program.

the simple example illustrated in Figure 7.8, which shows the constraint set $\mathcal{U}(x)$ for various values of the parameter $x$ in the interval $[x_1, x_3]$. The set $\mathcal{U}(x_1)$ has six faces: $F_1(x_1), F_2(x_1), F_3(x_1), F_4(x_1), F_5(x_1)$ and $F_6(x_1)$. Face $F_1(x)$ lies in the hyperplane $\mathcal{H}_1(x)$ that varies linearly with $x$; each face $F_i(x)$, $i = 2, \ldots, 6$, lies in the hyperplane $\mathcal{H}_i$ that does *not* vary with $x$. All the faces vary with $x$ as shown so that $\mathcal{U}(x_2)$ has four faces: $F_1(x_2), F_3(x_2), F_4(x_2)$ and $F_5(x_2)$; and $\mathcal{U}(x_3)$ has three faces: $F_1(x_3)$, $F_4(x_3)$ and $F_5(x_3)$. The face $F_1(x)$ is shown for three values of $x$: $x = x_1$ (the bold line), and $x = x_2$ and $x = x_3$ (dotted lines). It is apparent that for $x \in [x_1, x_2]$, $u^0(x) = u_{2,3}$ in which $u_{2,3}$ is the intersection of $\mathcal{H}_2$ and $\mathcal{H}_3$, and $u^0(x_3) = u_{3,4}$, in which $u_{3,4}$ is the intersection of $\mathcal{H}_3$ and $\mathcal{H}_4$. It can also be seen that $u^0(x)$ is unique for all $x \in \mathcal{X}$.

We now return to the general case. Suppose, for some $\in \mathcal{X}$, $u^0(x)$ is the unique solution of $\mathbb{P}(x)$; $u^0(x)$ is the unique solution of

$$M_x^0 u = N_x^0 x + p_x^0$$

It follows that $u^0(x)$ is the trivial solution of the simple equality constrained problem defined by

$$V^0(x) = \min_u \{V(x, u) \mid M_x^0 u = N_x^0 x + p_x^0\} \tag{7.20}$$

The solution $u^0(x)$ of this equality constrained problem is trivial because it is determined entirely by the equality constraints; the cost plays no part.

The optimization problem (7.20) motivates us, as in parametric quadratic programming, to consider, for any parameter $w$ "close" to $x$, the simpler equality constrained problem $\mathbb{P}_x(w)$ defined by

$$V_x^0(w) = \min_u \{V(w,u) \mid M_x^0 u = N_x^0 w + p_x^0\}$$
$$u_x^0(w) = \arg\min_u \{V(w,u) \mid M_x^0 u = N_x^0 w + p_x^0\}$$

Let $u_x^0(w)$ denote the solution of $\mathbb{P}_x(w)$. Because, for each $x \in \mathcal{X}$, the matrix $M_x^0$ has full rank $m$, there exists an index set $I_x$ such that $M_{I_x} \in \mathbb{R}^{m\times m}$ is invertible. Hence, for each $w$, $u_x^0(w)$ is the unique solution of

$$M_{I_x} u = N_{I_x} w + p_{I_x}$$

so that for all $x \in \mathcal{X}$, all $w \in \mathbb{R}^m$

$$u_x^0(w) = K_x w + k_x \tag{7.21}$$

where $K_x := (M_{I_x})^{-1} N_{I_x}$ and $k_x := (M_{I_x})^{-1} p_{I_x}$. In particular, $u^0(x) = u_x^0(x) = K_x x + k_x$. Since $V_x^0(x) = V_x(x, u_x^0(w)) = q'x + r' u_x^0(w)$, it follows that

$$V_x^0(x) = (q' + r' K_x) x + r' k_x$$

for all $x \in \mathcal{X}$, all $w \in \mathbb{R}^m$. Both $V_x^0(\cdot)$ and $u_x^0(\cdot)$ are affine in $x$.

It follows from Proposition 7.19 that $-r \in C^*(x, u^0(x)) = \text{cone}\{M_i' \mid i \in I^0(x) = I(x, u^0(x))\} = \text{cone}\{M_i' \mid i \in I_x\}$. Since $\mathbb{P}_x(w)$ satisfies the conditions of Proposition 7.8, we may proceed as in Section 7.3.4 and define, for each $x \in \mathcal{X}$, the set $R_x^0$ as in (7.5)

$$R_x^0 := \left\{ w \in \mathbb{R}^n \;\middle|\; \begin{matrix} u_x^0(w) \in \mathcal{U}(w) \\ -\nabla_u V(w, u_x^0(w)) \in C^*(x, u^0(x)) \end{matrix} \right\}$$

It then follows, as shown in Proposition 7.9, that for any $x \in \mathcal{X}$, $u_x^0(w)$ is optimal for $\mathbb{P}(w)$ for all $w \in R_x^0$. Because $\mathbb{P}(w)$ is a parametric linear program, however, rather than a parametric quadratic program, it is possible to simplify the definition of $R_x^0$. We note that $\nabla_u V(w, u_x^0(w)) = r$ for all $x \in \mathcal{X}$, all $w \in \mathbb{R}^m$. Also, it follows from Proposition 7.8, since $u^0(x)$ is optimal for $\mathbb{P}(x)$, that $-\nabla_u V(x, u^0(x)) = -r \in C^*(x)$ so that the second condition in the definition above for

$R_x^0$ is automatically satisfied. Hence we may simplify our definition for $R_x^0$; for the parametric linear program, $R_x^0$ may be defined by

$$R_x^0 := \{w \in \mathbb{R}^n \mid u_x^0(w) \in \mathcal{U}(w)\} \tag{7.22}$$

Because $u_x^0(\cdot)$ is affine, it follows from the definition of $\mathcal{U}(w)$ that $R_x^0$ is polyhedral. The next result follows from the discussion in Section 7.3.4:

**Proposition 7.20** (Solution of $\mathbb{P}$)**.** *For any $x \in \mathcal{X}$, $u_x^0(w)$ is optimal for $\mathbb{P}(w)$ for all $w$ in the set $R_x^0$ defined in* (7.22)*.*

Finally, the next result characterizes the solution of the parametric linear program $\mathbb{P}(x)$ when the minimizer is unique.

**Proposition 7.21** (Piecewise affine cost and solution)**.**

*(a) There exists a finite set of points $X$ in $\mathcal{X}$ such that $\{R_x^0 \mid x \in X\}$ is a polyhedral partition of $\mathcal{X}$.*

*(b) The value function $V^0(\cdot)$ for $\mathbb{P}(x)$ and the minimizer $u^0(\cdot)$ are piecewise affine in $\mathcal{X}$ being equal, respectively, to the affine functions $V_x^0(\cdot)$ and $u_x^0(\cdot)$ in each region $R_x$, $x \in \mathcal{X}$.*

*(c) The value function $V^0(\cdot)$ and the minimizer $u^0(\cdot)$ are continuous in $\mathcal{X}$.*

*Proof.* The proof of parts (a) and (b) follows, apart from minor changes, the proof of Proposition 7.10. The proof of part (c) uses the fact that $u^0(x)$ is unique, by assumption, for all $x \in \mathcal{X}$ and is similar to the proof of Proposition 7.13. ■

## 7.8 Constrained Linear Control

The previous results on parametric linear programming may be applied to obtain the optimal receding horizon control law when the system is linear, the constraints polyhedral, and the cost linear as is done in a similar fashion in Section 7.4 where the cost is quadratic. The optimal control problem is therefore defined as in Section 7.4, except that the stage cost $\ell(\cdot)$ and the terminal cost $V_f(\cdot)$ are now defined by

$$\ell(x, u) := q'x + r'u, \quad V_f(x) := q_f'x$$

As in Section 7.4, the optimal control problem $\mathbb{P}_N(x)$ may be expressed as

$$V_N^0(x) = \min_{\mathbf{u}} \{V_N(x, \mathbf{u}) \mid \mathbf{M}\mathbf{u} \le \mathbf{N}x + \mathbf{p}\}$$

where, now

$$V_N(x, \mathbf{u}) = \mathbf{q}'x + \mathbf{r}'\mathbf{u}$$

Hence the problem has the same form as that discussed in Section 7.7 and may be solved as shown there.

It is possible, using a simple transcription, to use the solution of $\mathbb{P}_N(x)$ to solve the optimal control problem when the stage cost and terminal cost are defined by

$$\ell(x, u) := |Qx|_p + |Ru|_p, \quad V_f(x) := |Q_f x|_p$$

where $|\cdot|_p$ denotes the $p$-norm and $p$ is either 1 or $\infty$.

## 7.9 Computation

Our main purpose above was to establish the structure of the solution of parametric linear or quadratic programs and, hence, of the solutions of constrained linear optimal control problems when the cost is quadratic or linear and we have not presented algorithms for solving these problems. A naive approach to computation would be to generate points $x$ in $\mathcal{X}$ randomly and to compute the corresponding polyhedral sets $R_x$ using the formula given previously. But, due to numerical errors, these regions would either overlap or leave gaps or both, rendering the solution useless. Methods for low-dimensional problems are described in the survey paper by Alessio and Bemporad (2008). The preferred methods generate new regions adjacent to a set of regions already determined. In this class is the lexicographic perturbation algorithm described by Jones, Kerrigan, and Maciejowski (2007) for parametric linear programs and by Jones and Morari (2006) for parametric linear complementarity problems and for quadratic programs. The toolboxes (Bemporad, 2004) and (Kvasnica, Grieder, and Baotić, 2006) provide tools for the determination of feedback control laws for relatively simple linear systems with polyhedral constraints.

## 7.10 Notes

Early work on parametric programming, e.g. (Dantzig, Folkman, and Shapiro, 1967) and (Bank, Guddat, Klatte, Kummer, and Tanner, 1983), was concerned with the sensitivity of optimal solutions to parameter variations. Solutions to the parametric linear programming problem

were obtained relatively early (Gass and Saaty, 1955) and (Gal and Nedoma, 1972). Solutions to parametric quadratic programs were obtained in (Seron, De Doná, and Goodwin, 2000) and (Bemporad, Morari, Dua, and Pistikopoulos, 2002) and applied to the determination of optimal control laws for linear systems with polyhedral constraints. Since then a large number of papers on this topic have appeared, many of which are reviewed in (Alessio and Bemporad, 2008). Most papers employ the Kuhn-Tucker conditions of optimality in deriving the regions $R_x$, $x \in X$. Use of the polar cone condition was advocated in (Mayne and Raković, 2002) in order to focus on the geometric properties of the parametric optimization problem and avoid degeneracy problems. Section 7.5, on parametric piecewise quadratic programming, is based on (Mayne, Raković, and Kerrigan, 2007). The examples in Section 7.4 were computed by Raković. That uniqueness of the minimizer can be employed, instead of maximum theorems, to establish, as in Section 7.3.5, continuity of $u^0(\cdot)$ and, hence, of $V^0(\cdot)$, was pointed out by Bemporad et al. (2002) and Borrelli (2003, p. 37).

## 7.11 Exercises

### Exercise 7.1: Quadratic program with equality constraints

Obtain the solution $u^0$ and the value $V^0$ of the equality constrained optimization problem $V^0 = \min_u\{V(u) \mid h(u) = 0\}$ where $V(u) = (1/2)u'Ru + r'u + c$ and $h(u) := Mu - p$.

### Exercise 7.2: Parametric quadratic program with equality constraints

Show that the solution $u^0(x)$ and the value $V^0(x)$ of the parametric optimization problem $V^0(x) = \min_u\{V(x,u) \mid h(x,u) = 0\}$ where $V(x,u) := (1/2)x'Qx + u'Sx + (1/2)u'Ru + q'x + r'u + c$ and $h(x,u) := Mu - Nx - p$ have the form $u^0(x) = Kx + k$ and $V^0(x) = (1/2)x'\bar{Q}x + \bar{q}'x + s$. Determine $\bar{Q}$, $\bar{q}$, $s$, $K$ and $k$.

### Exercise 7.3: State and input trajectories in constrained LQ problem

For the constrained linear quadratic problem defined in Section 7.4, show that $\mathbf{u} := \{u(0), u(1), \ldots, u(N-1)\}$ and $\mathbf{x} := \{x(0), x(1), \ldots, x(N)\}$, where $x(0) = x$ and $x(i) = \phi(i; x, \mathbf{u})$, $i = 0, 1, \ldots, N$, satisfy:

$$\mathbf{x} = \mathbf{F}x + \mathbf{G}\mathbf{u}$$

and determine the matrices $\mathbf{F}$ and $\mathbf{G}$; in this equation $\mathbf{u}$ and $\mathbf{x}$ are column vectors. Hence show that $V_N(x, \mathbf{u})$ and $\mathbb{Z}$, defined respectively in (7.9) and (7.11), satisfy (7.12) and (7.13), and determine $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{M}$, $\mathbf{N}$ and $\mathbf{p}$.

### Exercise 7.4: The parametric linear program with unique minimizer

For the example of Figure 7.8, determine $u^0(x)$, $V^0(x)$, $I^0(x)$ and $C^*(x)$ for all $x$ in the interval $[x_1, x_3]$. Show that $-r$ lies in $C^*(x)$ for all $x$ in $[x_1, x_3]$.

### Exercise 7.5: Cost function and constraints in constrained LQ control problem

For the constrained linear control problem considered in Section 7.8, determine the matrices $\mathbf{M}$, $\mathbf{N}$ and $\mathbf{p}$ that define the constraint set $\mathbb{Z}$, and the vectors $\mathbf{q}$ and $\mathbf{r}$ that define the cost $V_N(\cdot)$.

### Exercise 7.6: Cost function in constrained linear control problem

Show that $|x|_p$, $p = 1$ and $p = \infty$, may be expressed as $\max_j\{s_j'x \mid j \in J\}$ and determine $s_i$, $i \in I$ for the two cases $p = 1$ and $p = \infty$. Hence show that the optimal control problem in Section 7.8 may be expressed as

$$V_N^0(x) = \min_{\mathbf{v}}\{V_N(x, \mathbf{v}) \mid \mathbf{M}\mathbf{v} \leq \mathbf{N}x + \mathbf{p}\}$$

where, now, $\mathbf{v}$ is a column vector whose components are $u(0), u(1), \ldots, u(N-1)$, $\ell_x(0), \ell_x(1), \ldots, \ell_x(N)$, $\ell_u(0), \ell_u(1), \ldots, \ell_u(N-1)$ and $f$; the cost $V_N(x, \mathbf{v})$ is now defined by

$$V_N(x, \mathbf{v}) = \sum_{i=0}^{N-1} (\ell_x(i) + \ell_u(i)) + f$$

Finally, $\mathbf{Mv} \le \mathbf{N}x + \mathbf{p}$ now specifies the constraints $u(i) \in \mathbb{U}$ and $x(i) \in \mathbb{X}$, $|Ru(i)|_p \le \ell_u(i)$, $|Qx(i)|_p \le \ell_x(i)$, $i = 0, 1, \ldots, N-1$, $x(N) \in \mathbb{X}_f$, and $|Q_f x(N)| \le f$. As before, $\mathbf{x}^+ = \mathbf{Fx} + \mathbf{Gu}$.

### Exercise 7.7: Is QP constraint qualification relevant to MPC?

Continuity properties of the MPC control law are often used to establish robustness properties of MPC such as robust asymptotic stability. In early work on continuity properties of linear model MPC, Scokaert, Rawlings, and Meadows (1997) used results on continuity of QPs with respect to parameters to establish MPC stability under perturbations. For example, Hager (1979) considered the following quadratic program

$$\min_u (1/2)u'Hu + h'u + c$$

subject to

$$Du \le d$$

and established that the QP solution $u^0$ and cost $V^0$ are Lipschitz continuous in the data of the QP, namely the parameters $H, h, D, d$. To establish this result Hager (1979) made the following assumptions.

- The solution is unique for all $H, h, D, d$ in a chosen set of interest.
- The rows of $D$ corresponding to the constraints active at the solution are linearly independent. The assumption of linear independence of active constraints is a form of *constraint qualification*.

(a) First we show that some form of constraint qualification is required to establish continuity of the QP solution with respect to matrix $D$. Consider the following QP example that does not satisfy Hager's constraint qualification assumption.

$$H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \qquad d = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad h = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \qquad c = 1$$

Find the solution $u^0$ for this problem.

Next perturb the $D$ matrix to

$$D = \begin{bmatrix} 1 & 1 \\ -1+\epsilon & -1 \end{bmatrix}$$

in which $\epsilon > 0$ is a small perturbation. Find the solution to the perturbed problem. Are $V^0$ and $u^0$ continuous in parameter $D$ for this QP? Draw a sketch of the feasible region and cost contours for the original and perturbed problems. What happens to the feasible set when $D$ is perturbed?

(b) Next consider MPC control of the following system with state inequality constraint and no input constraints

$$A = \begin{bmatrix} -1/4 & 1 \\ -1 & 1/2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \qquad x(k) \le \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad k \in \mathbb{I}_{0:N}$$

Using a horizon $N = 1$, eliminate the state $x(1)$ and write out the MPC QP for the input $u(0)$ in the form given above for $Q = R = I$ and zero terminal penalty. Find an initial condition $x_0$ such that the MPC constraint matrix $D$ and vector $d$ are identical to those given in the previous part. Is this $x_0 \in \mathcal{X}_N$?

Are the rows of the matrix of active constraints linearly independent in this MPC QP on the set $\mathcal{X}_N$? Are the MPC control law $\kappa_N(x)$ and optimal value function $V_N^0(x)$ Lipschitz continuous on the set $\mathcal{X}_N$ for this system? Explain the reason if these two answers differ.

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# A
# Mathematical Background

## A.1 Introduction

In this appendix we give a brief review of some concepts that we need. It is assumed that the reader has had at least a first course on linear systems and has some familiarity with linear algebra and analysis. The appendices of Polak (1997); Nocedal and Wright (1999); Boyd and Vandenberghe (2004) provide useful summaries of the results we require. The material presented in Sections A.2–A.14 follows closely Polak (1997) and earlier lecture notes of Professor Polak.

## A.2 Vector Spaces

The Euclidean space $\mathbb{R}^n$ is an example of a vector space that satisfies a set of axioms the most significant being: if $x$ and $z$ are two elements of a vector space $\mathcal{V}$, then $\alpha x + \beta z$ is also an element of $\mathcal{V}$ for all $\alpha, \beta \in \mathbb{R}$. This definition presumes addition of two elements of $\mathcal{V}$ and multiplication of any element of $\mathcal{V}$ by a scalar are defined. Similarly, $S \subset \mathcal{V}$ is a linear subspace[1] of $\mathcal{V}$ if any two elements of $x$ and $z$ of $S$ satisfy $\alpha x + \beta z \in S$ for all $\alpha, \beta \in \mathbb{R}$. Thus, in $\mathbb{R}^3$, the origin, a line or a plane passing through the origin, the whole set $\mathbb{R}^3$, and even the empty set are all subspaces.

## A.3 Range and Nullspace of Matrices

Suppose $A \in \mathbb{R}^{m \times n}$. Then $\mathcal{R}(A)$, the *range* of $A$, is the set $\{Ax \mid x \in \mathbb{R}^n\}$; $\mathcal{R}(A)$ is a subspace of $\mathbb{R}^m$ and its dimension, i.e., the number of linearly independent vectors that span $\mathcal{R}(A)$, is the rank of $A$. For

[1] All of the subspaces used in this text are linear subspaces, so we often omit the adjective linear.

example, if $A$ is the column vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then $\mathcal{R}(A)$ is the subspace spanned by the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and the rank of $A$ is 1. The *nullspace* $\mathcal{N}(A)$ is the set of vectors in $\mathbb{R}^n$ that are mapped to zero by $A$ so that $\mathcal{N}(A) = \{x \mid Ax = 0\}$. The nullspace $\mathcal{N}(A)$ is a subspace of $\mathbb{R}^n$. For the example above, $\mathcal{N}(A)$ is the subspace spanned by the vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. It is an important fact that $\mathcal{R}(A') \oplus \mathcal{N}(A) = \mathbb{R}^n$ or, equivalently, that $\mathcal{N}(A) = (\mathcal{R}(A'))^\perp$ where $A' \in \mathbb{R}^{n \times m}$ is the transpose of $A$ and $S^\perp$ denotes the orthogonal complement of any subspace $S$; a consequence is that the sum of the dimensions $\mathcal{R}(A)$ and $\mathcal{N}(A)$ is $n$. If $A$ is square and invertible, then $n = m$ and the dimension of $\mathcal{R}(A)$ is $n$ so that the dimension of $\mathcal{N}(A)$ is 0, i.e., the nullspace contains only the zero vector, $\mathcal{N}(A) = \{0\}$.

## A.4 Linear Equations — Existence and Uniqueness

Let $A \in \mathbb{R}^{m \times n}$ be a real-valued matrix with $m$ rows and $n$ columns. We are often interested in solving linear equations of the type

$$Ax = b$$

in which $b \in \mathbb{R}^m$ is given, and $x \in \mathbb{R}^n$ is the unknown. The fundamental theorem of linear algebra gives a complete characterization of the existence and uniqueness of solutions to $Ax = b$ (Strang, 1980, pp.87–88). Every matrix $A$ decomposes the spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ into the four fundamental subspaces depicted in Figure A.1. A solution to $Ax = b$ *exists* for every $b$ if and only if the *rows* of $A$ are linearly independent. A solution to $Ax = b$ is *unique* if and only if the *columns* of $A$ are linearly independent.

## A.5 Pseudo-Inverse

The solution of $Ax = y$ when $A$ is invertible is $x = A^{-1}y$ where $A^{-1}$ is the inverse of $A$. Often an approximate inverse of $y = Ax$ is required when $A$ is *not* invertible. This is yielded by the pseudo-inverse $A^\dagger$ of $A$; if $A \in \mathbb{R}^{m \times n}$, then $A^\dagger \in \mathbb{R}^{n \times m}$. The properties of the pseudo-inverse are illustrated in Figure A.2 for the case when $A \in \mathbb{R}^{2 \times 2}$ where both $\mathcal{R}(A)$ and $\mathcal{N}(A)$ have dimension 1. Suppose we require a solution to the equation $Ax = y$. Since every $x \in \mathbb{R}^2$ is mapped into $\mathcal{R}(A)$, we see that a solution may only be obtained if $y \in \mathcal{R}(A)$. Suppose this is not the case, as in Figure A.2. Then the closest point, in the Euclidean sense, to $y$ in $\mathcal{R}(A)$ is the point $y^*$ which is the orthogonal projection

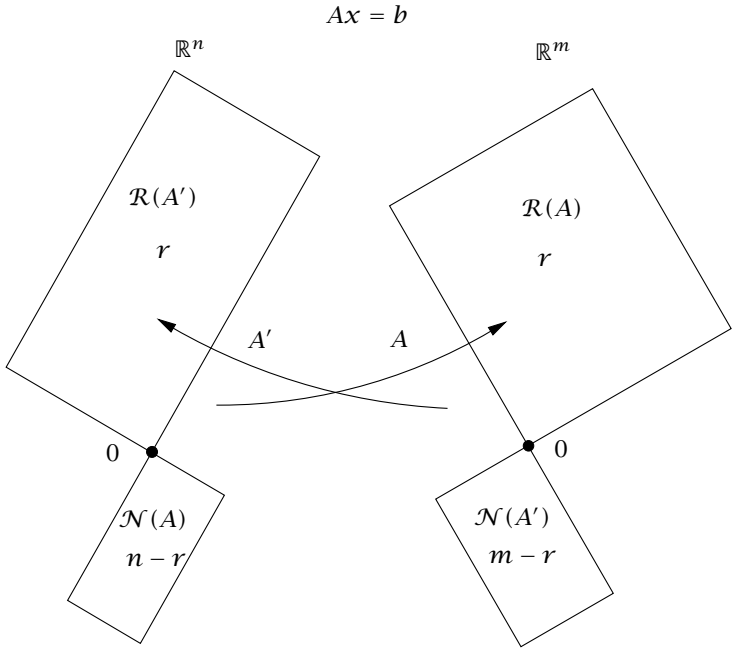


**Figure A.1:** The four fundamental subspaces of matrix $A$ (after (Strang, 1980, p.88)). The dimension of the range of $A$ and $A'$ is $r$, the rank of matrix $A$. The nullspace of $A$ and range of $A'$ are orthogonal as are the nullspace of $A'$ and range of $A$. Solutions to $Ax = b$ exist for all $b$ if and only if $m = r$ (rows independent). A solution to $Ax = b$ is unique if and only if $n = r$ (columns independent).

of $y$ onto $\mathcal{R}(A)$, i.e., $y - y^*$ is orthogonal to $\mathcal{R}(A)$. Since $y^* \in \mathcal{R}(A)$, there exists a point in $\mathbb{R}^2$ that $A$ maps into $y^*$. Now $A$ maps any point of the form $x + h$ where $h \in \mathcal{N}(A)$ into $A(x + h) = Ax + Ah = Ax$ so that there must exist a point $x^* \in (\mathcal{N}(A))^\perp = \mathcal{R}(A')$ such that $Ax^* = y^*$, as shown in Figure A.2. All points of the form $x = x^* + h$ where $h \in \mathcal{N}(A)$ are also mapped into $y^*$; $x^*$ is the point of least norm that satisfies $Ax^* = y^*$ where $y^*$ is that point in $\mathcal{R}(A)$ closest, in the Euclidean sense, to $y$.

The pseudo-inverse $A^\dagger$ of a matrix $A \in \mathbb{R}^{m \times n}$ is a matrix in $\mathbb{R}^{n \times m}$ that maps every $y \in \mathbb{R}^m$ to that point $x \in \mathcal{R}(A')$ of least Euclidean norm that minimizes $|y - Ax|_2$. The operation of $A^\dagger$ is illustrated in

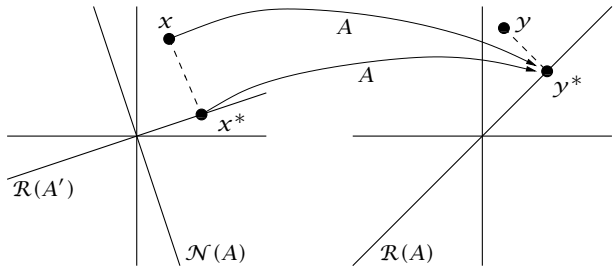


**Figure A.2:** Matrix $A$ maps into $\mathcal{R}(A)$.

Figure A.3. Hence $AA^\dagger$ projects any point $y \in \mathbb{R}^m$ orthogonally onto $\mathcal{R}(A)$, i.e., $AA^\dagger y = y^*$, and $A^\dagger A$ projects any $x \in \mathbb{R}^n$ orthogonally onto $\mathcal{R}(A')$, i.e., $A^\dagger Ax = x^*$.

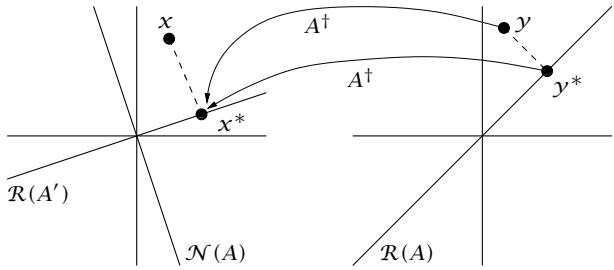


**Figure A.3:** Pseudo-inverse of $A$ maps into $\mathcal{R}(A')$.

If $A \in \mathbb{R}^{m\times n}$ where $m < n$ has maximal rank $m$, then $AA' \in \mathbb{R}^{m\times m}$ is invertible and $A^\dagger = A'(AA')^{-1}$; in this case, $\mathcal{R}(A) = \mathbb{R}^m$ and every $y \in \mathbb{R}^m$ lies in $\mathcal{R}(A)$. Similarly, if $n < m$ and $A$ has maximal rank $n$, then $A'A \in \mathbb{R}^{n\times n}$ is invertible and $A^\dagger = (A'A)^{-1}A'$; in this case, $\mathcal{R}(A') = \mathbb{R}^n$ and every $x \in \mathbb{R}^n$ lies in $\mathcal{R}(A')$. More generally, if $A \in \mathbb{R}^{m\times n}$ has rank $r$, then $A$ has the *singular-value decomposition* $A = U\Sigma V'$ where $U \in \mathbb{R}^{m\times r}$ and $V \in \mathbb{R}^{r\times n}$ are orthogonal matrices, i.e., $U'U = I_r$ and $V'V = I_r$, and $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r) \in \mathbb{R}^{r\times r}$ where $\sigma_1 > \sigma_2 \ldots > \sigma_r > 0$. The pseudo-inverse of $A$ is then

$$A^\dagger = V\Sigma^{-1}U'$$

## A.6 Partitioned Matrix Inversion Theorem

Let matrix $Z$ be partitioned into

$$Z = \begin{bmatrix} B & C \\ D & E \end{bmatrix}$$

and assume $Z^{-1}, B^{-1}$ and $E^{-1}$ exist. Performing row elimination gives

$$Z^{-1} = \begin{bmatrix} B^{-1} + B^{-1}C(E - DB^{-1}C)^{-1}DB^{-1} & -B^{-1}C(E - DB^{-1}C)^{-1} \\ -(E - DB^{-1}C)^{-1}DB^{-1} & (E - DB^{-1}C)^{-1} \end{bmatrix}$$

Note that this result is still valid if $E$ is singular. Performing column elimination gives

$$Z^{-1} = \begin{bmatrix} (B - CE^{-1}D)^{-1} & -(B - CE^{-1}D)^{-1}CE^{-1} \\ -E^{-1}D(B - CE^{-1}D)^{-1} & E^{-1} + E^{-1}D(B - CE^{-1}D)^{-1}CE^{-1} \end{bmatrix}$$

Note that this result is still valid if $B$ is singular. A host of other useful control-related inversion formulas follow from these results. Equating the (1,1) or (2,2) entries of $Z^{-1}$ gives the identity

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(DA^{-1}B + C^{-1})^{-1}DA^{-1}$$

A useful special case of this result is

$$(I + X^{-1})^{-1} = I - (I + X)^{-1}$$

Equating the (1,2) or (2,1) entries of $Z^{-1}$ gives the identity

$$(A + BCD)^{-1}BC = A^{-1}B(DA^{-1}B + C^{-1})^{-1}$$

**Determinants.** We require some results on determinants of partitioned matrices when using normal distributions in the discussion of probability. If $E$ is nonsingular

$$\det(A) = \det(E)\det(B - CE^{-1}D)$$

If $B$ is nonsingular

$$\det(A) = \det(B)\det(E - DB^{-1}C)$$

## A.7 Quadratic Forms

Positive definite and positive semidefinite matrices show up often in LQ problems. Here are some basic facts about them. In the following $Q$ is real and symmetric and $R$ is real.

The matrix $Q$ is positive definite ($Q > 0$), if

$$x'Qx > 0, \quad \forall \text{ nonzero } x \in \mathbb{R}^n$$

The matrix $Q$ is positive semidefinite ($Q \geq 0$), if

$$x'Qx \geq 0, \quad \forall x \in \mathbb{R}^n$$

You should be able to prove the following facts.

1. $Q > 0$ if and only if $\lambda(Q) > 0, \quad \lambda \in \text{eig}(Q)$.
2. $Q \geq 0$ if and only if $\lambda(Q) \geq 0, \quad \lambda \in \text{eig}(Q)$.
3. $Q \geq 0 \Rightarrow R'QR \geq 0 \quad \forall R$.
4. $Q > 0$ and $R$ nonsingular $\Rightarrow R'QR > 0$.
5. $Q > 0$ and $R$ full column rank $\Rightarrow R'QR > 0$.
6. $Q_1 > 0, Q_2 \geq 0 \Rightarrow Q = Q_1 + Q_2 > 0$.
7. $Q > 0 \Rightarrow z^*Qz > 0 \quad \forall$ nonzero $z \in \mathbb{C}^n$.
8. Given $Q \geq 0$, $x'Qx = 0$ if and only if $Qx = 0$.

You may want to use the Schur decomposition (Schur, 1909) of a matrix in establishing some of these eigenvalue results. Golub and Van Loan (1996, p.313) provide the following theorem

**Theorem A.1** (Schur decomposition)**.** *If $A \in \mathbb{C}^{n\times n}$ then there exists a unitary $Q \in \mathbb{C}^{n\times n}$ such that*

$$Q^*AQ = T$$

*in which $T$ is upper triangular.*

Note that because $T$ is upper triangular, its diagonal elements are the eigenvalues of $A$. Even if $A$ is a real matrix, $T$ can be complex because the eigenvalues of a real matrix may come in complex conjugate pairs. Recall a matrix $Q$ is unitary if $Q^*Q = I$. You should also be able to prove the following facts (Horn and Johnson, 1985).

1. If $A \in \mathbb{C}^{n\times n}$ and $BA = I$ for some $B \in \mathbb{C}^{n\times n}$, then
   (a) $A$ is nonsingular
   (b) $B$ is unique
   (c) $AB = I$
2. The matrix $Q$ is unitary if and only if
   (a) $Q$ is nonsingular and $Q^* = Q^{-1}$
   (b) $QQ^* = I$
   (c) $Q^*$ is unitary
   (d) The rows of $Q$ form an orthonormal set
   (e) The columns of $Q$ form an orthonormal set
3. If $A$ is real and symmetric, then $T$ is real and diagonal and $Q$ can be chosen real and orthogonal. It does not matter if the eigenvalues of $A$ are repeated.

For real, but not necessarily symmetric, $A$ you can restrict yourself to real matrices, by using the real Schur decomposition (Golub and Van Loan, 1996, p.341), but the price you pay is that you can achieve only block upper triangular $T$, rather than strictly upper triangular $T$.

**Theorem A.2** (Real Schur decomposition)**.** *If $A \in \mathbb{R}^{n\times n}$ then there exists an orthogonal $Q \in \mathbb{R}^{n\times n}$ such that*

$$Q'AQ = \begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1m} \\ 0 & R_{22} & \cdots & R_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_{mm} \end{bmatrix}$$

*in which each $R_{ii}$ is either a real scalar or a 2×2 real matrix having complex conjugate eigenvalues; the eigenvalues of $R_{ii}$ are the eigenvalues of $A$.*

If the eigenvalues of $R_{ii}$ are disjoint (i.e., the eigenvalues are not repeated), then $R$ can be taken block diagonal instead of block triangular (Golub and Van Loan, 1996, p.366).

## A.8 Norms in $\mathbb{R}^n$

A norm in $\mathbb{R}^n$ is a function $|\cdot| : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that

(a) $|x| = 0$ if and only if $x = 0$;

(b) $|\lambda x| = |\lambda|\,|x|$, for all $\lambda \in \mathbb{R}, x \in \mathbb{R}^n$;

(c) $|x + y| \leq |x| + |y|$, for all $x, y \in \mathbb{R}^n$.

Let $\mathcal{B} := \{x \mid |x| \leq 1\}$ denote the *closed* ball of radius 1 centered at the origin. For any $x \in \mathbb{R}^n$ and $\rho > 0$, we denote by $x \oplus \rho\mathcal{B}$ or $B(x, \rho)$ the *closed* ball $\{z \mid |z - x| \leq \rho\}$ of radius $\rho$ centered at $x$. Similarly $\{x \mid |x| < 1\}$ denotes the *open* ball of radius 1 centered at the origin and $\{z \mid |z - x| < \rho\}$ the *open* ball of radius $\rho$ centered at $x$; closed and open sets are defined below.

## A.9 Sets in $\mathbb{R}^n$

The complement of $S \subset \mathbb{R}^n$ in $\mathbb{R}^n$, is the set $S^c := \{x \in \mathbb{R}^n \mid x \notin S\}$. A set $X \subset \mathbb{R}^n$ is said to be *open*, if for every $x \in X$, there exists a $\rho > 0$ such that $B(x, \rho) \subseteq X$. A set $X \subset \mathbb{R}^n$ is said to be *closed* if $X^c$, its complement in $\mathbb{R}^n$, is open.

A set $X \subset \mathbb{R}^n$ is said to be *bounded* if there exists an $M < \infty$ such that $|x| \leq M$ for all $x \in X$. A set $X \subset \mathbb{R}^n$ is said to be *compact* if $X$ is closed and bounded. An element $x \in S \subseteq \mathbb{R}^n$ is an *interior* point of the set $S$ if there exists a $\rho > 0$ such that $z \in S$, for all $|z - x| < \rho$. The interior of a set $S \subset \mathbb{R}^n$, $\text{int}(S)$, is the set of all interior points of $S$; $\text{int}(S)$ is an open set, the *largest*[2] open subset of $S$. For example, if $S = [a, b] \subset \mathbb{R}$, then $\text{int}(S) = (a, b)$; as another example, $\text{int}(B(x, \rho)) = \{z \mid |z - x| < \rho\}$. The closure of a set $S \subset \mathbb{R}^n$, denoted $\bar{S}$, is the *smallest*[3] closed set containing $S$. For example, if $S = (a, b] \subset \mathbb{R}$, then $\bar{S} = [a, b]$. The boundary of $S \subset \mathbb{R}^n$, is the set $\delta S := \bar{S} \setminus \text{int}(S) = \{s \in \bar{S} \mid s \notin \text{int}(S)\}$. For example, if $S = (a, b] \subset \mathbb{R}$, then $\text{int}(S) = (a, b), \bar{S} = [a, b], \partial S = \{a, b\}$.

An *affine* set $S \subset \mathbb{R}^n$ is a set that can be expressed in the form $S = \{x\} \oplus \mathcal{V} := \{x + v \mid v \in \mathcal{V}\}$ for some $x \in \mathbb{R}^n$ and some subspace $\mathcal{V}$ of $\mathbb{R}^n$. An example is a line in $\mathbb{R}^n$ not passing through the origin. The *affine hull* of a set $S \subset \mathbb{R}^n$, denoted $\text{aff}(S)$, is the smallest[4] affine set that contains $S$. That is equivalent to the intersection of all affine sets containing $S$.

[2] Largest in the sense that every open subset of $S$ is a subset of $\text{int}(S)$.

[3] Smallest in the sense that $\bar{S}$ is a subset of any closed set containing $S$.

[4] In the sense that $\text{aff}(S)$ is a subset of any other affine set containing $S$.

Some sets $S$, such as a line in $\mathbb{R}^n, n \geq 2$, do not have an interior, but do have an interior *relative* to the smallest affine set in which $S$ lies, which is $\text{aff}(S)$ defined above. The *relative interior* of $S$ is the set $\{x \in S \mid \exists \rho > 0 \text{ such that } \text{int}(B(x,\rho)) \cap \text{aff}(S) \subset S\}$. Thus the line segment, $S := \{x \subset \mathbb{R}^2 | x = \lambda \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (1-\lambda) \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \lambda \in [0,1]\}$ does not have an interior, but does have an interior relative to the line containing it, $\text{aff}(S)$. The relative interior of $S$ is the open line segment $\{x \in \mathbb{R}^2 | x = \lambda \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (1-\lambda) \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \lambda \in (0,1)\}$.

## A.10 Sequences

Let the set of nonnegative integers be denoted by $\mathbb{I}_{\geq 0}$. A *sequence* is a function from $\mathbb{I}_{\geq 0}$ into $\mathbb{R}^n$. We denote a sequence by the set of its values, $\{x_i \mid i \in \mathbb{I}_{\geq 0}\}$. A *subsequence* of $\{x_i \mid i \in \mathbb{I}_{\geq 0}\}$ is a sequence of the form $\{x_i \mid i \in K\}$, where $K$ is an infinite subset of $\mathbb{I}_{\geq 0}$.

A sequence $\{x_i \mid i \in \mathbb{I}_{\geq 0}\}$ in $\mathbb{R}^n$ is said to *converge* to a point $\hat{x}$ if $\lim_{i \to \infty} |x_i - \hat{x}| = 0$, i.e., if, for all $\delta > 0$, there exists an integer $k$ such that $|x_i - \hat{x}| \leq \delta$ for all $i \geq k$; we write $x_i \to \hat{x}$ as $i \to \infty$ to denote the fact that the sequence $\{x_i\}$ converges to $\hat{x}$. The point $\hat{x}$ is called a *limit* of the sequence $\{x_i\}$. A point $x^*$ is said to be an *accumulation* point of a sequence $\{x_i \mid i \in \mathbb{I}_{\geq 0}\}$ in $\mathbb{R}^n$, if there exists an infinite subset $K \subset \mathbb{I}_{\geq 0}$ such that $x_i \to x^*$ as $i \to \infty, i \in K$ in which case we say $x_i \overset{K}{\to} x^*$.[5]

Let $\{x_i\}$ be a bounded infinite sequence in $\mathbb{R}$ and let the $S$ be the set of all accumulation points of $\{x_i\}$. Then $S$ is compact and $\limsup x_i$ is the largest and $\liminf x_i$ the smallest accumulation point of $\{x_i\}$:

$$\limsup_{i \to \infty} x_i := \max\{x \mid x \in S\}, \text{ and}$$
$$\liminf_{i \to \infty} x_i := \min\{x \mid x \in S\}$$

**Theorem A.3** (Bolzano-Weierstrass)**.** *Suppose $X \subset \mathbb{R}^n$ is compact and $\{x_i \mid i \in \mathbb{I}_{\geq 0}\} \subseteq X$. Then $\{x_i \mid i \in \mathbb{I}_{\geq 0}\}$ must have at least one accumulation point.*

From Exercise A.7, it follows that the accumulation point postulated by Theorem A.3 lies in $X$. In proving asymptotic stability we need the following property of monotone sequences.

[5] Be aware of inconsistent usage of the term *limit point.* Some authors use limit point as synonymous with limit. Others use limit point as synonymous with accumulation point. For this reason we avoid the term limit point.

**Proposition A.4** (Convergence of monotone sequences)**.** *Suppose that* $\{x_i \mid i \in \mathbb{I}_{\geq 0}\}$ *is a sequence in* $\mathbb{R}$ *such that* $x_0 \geq x_1 \geq x_2 \geq \ldots$*, i.e., suppose the sequence is monotone nonincreasing. If* $\{x_i\}$ *has an accumulation point* $x^*$*, then* $x_i \to x^*$ *as* $i \to \infty$*, i.e.,* $x^*$ *is a limit.*

*Proof.* For the sake of contradiction, suppose that $\{x_i \mid i \in \mathbb{I}_{\geq 0}\}$ does not converge to $x^*$. Then, for some $\rho > 0$, there exists a subsequence $\{x_i \mid i \in K\}$ such that $x_i \notin B(x^*, \rho)$ for all $i \in K$, i.e., $|x_i - x^*| > \rho$ for all $i \in K$. Since $x^*$ is an accumulation point, there exists a subsequence $\{x_i \mid i \in K^*\}$ such that $x_i \overset{K^*}{\to} x^*$. Hence there is an $i_1 \in K^*$ such that $|x_i - x^*| \leq \rho/2$, for all $i \geq i_1, i \in K^*$. Let $i_2 \in K$ be such that $i_2 > i_1$. Then we must have that $x_{i_2} \leq x_{i_1}$ and $|x_{i_2} - x^*| > \rho$, which leads to the conclusion that $x_{i_2} < x^* - \rho$. Now let $i_3 \in K^*$ be such that $i_3 > i_2$. Then we must have that $x_{i_3} \leq x_{i_2}$ and hence that $x_{i_3} < x^* - \rho$ which implies that $|x_{i_3} - x^*| > \rho$. But this contradicts the fact that $|x_{i_3} - x^*| \leq \rho/2$, and hence we conclude that $x_i \to x^*$ as $i \to \infty$. ■

It follows from Proposition A.4 that if $\{x_i \mid i \in \mathbb{I}_{\geq 0}\}$ is a monotone decreasing sequence in $\mathbb{R}$ bounded below by $b$, then the sequence $\{x_i \mid i \in \mathbb{I}_{\geq 0}\}$ converges to some $x^* \in \mathbb{R}$ where $x^* \geq b$.

## A.11 Continuity

We now summarize some essential properties of continuous functions.

1. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be *continuous at a point* $x \in \mathbb{R}^n$, if for every $\delta > 0$ there exists a $\rho > 0$ such that

$$|f(x') - f(x)| < \delta \ \forall x' \in \text{int}(B(x, \rho))$$

   A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be *continuous* if it is continuous at all $x \in \mathbb{R}^n$.

2. Let $X$ be a closed subset of $\mathbb{R}^n$. A function $f : X \to \mathbb{R}^m$ is said to be *continuous at a point* $x$ in $X$ if for every $\delta > 0$ there exists a $\rho > 0$ such that

$$|f(x') - f(x)| < \delta \ \forall x' \in \text{int}(B(x, \rho)) \cap X$$

   A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be *continuous* on $X$ if it is continuous at all $x$ in $X$.

3. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be *upper semicontinuous at a point* $x \in \mathbb{R}^n$, if for every $\delta > 0$ there exists a $\rho > 0$ such that

$$f(x') - f(x) < \delta \ \forall x' \in \text{int}(B(x, \rho))$$

A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be *upper semicontinuous* if it is upper semicontinuous at all $x \in \mathbb{R}^n$.

4. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be *lower semicontinuous at a point* $x \in \mathbb{R}^n$, if for every $\delta > 0$ there exists a $\rho > 0$ such that

$$f(x') - f(x) > -\delta \ \forall x' \in \text{int}(B(x, \rho))$$

A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be *lower semicontinuous* if it is continuous at all $x \in \mathbb{R}^n$.

5. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be *uniformly continuous* on a subset $X \subset \mathbb{R}^n$ if for any $\delta > 0$ there exists a $\rho > 0$ such that for any $x', x'' \in X$ satisfying $|x' - x''| < \rho$,

$$|f(x') - f(x'')| < \delta$$

**Proposition A.5** (Uniform continuity)**.** *Suppose that $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous and that $X \subset \mathbb{R}^n$ is compact. Then $f$ is uniformly continuous on $X$.*

*Proof.* For the sake of contradiction, suppose that $f$ is *not* uniformly continuous on $X$. Then, for some $\delta > 0$, there exist sequences $\{x_i'\}, \{x_i''\}$ in $X$ such that

$$|x_i' - x_i''| < (1/i), \text{ for all } i \in \mathbb{I}_{\geq 0}$$

but

$$|f(x_i') - f(x_i'')| > \delta, \text{ for all } i \in \mathbb{I}_{\geq 0} \tag{A.1}$$

Since $X$ is compact, there must exist a subsequence $\{x_i'\}_{i \in K}$ such that $x_i' \overset{K}{\to} x^* \in X$ as $i \to \infty$. Furthermore, because of (A.1), $x_i'' \overset{K}{\to} x^*$ also holds. Hence, since $f(\cdot)$ is continuous, we must have $f(x_i') \overset{K}{\to} f(x^*)$ and $f(x_i'') \overset{K}{\to} f(x^*)$. Therefore, there exists a $i_0 \in K$ such that for all $i \in K, i \geq i_0$

$$|f(x_i') - f(x_i'')| \leq |f(x_i') - f(x^*)| + |f(x^*) - f(x_i'')| < \delta/2$$

contradicting (A.1). This completes our proof. ∎

**Proposition A.6** (Compactness of continuous functions of compact sets)**.** *Suppose that $X \subset \mathbb{R}^n$ is compact and that $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous. Then the set*

$$f(X) := \{f(x) \mid x \in X\}$$

*is compact.*

*Proof.*

(a) First we show that $f(X)$ is closed. Thus, let $\{f(x_i) \mid i \in \mathbb{I}_{\geq 0}\}$, with $x_i \in X$, be any sequence in $f(X)$ such that $f(x_i) \to y$ as $i \to \infty$. Since $\{x_i\}$ is in a compact set $X$, there exists a subsequence $\{x_i \mid i \in K\}$ such that $x_i \overset{K}{\to} x^* \in X$ as $i \to \infty$. Since $f(\cdot)$ is continuous, $f(x_i) \overset{K}{\to} f(x^*)$ as $i \to \infty$. But $y$ is the limit of $\{f(x_i) \mid i \in \mathbb{I}_{\geq 0}\}$ and hence it is the limit of any subsequence of $\{f(x_i)\}$. We conclude that $y = f(x^*)$ and hence that $y \in f(X)$, i.e., $f(X)$ is closed.

(b) Next, we prove that $f(X)$ is bounded. Suppose $f(X)$ is not bounded. Then there exists a sequence $\{x_i\}$ such that $|f(x_i)| \geq i$ for all $i \in \mathbb{I}_{\geq 0}$. Now, since $\{x_i\}$ is in a compact set, there exists a subsequence $\{x_i \mid i \in K\}$ such that $x_i \overset{K}{\to} x^*$ with $x^* \in X$, and $f(x_i) \overset{K}{\to} f(x^*)$ by continuity of $f(\cdot)$. Hence there exists an $i_0$ such that for any $j > i > i_0, j, i \in K$

$$|f(x_j) - f(x_i)| \leq |f(x_j) - f(x^*)| + |f(x_i) - f(x^*)| < 1/2 \qquad \text{(A.2)}$$

Let $i \geq i_0$ be given. By hypothesis there exists a $j \in K, j \geq i$ such that $|f(x_j)| \geq j \geq |f(x_i)| + 1$. Hence

$$|f(x_j) - f(x_i)| \geq \big|\, |f(x_j)| - |f(x_i)| \,\big| \geq 1$$

which contradicts (A.2). Thus $f(X)$ must be bounded, which completes the proof. ∎

Let $Y \subset \mathbb{R}$. Then $\inf(Y)$, the *infimum* of $Y$, is defined to be the greatest lower bound[6] of $Y$. If $\inf(Y) \in Y$, then $\min(Y) := \min\{y \mid y \in Y\}$, the minimum of the set $Y$, exists and is equal to $\inf(Y)$. The infimum of a set $Y$ always exists if $Y$ is not empty and is bounded from below, in which case there always exist sequences $\{y_i\} \in Y$ such that $y_i \searrow \beta := \inf(Y)$ as $i \to \infty$. Note that $\beta := \inf(Y)$ does not necessarily lie in the set $Y$.

[6]The value $\alpha \in \mathbb{R}$ is the greatest lower bound of $Y$ if $y \geq \alpha$ for all $y \in Y$, and $\beta > \alpha$ implies that $\beta$ is *not* a lower bound for $Y$.

**Proposition A.7** (Weierstrass). *Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is continuous and that $X \subset \mathbb{R}^n$ is compact. Then there exists an $\hat{x} \in X$ such that*

$$f(\hat{x}) = \inf_{x \in X} f(x)$$

*i.e.,* $\min_{x \in X} f(x)$ *is well defined.*

*Proof.* Since $X$ is compact, $f(X)$ is bounded. Hence $\inf_{x \in X} f(x) = \alpha$ is finite. Let $\{x_i\}$ be an infinite sequence in $X$ such that $f(x_i) \searrow \alpha$ as $i \to \infty$. Since $X$ is compact, there exists a converging subsequence $\{x_i \mid i \in K\}$ such that $x_i \overset{K}{\to} \hat{x} \in X$. By continuity, $f(x_i) \overset{K}{\to} f(\hat{x})$ as $i \to \infty$. Because $\{f(x_i)\}$ is a monotone nonincreasing sequence that has an accumulation point $f(\hat{x})$, it follows from Proposition A.4 that $f(x_i) \to f(\hat{x})$ as $i \to \infty$. Since the limit of the sequence $\{f(x_i)\}$ is unique, we conclude that $f(\hat{x}) = \alpha$. ∎

## A.12 Derivatives

We first define some notation. If $f : \mathbb{R}^n \to \mathbb{R}$, then $(\partial/\partial x)f(x)$ is a *row* vector defined by

$$(\partial/\partial x)f(x) := [(\partial/\partial x_1)f(x), \ldots, (\partial/\partial x_n)f(x)]$$

provided the partial derivatives $(\partial/\partial x_i)f(x)$, $i = 1, 2, \ldots, n$ exist. Similarly, if $f : \mathbb{R}^n \to \mathbb{R}^m$, $(\partial/\partial x)f(x)$ is defined to be the matrix

$$(\partial/\partial x)f(x) := \begin{bmatrix} (\partial/\partial x_1)f_1(x) & (\partial/\partial x_2)f_1(x) & \ldots & (\partial/\partial x_n)f_1(x) \\ (\partial/\partial x_1)f_2(x) & (\partial/\partial x_2)f_2(x) & \ldots & (\partial/\partial x_n)f_2(x) \\ \vdots & \vdots & \vdots & \vdots \\ (\partial/\partial x_1)f_m(x) & (\partial/\partial x_2)f_m(x) & \ldots & (\partial/\partial x_n)f_m(x) \end{bmatrix}$$

where $x_i$ and $f_i$ denote, respectively, the $i$th component of the vectors $x$ and $f$. We sometimes use $f_x(x)$ in place of $(\partial/\partial x)f(x)$. If $f : \mathbb{R}^n \to \mathbb{R}$, then its *gradient* $\nabla f(x)$ is a *column* vector defined by

$$\nabla f(x) := \begin{bmatrix} (\partial/\partial x_1)f(x) \\ (\partial/\partial x_2)f(x) \\ \vdots \\ (\partial/\partial x_n)f(x) \end{bmatrix}$$

and its *Hessian* is $\nabla^2 f(x) = (\partial^2/\partial x^2) f(x) = f_{xx}(x)$ defined by

$$\nabla^2 f(x) := \begin{bmatrix} (\partial^2/\partial x_1^2) f(x) & (\partial^2/\partial x_1 \partial x_2) f(x) & \dots & (\partial^2/\partial x_1 \partial x_n) f(x) \\ (\partial^2/\partial x_2 \partial x_1) f(x) & (\partial x_2^2) f(x) & \dots & (\partial^2/\partial x_2 \partial x_n) f(x) \\ \vdots & \vdots & \ddots & \vdots \\ (\partial^2/\partial x_n \partial x_1) f(x) & (\partial^2/\partial x_n \partial x_2) f(x) & \dots & (\partial^2/\partial x_n^2) f(x) \end{bmatrix}$$

We note that $\nabla f(x) = [(\partial/\partial x) f(x)]' = f_x'(x)$.

We now define what we mean by the derivative of $f(\cdot)$. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a continuous function with domain $\mathbb{R}^n$. We say that $f(\cdot)$ is differentiable at $\hat{x}$ if there exists a matrix $Df(\hat{x}) \in \mathbb{R}^{m \times n}$ (the Jacobian) such that

$$\lim_{h \to 0} \frac{|f(\hat{x} + h) - f(\hat{x}) - Df(\hat{x})h|}{|h|} = 0$$

in which case $Df(\cdot)$ is called the derivative of $f(\cdot)$ at $\hat{x}$. When $f(\cdot)$ is differentiable at all $x \in \mathbb{R}^n$, we say that $f$ is *differentiable*.

We note that the affine function $h \mapsto f(\hat{x}) + Df(\hat{x})h$ is a first order approximation of $f(\hat{x} + h)$. The Jacobian can be expressed in terms of the partial derivatives of $f(\cdot)$.

**Proposition A.8** (Derivative and partial derivative)**.** *Suppose that the function $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\hat{x}$. Then its derivative $Df(\hat{x})$ satisfies*

$$Df(\hat{x}) = f_x(\hat{x}) := \partial f(\hat{x})/\partial x$$

*Proof.* From the definition of $Df(\hat{x})$ we deduce that for each $i \in \{1, 2, \dots, m\}$

$$\lim_{h \to 0} \frac{|f_i(\hat{x} + h) - f_i(\hat{x}) - Df_i(\hat{x})h|}{|h|} = 0$$

where $f_i$ is the $i$th element of $f$ and $(Df)_i$ the $i$th row of $Df$. Set $h = te_j$, where $e_j$ is the $j$-th unit vector in $\mathbb{R}^n$ so that $|h| = t$. Then $(Df)_i(\hat{x})h = t(Df)_i(\hat{x})e_j = (Df)_{ij}(\hat{x})$, the $ij$th element of the matrix $Df(\hat{x})$. It then follows that

$$\lim_{t \searrow 0} \frac{|f^i(\hat{x} + te_j) - f(\hat{x}) - t(Df)_{ij}(\hat{x})|}{t} = 0$$

which shows that $(Df)_{ij}(\hat{x}) = \partial f_i(\hat{x})/\partial x_j$. ∎

A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is *locally Lipschitz continuous* at $\hat{x}$ if there exist $L \in [0, \infty)$, $\hat{\rho} > 0$ such that

$$|f(x) - f(x')| \leq L|x - x'|, \text{ for all } x, x' \in B(\hat{x}, \hat{\rho})$$

The function $f$ is globally Lipschitz continuous if the inequality holds for all $x, x' \in \mathbb{R}^n$. The constant $L$ is called the *Lipschitz constant* of $f$. It should be noted that the existence of partial derivatives of $f(\cdot)$ does not ensure the existence of the derivative $Df(\cdot)$ of $f(\cdot)$; see e.g. Apostol (1974, p.103). Thus consider the function

$$f(x, y) = x + y \text{ if } x = 0 \text{ or } y = 0$$

$$f(x, y) = 1 \text{ otherwise}$$

In this case

$$\frac{\partial f(0,0)}{\partial x} = \lim_{t \to 0} \frac{f(t,0) - f(0,0)}{t} = 1$$
$$\frac{\partial f(0,0)}{\partial y} = \lim_{t \to 0} \frac{f(0,t) - f(0,0)}{t} = 1$$

but the function is not even continuous at $(0,0)$. In view of this, the following result is relevant.

**Proposition A.9** (Continuous partial derivatives)**.** *Consider a function $f : \mathbb{R}^n \to \mathbb{R}^m$ such that the partial derivatives $\partial f^i(x)/dx^j$ exist in a neighborhood of $\hat{x}$, for $i = 1, 2, \ldots, n, j = 1, 2, \ldots, m$. If these partial derivatives are continuous at $\hat{x}$, then the derivative $Df(\hat{x})$ exists and is equal to $f_x(\hat{x})$.*

The following *chain rule* holds.

**Proposition A.10** (Chain rule)**.** *Suppose that $f : \mathbb{R}^n \to \mathbb{R}^m$ is defined by $f(x) = h(g(x))$ with both $h : \mathbb{R}^l \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^l$ differentiable. Then*

$$\frac{\partial f(\hat{x})}{\partial x} = \frac{\partial h(g(\hat{x}))}{\partial y} \frac{\partial g(\hat{x})}{\partial x}$$

The following result Dieudonne (1960), replaces, *inter alia*, the mean value theorem for functions $f : \mathbb{R}^n \to \mathbb{R}^m$ when $m > 1$.

**Proposition A.11** (Mean value theorem for vector functions)**.**

*(a) Suppose that $f : \mathbb{R}^n \to \mathbb{R}^m$ has continuous partial derivatives at each point $x$ of $\mathbb{R}^n$. Then for any $x, y \in \mathbb{R}^n$,*

$$f(y) = f(x) + \int_0^1 f_x(x + s(y - x))(y - x)ds$$

*(b) Suppose that $f : \mathbb{R}^n \to \mathbb{R}^m$ has continuous partial derivatives of order two at each point $x$ of $\mathbb{R}^n$. Then for any $x, y \in \mathbb{R}^n$,*

$$f(y) = f(x)+f_x(x)(y-x)+\int_0^1 (1-s)(y-x)' f_{xx}(x+s(y-x))(y-x)ds$$

*Proof.*

(a) Consider the function $g(s) = f(x + s(y - x))$ where $f : \mathbb{R}^n \to \mathbb{R}^m$. Then $g(1) = f(y), g(0) = f(x)$ and

$$\begin{aligned} g(1) - g(0) &= \int_0^1 g'(s)ds \\ &= \int_0^1 Df(x + s(y - x))(y - x)ds \end{aligned}$$

which completes the proof for $p = 1$.

(b) Consider the function $g(s) = f(x + s(y - x))$ where $f : \mathbb{R}^n \to \mathbb{R}$. Then

$$\frac{d}{ds}[g'(s)(1 - s) + g(s)] = g''(s)(1 - s)$$

Integrating from 0 to 1 yields

$$g(1) - g(0) - g'(0) = \int_0^1 (1 - s)g''(s)ds$$

But $g''(s) = (y - x)' f_{xx}(x + s(y - x))(y - x)$ so that the last equation yields

$$f(y)-f(x) = f_x(x)(y-x)+\int_0^1 (1-s)(y-x)' f_{xx}(x+s(y-x))(y-x)ds$$

when $g(s)$ is replaced by $f(x + s(y - x))$.

■

Finally, we define directional derivatives which may exist even when a function fails to have a derivative. Let $f : \mathbb{R}^n \to \mathbb{R}^m$. We define the *directional derivative* of $f$ at a point $\hat{x} \in \mathbb{R}^n$ in the direction $h \in \mathbb{R}^n (h \neq 0)$ by

$$df(\hat{x}; h) := \lim_{t \searrow 0} \frac{f(\hat{x} + th) - f(\hat{x})}{t}$$

if this limit exists (note that $t > 0$ is required). The directional derivative is positively homogeneous, i.e., $df(x; \lambda h) = \lambda df(x; h)$ for all $\lambda > 0$.

Not all the functions we discuss are differentiable everywhere. Examples include the max function $\psi(\cdot)$ defined by $\psi(x) := \max_i\{f^i(x) \mid i \in I\}$ where each function $f^i : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable everywhere. The function $\psi(\cdot)$ is not differentiable at those $x$ for which the active set $I^0(x) := \{i \in I \mid f^i(x) = \psi(x)\}$ has more than one element. The directional derivative $d(x;h)$ exists for all $x, h$ in $\mathbb{R}^n$, however, and is given by

$$d\psi(x;h) = \max_i\{df_i(x;h) \mid i \in I^0(x)\} = \max_i\{\langle \nabla f_i(x), h\rangle \mid i \in I^0(x)\}$$

When, as in this example, the directional derivative exists for all $x, h$ in $\mathbb{R}^n$ we can define a generalization, called the *subgradient*, of the conventional gradient. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ has a directional derivative for all $x, h$ in $\mathbb{R}^n$. The $f(\cdot)$ has a subgradient $\partial f(\cdot)$ defined by

$$\partial\psi(x) := \{g \in \mathbb{R}^n \mid df(x;h) \geq \langle g, h\rangle \ \forall h \in \mathbb{R}^n\}$$

The subgradient at a point $x$ is, unlike the ordinary gradient, a set. For our max example ($f(x) = \psi(x) = \max_i\{f_i(x) \mid i \in I\}$) we have $d\psi(x;h) = \max_i\{\langle \nabla f^i(x), h\rangle \mid i \in I^0(x)\}$. In this case, it can be shown that

$$\partial\psi(x) = \text{co}\{\nabla f^i(x) \mid i \in I^0(x)\}$$

If the directional derivative $h \mapsto df(x;h)$ is convex, then the subgradient $\partial f(x)$ is nonempty and the directional derivative $df(x;h)$ may be expressed as

$$df(x;h) = \max_g\{\langle g, h\rangle \mid g \in \partial f(x)\}$$

Figure A.4 illustrates this for the case when $\psi(x) := \max\{f_1(x), f_2(x)\}$ and $I^0(x) = \{1, 2\}$.

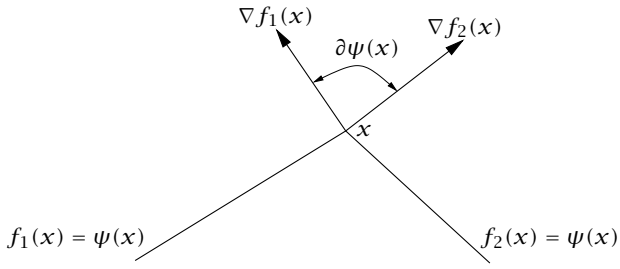


**Figure A.4:** Subgradient.

## A.13 Convex Sets and Functions

Convexity is an enormous subject. We collect here only a few essential results that we will need in our study of optimization; for further details see Rockafellar (1970). We begin with convex sets.

### A.13.1 Convex Sets

**Definition A.12** (Convex set)**.** A set $S \in \mathbb{R}^n$ is said to be *convex* if, for any $x', x'' \in S$ and $\lambda \in [0,1]$, $(\lambda x' + (1-\lambda)x'') \in S$.

Let $S$ be a subset of $\mathbb{R}^n$. We say that $\text{co}(S)$ is the *convex hull* of $S$ if it is the smallest[7] convex set containing $S$.

**Theorem A.13** (Caratheodory)**.** *Let $S$ be a subset of $\mathbb{R}^n$. If $\bar{x} \in co(S)$, then it may be expressed as a convex combination of no more than $n+1$ points in $S$, i.e., there exist $m \leq n+1$ distinct points, $\{x_i\}_{i=1}^m$, in $S$ such that $\bar{x} = \sum_{i=1}^m \mu^i x_i, \mu^i > 0, \sum_{i=1}^m \mu^i = 1$.*

*Proof.* Consider the set

$$C_s := \{x \mid x = \sum_{i=1}^{k_x} \mu^i x_i, x_i \in S, \mu^i \geq 0, \sum_{i=1}^{k_x} \mu^i = 1, k_x \in \mathbb{I}_{\geq 0}\}$$

First, it is clear that $S \subset C_s$. Next, since for any $x', x'' \in C_s, \lambda x' + (1-\lambda x'') \in C_s$, for $\lambda \in [0,1]$, it follows that $C_s$ is convex. Hence we must have that $\text{co}(S) \subset C_s$. Because $C_s$ consists of all the convex combinations of points in $S$, however, we must also have that $C_s \subset \text{co}(S)$. Hence $C_s = \text{co}(S)$. Now suppose that

$$\bar{x} = \sum_{i=1}^{\bar{k}} \bar{\mu}^i x_i$$

with $\bar{\mu}^i \geq 0, i = 1, 2, \ldots, \bar{k}, \sum_{i=1}^{\bar{k}} \bar{\mu}^i = 1$. Then the following system of equations is satisfied

$$\sum_{i=1}^{\bar{k}} \bar{\mu}^i \begin{bmatrix} x_i \\ 1 \end{bmatrix} = \begin{bmatrix} \bar{x} \\ 1 \end{bmatrix} \tag{A.3}$$

with $\bar{\mu}^i \geq 0$. Suppose that $\bar{k} > n+1$. Then there exist coefficients $\alpha^j$, $j = 1, 2, \ldots, \bar{k}$, not all zero, such that

$$\sum_{i=1}^{\bar{k}} \alpha^i \begin{bmatrix} x_i \\ 1 \end{bmatrix} = 0 \tag{A.4}$$

[7] Smallest in the sense that any other convex set containing $S$ also contains $\text{co}(S)$.

Adding (A.4) multiplied by $\theta$ to (A.3) we get

$$\sum_{i=1}^{\bar{k}} (\bar{\mu}^i + \theta\alpha^i) \begin{bmatrix} x_i \\ 1 \end{bmatrix} = \begin{bmatrix} \bar{x} \\ 1 \end{bmatrix}$$

Suppose, without loss of generality, that at least one $\alpha^i < 0$. Then there exists a $\bar{\theta} > 0$ such that $\bar{\mu}^j + \bar{\theta}\alpha^j = 0$ for some $j$ while $\bar{\mu}^i + \bar{\theta}\alpha^i \geq 0$ for all other $i$. Thus we have succeeded in expressing $\bar{x}$ as a convex combination of $\bar{k} - 1$ vectors in $S$. Clearly, these reductions can go on as long as $\bar{x}$ is expressed in terms of more than $(n + 1)$ vectors in $S$. This completes the proof. ■

Let $S_1, S_2$ be any two sets in $\mathbb{R}^n$. We say that the hyperplane

$$H = \{x \in \mathbb{R}^n \mid \langle x, v \rangle = \alpha\}$$

*separates* $S_1$ and $S_2$ if

$$\langle x, v \rangle \geq \alpha \text{ for all } x \in S_1$$
$$\langle y, v \rangle \leq \alpha \text{ for all } y \in S_2$$

The separation is said to be *strong* if there exists an $\varepsilon > 0$ such that

$$\langle x, v \rangle \geq \alpha + \varepsilon \text{ for all } x \in S_1$$
$$\langle y, v \rangle \leq \alpha - \varepsilon \text{ for all } y \in S_2$$

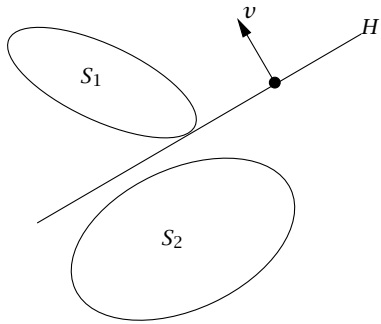


**Figure A.5:** Separating hyperplane.

**Theorem A.14** (Separation of convex sets)**.** *Let $S_1, S_2$ be two convex sets in $\mathbb{R}^n$ such that $S_1 \cap S_2 = \emptyset$. Then there exists a hyperplane which separates $S_1$ and $S_2$. Furthermore, if $S_1$ and $S_2$ are closed and either $S_1$ or $S_2$ is compact, then the separation can be made strict.*

**Theorem A.15** (Separation of convex set from zero)**.** *Suppose that* $S \subset \mathbb{R}^n$ *is closed and convex and* $0 \notin S$*. Let*

$$\hat{x} = \arg\min\{|x|^2 \mid x \in S\}$$

*Then*

$$H = \{x \mid \langle \hat{x}, x \rangle = |\hat{x}|^2\}$$

*separates* $S$ *from 0, i.e.,* $\langle \hat{x}, x \rangle \geq |\hat{x}|^2$ *for all* $x \in S$*.*

*Proof.* Let $x \in S$ be arbitrary. Then, since $S$ is convex, $[\hat{x} + \lambda(x - \hat{x})] \in S$ for all $\lambda \in [0, 1]$. By definition of $\hat{x}$, we must have

$$\begin{aligned} 0 < |\hat{x}|^2 &\leq |\hat{x} + \lambda(x - \hat{x})|^2 \\ &= |\hat{x}|^2 + 2\lambda\langle \hat{x}, x - \hat{x} \rangle + \lambda^2 |x - \hat{x}|^2 \end{aligned}$$

Hence, for all $\lambda \in (0, 1]$,

$$0 \leq 2\langle \hat{x}, x - \hat{x} \rangle + \lambda |x - \hat{x}|^2$$

Letting $\lambda \to 0$ we get the desired result. ∎

Theorem A.15 can be used to prove the following special case of Theorem A.14:

**Corollary A.16** (Existence of separating hyperplane)**.** *Let* $S_1, S_2$ *be two compact convex sets in* $\mathbb{R}^n$ *such that* $S_1 \cap S_2 = \emptyset$*. Then there exists a hyperplane which separates* $S_1$ *and* $S_2$*.*

*Proof.* Let $C = S_1 - S_2 := \{x_1 - x_2 \mid x_1 \in S_1, x_2 \in S_2\}$. Then $C$ is convex and compact and $0 \notin C$. Let $\hat{x} = (\hat{x}_1 - \hat{x}_2) = \arg\min\{|x|^2 \mid x \in C\}$, where $\hat{x}_1 \in S_1$ and $\hat{x}_2 \in S_2$. Then, by Theorem A.15

$$\langle x - \hat{x}, \hat{x} \rangle \geq 0, \text{ for all } x \in C \tag{A.5}$$

Let $x = x_1 - \hat{x}_2$, with $x_1 \in S_1$. Then (A.5) leads to

$$\langle x_1 - \hat{x}_2, \hat{x} \rangle \geq |\hat{x}|^2 \tag{A.6}$$

for all $x_1 \in S_1$. Similarly, letting $x = \hat{x}_1 - x_2$, in (A.5) yields

$$\langle \hat{x}_1 - x_2, \hat{x} \rangle \geq |\hat{x}|^2 \tag{A.7}$$

for all $x_2 \in S_2$. The inequality in (A.7) implies that

$$\langle \hat{x}_1 - \hat{x}_2 + \hat{x}_2 - x_2, \hat{x} \rangle \geq |\hat{x}|^2$$

Since $\hat{x}_1 - \hat{x}_2 = \hat{x}$, we obtain

$$\langle x_2 - \hat{x}_2, \hat{x} \rangle \leq 0 \tag{A.8}$$

for all $x_2 \in S_2$. The desired result follows from (A.6) and (A.8), the separating hyperplane $H$ being $\{x \in \mathbb{R}^n \mid \langle \hat{x}, x - \hat{x}_2 \rangle = 0\}$. ■

**Definition A.17** (Support hyperplane)**.** Suppose $S \subset \mathbb{R}^n$ is convex. We say that $H = \{x \mid \langle x - \bar{x}, v \rangle = 0\}$ is a *support hyperplane* to $S$ through $\bar{x}$ with *inward (outward) normal* $v$ if $\bar{x} \in \overline{S}$ and

$$\langle x - \bar{x}, v \rangle \geq 0 \ (\leq 0) \text{ for all } x \in S$$

**Theorem A.18** (Convex set and halfspaces)**.** *A closed convex set is equal to the intersection of the halfspaces which contain it.*

*Proof.* Let $C$ be a closed convex set and $A$ the intersection of halfspaces containing $C$. Then clearly $C \subset A$. Now suppose $\bar{x} \notin C$. Then there exists a support hyperplane $H$ which separates strictly $\bar{x}$ and $C$ so that $\bar{x}$ does not belong to one halfspace containing $C$. It follows that $\bar{x} \notin A$. Hence $C^c \subset A^c$ which leads to the conclusion that $A \subset C$. ■

An important example of a convex set is a convex *cone*.

**Definition A.19** (Convex cone)**.** A subset $C$ of $\mathbb{R}^n$, $C \neq \emptyset$, is called a *cone* if $x \in C$ implies $\lambda x \in C$ for all $\lambda \geq 0$. A cone $C$ is *pointed* if $C \cap -C = \{0\}$. A *convex* cone is a cone that is convex.

An example of a cone is a halfspaces with a boundary that is a hyperplane passing through the origin; an example of a pointed cone is the positive orthant. A polyhedron $C$ defined by $C := \{x \mid \langle a_i, x \rangle \leq 0, \ i \in \mathcal{I}\}$ is a convex cone that is pointed

**Definition A.20** (Polar cone)**.** Given a cone $C \subset \mathbb{R}^n$, the cone $C^*$ defined by

$$C^* := \{h \mid \langle h, x \rangle \leq 0 \ \forall x \in C\}$$

is called the *polar cone* of $C$.

An illustration of this definition when $C$ is a polyhedron containing the origin is given in Figure A.6. In this figure, $H$ is the hyperplane with normal $h$ passing through the origin.

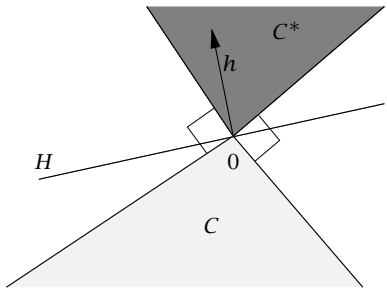


**Figure A.6:** Polar cone.

**Definition A.21** (Cone generator)**.** A cone $K$ is said to be *generated* by a set $\{a_i \mid i \in \mathcal{I}\}$ where $\mathcal{I}$ is an index set if

$$K = \left\{ \sum_{i \in \mathcal{I}} \mu_i a_i \mid \mu_i \geq 0, i \in \mathcal{I} \right\}$$

in which case we write $K = \text{cone}\{a_i \mid i \in \mathcal{I}\}$.

We make use of the following result:

**Proposition A.22** (Cone and polar cone generator)**.**
*(a) Suppose $C$ is a convex cone containing the origin and defined by*

$$C := \{x \in \mathbb{R}^n \mid \langle a_i, x \rangle \leq 0,\ i \in \mathcal{I}\}$$

*Then*

$$C^* = \text{cone}\{a_i \mid i \in \mathcal{I}\}$$

*(b) If $C$ is a closed convex cone, then $(C^*)^* = C$.*

*(c) If $C_1 \subset C_2$, then $C_2^* \subset C_1^*$.*

*Proof.*
(a) Let the convex set $K$ be defined by

$$K := \text{cone}\{a_i \mid i \in \mathcal{I}\}$$

We wish to prove $C^* = K$. To prove $K \subset C^*$, suppose $h$ is an arbitrary point in $K := \text{cone}\{a_i \mid i \in \mathcal{I}\}$. Then $h = \sum_{i \in \mathcal{I}} \mu_i a_i$ where $\mu_i \geq 0$ for all $i \in \mathcal{I}$. Let $x$ be an arbitrary point in $C$ so that $\langle a_i, x \rangle \leq 0$ for all $i \in \mathcal{I}$. Hence

$$\langle h, x \rangle = \langle \sum_{i \in \mathcal{I}} \mu_i a_i, x \rangle = \sum_{i \in \mathcal{I}} \mu_i \langle a_i, x \rangle \leq 0$$

so that $h \in C^*$. This proves that $K \subset C^*$. To prove that $C^* \subset K$, assume that $h \in C^*$ but that, contrary to what we wish to prove, $h \notin K$. Hence $h = \sum_{i \in \mathcal{I}} \mu_i a_i + \widetilde{h}$ where either $\mu_j > 0$ for at least one $j \in \mathcal{I}$, or $\widetilde{h}$, which is orthogonal to $a_i, i \in \mathcal{I}$, is not zero, or both. If $\mu_j < 0$, let $x \in C$ be such that $\langle a_i, x \rangle = 0$ for all $i \in \mathcal{I}, i \neq j$ and $\langle a_j, x \rangle < 0$; if $\widetilde{h} \neq 0$, let $x \in C$ be such that $\langle \widetilde{h}, x \rangle > 0$ (both conditions can be satisfied). Then

$$\langle h, x \rangle = \langle \mu_j a_j, x \rangle + \langle \widetilde{h}, x \rangle = \mu_j \langle a_j, x \rangle + \langle \widetilde{h}, x \rangle > 0$$

since either both $\mu_j$ and $\langle a_j, x \rangle$ are strictly negative or $\widetilde{h} \neq 0$ or both. This contradicts the fact that $x \in C$ and $h \in C^*$ (so that $\langle h, x \rangle \leq 0$). Hence $h \in K$ so that $C^* \subset K$. It follows that $C^* = \text{cone}\{a_i \mid i \in \mathcal{I}\}$.

(b) That $(C^*)^* = C$ when $C$ is a closed convex cone is given in Rockafellar and Wets (1998), Corollary 6.21.

(c) This result follows directly from the definition of a polar cone.

■

### A.13.2 Convex Functions

Next we turn to convex functions. For an example see Figure A.7.

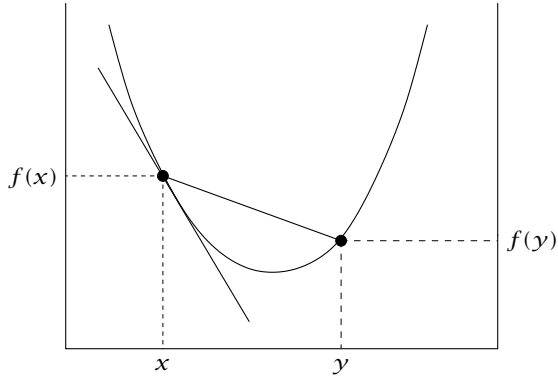


**Figure A.7:** A convex function.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be *convex* if for any $x', x'' \in \mathbb{R}^n$ and $\lambda \in [0, 1]$,

$$f(\lambda x' + (1 - \lambda)x'') \leq \lambda f(x') + (1 - \lambda) f(x'')$$

A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be *concave* if $-f$ is convex.

The *epigraph* of a function $f : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\text{epi}(f) := \{(x, y) \in \mathbb{R}^n \times \mathbb{R} \mid y \geq f(x)\}$$

**Theorem A.23** (Convexity implies continuity)**.** *Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is convex. Then $f$ is continuous in the interior of it domain.*

The following property is illustrated in Figure A.7.

**Theorem A.24** (Differentiability and convexity)**.** *Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable. Then $f$ is convex if and only if*

$$f(y) - f(x) \geq \langle \nabla f(x), y - x \rangle \text{ for all } x, y \in \mathbb{R}^n \tag{A.9}$$

*Proof.* $\Rightarrow$ Suppose $f$ is convex. Then for any $x, y \in \mathbb{R}^n$, and $\lambda \in [0, 1]$

$$f(x + \lambda(y - x)) \leq (1 - \lambda) f(x) + \lambda f(y) \tag{A.10}$$

Rearranging (A.10) we get

$$\frac{f(x + \lambda(y - x)) - f(x)}{\lambda} \leq f(y) - f(x) \text{ for all } \lambda \in [0, 1]$$

Taking the limit as $\lambda \to 0$ we get (A.9).

$\Leftarrow$ Suppose (A.9) holds. Let $x$ and $y$ be arbitrary points in $\mathbb{R}^n$ and let $\lambda$ be an arbitrary point in $[0, 1]$. Let $z = \lambda x + (1 - \lambda) y$. Then

$$\begin{aligned} f(x) &\geq f(z) + f'(z)(x - z), \text{ and} \\ f(y) &\geq f(z) + f'(z)(y - z) \end{aligned}$$

Multiplying the first equation by $\lambda$ and the second by $(1 - \lambda)$, adding the resultant equations, and using the fact that $z = \lambda x + (1 - \lambda) y$ yields

$$\lambda f(x) + (1 - \lambda) f(y) \geq f(z) = f(\lambda x + (1 - \lambda) y)$$

Since $x$ and $y$ in $\mathbb{R}^n$ and $\lambda$ in $[0, 1]$ are all arbitrary, the convexity of $f(\cdot)$ is established. ∎

**Theorem A.25** (Second derivative and convexity)**.** *Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable. Then $f$ is convex if and only if the Hessian (second derivative) matrix $\partial^2 f(x)/\partial x^2$ is positive semidefinite for all $x \in \mathbb{R}^n$, i.e., $\langle y, \partial^2 f(x)/\partial x^2 y \rangle \geq 0$ for all $x, y \in \mathbb{R}^n$.*

*Proof.* ⇒ Suppose $f$ is convex. Then for any $x, y \in \mathbb{R}^n$, because of Theorem A.24 and Proposition A.11

$$\begin{aligned} 0 &\le f(y) - f(x) - \langle \nabla f(x), y - x \rangle \\ &= \int_0^1 (1-s) \left\langle y - x, \frac{\partial^2 f(x + s(y-x))}{\partial x^2}(y-x) \right\rangle ds \end{aligned} \tag{A.11}$$

Hence, dividing by $|y-x|^2$ and letting $y \to x$, we obtain that $\partial^2 f(x)/\partial x^2$ is positive semidefinite.

⇐ Suppose that $\partial^2 f(x)/\partial x^2$ is positive semidefinite for all $x \in \mathbb{R}$. Then it follows directly from the equality in (A.11) and Theorem A.24 that $f$ is convex. ■

**Definition A.26** (Level set)**.** Suppose $f : \mathbb{R}^n \to \mathbb{R}$. A *level set* of $f$ is a set of the form $\{x \mid f(x) = \alpha\}, \alpha \in \mathbb{R}$.

**Definition A.27** (Sublevel set)**.** Suppose $f : \mathbb{R}^n \to \mathbb{R}$. A *sublevel set* $\mathbb{X}$ of $f$ is a set of the form $\mathbb{X} = \{x \mid f(x) \le \alpha\}, \alpha \in \mathbb{R}$. We also write the sublevel set as $\mathbb{X} = \text{lev}_\alpha f$.

**Definition A.28** (Support function)**.** Suppose $Q \subset \mathbb{R}^n$. The support function $\sigma_Q : \mathbb{R}^n \to \mathbb{R}_e = \mathbb{R} \cup \{+\infty\}$ is defined by:

$$\sigma_Q(p) = \sup_x \{\langle p, x \rangle \mid x \in Q\}$$

$\sigma_Q(p)$ measures how far $Q$ extends in direction $p$.

**Proposition A.29** (Set membership and support function)**.** *Suppose $Q \subset \mathbb{R}^n$ is a closed and convex set. Then $x \in Q$ if and only if $\sigma_Q(p) \ge \langle p, x \rangle$ for all $p \in \mathbb{R}^n$*

**Proposition A.30** (Lipschitz continuity of support function)**.** *Suppose $Q \subset \mathbb{R}^n$ is bounded. Then $\sigma_Q$ is bounded and Lipschitz continuous $|\sigma_Q(p) - \sigma_Q(q)| \le K|p-q|$ for all $p, q \in \mathbb{R}^n$, where $K := \sup\{|x| \mid x \in Q\} < \infty$.*

## A.14 Differential Equations

Although difference equation models are employed extensively in this book, the systems being controlled are most often described by differential equations. Thus, if the system being controlled is described by the differential equation $\dot{x} = f_c(x, u)$, as is often the case, and if it

is decided to control the system using piecewise constant control with period $\Delta$, then, at sampling instants $k\Delta$ where $k \in \mathbb{I}$, the system is described by the difference equation

$$x^+ = f(x, u)$$

then $f(\cdot)$ may be derived from $f_c(\cdot)$ as follows

$$f(x, u) = x + \int_0^\Delta f_c(\phi_c(s; x, u), u) ds$$

where $\phi_c(s; x, u)$ is the solution of $\dot{x} = f_c(x, u)$ at time $s$ if its initial state at time 0 is $x$ and the control has a constant value $u$ in the interval $[0, \Delta]$. Thus $x$ in the difference equation is the state at time $k$, say, $u$ is the control in the interval $[0, \Delta]$, and $x^+$ is the state at time $k + 1$.

Because the discrete time system is most often obtained by a continuous time system, we must be concerned with conditions which guarantee the existence and uniqueness of solutions of the differential equation describing the continuous time system. For excellent expositions of the theory of ordinary differential equations see the books by Hale (1980), McShane (1944), Hartman (1964), and Coddington and Levinson (1955).

Consider, first, the unforced system described by

$$(d/dt)x(t) = f(x(t), t) \text{ or } \dot{x} = f(x, t) \tag{A.12}$$

with initial condition

$$x(t_0) = x_0 \tag{A.13}$$

Suppose $f : D \to \mathbb{R}^n$, where $D$ is an open set in $\mathbb{R}^n \times \mathbb{R}$, is continuous. A function $x : T \to \mathbb{R}^n$, where $T$ is an interval in $\mathbb{R}$, is said to be a (conventional) solution of (A.12) with initial condition (A.13) (or passing through $(x_0, t_0)$) if:

(a) $x$ is continuously differentiable and $x$ satisfies (A.12) on $T$,

(b) $x(t_0) = x_0$,

and $(x(t), t) \in D$ for all $t$ in $T$. It is easily shown, when $f$ is continuous, that $x$ satisfies (A.12) and (A.13) if and only if:

$$x(t) = x_0 + \int_{t_0}^{t} f(x(s), s) ds \tag{A.14}$$

Peano's existence theorem states that if $f$ is continuous on $D$, then, for all $(x_0, t_0) \in D$ there exists at least one solution of (A.12)) passing through $(x_0, t_0)$. The solution is not necessarily unique - a counter example being $\dot{x} = \sqrt{x}$ for $x \geq 0$. To proceed we need to be able to deal with systems for which $f(\cdot)$ is not necessarily continuous for the following reason. If the system is described by $\dot{x} = f(x, u, t)$ where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous, and the control $u : \mathbb{R} \to \mathbb{R}^m$ is continuous, then, for given $u(\cdot)$, the function $f^u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ defined by:

$$f^u(x, t) := f(x, u(t), t)$$

is continuous in $t$. We often encounter controls that are not continuous, however, in which case $f^u(\cdot)$ is also not continuous. We need a richer class of controls. A suitable class is the class of *measurable* functions which, for the purpose of this book, we may take to be a class rich enough to include all controls, such as those that are merely piecewise continuous, that we may encounter. If the control $u(\cdot)$ is measurable and $f(\cdot)$ is continuous, then $f^u(\cdot)$, defined above, is continuous in $x$ but measurable in $t$, so we are forced to study such functions. Suppose, as above, $D$ is an open set in $\mathbb{R}^n \times \mathbb{R}$. The function $f : D \to \mathbb{R}^n$ is said to satisfy the *Caratheodory* conditions in $D$ if:

(a) $f$ is measurable in $t$ for each fixed $x$,

(b) $f$ is continuous in $x$ for each fixed $t$,

(c) for each compact set $F$ in $D$ there exists a measurable function $t \mapsto m_F(t)$ such that

$$|f(x, t)| \leq m_F(t)$$

for all $(x, t) \in F$. We now make use of the fact that if $t \mapsto h(t)$ is measurable, its integral $t \mapsto H(t) \stackrel{\Delta}{=} \int_{t_0}^{t} h(s)ds$ is absolutely continuous and, therefore, has a derivative almost everywhere. Where $H(\cdot)$ is differentiable, its derivative is equal to $h(\cdot)$. Consequently, if $f(\cdot)$ satisfies the Caratheodory conditions, then the solution of (A.14), i.e., a function $\phi(\cdot)$ satisfying (A.14) everywhere does not satisfy (A.12) everywhere but only almost everywhere, at the points where $\phi(\cdot)$ is differentiable. In view of this, we may speak *either* of a solution of (A.14) *or* of a solution of (A.12) provided we interpret the latter as an absolutely continuous function which satisfies (A.12)) almost everywhere. The appropriate generalization of Peano's existence theorem is the following result due to Caratheodory:

**Theorem A.31** (Existence of solution to differential equations)**.** *If $D$ is an open set in $\mathbb{R}^n \times \mathbb{R}$ and $f(\cdot)$ satisfies the Caratheodory conditions on $D$, then, for any $(x_0, t_0)$ in $D$, there exists a solution of* (A.14) *or* (A.12) *passing through $(x_0, t_0)$.*

Two other classical theorems on ordinary differential equations that are relevant are:

**Theorem A.32** (Maximal interval of existence)**.** *If $D$ is an open set in $\mathbb{R}^n \times \mathbb{R}$, $f(\cdot)$ satisfies the Caratheodory conditions on $D$, and $\phi(\cdot)$ is a solution of* (A.10) *on some interval, then there is a continuation $\phi'(\cdot)$ of $\phi(\cdot)$ to a maximal interval $(t_a, t_b)$ of existence. The solution $\phi'(\cdot)$, the continuation of $\phi(\cdot)$, tends to the boundary of $D$ as $t \searrow t_a$ and $t \nearrow t_b$.*

**Theorem A.33** (Continuity of solution to differential equation)**.** *Suppose $D$ is an open set in $\mathbb{R}^n \times \mathbb{R}$, $f$ satisfies the Caratheodory condition and, for each compact set $U$ in $D$, there exists an integrable function $t \mapsto k_u(t)$ such that*

$$|f(x,t) - f(y,t)| \leq k_u(t)|x - y|$$

*for all $(x,t), (y,t)$ in $U$. Then, for any $(x_0, t_0)$ in $U$ there exists a unique solution $\phi(\cdot; x_0, t_0)$ passing through $(x_0, t_0)$. The function $(t, x_0, t_0) \mapsto \phi(t; x_0, t_0) : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is continuous in its domain $E$ which is open.*

Note that $D$ is often $\mathbb{R}^n \times \mathbb{R}$, in which case Theorem A.32 states that a solution $x(\cdot)$ of (A.14) escapes, i.e., $|x(t)| \to \infty$ as $t \searrow t_a$ or $t \nearrow t_b$ if $t_a$ and $t_b$ are finite; $t_a$ and $t_b$ are the escape times. An example of a differential equation with finite escape time is $\dot{x} = x^2$ which has, if $x_0 > 0, t_0 = 0$, a solution $x(t) = x_0[1 - (t - t_0)x_0]^{-1}$ and the maximal interval of existence is $(t_a, t_b) = (-\infty, t_0 + 1/x_0)$.

These results, apart from absence of a control $u$ which is trivially corrected, do not go far enough. We require solutions on an interval $[t_0, t_f]$ given a priori. Further assumptions are needed for this. A useful tool in developing the required results is the Bellman-Gronwall Lemma:

**Theorem A.34** (Bellman-Gronwall)**.** *Suppose that $c \in (0, \infty)$ and that $\alpha : [0,1] \to \mathbb{R}_+$ is a bounded, integrable function, and that the integrable function $y : [0,1] \to \mathbb{R}$ satisfies the inequality*

$$y(t) \leq c + \int_0^t \alpha(s) y(s) ds \tag{A.15}$$

*for all* $t \in [0,1]$. *Then*

$$y(t) \leq ce^{\int_0^t \alpha(s)ds} \tag{A.16}$$

*for all* $t \in [0,1]$.

Note that, if the inequality in (A.15) were replaced by an equality, (A.15) could be integrated to yield (A.16).

*Proof.* Let the function $Y : [0,1] \to \mathbb{R}$ be defined by

$$Y(t) = \int_0^t \alpha(s)y(s)ds \tag{A.17}$$

so that $\dot{Y}(t) = \alpha(t)y(t)$ almost everywhere on $[0,1]$. It follows from (A.15) and (A.17) that:

$$y(t) \leq c + Y(t) \ \forall t \in [0,1]$$

Hence

$$\begin{aligned}(d/dt)[e^{-\int_0^t \alpha(s)ds}Y(t)] &= e^{-\int_0^t \alpha(s)ds}(\dot{Y}(t) - \alpha(t)Y(t)) \\ &= (e^{-\int_0^t \alpha(s)ds})\alpha(t)(y(t) - Y(t)) \\ &\leq c(e^{-\int_0^t \alpha(s)ds})\alpha(t) \end{aligned} \tag{A.18}$$

almost everywhere on $[0,1]$. Integrating both sides of (A.18) from 0 to $t$ yields

$$e^{-\int_0^t \alpha(s)ds}Y(t) \leq c[1 - e^{-\int_0^t \alpha(s)ds}]$$

for all $t \in [0,1]$. Hence

$$Y(t) \leq c[e^{\int_0^t \alpha(s)ds} - 1]$$

and

$$y(t) \leq ce^{\int_0^t \alpha(s)ds}$$

for all $t \in [0,1]$. ∎

The interval $[0,1]$ may, of course, be replaced by $[t_0, t_f]$ for arbitrary $t_0, t_f \in (-\infty, \infty)$. Consider now the forced system described by

$$\dot{x}(t) = f(x(t), u(t), t) \text{ a.e} \tag{A.19}$$

with initial condition

$$x(0) = 0$$

The period of interest is now $T := [0,1]$ and "a.e." denotes "almost everywhere on $T$." Admissible controls $u(\cdot)$ are measurable and satisfy the control constraint

$$u(t) \in \Omega \text{ a.e.}$$

where $\Omega \subset \mathbb{R}^m$ is compact. For convenience, we denote the set of admissible controls by

$$\mathcal{U} := \{u : T \to \mathbb{R}^m \mid u(\cdot) \text{ is measurable}, u(t) \in \Omega \text{ a.e.}\}$$

Clearly $\mathcal{U}$ is a subset of $L_\infty$. For simplicity we assume, in the sequel, that $f$ is continuous; this is not restrictive. For each $u$ in $\mathcal{U}$, $x$ in $\mathbb{R}^n$, the function $t \mapsto f^u(x,t) := f(x,u(t),t)$ is measurable so that $f^u$ satisfies the Caratheodory conditions and our previous results, Theorems A.31–A.33, apply. Our concern now is to show that, with additional assumptions, for each $u$ in $\mathcal{U}$, a solution to (A.12) or (A.13) exists on $T$, rather than on some maximal interval that may be a subset of $T$, and that this solution is unique and bounded.

**Theorem A.35** (Existence of solutions to forced systems)**.** *Suppose:*

*(a) $f$ is continuous and*

*(b) there exists a positive constant $c$ such that*

$$|f(x',u,t) - f(x,u,t)| \le c|x' - x|$$

*for all $(x,u,t) \in \mathbb{R}^n \times \Omega \times T$. Then, for each $u$ in $\mathcal{U}$, there exists a unique, absolutely continuous solution $x^u : T \to \mathbb{R}^n$ of* (A.19) *on the interval $T$ passing through $(x_0,0)$. Moreover, there exists a constant $K$ such that*

$$|x^u(t)| \le K$$

*for all $t \in T$, all $u \in \mathcal{U}$.*

*Proof.* A direct consequence of (b) is the existence of a constant which, without loss of generality, we take to be $c$, satisfying

(c) $|f(x,u,t)| \le c(1 + |x|)$ for all $(x,u,t) \in \mathbb{R}^n \times \Omega \times T$.

Assumptions (a) and (b) and their corollary (c), a growth condition on $f(\cdot)$, ensure that $f^u(\cdot)$ satisfies the Caratheodory conditions stated earlier. Hence, our previous results apply, and there exists an interval $[0,t_b]$ on which a unique solution $x^u(\cdot)$ exists; moreover $|x^u(t)| \to \infty$ as $t \nearrow t_b$. Since $x^u(\cdot)$ satisfies

$$x^u(t) = x_0 + \int_0^t f(x^u(s),u(s),s)ds$$

it follows from the growth condition that

$$\begin{aligned} |x^u(t)| &\le |x_0| + \int_0^t |f(x^u(s), u(s), s)| ds \\ &\le |x_0| + c \int_0^t (1 + |x^u(s)|) ds \\ &\le (|x_0| + c) + c \int_0^t |x^u(s)| ds \end{aligned}$$

Applying the Bellman-Gronwall Lemma yields

$$|x^u(t)| \le (c + |x_0|) e^{ct}$$

for all $t \in [0, t_b), u \in \mathcal{U}$. If follows that the escape time $t_b$ cannot be finite, so that, for all $u$ in $\mathcal{U}$, there exists a unique absolutely continuous solution $x^u(\cdot)$ on $T$ passing through $(x_0, (0))$. Moreover, for all $u$ in $\mathcal{U}$, all $t \in T$

$$|x^u(t)| \le K$$

where $K := (c + |x_0|)e^c$. ■

## A.15 Random Variables and the Probability Density

Let $\xi$ be a random variable taking values in the field of real numbers and the function $F_\xi(x)$ denote the **probability distribution function** of the random variable so that

$$F_\xi(x) = \Pr(\xi \le x)$$

i.e., $F_\xi(x)$ is the probability that the random variable $\xi$ takes on a value less than or equal to $x$. $F_\xi$ is obviously a nonnegative, nondecreasing function and has the following properties due to the axioms of probability

$$F_\xi(x_1) \le F_\xi(x_2) \quad \text{if } x_1 < x_2$$

$$\lim_{x \to -\infty} F_\xi(x) = 0$$

$$\lim_{x \to \infty} F_\xi(x) = 1$$

We next define the **probability density function**, denoted $p_\xi(x)$, such that

$$F_\xi(x) = \int_{-\infty}^{x} p_\xi(s) ds, \qquad -\infty < x < \infty \tag{A.20}$$

We can allow discontinuous $F_\xi$ if we are willing to accept generalized functions (delta functions and the like) for $p_\xi$. Also, we can define the density function for discrete as well as continuous random variables if we allow delta functions. Alternatively, we can replace the integral in (A.20) with a sum over a discrete density function. The random variable may be a coin toss or a dice game, which takes on values from a discrete set contrasted to a temperature or concentration measurement, which takes on a values from a continuous set. The density function has the following properties

$$p_\xi(x) \geq 0$$

$$\int_{-\infty}^{\infty} p_\xi(x)dx = 1$$

and the interpretation in terms of probability

$$\Pr(x_1 \leq \xi \leq x_2) = \int_{x_1}^{x_2} p_\xi(x)dx$$

The **mean** or **expectation** of a random variable $\xi$ is defined as

$$\mathcal{E}(\xi) = \int_{-\infty}^{\infty} x p_\xi(x)dx$$

The moments of a random variable are defined by

$$\mathcal{E}(\xi^n) = \int_{-\infty}^{\infty} x^n p_\xi(x)dx$$

and it is clear that the mean is the zeroth moment. Moments of $\xi$ about the mean are defined by

$$\mathcal{E}((\xi - \mathcal{E}(\xi))^n) = \int_{-\infty}^{\infty} (x - \mathcal{E}(\xi))^n p_\xi(x)dx$$

and the variance is defined as the second moment about the mean

$$\text{var}(\xi) = \mathcal{E}((\xi - \mathcal{E}(\xi))^2) = \mathcal{E}(\xi^2) - \mathcal{E}^2(\xi)$$

The standard deviation is the square root of the variance

$$\sigma(\xi) = (\text{var}(\xi))^{1/2}$$

**Normal distribution.** The normal or Gaussian distribution is ubiquitous in applications. It is characterized by its mean, $m$ and variance, $\sigma^2$, and is given by

$$p_\xi(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}\frac{(x-m)^2}{\sigma^2}\right) \tag{A.21}$$

We proceed to check that the mean of this distribution is indeed $m$ and the variance is $\sigma^2$ as claimed and that the density is normalized so that its integral is one. We require the definite integral formulas

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$
$$\int_0^{\infty} x^{1/2} e^{-x} dx = \Gamma(3/2) = \frac{\sqrt{\pi}}{2}$$

The first formula may also be familiar from the error function in transport phenomena

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du$$
$$\operatorname{erf}(\infty) = 1$$

We calculate the integral of the normal density as follows

$$\int_{-\infty}^{\infty} p_\xi(x) dx = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\frac{(x-m)^2}{\sigma^2}\right) dx$$

Define the change of variable

$$u = \frac{1}{\sqrt{2}}\left(\frac{x-m}{\sigma}\right)$$

which gives

$$\int_{-\infty}^{\infty} p_\xi(x) dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left(-u^2\right) du = 1$$

and (A.21) does have unit area. Computing the mean gives

$$\mathcal{E}(\xi) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x \exp\left(-\frac{1}{2}\frac{(x-m)^2}{\sigma^2}\right) dx$$

using the same change of variables as before yields

$$\mathcal{E}(\xi) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} (\sqrt{2}u\sigma + m) e^{-u^2} du$$

The first term in the integral is zero because $u$ is an odd function, and the second term produces

$$\mathcal{E}(\xi) = m$$

as claimed. Finally the definition of the variance of $\xi$ gives

$$\text{var}(\xi) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} (x-m)^2 \exp\left(-\frac{1}{2}\frac{(x-m)^2}{\sigma^2}\right) dx$$

Changing the variable of integration as before gives

$$\text{var}(\xi) = \frac{2}{\sqrt{\pi}}\sigma^2 \int_{-\infty}^{\infty} u^2 e^{-u^2} du$$

and because the integrand is an even function,

$$\text{var}(\xi) = \frac{4}{\sqrt{\pi}}\sigma^2 \int_{0}^{\infty} u^2 e^{-u^2} du$$

Now changing the variable of integration again using $s = u^2$ gives

$$\text{var}(\xi) = \frac{2}{\sqrt{\pi}\sigma^2} \int_{0}^{\infty} s^{1/2} e^{-s} ds$$

The second integral formula then gives

$$\text{var}(\xi) = \sigma^2$$

Shorthand notation for the random variable $\xi$ having a normal distribution with mean $m$ and variance $\sigma^2$ is

$$\xi \sim N(m, \sigma^2)$$

Figure A.8 shows the normal distribution with a mean of one and variances of 1/2, 1 and 2. Notice that a large variance implies that the random variable is likely to take on large values. As the variance shrinks to zero, the probability density becomes a delta function and the random variable approaches a deterministic value.

**Central limit theorem.**

> The central limit theorem states that if a set of $n$ random variables $x_i, i = 1, 2, \ldots, n$ are independent, then under general conditions the density $p_y$ of their sum
>
> $$y = x_1 + x_2 + \cdots + x_n$$
>
> properly normalized, tends to a normal density as $n \to \infty$. (Papoulis, 1984, p. 194).

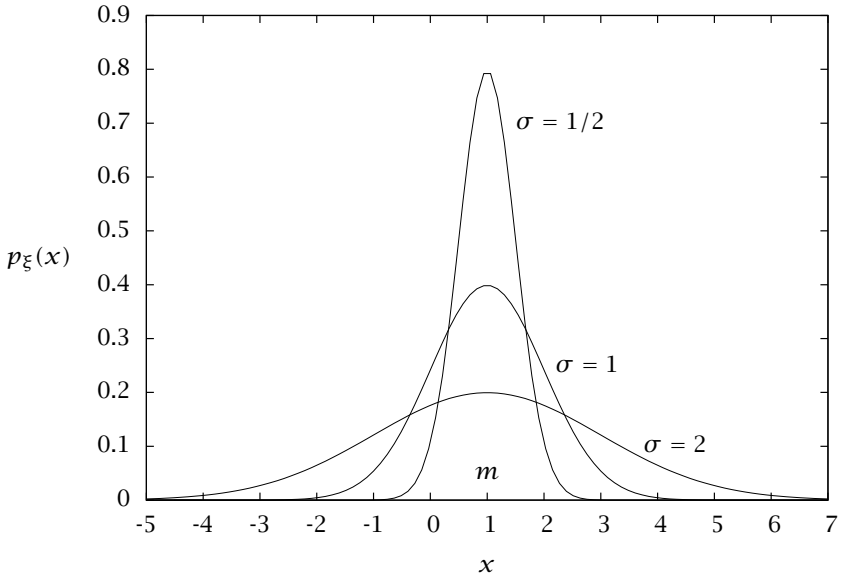


**Figure A.8:** Normal distribution, $p_\xi(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}\frac{(x-m)^2}{\sigma^2}\right)$. Mean is one and standard deviations are 1/2, 1 and 2.

Notice that we require only mild restrictions on how the $x_i$ themselves are distributed for the sum $y$ to tend to a normal. See Papoulis (1984, p. 198) for one set of sufficient conditions and a proof of this theorem.

**Fourier transform of the density function.** It is often convenient to handle the algebra of density functions, particularly normal densities, by using the Fourier transform of the density function rather than the density itself. The transform, which we denote as $\varphi_\xi(u)$, is often called the characteristic function or generating function in the statistics literature. From the definition of the Fourier transform

$$\varphi_\xi(u) = \int_{-\infty}^{\infty} e^{iux} p_\xi(x) dx$$

The transform has a one-to-one correspondence with the density function, which can be seen from the inverse transform formula

$$p_\xi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iux} \varphi_\xi(u) du$$

**Example A.36: Fourier transform of the normal density.**

Show the Fourier transform of the normal density is $\varphi_\xi(u) = \exp\left(ium - \frac{1}{2}u^2\sigma^2\right)$. □

## A.16 Multivariate Density Functions

In applications we normally do not have a single random variable but a collection of random variables. We group these variables together in a vector and let random variable $\xi$ now take on values in $\mathbb{R}^n$. The probability density function is still a nonnegative scalar function

$$p_\xi(x) : \mathbb{R}^n \to \mathbb{R}^+$$

which is sometimes called the **joint density function**. As in the scalar case, the probability that the $n$-dimensional random variable $\xi$ takes on values between $a$ and $b$ is given by

$$\Pr(a \le \xi \le b) = \int_{a_n}^{b_n} \dots \int_{a_1}^{b_1} p_\xi(x) dx_1 \cdots dx_n$$

**Marginal density functions.** We are often interested in only some subset of the random variables in a problem. Consider two vectors of random variables, $\xi \in \mathbb{R}^n$ and $\eta \in \mathbb{R}^m$. We can consider the joint distribution of both of these random variables $p_{\xi,\eta}(x, y)$ or we may only be interested in the $\xi$ variables, in which case we can integrate out the $m$ $\eta$ variables to obtain the marginal density of $\xi$

$$p_\xi(x) = \int_{-\infty}^{\infty} \cdots \int p_{\xi,\eta}(x, y) dy_1 \cdots dy_m$$

Analogously to produce the marginal density of $\eta$ we use

$$p_\eta(y) = \int_{-\infty}^{\infty} \cdots \int p_{\xi,\eta}(x, y) dx_1 \cdots dx_n$$

**Multivariate normal density.** We define the multivariate normal density of the random variable $\xi \in \mathbb{R}^n$ as

$$p_\xi(x) = \frac{1}{(2\pi)^{n/2}(\det P)^{1/2}} \exp\left[-\frac{1}{2}(x - m)' P^{-1}(x - m)\right] \quad \text{(A.22)}$$

in which $m \in \mathbb{R}^n$ is the mean and $P \in \mathbb{R}^{n \times n}$ is the covariance matrix. The notation $\det P$ denotes determinant of $P$. As noted before, $P$ is a

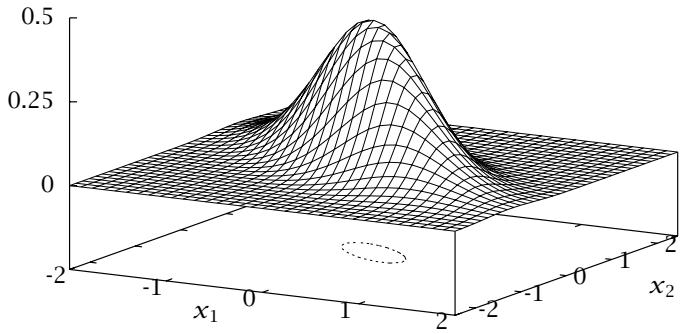


**Figure A.9:** Multivariate normal in two dimensions.

real, symmetric matrix. The multivariate normal density is well-defined only for $P > 0$. The singular, or degenerate, case $P \geq 0$ is discussed subsequently. Shorthand notation for the random variable $\xi$ having a normal distribution with mean $m$ and covariance $P$ is

$$\xi \sim N(m, P)$$

The matrix $P$ is a real, symmetric matrix. Figure A.9 displays a multivariate normal for

$$P^{-1} = \begin{bmatrix} 3.5 & 2.5 \\ 2.5 & 4.0 \end{bmatrix} \qquad m = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

As displayed in Figure A.9, lines of constant probability in the multivariate normal are lines of constant

$$(x - m)' P^{-1} (x - m)$$

To understand the geometry of lines of constant probability (ellipses in two dimensions, ellipsoids or hyperellipsoids in three or more dimensions) we examine the eigenvalues and eigenvectors of the $P^{-1}$ matrix.

$$x'Ax = b$$
$$Av_i = \lambda_i v_i$$

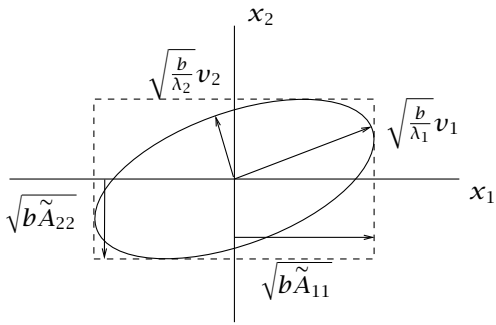


**Figure A.10:** The geometry of quadratic form $x'Ax = b$.

Consider the quadratic function $x'Ax$ depicted in Figure A.10. Each eigenvector of $A$ points along one of the axes of the ellipse $x'Ax = b$. The eigenvalues show us how stretched the ellipse is in each eigenvector direction. If we want to put simple bounds on the ellipse, then we draw a box around it as shown in Figure A.10. Notice the box contains much more area than the corresponding ellipse and we have lost the correlation between the elements of $x$. This loss of information means we can put different tangent ellipses of quite different shapes inside the same box. The size of the bounding box is given by

$$\text{length of } i\text{th side} = \sqrt{b\widetilde{A}_{ii}}$$

in which

$$\widetilde{A}_{ii} = (i,i) \text{ element of } A^{-1}$$

See Exercise A.45 for a derivation of the size of the bounding box. Figure A.10 displays these results: the eigenvectors are aligned with the ellipse axes and the eigenvalues scale the lengths. The lengths of the sides of the box that are tangent to the ellipse are proportional to the square root of the diagonal elements of $A^{-1}$.

**Singular or degenerate normal distributions.** It is often convenient to extend the definition of the normal distribution to admit positive *semidefinite* covariance matrices. The distribution with a semidefinite covariance is known as a singular or degnerate normal distribution (An-

derson, 2003, p. 30). Figure A.11 shows a nearly singular normal distribution.

To see how the singular normal arises, let the scalar random variable $\xi$ be distributed normally with zero mean and positive definite covariance, $\xi \sim N(0, P_x)$, and consider the simple linear transformation

$$\eta = A\xi \qquad A = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

in which we have created two identical copies of $\xi$ for the two components $\eta_1$ and $\eta_2$ of $\eta$. Now consider the density of $\eta$. If we try to use the standard formulas for transformation of a normal, we would have

$$\eta \sim N(0, P_y) \qquad P_y = AP_xA' = \begin{bmatrix} P_x & P_x \\ P_x & P_x \end{bmatrix}$$

and $P_y$ is singular since its rows are linearly dependent. Therefore one of the eigenvalues of $P_y$ is zero and $P_y$ is positive semidefinite and not positive definite. Obviously we cannot use (A.22) for the density in this case because the inverse of $P_y$ does not exist. To handle these cases, we first provide an interpretation that remains valid when the covariance matrix is singular and semidefinite.

**Definition A.37** (Density of a singular normal)**.** A singular joint normal density of random variables $(\xi_1, \xi_2)$, $\xi_1 \in \mathbb{R}^{n_1}$, $\xi_2 \in \mathbb{R}^{n_2}$, is denoted

$$\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \sim N\left[ \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}, \begin{bmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{bmatrix} \right]$$

with $\Lambda_1 > 0$. The density is defined by

$$p_\xi(x_1, x_2) = \frac{1}{(2\pi)^{\frac{n_1}{2}} (\det \Lambda_1)^{\frac{1}{2}}} \exp\left[ -\frac{1}{2} |x_1 - m_1)|^2_{\Lambda_1^{-1}} \right] \delta(x_2 - m_2) \tag{A.23}$$

In this limit, the "random" variable $\xi_2$ becomes deterministic and equal to its mean $m_2$. For the case $n_1 = 0$, we have the completely degenerate case in which $p_{\xi_2}(x_2) = \delta(x_2 - m_2)$, which describes the completely deterministic case $\xi_2 = m_2$ and there is no random component $\xi_1$. This expanded definition enables us to generalize the important result that the linear transformation of a normal is normal, so that it holds for *any* linear transformation, including rank deficient transformations such as the $A$ matrix given above in which the rows

are not independent (see Exercise 1.40). Starting with the definition of a singular normal, we can obtain the density for $\xi \sim N(m_x, P_x)$ for any positive semidefinite $P_x \geq 0$. The result is

$$p_\xi(x) = \frac{1}{(2\pi)^{\frac{r}{2}}(\det \Lambda_1)^{\frac{1}{2}}} \exp\left[ -\frac{1}{2} |(x - m_x)|^2_{Q_1} \right] \delta(Q_2'(x - m_x)) \tag{A.24}$$

in which matrices $\Lambda \in \mathbb{R}^{r \times r}$ and orthonormal $Q \in \mathbb{R}^{n \times n}$ are obtained from the eigenvalue decomposition of $P_x$

$$P_x = Q\Lambda Q' = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Q_1' \\ Q_2' \end{bmatrix}$$

and $\Lambda_1 > 0 \in \mathbb{R}^{r \times r}$, $Q_1 \in \mathbb{R}^{n \times r}$, $Q_2 \in \mathbb{R}^{n \times (n-r)}$. This density is nonzero for $x$ satisfying $Q_2'(x - m_x) = 0$. If we let $N(Q_2')$ denote the $r$ dimensional nullspace of $Q_2'$, we have that the density is nonzero for $x \in N(Q_2') \oplus \{m_x\}$ in which $\oplus$ denotes set addition.

**Example A.38: Marginal normal density**

Given that $\xi$ and $\eta$ are jointly, normally distributed with mean

$$m = \begin{bmatrix} m_x \\ m_y \end{bmatrix}$$

and covariance matrix

$$P = \begin{bmatrix} P_x & P_{xy} \\ P_{yx} & P_y \end{bmatrix}$$

show that the marginal density of $\xi$ is normal with the following parameters

$$\xi \sim N(m_x, P_x) \tag{A.25}$$

**Solution**

As a first approach to establish (A.25), we directly integrate the $y$ variables. Let $\bar{x} = x - m_x$ and $\bar{y} = y - m_y$, and $n_x$ and $n_y$ be the dimension of the $\xi$ and $\eta$ variables, respectively, and $n = n_x + n_y$. Then the definition of the marginal density gives

$$p_\xi(x) = \frac{1}{(2\pi)^{n/2}(\det P)^{1/2}} \int_{-\infty}^{\infty} \exp\left[ -\frac{1}{2} \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix}' \begin{bmatrix} P_x & P_{xy} \\ P_{yx} & P_y \end{bmatrix}^{-1} \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} \right] d\bar{y}$$

Let the inverse of $P$ be denoted as $\widetilde{P}$ and partition $\widetilde{P}$ as follows

$$\begin{bmatrix} P_x & P_{xy} \\ P_{yx} & P_y \end{bmatrix}^{-1} = \begin{bmatrix} \widetilde{P}_x & \widetilde{P}_{xy} \\ \widetilde{P}_{yx} & \widetilde{P}_y \end{bmatrix} \tag{A.26}$$

Substituting (A.26) into the definition of the marginal density and expanding the quadratic form in the exponential yields

$$(2\pi)^{n/2}(\det P)^{1/2} p_\xi(x) =$$
$$\exp\left(-(1/2)\bar{x}'\widetilde{P}_x\bar{x}\right)\int_{-\infty}^{\infty}\exp\left(-(1/2)(2\bar{y}'\widetilde{P}_{yx}\bar{x}+\bar{y}'\widetilde{P}_y\bar{y})\right)d\bar{y}$$

We complete the square on the term in the integral by noting that

$$(\bar{y}+\widetilde{P}_y^{-1}\widetilde{P}_{yx}\bar{x})'\widetilde{P}_y(\bar{y}+\widetilde{P}_y^{-1}\widetilde{P}_{yx}\bar{x}) = \bar{y}'\widetilde{P}_y\bar{y}+2\bar{y}'\widetilde{P}_{yx}\bar{x}+\bar{x}'\widetilde{P}_{yx}'\widetilde{P}_y^{-1}\widetilde{P}_{yx}\bar{x}$$

Substituting this relation into the previous equation gives

$$(2\pi)^{n/2}(\det P)^{1/2} p_\xi(x) = \exp\left(-(1/2)\bar{x}'(\widetilde{P}_x-\widetilde{P}_{yx}'\widetilde{P}_y^{-1}\widetilde{P}_{yx})\bar{x}\right)$$
$$\int_{-\infty}^{\infty}\exp\left(-(1/2)(\bar{y}+a)'\widetilde{P}_y(\bar{y}+a)\right)d\bar{y}$$

in which $a = \widetilde{P}_y^{-1}\widetilde{P}_{yx}\bar{x}$. Using (A.22) to evaluate the integral gives

$$p_\xi(x) = \frac{1}{(2\pi)^{n_x/2}\left(\det(P)\det(\widetilde{P}_y)\right)^{1/2}}\exp\left(-(1/2)\bar{x}'(\widetilde{P}_x-\widetilde{P}_{yx}'\widetilde{P}_y^{-1}\widetilde{P}_{yx})\bar{x}\right)$$

From the matrix inversion formula we conclude

$$\widetilde{P}_x - \widetilde{P}_{xy}'\widetilde{P}_y^{-1}\widetilde{P}_{yx} = P_x^{-1}$$

and

$$\det(P) = \det(P_x)\det(P_y - P_{yx}P_x^{-1}P_{xy}) = \det P_x \det\widetilde{P}_y^{-1} = \frac{\det P_x}{\det\widetilde{P}_y}$$

Substituting these results into the previous equation gives

$$p_\xi(x) = \frac{1}{(2\pi)^{n_x/2}(\det P_x)^{1/2}}\exp\left(-(1/2)\bar{x}'P_x^{-1}\bar{x}\right)$$

Therefore

$$\xi \sim N(m_x, P_x)$$

□

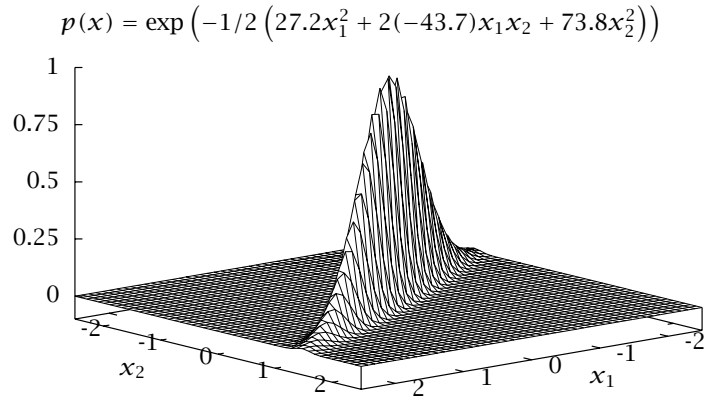


**Figure A.11:** A nearly singular normal density in two dimensions.

**Functions of random variables.** In stochastic dynamical systems we need to know how the density of a random variable is related to the density of a function of that random variable. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a mapping of the random variable $\xi$ into the random variable $\eta$ and assume that the inverse mapping also exits

$$\eta = f(\xi), \qquad \xi = f^{-1}(\eta)$$

Given the density of $\xi, p_\xi(x)$, we wish to compute the density of $\eta$, $p_\eta(y)$, induced by the function $f$. Let $S$ denote an arbitrary region of the field of the random variable $\xi$ and define the set $S'$ as the transform of this set under the function $f$

$$S' = \{y | y = f(x), x \in S\}$$

Then we seek a function $p_\eta(y)$ such that

$$\int_S p_\xi(x)dx = \int_{S'} p_\eta(y)dy \tag{A.27}$$

for every admissible set $S$. Using the rules of calculus for transforming a variable of integration we can write

$$\int_S p_\xi(x)dx = \int_{S'} p_\xi(f^{-1}(y)) \left| \det\left(\frac{\partial f^{-1}(y)}{\partial y}\right) \right| dy \tag{A.28}$$

in which $\left|\det(\partial f^{-1}(y)/\partial y)\right|$ is the absolute value of the determinant of the Jacobian matrix of the transformation from $\eta$ to $\xi$. Subtracting (A.28) from (A.27) gives

$$\int_{S'} \left( p_\eta(y) - p_\xi(f^{-1}(y)) \left|\det(\partial f^{-1}(y)/\partial y)\right| \right) dy = 0 \tag{A.29}$$

Because (A.29) must be true for any set $S'$, we conclude (a proof by contradiction is immediate)[8]

$$\boxed{p_\eta(y) = p_\xi(f^{-1}(y)) \left|\det(\partial f^{-1}(y)/\partial y)\right|} \tag{A.30}$$

**Example A.39: Nonlinear transformation**

Show that

$$p_\eta(y) = \frac{1}{3\sqrt{2\pi}\sigma y^{2/3}} \exp\left[ -\frac{1}{2}\left(\frac{y^{1/3} - m}{\sigma}\right)^2 \right]$$

is the density function of the random variable $\eta$ under the transformation

$$\eta = \xi^3$$

for $\xi \sim N(m, \sigma^2)$. Notice that the density $p_\eta$ is singular at $y = 0$. □

**Noninvertible transformations.** Given $n$ random variables $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ with joint density $p_\xi$ and $k$ random variables $\eta = (\eta_1, \eta_2, \ldots, \eta_k)$ defined by the transformation $\eta = f(\xi)$

$$\eta_1 = f_1(\xi) \quad \eta_2 = f_2(\xi) \quad \cdots \quad \eta_k = f_k(\xi)$$

We wish to find $p_\eta$ in terms of $p_\xi$. Consider the region generated in $\mathbb{R}^n$ by the vector inequality

$$f(x) \le c$$

[8] Some care should be exercised if one has generalized functions in mind for the conditional density.

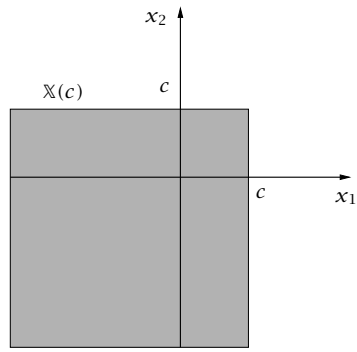


**Figure A.12:** The region $\mathbb{X}(c)$ for $y = \max(x_1, x_2) \leq c$.

Call this region $\mathbb{X}(c)$, which is by definition

$$\mathbb{X}(c) = \{x | f(x) \leq c\}$$

Note $\mathbb{X}$ is not necessarily simply connected. The probability distribution (not density) for $\eta$ then satisfies

$$P_\eta(y) = \int_{\mathbb{X}(y)} p_\xi(x)dx \tag{A.31}$$

If the density $p_\eta$ is of interest, it can be obtained by differentiating $P_\eta$.

**Example A.40: Maximum of two random variables**

Given two independent random variables, $\xi_1, \xi_2$ and the new random variable defined by the noninvertible, nonlinear transformation

$$\eta = \max(\xi_1, \xi_2)$$

Show that $\eta$'s density is given by

$$p_\eta(y) = p_{\xi_1}(y) \int_{-\infty}^{y} p_{\xi_2}(x)dx + p_{\xi_2}(y) \int_{-\infty}^{y} p_{\xi_1}(x)dx$$

**Solution**

The region $\mathbb{X}(c)$ generated by the inequality $y = \max(x_1, x_2) \leq c$ is sketched in Figure A.12. Applying (A.31) then gives

$$\begin{aligned} P_\eta(y) &= \int_{-\infty}^{y} \int_{-\infty}^{y} p_\xi(x_1, x_2)dx_1dx_2 \\ &= P_\xi(y, y) \\ &= P_{\xi_1}(y)P_{\xi_2}(y) \end{aligned}$$

which has a clear physical interpretation. It says the probability that the *maximum* of two independent random variables is less than some value is equal to the probability that *both* random variables are less than that value. To obtain the density, we differentiate

$$\begin{aligned} p_\eta(y) &= p_{\xi_1}(y)P_{\xi_2}(y) + P_{\xi_1}(y)p_{\xi_2}(y) \\ &= p_{\xi_1}(y)\int_{-\infty}^{y} p_{\xi_2}(x)dx + p_{\xi_2}(y)\int_{-\infty}^{y} p_{\xi_1}(x)dx \end{aligned}$$

□

### A.16.1 Statistical Independence and Correlation

We say two random variables $\xi, \eta$ are **statistically independent** or simply independent if

$$p_{\xi,\eta}(x,y) = p_\xi(x)p_\eta(y), \qquad \text{all } x, y$$

The covariance of two random variables $\xi, \eta$ is defined as

$$\text{cov}(\xi,\eta) = \mathcal{E}\left((\xi - \mathcal{E}(\xi))\,(\eta - \mathcal{E}(\eta))\right)$$

The covariance of the vector-valued random variable $\xi$ with components $\xi_i, i = 1,\ldots n$ can be written as

$$P_{ij} = \text{cov}(\xi_i, \xi_j)$$

$$P = \begin{bmatrix} \text{var}(\xi_1) & \text{cov}(\xi_1,\xi_2) & \cdots & \text{cov}(\xi_1,\xi_n) \\ \text{cov}(\xi_2,\xi_1) & \text{var}(\xi_2) & \cdots & \text{cov}(\xi_2,\xi_n) \\ \vdots & \vdots & \ddots & \vdots \\ \text{cov}(\xi_n,\xi_1) & \text{cov}(\xi_n,\xi_2) & \cdots & \text{var}(\xi_n) \end{bmatrix}$$

We say two random variables, $\xi$ and $\eta$, are **uncorrelated** if

$$\text{cov}(\xi,\eta) = 0$$

**Example A.41: Independent implies uncorrelated**

Prove that if $\xi$ and $\eta$ are statistically independent, then they are uncorrelated.

**Solution**

The definition of covariance gives

$$\begin{aligned}\text{cov}(\xi,\eta) &= \mathcal{E}((\xi-\mathcal{E}(\xi))(\eta-\mathcal{E}(\eta)))\\ &= \mathcal{E}(\xi\eta-\xi\mathcal{E}(\eta)-\eta\mathcal{E}(\xi)+\mathcal{E}(\xi)\mathcal{E}(\eta))\\ &= \mathcal{E}(\xi\eta)-\mathcal{E}(\xi)\mathcal{E}(\eta)\end{aligned}$$

Taking the expectation of the product $\xi\eta$ and using the fact that $\xi$ and $\eta$ are independent gives

$$\begin{aligned}\mathcal{E}(\xi\eta) &= \int_{-\infty}^{\infty}\!\!\int xyp_{\xi,\eta}(x,y)dxdy\\ &= \int_{-\infty}^{\infty}\!\!\int xyp_{\xi}(x)p_{\eta}(y)dxdy\\ &= \int_{-\infty}^{\infty} xp_{\xi}(x)dx\int_{-\infty}^{\infty} yp_{\eta}(y)dy\\ &= \mathcal{E}(\xi)\mathcal{E}(\eta)\end{aligned}$$

Substituting this fact into the covariance equation gives

$$\text{cov}(\xi,\eta)=0$$

□

**Example A.42: Does uncorrelated imply independent?**

Let $\xi$ and $\eta$ be jointly distributed random variables with probability density function

$$p_{\xi,\eta}(x,y)=\begin{cases}\frac{1}{4}[1+xy(x^2-y^2)], & |x|<1, \quad |y|<1\\ 0, & \text{otherwise}\end{cases}$$

(a) Compute the marginals $p_{\xi}(x)$ and $p_{\eta}(y)$. Are $\xi$ and $\eta$ independent?

(b) Compute $\text{cov}(\xi,\eta)$. Are $\xi$ and $\eta$ uncorrelated?

(c) What is the relationship between independent and uncorrelated? Are your results on this example consistent with this relationship? Why or why not?

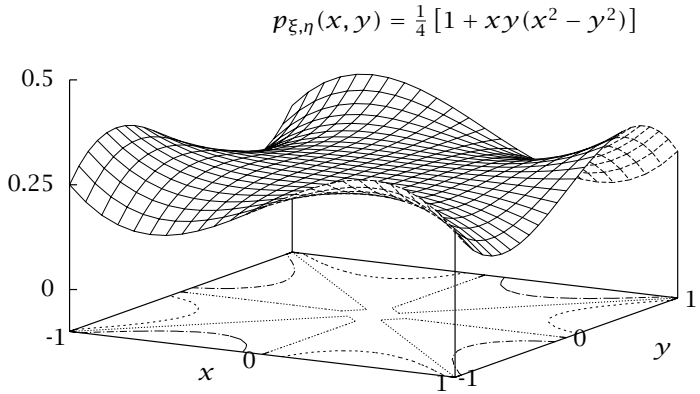


**Figure A.13:** A joint density function for the two uncorrelated random variables in Example A.42.

**Solution**

The joint density is shown in Figure A.13.

(a) Direct integration of the joint density produces

$$p_\xi(x) = (1/2), \quad |x| < 1 \qquad \mathcal{E}(\xi) = 0$$
$$p_\eta(y) = (1/2), \quad |y| < 1 \qquad \mathcal{E}(\eta) = 0$$

and we see that both marginals are zero mean, uniform densities. Obviously $\xi$ and $\eta$ are not independent because the joint density is not the product of the marginals.

(b) Performing the double integral for the expectation of the product term gives

$$\mathcal{E}(\xi\eta) = \int_{-1}^{1}\int xy + (xy)^2(x^2 - y^2)dxdy$$
$$= 0$$

and the covariance of $\xi$ and $\eta$ is therefore

$$\begin{aligned}\operatorname{cov}(\xi,\eta) &= \mathcal{E}(\xi\eta) - \mathcal{E}(\xi)\mathcal{E}(\eta)\\ &= 0\end{aligned}$$

and $\xi$ and $\eta$ are uncorrelated.

(c) We know independent implies uncorrelated. This example does not contradict that relationship. This example shows uncorrelated does not imply independent, in general, but see the next example for normals.

□

**Example A.43: Independent and uncorrelated are equivalent for normals**

If two random variables are jointly normally distributed,

$$\begin{bmatrix}\xi\\ \eta\end{bmatrix} \sim N\left(\begin{bmatrix}m_x\\ m_y\end{bmatrix}, \begin{bmatrix}P_x & P_{xy}\\ P_{yx} & P_y\end{bmatrix}\right)$$

Prove $\xi$ and $\eta$ are statistically independent if and only if $\xi$ and $\eta$ are uncorrelated, or, equivalently, $P$ is block diagonal.

**Solution**

We have already shown that independent implies uncorrelated for any density, so we now show that, *for normals*, uncorrelated implies independent. Given $\operatorname{cov}(\xi,\eta) = 0$, we have

$$P_{xy} = P'_{yx} = 0 \qquad \det P = \det P_x \det P_y$$

so the density can be written

$$p_{\xi,\eta}(x,y) = \frac{\exp\left(-\frac{1}{2}\begin{bmatrix}\bar{x}\\ \bar{y}\end{bmatrix}'\begin{bmatrix}P_x & 0\\ 0 & P_y\end{bmatrix}^{-1}\begin{bmatrix}\bar{x}\\ \bar{y}\end{bmatrix}\right)}{(2\pi)^{(n_x+n_y)/2}\left(\det P_x \det P_y\right)^{1/2}} \tag{A.32}$$

For any joint normal, we know the marginals are simply

$$\xi \sim N(m_x, P_x) \qquad \eta \sim N(m_y, P_y)$$

so we have

$$p_\xi(x) = \frac{1}{(2\pi)^{n_x/2}(\det P_x)^{1/2}} \exp\left(-(1/2)\bar{x}' P_x^{-1}\bar{x}\right)$$
$$p_\eta(y) = \frac{1}{(2\pi)^{n_y/2}(\det P_y)^{1/2}} \exp\left(-(1/2)\bar{y}' P_y^{-1}\bar{y}\right)$$

Forming the product and combining terms gives

$$p_\xi(x)p_\eta(y) = \frac{\exp\left(-\frac{1}{2}\begin{bmatrix}\bar{x}\\ \bar{y}\end{bmatrix}'\begin{bmatrix}P_x^{-1} & 0\\ 0 & P_y^{-1}\end{bmatrix}\begin{bmatrix}\bar{x}\\ \bar{y}\end{bmatrix}\right)}{(2\pi)^{(n_x+n_y)/2}\left(\det P_x \det P_y\right)^{1/2}}$$

Comparing this equation to (A.32), and using the inverse of a block-diagonal matrix, we have shown that $\xi$ and $\eta$ are statistically independent. □

## A.17 Conditional Probability and Bayes's Theorem

Let $\xi$ and $\eta$ be jointly distributed random variables with density $p_{\xi,\eta}(x, y)$. We seek the density function of $\xi$ given a specific realization $y$ of $\eta$ has been observed. We define the conditional density function as

$$p_{\xi|\eta}(x|y) = \frac{p_{\xi,\eta}(x, y)}{p_\eta(y)}$$

Consider a roll of a single die in which $\eta$ takes on values E or O to denote whether the outcome is even or odd and $\xi$ is the integer value of the die. The twelve values of the joint density function are simply computed

$$\begin{array}{rclrcl}
p_{\xi,\eta}(1,\mathrm{E}) &=& 0 & p_{\xi,\eta}(1,\mathrm{O}) &=& 1/6\\
p_{\xi,\eta}(2,\mathrm{E}) &=& 1/6 & p_{\xi,\eta}(2,\mathrm{O}) &=& 0\\
p_{\xi,\eta}(3,\mathrm{E}) &=& 0 & p_{\xi,\eta}(3,\mathrm{O}) &=& 1/6\\
p_{\xi,\eta}(4,\mathrm{E}) &=& 1/6 & p_{\xi,\eta}(4,\mathrm{O}) &=& 0\\
p_{\xi,\eta}(5,\mathrm{E}) &=& 0 & p_{\xi,\eta}(5,\mathrm{O}) &=& 1/6\\
p_{\xi,\eta}(6,\mathrm{E}) &=& 1/6 & p_{\xi,\eta}(6,\mathrm{O}) &=& 0
\end{array} \tag{A.33}$$

The marginal densities are then easily computed; we have for $\xi$

$$p_\xi(x) = \sum_{y=\mathrm{O}}^{\mathrm{E}} p_{\xi,\eta}(x, y)$$

which gives by summing across rows of (A.33)

$$p_\xi(x) = 1/6, \qquad x = 1, 2, \ldots 6$$

Similarly, we have for $\eta$

$$p_\eta(y) = \sum_{x=1}^{6} p_{\xi,\eta}(x, y)$$

which gives by summing down the columns of (A.33)

$$p_\eta(y) = 1/2, \qquad y = \mathrm{E}, \mathrm{O}$$

These are both in accordance of our intuition on the rolling of the die: uniform probability for each value 1 to 6 and equal probability for an even or an odd outcome. Now the conditional density is a different concept. The conditional density $p_{\xi|\eta}(x, y)$ tells us the density of $x$ given that $\eta = y$ has been observed. So consider the value of this function

$$p_{\xi|\eta}(1|\mathrm{O})$$

which tells us the probability that the die has a 1 given that we know that it is odd. We expect that the additional information on the die being odd causes us to revise our probability that it is 1 from 1/6 to 1/3. Applying the defining formula for conditional density indeed gives

$$p_{\xi|\eta}(1|\mathrm{O}) = p_{\xi,\eta}(1, \mathrm{O})/p_\eta(\mathrm{O}) = \frac{1/6}{1/2} = 1/3$$

Consider the reverse question, the probability that we have an odd given that we observe a 1. The definition of conditional density gives

$$p_{\eta,\xi}(\mathrm{O}|1) = p_{\eta,\xi}(\mathrm{O}, 1)/p_\xi(1) = \frac{1/6}{1/6} = 1$$

i.e., we are sure the die is odd if it is 1. Notice that the arguments to the conditional density do not commute as they do in the joint density.

This fact leads to a famous result. Consider the definition of conditional density, which can be expressed as

$$p_{\xi,\eta}(x, y) = p_{\xi|\eta}(x|y)p_\eta(y)$$

or

$$p_{\eta,\xi}(y, x) = p_{\eta|\xi}(y|x)p_\xi(x)$$

Because $p_{\xi,\eta}(x,y) = p_{\eta,\xi}(y,x)$, we can equate the right-hand sides and deduce

$$p_{\xi|\eta}(x|y) = \frac{p_{\eta|\xi}(y|x)p_{\xi}(x)}{p_{\eta}(y)}$$

which is known as Bayes's theorem (Bayes, 1763). Notice that this result comes in handy whenever we wish to switch the variable that is known in the conditional density, which we will see is a key step in state estimation problems.

**Example A.44: Conditional normal density**

Show that if $\xi$ and $\eta$ are jointly normally distributed as

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} \sim N\left(\begin{bmatrix} m_x \\ m_y \end{bmatrix}, \begin{bmatrix} P_x & P_{xy} \\ P_{yx} & P_y \end{bmatrix}\right)$$

then the conditional density of $\xi$ given $\eta$ is also normal

$$(\xi|\eta) \sim N(m,P)$$

in which the mean is

$$m = m_x + P_{xy}P_y^{-1}(y - m_y) \tag{A.34}$$

and the covariance is

$$P = P_x - P_{xy}P_y^{-1}P_{yx} \tag{A.35}$$

**Solution**

The definition of conditional density gives

$$p_{\xi|\eta}(x|y) = \frac{p_{\xi,\eta}(x,y)}{p_{\eta}(y)}$$

Because $(\xi,\eta)$ is jointly normal, we know from Example A.38

$$p_{\eta}(y) = \frac{1}{(2\pi)^{n_\eta/2}(\det P_y)^{1/2}} \exp\left(-(1/2)(y - m_y)'P_y^{-1}(y - m_y)\right)$$

and therefore

$$p_{\xi|\eta}(x|y) = \frac{(\det P_y)^{1/2}}{(2\pi)^{n_\xi/2}\left(\det\begin{bmatrix} P_x & P_{xy} \\ P_{yx} & P_y \end{bmatrix}\right)^{1/2}} \exp(-1/2a) \tag{A.36}$$

in which the argument of the exponent is

$$a = \begin{bmatrix} x - m_x \\ y - m_y \end{bmatrix}' \begin{bmatrix} P_x & P_{xy} \\ P_{yx} & P_y \end{bmatrix}^{-1} \begin{bmatrix} x - m_x \\ y - m_y \end{bmatrix} - (y - m_y)' P_y^{-1} (y - m_y)$$

If we use $P = P_x - P_{xy} P_y^{-1} P_{yx}$ as defined in (A.35) then we can use the partitioned matrix inversion formula to express the matrix inverse in the previous equation as

$$\begin{bmatrix} P_x & P_{xy} \\ P_{yx} & P_y \end{bmatrix}^{-1} = \begin{bmatrix} P^{-1} & -P^{-1} P_{xy} P_y^{-1} \\ -P_y^{-1} P_{yx} P^{-1} & P_y^{-1} + P_y^{-1} P_{yx} P^{-1} P_{xy} P_y^{-1} \end{bmatrix}$$

Substituting this expression and multiplying out terms yields

$$\begin{aligned} a = {} & (x - m_x)' P^{-1} (x - m_x) - 2(y - m_y)' (P_y^{-1} P_{yx} P^{-1})(x - m_x) \\ & + (y - m_y)' (P_y^{-1} P_{yx} P^{-1} P_{xy} P_y^{-1})(y - m_y) \end{aligned}$$

which is the expansion of the following quadratic term

$$a = \left[ (x - m_x) - P_{xy} P_y^{-1} (y - m_y) \right]' P^{-1} \left[ (x - m_x) - P_{xy} P_y^{-1} (y - m_y) \right]$$

in which we use the fact that $P_{xy} = P_{yx}'$. Substituting (A.34) into this expression yields

$$a = (x - m)' P^{-1} (x - m) \tag{A.37}$$

Finally noting that for the partitioned matrix

$$\det \begin{bmatrix} P_x & P_{xy} \\ P_{yx} & P_y \end{bmatrix} = \det P_y \det P \tag{A.38}$$

and substitution of equations (A.38) and (A.37) into (A.36) yields

$$p_{\xi|\eta}(x|y) = \frac{1}{(2\pi)^{n_\xi/2} (\det P)^{1/2}} \exp\left( -\frac{1}{2} (x - m)' P^{-1} (x - m) \right)$$

which is the desired result. □

**Example A.45: More normal conditional densities**

Let the joint conditional of random variables $a$ and $b$ given $c$ be a normal distribution with

$$p(a, b|c) \sim N\left( \begin{bmatrix} m_a \\ m_b \end{bmatrix}, \begin{bmatrix} P_a & P_{ab} \\ P_{ba} & P_b \end{bmatrix} \right) \tag{A.39}$$

Then the conditional density of $a$ given $b$ and $c$ is also normal

$$p(a|b,c) \sim N(m,P)$$

in which the mean is

$$m = m_a + P_{ab}P_b^{-1}(b - m_b)$$

and the covariance is

$$P = P_a - P_{ab}P_b^{-1}P_{ba}$$

**Solution**

From the definition of joint density we have

$$p(a|b,c) = \frac{p(a,b,c)}{p(b,c)}$$

Multiplying the top and bottom of the fraction by $p(c)$ yields

$$p(a|b,c) = \frac{p(a,b,c)}{p(c)}\frac{p(c)}{p(b,c)}$$

or

$$p(a|b,c) = \frac{p(a,b|c)}{p(b|c)}$$

Substituting the distribution given in (A.39) and using the result in Example A.38 to evaluate $p(b|c)$ yields

$$p(a|b,c) = \frac{N\left(\begin{bmatrix} m_a \\ m_b \end{bmatrix}, \begin{bmatrix} P_a & P_{ab} \\ P_{ba} & P_b \end{bmatrix}\right)}{N(m_b, P_b)}$$

And now applying the methods of Example A.44 this ratio of normal distributions reduces to the desired expression. □

**Adjoint operator.** Given a linear operator $\mathcal{G} : \mathbb{U} \to \mathbb{V}$ and inner products for the spaces $\mathbb{U}$ and $\mathbb{V}$, the adjoint of $\mathcal{G}$, denoted by $\mathcal{G}^*$ is the linear operator $\mathcal{G}^* : \mathbb{V} \to \mathbb{U}$ such that

$$\langle u, \mathcal{G}^* v\rangle = \langle \mathcal{G}u, v\rangle, \qquad \forall u \in \mathbb{U}, v \in \mathbb{V} \tag{A.40}$$

**Dual dynamic system (Callier and Desoer, 1991).** The dynamic system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k), \qquad k = 0, \ldots, N-1 \\ y(k) &= Cx(k) + Du(k) \end{aligned}$$

maps an initial condition and input sequence $(x(0), u(0), \ldots, u(N-1))$ into a final condition and an output sequence $(x(N), y(0), \ldots, y(N-1))$. Call this linear operator $\mathcal{G}$

$$\begin{bmatrix} x(N) \\ y(0) \\ \vdots \\ y(N-1) \end{bmatrix} = \mathcal{G} \begin{bmatrix} x(0) \\ u(0) \\ \vdots \\ u(N-1) \end{bmatrix}$$

The dual dynamic system represents the adjoint operator $\mathcal{G}^*$

$$\begin{bmatrix} \overline{x}(0) \\ \overline{y}(1) \\ \vdots \\ \overline{y}(N) \end{bmatrix} = \mathcal{G}^* \begin{bmatrix} \overline{x}(N) \\ \overline{u}(1) \\ \vdots \\ \overline{u}(N) \end{bmatrix}$$

We define the usual inner product, $\langle a, b \rangle = a'b$, and substitute into (A.40) to obtain

$$\underbrace{x(0)'\overline{x}(0) + u(0)'\overline{y}(1) + \cdots + u(N-1)'\overline{y}(N)}_{\langle u, G^* v \rangle} - \underbrace{x(N)'\overline{x}(N) + y(0)'\overline{u}(1) + \cdots + y(N-1)'\overline{u}(N)}_{\langle Gu, v \rangle} = 0$$

If we express the $y(k)$ in terms of $x(0)$ and $u(k)$ and collect terms we obtain

$$\begin{aligned} 0 = {} & x(0)' \left[ \overline{x}(0) - C'\overline{u}(1) - A'C'\overline{u}(2) - \cdots - A'^N \overline{x}(N) \right] \\ & + u(0)' \left[ \overline{y}(1) - D'\overline{u}(1) - B'C'\overline{u}(2) - \cdots - B'A'^{(N-2)}C'\overline{u}(N) - B'A'^{(N-1)}\overline{x}(N) \right] \\ & + \cdots \\ & + u(N-2)' \left[ \overline{y}(N-1) - D'\overline{u}(N-1) - B'C'\overline{u}(N) - B'A'\overline{x}(N) \right] \\ & + u(N-1)' \left[ \overline{y}(N) - D'\overline{u}(N) - B'\overline{x}(N) \right] \end{aligned}$$

Since this equation must hold for all $(x(0), u(0), \ldots, u(N-1))$, each term in brackets must vanish. From the $u(N-1)$ term we conclude

$$\overline{y}(N) = B'\overline{x}(N) + D'\overline{u}(N)$$

Using this result, the $u(N-2)$ term gives

$$B'\left(\overline{x}(N-1) - \left(A'\overline{x}(N) + C'\overline{u}(N)\right)\right) = 0$$

From which we find the state recursion for the dual system

$$\overline{x}(N-1) = A'\overline{x}(N) + C'\overline{u}(N)$$

Passing through each term then yields the dual state space description of the adjoint operator $\mathcal{G}^*$

$$\begin{aligned}\overline{x}(k-1) &= A'\overline{x}(k) + C'\overline{u}(k), \qquad k = N,\ldots,1\\ \overline{y}(k) &= B'\overline{x}(k) + D'\overline{u}(k)\end{aligned}$$

So the primal and dual dynamic systems change matrices in the following way

$$(A,B,C,D) \longrightarrow (A',C',B',D')$$

Notice this result produces the duality variables listed in Table A.1 if we first note that we have also renamed the regulator's input matrix $B$ to $G$ in the estimation problem. We also note that time runs in the opposite directions in the dynamic system and the dual dynamic system, which corresponds to the fact that the Riccati equation iterations run in opposite directions in the regulation and estimation problems.

## A.18 Exercises

**Exercise A.1: Norms in $\mathbb{R}^n$**

Show that the following three functions are all norms in $\mathbb{R}^n$

$$\begin{aligned}|x|_2 &:= \left(\sum_{i=1}^{n}(x^i)^2\right)^{1/2}\\ |x|_\infty &:= \max\{|x^1|,|x^2|,\ldots,|x^n|\}\\ |x|_1 &:= \sum_{i=1}^{n}|x^j|\end{aligned}$$

where $x^j$ denotes the $j$th component of the vector $x$.

**Exercise A.2: Equivalent norms**

Show that there are finite constants $K_{ij}, i,j = 1,2,\infty$ such that

$$|x|_i \le K_{ij}\,|x|_j\,, \text{ for all } i,j \in \{1,2,\infty\}.$$

This result shows that the norms are *equivalent* and may be used interchangeably for establishing that sequences are convergent, sets are open or closed, etc.

| Regulator | Estimator |
|---|---|
| $A$ | $A'$ |
| $B$ | $C'$ |
| $C$ | $G'$ |
| $k$ | $l = N - k$ |
| $\Pi(k)$ | $P^-(l)$ |
| $\Pi(k-1)$ | $P^-(l+1)$ |
| $\Pi$ | $P^-$ |
| $Q$ | $Q$ |
| $R$ | $R$ |
| $Q(N)$ | $Q(0)$ |
| $K$ | $-\widetilde{L}'$ |
| $A + BK$ | $(A - \widetilde{L}C)'$ |
| $x$ | $\varepsilon$ |

| Regulator | Estimator |
|---|---|
| $R > 0, \quad Q > 0$ | $R > 0, \quad Q > 0$ |
| $(A, B)$ stabilizable | $(A, C)$ detectable |
| $(A, C)$ detectable | $(A, G)$ stabilizable |

**Table A.1:** Duality variables and stability conditions for linear quadratic regulation and linear estimation.

### Exercise A.3: Open and closed balls

Let $x \in \mathbb{R}^n$ and $\rho > 0$ be given. Show that $\{z \mid |z - x| < \rho\}$ is open and that $B(x, \rho)$ is closed.

### Exercise A.4: Condition for closed set

Show that $X \subset \mathbb{R}^n$ is closed if and only if $\text{int}(B(x, \rho)) \cap X \neq \emptyset$ for all $\rho > 0$ implies $x \in X$.

### Exercise A.5: Convergence

Suppose that $x_i \to \hat{x}$ as $i \to \infty$; show that for every $\rho > 0$ there exists an $i_p \in \mathbb{I}_{\geq 0}$ such that $x_i \in B(\hat{x}, \rho)$ for all $i \geq i_p$.

### Exercise A.6: Limit is unique

Suppose that $\hat{x}, \hat{x}'$ are limits of a sequence $\{x_i\}_{i \in \mathbb{I}_{\geq 0}}$. Show that $\hat{x} = \hat{x}'$.

### Exercise A.7: Open and closed sets

(a) Show that a set $X \subset \mathbb{R}^n$ is open if and only if, for any $\hat{x} \in X$ and any sequence $\{x_i\} \subset \mathbb{R}^n$ such that $x_i \to \hat{x}$ as $i \to \infty$, there exists a $q \in \mathbb{I}_{\geq 0}$ such that $x_i \in X$ for all $i \geq q$.

(b) Show that a set $X \subset \mathbb{R}^n$ is closed if and only if for all $\{x_i\} \subset X$, if $x_i \to \hat{x}$ as $i \to \infty$, then $\hat{x} \in X$, i.e., a set $X$ is closed if and only if it contains the limit of every convergent sequences lying in $X$.

### Exercise A.8: Decreasing and bounded below

Prove the observation at the end of Section A.10 that a monotone decreasing sequence that is bounded below converges.

### Exercise A.9: Continuous function

Show that $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $\hat{x}$ implies $f(x_i) \to f(\hat{x})$ for any sequence $\{x_i\}$ satisfying $x_i \to \hat{x}$ as $i \to \infty$.

### Exercise A.10: Alternative proof of existence of minimum of continuous function on compact set

Prove Proposition A.7 by making use of the fact that $f(X)$ is compact.

### Exercise A.11: Differentiable implies Lipschitz

Suppose that $f : \mathbb{R}^n \to \mathbb{R}^m$ has a continuous derivative $f_x(\cdot)$ in a neighborhood of $\hat{x}$. Show that $f$ is locally Lipschitz continuous at $\hat{x}$.

### Exercise A.12: Continuous, Lipschitz continuous, and differentiable

Provide examples of functions meeting the following conditions.

1. Continuous but not Lipschitz continuous.
2. Lipschitz continuous but not differentiable.

### Exercise A.13: Differentiating quadratic functions and time-varying matrix inverses

(a) Show that $\nabla f(x) = Qx$ if $f(x) = (1/2)x'Qx$ and $Q$ is symmetric.

(b) Show that $(d/dt)A^{-1}(t) = -A^{-1}(t)\dot{A}(t)A^{-1}(t)$ if $A : \mathbb{R} \to \mathbb{R}^{n\times n}$, $A(t)$ is invertible for all $t \in \mathbb{R}$, and $\dot{A}(t) := (d/dt)A(t)$.

### Exercise A.14: Directional derivative

Suppose that $f : \mathbb{R}^n \to \mathbb{R}^m$ has a derivative $f_x(\hat{x})$ at $\hat{x}$. Show that for any $h$, the directional derivative $df(\hat{x}; h)$ exists and is given by

$$df(\hat{x}; h) = f_x(\hat{x})h = (\partial f(x)/\partial x)h.$$

### Exercise A.15: Convex combination

Suppose $S \subset \mathbb{R}^n$ is convex. Let $\{x_i\}_{i=1}^k$ be points in $S$ and let $\{\mu^i\}_{i=1}^k$ be scalars such that $\mu^i \geq 0$ for $i = 1, 2, \ldots, k$ and $\sum_{i=1}^k \mu^i = 1$. Show that

$$\left(\sum_{i=1}^{k} \mu^i x_i\right) \in S.$$

### Exercise A.16: Convex epigraph

Show that $f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if its epigraph is convex.

### Exercise A.17: Bounded second derivative and minimum

Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable and that for some $\infty > M \geq m > 0$, $M|y|^2 \geq \langle y, \partial^2 f / \partial x^2(x) y \rangle \geq m|y|^2$ for all $x, y \in \mathbb{R}^n$. Show that the sublevel sets of $f$ are convex and compact and that $f(\cdot)$ attains its infimum.

### Exercise A.18: Sum and max of convex functions are convex

Suppose that $f_i : \mathbb{R}^n \to \mathbb{R}, i = 1, 2, \ldots, m$ are convex. Show that

$$\psi^1(x) := \max_i \{f_i(x) \mid i \in \{1, 2, \ldots, m\}\},$$

$$\psi^2(x) := \sum_{i=1}^{m} f_i(x)$$

are both convex.

### Exercise A.19: Einige kleine Mathprobleme

(a) Prove that if $\lambda$ is an eigenvalue and $v$ is an eigenvector of $A$ ($Av = \lambda v$), then $\lambda$ is also an eigenvalue of $T$ in which $T$ is upper triangular and given by the Schur decomposition of $A$

$$Q^* A Q = T$$

What is the corresponding eigenvector?

(b) Prove statement 1. on positive definite matrices. Where is this fact needed?

(c) Prove statement 6. on positive definite matrices. Where is this fact needed?

(d) Prove statement 5. on positive definite matrices.

(e) Prove statement 8. on positive semidefinite matrices.

(f) Derive the two expressions for the partitioned $A^{-1}$.

### Exercise A.20: Positive definite but not symmetric matrices

Consider redefining the notation $A > 0$ for $A \in \mathbb{R}^{n \times n}$ to mean $x'Ax > 0$ for all $x \in \mathbb{R}^n \neq 0$. In other words, the restriction that $A$ is symmetric in the usual definition of positive definiteness is removed. Consider also $B := (A + A')/2$. Show the following hold for all $A$. (a) $A > 0$ if and only if $B$ is positive definite. (b) $\mathrm{tr}(A) = \mathrm{tr}(B)$. (Johnson, 1970; Johnson and Hillar, 2002)

### Exercise A.21: Trace of a matrix function

Derive the following formula for differentiating the trace of a function of a square matrix

$$\frac{d\mathrm{tr}(f(A))}{dA} = g(A') \qquad g(x) = \frac{df(x)}{dx}$$

in which $g$ is the usual scalar derivative of the scalar function $f$. This result proves useful in evaluating the change in the expectation of the stage cost in stochastic control problems.

### Exercise A.22: Some matrix differentiation

Derive the following formulas (Bard, 1974). $A, B \in \mathbb{R}^{n\times n}, a, x \in \mathbb{R}^n$.

(a)
$$\frac{\partial x' Ax}{\partial x} = Ax + A'x$$

(b)
$$\frac{\partial Axa'Bx}{\partial x'} = (a'Bx)A + Axa'B$$

(c)
$$\frac{\partial a'Ab}{\partial A} = ab'$$

### Exercise A.23: Partitioned matrix inversion formula

In deriving the partitioned matrix inversion formula we assumed A is partitioned into

$$A = \begin{bmatrix} B & C \\ D & E \end{bmatrix}$$

and that $A^{-1}, B^{-1}$ and $E^{-1}$ exist. In the final formula, the term

$$(E - DB^{-1}C)^{-1}$$

appears, but we did not assume this matrix is invertible. Did we leave out an assumption or can the existence of this matrix inverse be proven given the other assumptions? If we left out an assumption, provide an example in which this matrix is not invertible. If it follows from the other assumptions, prove this inverse exists.

### Exercise A.24: Partitioned positive definite matrices

Consider the partitioned positive definite, symmetric matrix

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}$$

Prove that the following matrices are also positive definite

1. $H_{11}$
2. $H_{22}$
3. $\overline{H}$ in which
$$\overline{H} = \begin{bmatrix} H_{11} & -H_{12} \\ -H_{21} & H_{22} \end{bmatrix}$$
4. $H_{11} - H_{12}H_{22}^{-1}H_{21}$ and $H_{22} - H_{21}H_{11}^{-1}H_{12}$

### Exercise A.25: Properties of the matrix exponential

Prove that the following properties of the matrix exponential, which are useful for dealing with continuous time linear systems. The matrix $A$ is a real-valued $n \times n$ matrix, and $t$ is real.

(a)
$$\text{rank}\left(e^{At}\right) = n \quad \forall t$$

(b)

$$\text{rank}\left(\int_0^t e^{A\tau}d\tau\right) = n \quad \forall t > 0$$

### Exercise A.26: Controllability in continuous time

A linear, time-invariant, continuous time system

$$\begin{aligned}\frac{dx}{dt} &= Ax + Bu\\ x(0) &= x_0\end{aligned} \tag{A.41}$$

is **controllable** if there exists an input $u(t), 0 \le t \le t_1, t_1 > 0$ that takes the system from any $x_0$ at time zero to any $x_1$ at some finite time $t_1$.

(a) Prove that the system in (A.41) is controllable if and only if

$$\text{rank}\,(C) = n$$

in which $C$ is, remarkably, the same controllability matrix that was defined for discrete time systems 1.17

$$C = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

(b) Describe a calculational procedure for finding this required input.

### Exercise A.27: Reachability Gramian in continuous time

Consider the symmetric, $n \times n$ matrix $W$ defined by

$$W(t) = \int_0^t e^{(t-\tau)A}BB'e^{(t-\tau)A'}d\tau$$

The matrix $W$ is known as the reachability Gramian of the linear, time-invariant system. The reachability Gramian proves useful in analyzing controllability and reachability. Prove the following important properties of the reachability Gramian.

(a) The reachability Gramian satisfies the following matrix differential equation

$$\begin{aligned}\frac{dW}{dt} &= BB' + AW + WA'\\ W(0) &= 0\end{aligned}$$

which provides one useful way to calculate its values.

(b) The reachability Gramian $W(t)$ is full rank for all $t > 0$ if and only if the system is controllable.

### Exercise A.28: Differences in continuous time and discrete time systems

Consider the definition that a system is controllable if there exists an input that takes the system from any $x_0$ at time zero to any $x_1$ at some finite time $t_1$.

(a) Show that $x_1$ can be taken as zero without changing the meaning of controllability for a linear continuous time system.

(b) In linear discrete time systems, $x_1$ cannot be taken as zero without changing the meaning of controllability. Why not? Which $A$ require a distinction in discrete time. What are the eigenvalues of the corresponding $A$ in continuous time?

### Exercise A.29: Observability in continuous time

Consider the linear time-invariant continuous time system

$$\begin{aligned} \frac{dx}{dt} &= Ax \\ x(0) &= x_0 \\ y &= Cx \end{aligned} \tag{A.42}$$

and let $y(t;x_0)$ represent the solution to (A.42) as a function of time $t$ given starting state value $x_0$ at time zero. Consider the output from two different initial conditions $y(t;w)$, $y(t;z)$ on the time interval $0 \le t \le t_1$ with $t_1 > 0$.

The system in (A.42) is **observable** if

$$y(t;w) = y(t;z), \quad 0 \le t \le t_1 \Longrightarrow w = z$$

In other words, if two output measurement trajectories agree, the initial conditions that generated the output trajectories must agree, and hence, the initial condition is unique. This uniqueness of the initial condition allows us to consider building a state estimator to reconstruct $x(0)$ from $y(t;x_0)$. After we have found the unique $x(0)$, solving the model provides the rest of the state trajectory $x(t)$. We will see later that this procedure is not the preferred way to build a state estimator; it simply shows that if the system is observable, the goal of state estimation is reasonable.

Show that the system in (A.42) is observable if and only if

$$\text{rank}\,(\mathcal{O}) = n$$

in which $\mathcal{O}$ is, again, the same observability matrix that was defined for discrete time systems 1.37

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Hint: what happens if you differentiate $y(t;w) - y(t;z)$ with respect to time? How many times is this function differentiable?

### Exercise A.30: Observability Gramian in continuous time

Consider the symmetric, $n \times n$ matrix $W_o$ defined by

$$W_o(t) = \int_0^t e^{A'\tau} C' C e^{A\tau} d\tau$$

The matrix $W_o$ is known as the observability Gramian of the linear, time-invariant system. Prove the following important properties of the observability Gramian.

(a) The observability Gramian $W_o(t)$ is full rank for all $t > 0$ if and only if the system is observable.

(b) Consider an observable linear time invariant system with $u(t) = 0$ so that $y(t) = Ce^{At}x_0$. Use the observability Gramian to solve this equation for $x_0$ as a function of $y(t), 0 \le t \le t_1$.

(c) Extend your result from the previous part to find $x_0$ for an arbitrary $u(t)$.

### Exercise A.31: Detectability of $(A, C)$ and output penalty

Given a system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \end{aligned}$$

Suppose $(A, C)$ is detectable and an input sequence has been found such that

$$u(k) \to 0 \qquad y(k) \to 0$$

Show that $x(k) \to 0$.

### Exercise A.32: Prove your favorite Hautus Lemma

Prove the Hautus Lemma for controllability, Lemma 1.2, or observability, Lemma 1.4.

### Exercise A.33: Positive semidefinite $Q$ penalty and its square root

Consider the linear quadratic problem with system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Q^{1/2}x(k) \end{aligned}$$

and infinite horizon cost function

$$\begin{aligned} \Phi &= \sum_{k=0}^{\infty} x(k)'Qx(k) + u(k)'Ru(k) \\ &= \sum_{k=0}^{\infty} y(k)'y(k) + u(k)'Ru(k) \end{aligned}$$

with $Q \geq 0$, $R > 0$, and $(A, B)$ stabilizable. In Exercise A.31 we showed that if $(A, Q^{1/2})$ is detectable and an input sequence has been found such that

$$u(k) \to 0 \qquad y(k) \to 0$$

then $x(k) \to 0$.

(a) Show that if $Q \geq 0$, then $Q^{1/2}$ is a well defined, real, symmetric matrix and $Q^{1/2} \geq 0$.

Hint: apply Theorem A.1 to $Q$, using fact 3..

(b) Show that $(A, Q^{1/2})$ is detectable (observable) if and only if $(A, Q)$ is detectable (observable). So we can express one of the LQ existence, uniqueness, and stability conditions using detectability of $(A, Q)$ instead of $(A, Q^{1/2})$.

### Exercise A.34: Probability density of the inverse function

Consider a scalar random variable $\xi \in \mathbb{R}$ and let the random variable $\eta$ be defined by the inverse function

$$\eta = \xi^{-1}$$

(a) If $\xi$ is distributed uniformly on $[a, 1]$ with $0 < a < 1$, what is the density of $\eta$?

(b) Is $\eta$'s density well defined if we allow $a = 0$? Explain your answer.

### Exercise A.35: Expectation as a linear operator

(a) Consider the random variable $x$ to be defined as a linear combination of the random variables $a$ and $b$

$$x = a + b$$

Show that

$$\mathcal{E}(x) = \mathcal{E}(a) + \mathcal{E}(b)$$

Do $a$ and $b$ need to be statistically independent for this statement to be true?

(b) Next consider the random variable $x$ to be defined as a scalar multiple of the random variable $a$

$$x = \alpha a$$

Show that

$$\mathcal{E}(x) = \alpha \mathcal{E}(a)$$

(c) What can you conclude about $\mathcal{E}(x)$ if $x$ is given by the linear combination

$$x = \sum_i \alpha_i v_i$$

in which $v_i$ are random variables and $\alpha_i$ are scalars.

### Exercise A.36: Minimum of two random variables

Given two independent random variables, $\xi_1, \xi_2$ and the random variable defined by the minimum operator

$$\eta = \min(\xi_1, \xi_2)$$

(a) Sketch the region $\mathbb{X}(c)$ for the inequality $\min(x_1, x_2) \leq c$.

(b) Find $\eta$'s probability density in terms of the probability densities of $\xi_1, \xi_2$.

### Exercise A.37: Maximum of $n$ normally distributed random variables

Given $n$ independent, identically distributed normal random variables, $\xi_1, \xi_2, \ldots, \xi_n$ and the random variable defined by the maximum operator

$$\eta = \max(\xi_1, \xi_2, \ldots \xi_n)$$

(a) Derive a formula for $\eta$'s density.

(b) Plot $p_\eta$ for $\xi_i \sim N(0,1)$ and $n = 1, 2, \ldots 5$. Describe the trend in $p_\eta$ as $n$ increases.

### Exercise A.38: Another picture of mean

Consider a scalar random variable $\xi$ with probability distribution $P_\xi$ shown in Figure A.14. Consider the inverse probability distribution, $P_\xi^{-1}$, also shown in Figure A.14.

(a) Show that the expectation of $\xi$ is equal to the following integral of the probability distribution (David, 1981, p. 38)

$$\mathcal{E}(\xi) = -\int_{-\infty}^{0} P_\xi(x)dx + \int_0^{\infty} (1 - P_\xi(x))dx \tag{A.43}$$

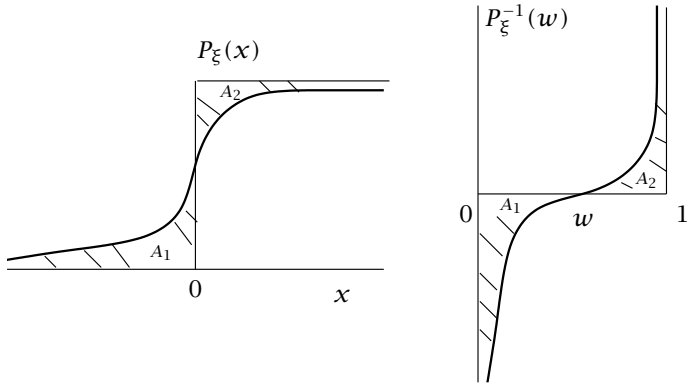


**Figure A.14:** The probability distribution and inverse distribution for random variable $\xi$. The mean of $\xi$ is given by the difference in the hatched areas, $\mathcal{E}(\xi) = A_2 - A_1$.

(b) Show that the expectation of $\xi$ is equal to the following integral of the inverse probability distribution

$$\mathcal{E}(\xi) = \int_0^1 P_\xi^{-1}(w)dw \tag{A.44}$$

These interpretations of mean are shown as the hatched areas in Figure A.14, $\mathcal{E}(\xi) = A_2 - A_1$.

### Exercise A.39: Ordering random variables

We can order two random variables $A$ and $B$ if they obey an inequality such as $A \geq B$. The frequency interpretation of the probability distribution, $P_A(c) = \Pr(A \leq c)$, then implies that $P_A(c) \leq P_B(c)$ for all $c$.

If $A \geq B$, show that

$$\mathcal{E}(A) \geq \mathcal{E}(B)$$

### Exercise A.40: Max of the mean and mean of the max

Given two random variables $A$ and $B$, establish the following inequality

$$\max(\mathcal{E}(A), \mathcal{E}(B)) \leq \mathcal{E}(\max(A, B))$$

In other words, the max of the mean is an underbound for the mean of the max.

### Exercise A.41: Observability

Consider the linear system with zero input

$$x(k+1) = Ax(k)$$
$$y(k) = Cx(k)$$

with

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

(a) What is the observability matrix for this system? What is its rank?

(b) Consider a string of data measurements

$$y(0) = y(1) = \cdots = y(n-1) = 0$$

Now $x(0) = 0$ is clearly consistent with these data. Is this $x(0)$ unique? If yes, prove it. If no, characterize the set of all $x(0)$ that are consistent with these data.

### Exercise A.42: Nothing is revealed

An agitated graduate student shows up at your office. He begins, "I am afraid I have discovered a deep contradiction in the foundations of systems theory." You ask him to calm down and tell you about it. He continues, "Well, we have the pole placement theorem that says if $(A, C)$ is observable, then there exists a matrix $L$ such that the eigenvalues of an observer

$$A - ALC$$

can be assigned arbitrarily."

You reply, "Well, they do have to be conjugate pairs because the matrices $A, L, C$ are real-valued, but yeah, sure, so what?"

He continues, "Well we also have the Hautus Lemma that says $(A, C)$ is observable if and only if

$$\operatorname{rank} \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n \qquad \forall \lambda \in \mathbb{C}$$

"You know, the Hautus Lemma has always been one of my favorite lemmas; I don't see a problem," you reply.

"Well," he continues, "isn't the innovations form of the system, $(A - ALC, C)$, observable if and only if the original system, $(A, C)$, is observable?"

"Yeah ...I seem to recall something like that," you reply, starting to feel a little uncomfortable.

"OK, how about if I decide to put all the observer poles at zero?" he asks, innocently.

You object, "Wait a minute, I guess you can do that, but that's not going to be a very good observer, so I don't think it matters if ...."

"Well," he interrupts, "how about we put all the eigenvalues of $A - ALC$ at zero, like I said, and then we check the Hautus condition at $\lambda = 0$? I get

$$\operatorname{rank} \begin{bmatrix} \lambda I - (A - ALC) \\ C \end{bmatrix} = \operatorname{rank} \begin{bmatrix} 0 \\ C \end{bmatrix} \qquad \lambda = 0$$

"So tell me, how is that matrix on the right ever going to have rank $n$ with that big, fat zero sitting there?" At this point, you start feeling a little dizzy.

What's causing the contradiction here: the pole placement theorem, the Hautus Lemma, the statement about equivalence of observability in innovations form, something else? How do you respond to this student?

### Exercise A.43: The sum of throwing two dice

Using (A.30), what is the probability density for the sum of throwing two dice? On what number do you want to place your bet? How often do you expect to win if you bet on this outcome?

Make the standard assumptions: the probability density for each die is uniform over the integer values from one to six, and the outcome of each die is independent of the other die.

### Exercise A.44: The product of throwing two dice

Using (A.30), what is the probability density for the product of throwing two dice? On what number do you want to place your bet? How often do you expect to win if you bet on this outcome?

Make the standard assumptions: the probability density for each die is uniform over the integer values from one to six, and the outcome of each die is independent of the other die.

### Exercise A.45: The size of an ellipse's bounding box

Here we derive the size of the bounding box depicted in Figure A.10. Consider a real, positive definite, symmetric matrix $A \in \mathbb{R}^{n \times n}$ and a real vector $x \in \mathbb{R}^n$. The set of $x$ for which the scalar $x'Ax$ is constant are $n$-dimensional ellipsoids. Find the length of the sides of the smallest box that contains the ellipsoid defined by

$$x'Ax = b$$

Hint: Consider the equivalent optimization problem to minimize the value of $x'Ax$ such that the $i$th component of $x$ is given by $x_i = c$. This problem defines the ellipsoid that is tangent to the plane $x_i = c$, and can be used to answer the original question.

### Exercise A.46: The tangent points of an ellipse's bounding box

Find the tangent points of an ellipsoid defined by $x'Ax = b$, and its bounding box as depicted in Figure A.10 for $n = 2$. For $n = 2$, draw the ellipse, bounding box and compute the tangent points for the following parameters taken from Figure A.10

$$A = \begin{bmatrix} 3.5 & 2.5 \\ 2.5 & 4.0 \end{bmatrix} \qquad b = 1$$

### Exercise A.47: Let's make a deal!

Consider the following contest of the American television game show of the 1960s, Let's Make a Deal. In the show's grand finale, a contestant is presented with three doors. Behind one of the doors is a valuable prize such as an all-expenses-paid vacation to Hawaii or a new car. Behind the other two doors are goats and donkeys. The contestant selects a door, say door number one. The game show host, Monty Hall, then says,

"Before I show you what is behind your door, let's reveal what is behind door number three!" Monty always chooses a door that has one of the booby prizes behind it. As the goat or donkey is revealed, the audience howls with laughter. Then Monty asks innocently,

"Before I show you what is behind your door, I will allow you one chance to change your mind. Do you want to change doors?" While the contestant considers this option, the audience starts screaming out things like,

"Stay with your door! No, switch, switch!" Finally the contestant chooses again, and then Monty shows them what is behind their chosen door.

Let's analyze this contest to see how to *maximize* the chance of winning. Define

$$p(i, j, y), \qquad i, j, y = 1, 2, 3$$

to be the probability that you chose door $i$, the prize is behind door $j$ and Monty showed you door $y$ (named after the data!) after your initial guess. Then you would want to

$$\max_j p(j|i, y)$$

for your optimal choice after Monty shows you a door.

(a) Calculate this conditional density and give the probability that the prize is behind door $i$, your original choice, and door $j \neq i$.

(b) You will need to specify a model of Monty's behavior. Please state the one that is appropriate to Let's Make a Deal.

(c) For what other model of Monty's behavior is the answer that it doesn't matter if you switch doors. Why is this a poor model for the game show?

### Exercise A.48: Norm of an extended state

Consider $x \in \mathbb{R}^n$ with a norm denoted $|\cdot|_\alpha$, and $u \in \mathbb{R}^m$ with a norm denoted $|\cdot|_\beta$. Now consider a proposed norm for the extended state $(x, u)$

$$|(x, u)|_\gamma := |x|_\alpha + |u|_\beta$$

Show that this proposal satisfies the definition of a norm given in Section A.8.

If the $\alpha$ and $\beta$ norms are chosen to be $p$-norms, is the $\gamma$ norm also a $p$-norm? Show why or why not.

### Exercise A.49: Distance of an extended state to an extended set

Let $x \in \mathbb{R}^n$ and $\mathbb{X}$ a set of elements in $\mathbb{R}^n$, and $u \in \mathbb{R}^m$ and $\mathbb{U}$ a set of elements in $\mathbb{R}^m$. Denote distances from elements to their respective sets as

$$|x|_\mathbb{X} := \inf_{y \in \mathbb{X}} |x - y|_\alpha \qquad |u|_\mathbb{U} := \inf_{v \in \mathbb{U}} |u - v|_\beta$$

$$|(x, u)|_{\mathbb{X}\times\mathbb{U}} := \inf_{(y,v)\in\mathbb{X}\times\mathbb{U}} |(x, u) - (y, v)|_\gamma$$

Use the norm of the extended state defined in Exercise A.48 to show that

$$|(x, u)|_{\mathbb{X}\times\mathbb{U}} = |x|_\mathbb{X} + |u|_\mathbb{U}$$

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# B
# Stability Theory

## B.1 Introduction

In this appendix we consider stability properties of discrete time systems. A good general reference for stability theory of continuous time systems is Khalil (2002). There are not many texts for stability theory of discrete time systems; a useful reference is LaSalle (1986). Recently stability theory for discrete time systems has received more attention in the literature. In the notes below we draw on Jiang and Wang (2001, 2002); Kellet and Teel (2004a,b).

We consider systems of the form

$$x^+ = f(x,u)$$

where the state $x$ lies in $\mathbb{R}^n$ and the control (input) $u$ lies in $\mathbb{R}^m$; in this formulation $x$ and $u$ denote, respectively, the current state and control, and $x^+$ the successor state. We assume in the sequel that the function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous. Let $\phi(k;x,\mathbf{u})$ denote the solution of $x^+ = f(x,u)$ at time $k$ if the initial state is $x(0) = x$ and the control sequence is $\mathbf{u} = \{u(0),u(1),u(2),\ldots\}$; the solution exists and is unique. If a state-feedback control law $u = \kappa(x)$ has been chosen, the closed-loop system is described by $x^+ = f(x,\kappa(x))$, which has the same form $x^+ = f_c(x)$ where $f_c(\cdot)$ is defined by $f_c(x) := f(x,\kappa(x))$. Let $\phi(k;x,\kappa(\cdot))$ denote the solution of this difference equation at time $k$ if the initial state at time 0 is $x(0) = x$; the solution exists and is unique (even if $\kappa(\cdot)$ is discontinuous). If $\kappa(\cdot)$ is not continuous, as may be the case when $\kappa(\cdot)$ is an implicit model predictive control (MPC) law, then $f_c(\cdot)$ may not be continuous. In this case we assume that $f_c(\cdot)$ is *locally bounded*.[1]

[1] A function $f : X \to X$ is locally bounded if, for any $x \in X$, there exists a neighborhood $\mathcal{N}$ of $x$ such that $f(\mathcal{N})$ is a bounded set, i.e., if there exists a $M > 0$ such that $|f(x)| \leq M$ for all $x \in \mathcal{N}$.

We would like to be sure that the controlled system is "stable", i.e., that small perturbations of the initial state do not cause large variations in the subsequent behavior of the system, and that the state converges to a desired state or, if this is impossible due to disturbances, to a desired set of states. These objectives are made precise in Lyapunov stability theory; in this theory, the system $x^+ = f(x)$ is assumed given and conditions ensuring the stability, or asymptotic stability of a specified state or set are sought; the terms *stability* and *asymptotic stability* are defined below. If convergence to a specified state, $x^*$ say, is sought, it is desirable for this state to be an *equilibrium* point:

**Definition B.1** (Equilibrium point)**.** A point $x^*$ is an equilibrium point of $x^+ = f(x)$ if $x(0) = x^*$ implies $x(k) = \phi(k; x^*) = x^*$ for all $k \geq 0$. Hence $x^*$ is an equilibrium point if it satisfies

$$x^* = f(x^*)$$

An equilibrium point $x^*$ is isolated if there are no other equilibrium points in a sufficiently small neighborhood of $x^*$. A linear system $x^+ = Ax + b$ has a single equilibrium point $x^* = (I - A)^{-1}b$ if $I - A$ is invertible; if not, the linear system has a continuum $\{x \mid (I - A)x = b\}$ of equilibrium points. A nonlinear system, unlike a linear system, may have several isolated equilibrium points.

In other situations, for example when studying the stability properties of an oscillator, convergence to a specified closed set $\mathcal{A} \subset \mathbb{R}^n$ is sought. In the case of a linear oscillator with state dimension 2, this set is an ellipse. If convergence to a set $\mathcal{A}$ is sought, it is desirable for the set $\mathcal{A}$ to be *positive invariant*:

**Definition B.2** (Positive invariant set)**.** A set $\mathcal{A}$ is positive invariant for the system $x^+ = f(x)$ if $x \in \mathcal{A}$ implies $f(x) \in \mathcal{A}$.

Clearly, any solution of $x^+ = f(x)$ with initial state in $\mathcal{A}$, remains in $\mathcal{A}$. The (closed) set $\mathcal{A} = \{x^*\}$ consisting of a (single) equilibrium point is a special case; $x \in \mathcal{A}$ $(x = x^*)$ implies $f(x) \in \mathcal{A}$ $(f(x) = x^*)$. Define $|x|_{\mathcal{A}} := \inf_{z \in \mathcal{A}} |x - z|$ to be the distance of a point $x$ from the set $\mathcal{A}$; if $\mathcal{A} = \{x^*\}$, then $|x|_{\mathcal{A}} = |x - x^*|$ which reduces to $|x|$ when $x^* = 0$.

Before introducing the concepts of stability and asymptotic stability and their characterization by Lyapunov functions, it is convenient to make a few definitions.

**Definition B.3** ($\mathcal{K}$ function)**.** A function $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class $\mathcal{K}$ if it is continuous, zero at zero, and strictly increasing; $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class $\mathcal{K}_\infty$ if it is a class $\mathcal{K}$ and unbounded ($\sigma(s) \to \infty$ as $s \to \infty$). A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{I}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class $\mathcal{KL}$ if it is continuous and if, for each $t \geq 0$, $\beta(\cdot, t)$ is a class $\mathcal{K}$ function and for each $s \geq 0$, $\beta(s, \cdot)$ is nonincreasing and satisfies $\lim_{t \to \infty} \beta(s, t) = 0$. A function $\gamma : \mathbb{R} \to \mathbb{R}_{\geq 0}$ belongs to class $\mathcal{PD}$ (is positive definite) if it is continuous, zero at zero, and positive everywhere else.

The following useful properties of these functions are established in Khalil (2002, Lemma 4.2): if $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ are $\mathcal{K}$ functions ($\mathcal{K}_\infty$ functions), then $\alpha_1^{-1}(\cdot)$ and $(\alpha_1 \circ \alpha_2)(\cdot)$[2] are $\mathcal{K}$ functions ($\mathcal{K}_\infty$ functions). Moreover, if $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ are $\mathcal{K}$ functions and $\beta(\cdot)$ is a $\mathcal{KL}$ function, then $\sigma(r, s) = \alpha_1(\beta(\alpha_2(r), s))$ is a $\mathcal{KL}$ function.

## B.2 Stability and Asymptotic Stability

In this section we consider the stability properties of the autonomous system $x^+ = f(x)$; we assume that $f(\cdot)$ is locally bounded, and that the set $\mathcal{A}$ is closed and positive invariant for $x^+ = f(x)$ unless otherwise stated.

**Definition B.4** (Local stability)**.** The (closed positive invariant) set $\mathcal{A}$ is *locally stable* for $x^+ = f(x)$ if, for all $\varepsilon > 0$, there exists a $\delta > 0$ such that $|x|_{\mathcal{A}} < \delta$ implies $|\phi(i; x)|_{\mathcal{A}} < \varepsilon$ for all $i \in \mathbb{I}_{\geq 0}$.

See Figure B.1 for an illustration of this definition when $\mathcal{A} = \{0\}$; in this case we speak of stability of the origin.

**Definition B.5** (Global attraction)**.** The (closed positive invariant) set $\mathcal{A}$ is *globally attractive* for the system $x^+ = f(x)$ if $|\phi(i; x)|_{\mathcal{A}} \to 0$ as $i \to \infty$ for all $x \in \mathbb{R}^n$.

**Definition B.6** (Global asymptotic stability (GAS))**.** The (closed positive invariant) set $\mathcal{A}$ is *globally asymptotically stable* (GAS) for $x^+ = f(x)$ if it is locally stable and globally attractive.

It is possible for the origin to be globally attractive but *not* locally stable. Consider a second order system

$$x^+ = Ax + \phi(x)$$

[2] $(\alpha_1 \circ \alpha_2)(\cdot)$ is the composition of the two functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ and is defined by $(\alpha_1 \circ \alpha_2)(s) := \alpha_1(\alpha_2(s))$.

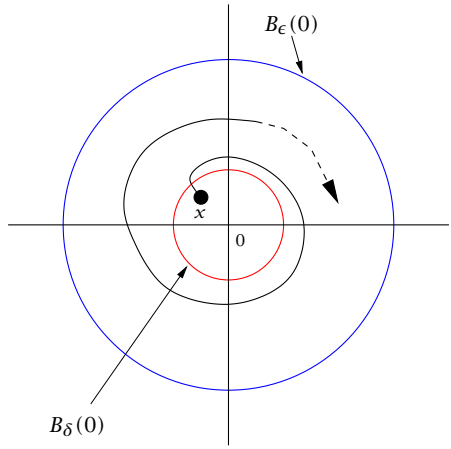


**Figure B.1:** Stability of the origin.

where $A$ has eigenvalues $\lambda_1 = 0.5$ and $\lambda_2 = 2$ with associated eigenvectors $w_1$ and $w_2$, shown in Figure B.2; $w_1$ is the "stable" and $w_2$ the "unstable" eigenvector; the smooth function $\phi(\cdot)$ satisfies $\phi(0) = 0$ and $(\partial/\partial x)\phi(0) = 0$ so that $x^+ = Ax + \phi(x)$ behaves like $x^+ = Ax$ near the origin. If $\phi(x) \equiv 0$, the motion corresponding to an initial state $\alpha w_1$, $\alpha \neq 0$, converges to the origin, whereas the motion corresponding to an initial state $\alpha w_2$ diverges. If $\phi(\cdot)$ is such that it steers nonzero states toward the horizontal axis, we get trajectories of the form shown in Figure B.2. All trajectories converge to the origin but the motion corresponding to an initial state $\alpha w_2$, *no matter how small*, is similar to that shown in Figure B.2 and cannot satisfy the $\varepsilon, \delta$ definition of local stability. The origin is globally attractive but not stable. A trajectory that joins an equilibrium point to itself, as in Figure B.2, is called a homoclinic orbit. We collect below a set of useful definitions:

**Definition B.7** (Various forms of stability)**.** The closed positive invariant set $\mathcal{A}$ is

(a) locally stable if, for each $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that $|x|_{\mathcal{A}} < \delta$ implies $|\phi(i;x)|_{\mathcal{A}} < \varepsilon$ for all $i \in \mathbb{I}_{\geq 0}$.

(b) unstable, if it is not locally stable.

(c) locally attractive if there exists $\eta > 0$ such that $|x|_{\mathcal{A}} < \eta$ implies $|\phi(i;x)|_{\mathcal{A}} \to 0$ as $i \to \infty$.

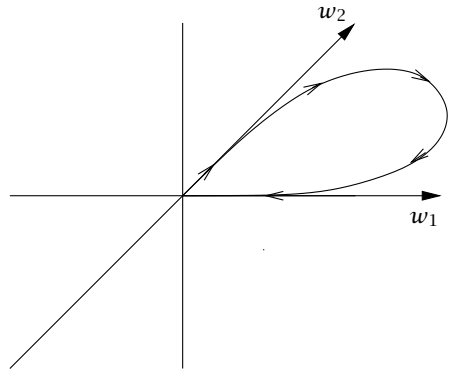


**Figure B.2:** An attractive but unstable origin.

(d) globally attractive if $|\phi(i;x)|_{\mathcal{A}} \to 0$ as $i \to \infty$ for all $x \in \mathbb{R}^n$.

(e) locally asymptotically stable if it is locally stable and locally attractive.

(f) globally asymptotically stable if it is locally stable and globally attractive.

(g) locally exponentially stable if there exist $\eta > 0$, $c > 0$, and $\gamma \in (0,1)$ such that $|x|_{\mathcal{A}} < \eta$ implies $|\phi(i;x)|_{\mathcal{A}} \leq c|x|_{\mathcal{A}}\gamma^i$ for all $i \in \mathbb{I}_{\geq 0}$.

(h) globally exponentially stable if there exists a $c > 0$ and a $\gamma \in (0,1)$ such that $|\phi(i;x)|_{\mathcal{A}} \leq c|x|_{\mathcal{A}}\gamma^i$ for all $x \in \mathbb{R}^n$, all $i \in \mathbb{I}_{\geq 0}$.

It is often convenient to characterize global asymptotic stability in terms of a *comparison* function $\beta(\cdot)$.

**Proposition B.8** (GAS and comparison function)**.** *Suppose $\mathcal{A}$ is compact (and positive invariant) and that $f(\cdot)$ is continuous. Then $\mathcal{A}$ is GAS for $x^+ = f(x)$ if and only if there exists a $\mathcal{KL}$ function $\beta(\cdot)$ such that, for each $x \in \mathbb{R}^n$*

$$|\phi(i;x)|_{\mathcal{A}} \leq \beta(|x|_{\mathcal{A}}, i) \qquad \forall\, i \in \mathbb{I}_{\geq 0} \tag{B.1}$$

That $\mathcal{A}$ is GAS for $x^+ = f(x)$ if there exists a $\mathcal{KL}$ function $\beta(\cdot)$ satisfying (B.1) follows directly from the definition of a class $\mathcal{KL}$ function. The converse is harder to prove but is stated in Jiang and Wang (2002) where Proposition 2.2 establishes the equivalence of the existence of a $\mathcal{KL}$ function satisfying (2) with UGAS (uniform global asymptotic stability), and Corollary 3.3 which establishes the equivalence, *when* $\mathcal{A}$ is compact, of UGAS and GAS.

In practice, global asymptotic stability of $\mathcal{A}$ cannot always be achieved because of state constraints. Hence we have to extend slightly the definitions given above.

**Definition B.9** (Various forms of stability (constrained))**.** Suppose $X \subset \mathbb{R}^n$ is positive invariant for $x^+ = f(x)$, that $\mathcal{A}$ is closed and positive invariant for $x^+ = f(x)$, and that $\mathcal{A}$ lies in the interior of $X$. Then $\mathcal{A}$ is

(a) locally stable in $X$ if, for each $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that $x \in X \cap (\mathcal{A} \oplus \delta\mathcal{B})$, implies $|\phi(i;x)|_{\mathcal{A}} < \varepsilon$ for all $i \in \mathbb{I}_{\geq 0}$.

(b) locally attractive in $X$ if there exists a $\eta > 0$ such that $x \in X \cap (\mathcal{A} \oplus \eta\mathcal{B})$ implies $|\phi(i;x)|_{\mathcal{A}} \to 0$ as $i \to \infty$.

(c) attractive in $X$ if $|\phi(i;x)|_{\mathcal{A}} \to 0$ as $i \to \infty$ for all $x \in X$.

(d) locally asymptotically stable in $X$ if it is locally stable in $X$ and locally attractive in $X$.

(e) asymptotically stable with a region of attraction $X$ if it is locally stable in $X$ and attractive in $X$.

(f) locally exponentially stable with a region of attraction $X$ if there exist $\eta > 0$, $c > 0$, and $\gamma \in (0,1)$ such that $x \in X \cap (\mathcal{A} \oplus \eta\mathcal{B})$ implies $|\phi(i;x)|_{\mathcal{A}} \leq c|x|_{\mathcal{A}}\gamma^i$ for all $i \in \mathbb{I}_{\geq 0}$.

(g) exponentially stable with a region of attraction $X$ if there exists a $c > 0$ and a $\gamma \in (0,1)$ such that $|\phi(i;x)|_{\mathcal{A}} \leq c|x|_{\mathcal{A}}\gamma^i$ for all $x \in X$, all $i \in \mathbb{I}_{\geq 0}$.

The assumption that $X$ is positive invariant $x^+ = f(x)$ ensures that $\phi(i;x) \in X$ for all $x \in X$, all $i \in \mathbb{I}_{\geq 0}$. Finally, we define the *domain of attraction* of an asymptotically stable set $\mathcal{A}$ for the system $x^+ = f(x)$ to be the set of all initial states $x$ such that $|\phi(i;x)|_{\mathcal{A}} \to 0$ as $i \to \infty$. We use the term *region of attraction* to denote any set of initial states $x$ such that $|\phi(i;x)|_{\mathcal{A}} \to 0$ as $i \to \infty$.

## B.3 Lyapunov Stability Theory

Energy in a passive electrical or mechanical system provides a useful analogy to Lyapunov stability theory. In a lumped mechanical system, the total stored energy is the sum of the potential and kinetic energies. As time proceeds, this energy is dissipated in friction and the total energy decays to zero at which point the system state is in equilibrium.

To establish stability or asymptotic stability, Lyapunov theory follows a similar path. If a real-valued function can be found which is positive and decreasing if the state does not lie in the set $\mathcal{A}$, then the state converges to this set. We now make this intuitive idea more precise.

**Definition B.10** (Lyapunov function)**.** A function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is said to be a Lyapunov function for the system $x^+ = f(x)$ and set $\mathcal{A}$ if there exist functions $\alpha_i \in \mathcal{K}_\infty$, $i = 1, 2$ and $\alpha_3 \in \mathcal{PD}$ such that for any $x \in \mathbb{R}^n$,

$$V(x) \geq \alpha_1(|x|_{\mathcal{A}}) \tag{B.2}$$

$$V(x) \leq \alpha_2(|x|_{\mathcal{A}}) \tag{B.3}$$

$$V(f(x)) - V(x) \leq -\alpha_3(|x|_{\mathcal{A}}) \tag{B.4}$$

If $V(\cdot)$ satisfies Equations B.2–B.4 for all $x \in X$ where $X \supset \mathcal{A}$ is a positive invariant set for $x^+ = f(x)$, then $V(\cdot)$ is said to be a Lyapunov function in $X$ for the system $x^+ = f(x)$ and set $\mathcal{A}$.

The existence of a Lyapunov function is a sufficient condition for global asymptotic stability as shown in the next result which we prove under the assumption, common in MPC, that $\alpha_3(\cdot)$ is $\mathcal{K}_\infty$ function.

**Theorem B.11** (Lyapunov function and GAS)**.** *Suppose $V(\cdot)$ is a Lyapunov function for $x^+ = f(x)$ and set $\mathcal{A}$ with $\alpha_3(\cdot)$ a $\mathcal{K}_\infty$ function. Then $\mathcal{A}$ is globally asymptotically stable.*

*Proof.* (i) Stability: Let $\varepsilon > 0$ be arbitrary and let $\delta := \alpha_2^{-1}(\alpha_1(\varepsilon))$. Suppose $|x|_{\mathcal{A}} < \delta$ so that, by (B.3), $V(x) \leq \alpha_2(\delta) = \alpha_1(\varepsilon)$. From (B.4), $\{V(x(i)) \mid i \in \mathbb{I}_{\geq 0}\}$, $x(i) := \phi(i; x)$, is a nonincreasing sequence so that, for all $i \in \mathbb{I}_{\geq 0}$, $V(x(i)) \leq V(x)$. From (B.2), $|x(i)|_{\mathcal{A}} \leq \alpha_1^{-1}(V(x)) \leq \alpha_1^{-1}(\alpha_1(\varepsilon)) = \varepsilon$ for all $i \in \mathbb{I}_{\geq 0}$. (ii) Attractivity: Let $x \in \mathbb{R}^n$ be arbitrary. From (B.3) $V(x)$ is finite, and from (B.2) and (B.4), the sequence $\{V(x(i)) \mid i \in \mathbb{I}_{\geq 0}\}$, $x(i) := \phi(i; x)$, is nonincreasing and bounded from below by zero. Hence both $V(x(i))$ and $V(x(i+1))$ converge to $\bar{V} \geq 0$ as $i \to \infty$. But $x(i+1) = f(x(i))$ so that, from (B.4), $\alpha_3(|x(i)|_{\mathcal{A}}) \to 0$ as $i \to \infty$. Since $|x(i)|_{\mathcal{A}} = \alpha_3^{-1}(\alpha_3(|x(i)|_{\mathcal{A}}))$ where $\alpha_3^{-1}$ is a $\mathcal{K}_\infty$ function, $|x(i)|_{\mathcal{A}} \to 0$ as $i \to \infty$. ∎

Theorem B.11 merely provides a sufficient condition for global asymptotic stability that might be thought to be conservative. The next result, a *converse* stability theorem by Jiang and Wang (2002), establishes necessity under a stronger hypothesis, namely that $f(\cdot)$ is continuous

rather than locally bounded and uses the fact, established in Jiang and Wang (2002), Lemma 2.7, that the existence of a continuous Lyapunov function for the system $x^+ = f(x)$ and set $\mathcal{A}$ implies the existence of a *smooth* Lyapunov function.[3]

**Theorem B.12** (Converse theorem for asymptotic stability)**.** *Let $\mathcal{A}$ be compact and $f(\cdot)$ continuous. Suppose that the set $\mathcal{A}$ is globally asymptotically stable for the system $x^+ = f(x)$. Then there exists a smooth Lyapunov function for the system $x^+ = f(x)$ and set $\mathcal{A}$.*

A proof of this result is given in Jiang and Wang (2002), Theorem 1, part 3, which establishes the result when $\mathcal{A}$ is UGAS (uniform globally asymptotically stable) for $x^+ = f(x)$ and $\mathcal{A}$ is merely closed and positive invariant and Corollary 3.3. which establishes the equivalence, when $\mathcal{A}$ is compact, of UGAS and GAS (globally asymptotically stable). It follows from Theorems B.11 and B.12 that if $\mathcal{A}$ is compact and $f(\cdot)$ continuous, the set $\mathcal{A}$ is globally asymptotically stable for $x^+ = f(x)$ if and only if there exists a smooth Lyapunov function for $x^+ = f(x)$ and set $\mathcal{A}$.

The appropriate generalization of Theorem B.11 for the constrained case is:

**Theorem B.13** (Lyapunov function for asymptotic stability (constrained case))**.** *Suppose $X \subset \mathbb{R}^n$ is positive invariant for $x^+ = f(x)$, that $\mathcal{A}$ is closed and positive invariant for $x^+ = f(x)$, and that $\mathcal{A}$ lies in the interior of $X$. If there exists a Lyapunov function in $X$ for the system $x^+ = f(x)$ and set $\mathcal{A}$ with $\alpha_3(\cdot)$ a $\mathcal{K}_\infty$ function, then $\mathcal{A}$ is asymptotically stable for $x^+ = f(x)$ with a region of attraction $X$.*

The proof of this result is similar to that of Theorem B.11 and is left as an exercise.

**Theorem B.14** (Lyapunov function for exponential stability)**.** *Suppose $X \subset \mathbb{R}^n$ is positive invariant for $x^+ = f(x)$, that $\mathcal{A}$ is closed and positive invariant for $x^+ = f(x)$, and that $\mathcal{A}$ lies in the interior of $X$. If there exists $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfying the following properties for all $x \in X$*

$$a_1 |x|_{\mathcal{A}}^{\sigma} \leq V(x) \leq a_2 |x|_{\mathcal{A}}^{\sigma}$$
$$V(f(x)) - V(x) \leq -a_3 |x|_{\mathcal{A}}^{\sigma}$$

*in which $a_1, a_2, a_3, \sigma > 0$, then $\mathcal{A}$ is exponentially stable for $x^+ = f(x)$ with a region of attraction $X$.*

[3] A smooth function has derivatives of all orders.

### B.3.1 Lyapunov Function for Linear Systems

We review some facts involving the discrete matrix Lyapunov equation and stability of the linear system

$$x^+ = Ax$$

in which $x \in \mathbb{R}^n$. The discrete time system is asymptotically stable if and only if the magnitudes of the eigenvalues of $A$ are strictly less than unity. Such an $A$ matrix is called stable, convergent, or discrete time Hurwitz.

In the following, $A, S, Q \in \mathbb{R}^{n\times n}$. The following matrix equation is known as a discrete matrix Lyapunov equation,

$$A'SA - S = -Q$$

The properties of solutions to this equation allow one to draw conclusions about the stability of $A$ without computing its eigenvalues. Sontag (1998a, p. 231) provides the following lemma

**Lemma B.15** (Lyapunov function for linear systems)**.** *The following statements are equivalent (Sontag, 1998a).*

*(a) $A$ is stable.*

*(b) For each $Q \in \mathbb{R}^{n\times n}$, there is a unique solution $S$ of the discrete matrix Lyapunov equation*

$$A'SA - S = -Q$$

*and if $Q > 0$ then $S > 0$.*

*(c) There is some $S > 0$ such that $A'SA - S < 0$.*

*(d) There is some $S > 0$ such that $V(x) = x'Sx$ is a Lyapunov function for the system $x^+ = Ax$.*

Exercise B.1 asks you to establish the equivalence of (a) and (b).

## B.4 Robust Stability

We now turn to the task of obtaining stability conditions for discrete time systems subject to disturbances. There are two separate questions that should be addressed. The first is *nominal* robustness; is asymptotic stability of a set $\mathcal{A}$ for a (nominal) system $x^+ = f(x)$ maintained in the presence of arbitrarily small disturbances? The second question is the determination of conditions for asymptotic stability of a set $\mathcal{A}$ for a system perturbed by disturbances lying in a given compact set.

### B.4.1 Nominal Robustness

Here we follow Teel (2004). The nominal system is $x^+ = f(x)$. Consider the perturbed system

$$x^+ = f(x + e) + w \tag{B.5}$$

where $e$ is the state error and $w$ the additive disturbance. Let $\mathbf{e} := \{e(0), e(1), \ldots\}$ and $\mathbf{w} := \{w(0), w(1), \ldots\}$ denote the disturbance sequences with norms $\|\mathbf{e}\| := \sup_{i\geq 0} |e(i)|$ and $\|\mathbf{w}\| := \sup_{i\geq 0} |w(i)|$. Let $M_\delta := \{(\mathbf{e}, \mathbf{w}) \mid \|\mathbf{e}\| \leq \delta, \|\mathbf{w}\| \leq \delta\}$ and, for each $x \in \mathbb{R}^n$, let $S_\delta$ denote the set of solutions $\phi(\cdot; x, \mathbf{e}, \mathbf{w})$ of (B.5) with initial state $x$ (at time 0) and perturbation sequences $(\mathbf{e}, \mathbf{w}) \in M_\delta$. A compact set $\mathcal{A}$ is *nominally* robust asymptotically stable for the (nominal) system $x^+ = f(x)$ if a small neighborhood of $\mathcal{A}$ is locally stable and attractive for all sufficiently small perturbation sequences. We use the adjective *nominal* to indicate that we are examining how a system $x^+ = f(x)$ for which $\mathcal{A}$ is known to be asymptotically stable behaves when subjected to small disturbances. More precisely Teel (2004):

**Definition B.16** (Nominal robust global asymptotic stability)**.** The compact set $\mathcal{A}$ is said to be nominally robustly globally asymptotically stable (nominally RGAS) for the system $x^+ = f(x)$ if there exists a $\mathcal{KL}$ function $\beta(\cdot)$ and, for each $\varepsilon > 0$ and each compact set $X$, there exists a $\delta > 0$ such that, for each $x \in X$ and each solution $\phi(\cdot)$ of the perturbed system lying in $S_\delta$, $|\phi(i)|_{\mathcal{A}} \leq \beta(|x|_{\mathcal{A}}, i) + \varepsilon$ for all $i \in \mathbb{I}_{\geq 0}$.

Thus, for each $\varepsilon > 0$, there exists a $\delta > 0$ such that each solution $\phi(\cdot)$ of $x^+ = f(x+e)+w$ starting in a $\delta$ neighborhood of $\mathcal{A}$ remains in a $\beta(\delta) + \varepsilon$ neighborhood of $\mathcal{A}$, and each solution starting anywhere in $\mathbb{R}^n$ converges to a $\varepsilon$ neighborhood of $\mathcal{A}$. These properties are a necessary relaxation (because of the perturbations) of local stability and global attractivity.

The following result, where we add the adjective "nominal", is established in Teel (2004), Theorem 2:

**Theorem B.17** (Nominal RGAS and Lyapunov function)**.** *Suppose $\mathcal{A}$ is compact and $f(\cdot)$ is locally bounded. Then the set $\mathcal{A}$ is nominally robust globally asymptotically stable for the system $x^+ = f(x)$ if and only if there exists a continuous (in fact, smooth) Lyapunov function for $x^+ = f(x)$ and set $\mathcal{A}$.*

The significance of this result is that while a nonrobust system, for which $\mathcal{A}$ is globally asymptotically stable, has a Lyapunov function,

that function is *not* continuous. For the globally asymptotically stable example $x^+ = f(x)$ discussed in Section 3.2 of Chapter 3, where $f(x) = (0, |x|)$ when $x_1 \neq 0$ and $f(x) = (0,0)$ otherwise, one Lyapunov function $V(\cdot)$ is $V(x) = 2|x|$ if $x_1 \neq 0$ and $V(x) = |x|$ if $x_1 = 0$. That $V(\cdot)$ is a Lyapunov function follows from the fact that it satisfies $V(x) \geq |x|$, $V(x) \leq 2|x|$ and $V(f(x)) - V(x) = -|x|$ for all $x \in \mathbb{R}^2$. It follows immediately from its definition that $V(\cdot)$ is not continuous; but we can also deduce from Theorem B.17 that every Lyapunov function for this system is not continuous since, as shown in Section 3.2 of Chapter 3, global asymptotic stability for this system is not robust. Theorem B.17 shows that existence of a continuous Lyapunov function guarantees nominal robustness. Also, it follows from Theorem B.12 that there exists a smooth Lyapunov function for $x^+ = f(x)$ if $f(\cdot)$ is continuous and $\mathcal{A}$ is GAS for $x^+ = f(x)$. Since $f(\cdot)$ is locally bounded if it is continuous, it then follows from Theorem B.17 that $\mathcal{A}$ is nominally robust GAS for $x^+ = f(x)$ if it is GAS and $f(\cdot)$ is continuous.

### B.4.2 Robustness

We turn now to stability conditions for systems subject to bounded disturbances (not vanishingly small) and described by

$$x^+ = f(x, w) \tag{B.6}$$

where the disturbance $w$ lies in the compact set $\mathbb{W}$. This system may equivalently be described by the difference inclusion

$$x^+ \in F(x) \tag{B.7}$$

where the set $F(x) := \{f(x, w) \mid w \in \mathbb{W}\}$. Let $S(x)$ denote the set of all solutions of (B.6) or (B.7) with initial state $x$. We require, in the sequel, that the set $\mathcal{A}$ is positive invariant for (B.6) (or for $x^+ \in F(x)$):

**Definition B.18** (Positive invariance with disturbances)**.** The set $\mathcal{A}$ is positive invariant for $x^+ = f(x, w)$, $w \in \mathbb{W}$ if $x \in \mathcal{A}$ implies $f(x, w) \in \mathcal{A}$ for all $w \in \mathbb{W}$; it is positive invariant for $x^+ \in F(x)$ if $x \in \mathcal{A}$ implies $F(x) \subseteq \mathcal{A}$.

Clearly the two definitions are equivalent; $\mathcal{A}$ is positive invariant for $x^+ = f(x, w)$, $w \in \mathbb{W}$, if and only if it is positive invariant for $x^+ \in F(x)$. In Definitions B.19-B.21, we use "positive invariant" to denote "positive invariant for $x^+ = f(x, w)$, $w \in \mathbb{W}$" or for $x^+ \in F(x)$.

**Definition B.19** (Local stability (disturbances))**.** The closed positive invariant set $\mathcal{A}$ is *locally stable* for $x^+ = f(x,w)$, $w \in \mathbb{W}$ (or for $x^+ \in F(x)$) if, for all $\varepsilon > 0$, there exists a $\delta > 0$ such that, for each $x$ satisfying $|x|_{\mathcal{A}} < \delta$, each solution $\phi \in S(x)$ satisfies $|\phi(i)|_{\mathcal{A}} < \varepsilon$ for all $i \in \mathbb{I}_{\geq 0}$.

**Definition B.20** (Global attraction (disturbances))**.** The closed positive invariant set $\mathcal{A}$ is *globally attractive* for the system $x^+ = f(x,w)$, $w \in \mathbb{W}$ (or for $x^+ \in F(x)$) if, for each $x \in \mathbb{R}^n$, each solution $\phi(\cdot) \in S(x)$ satisfies $|\phi(i)|_{\mathcal{A}} \to 0$ as $i \to \infty$.

**Definition B.21** (GAS (disturbances))**.** The closed positive invariant set $\mathcal{A}$ is *globally asymptotically stable* for $x^+ = f(x,w)$, $w \in \mathbb{W}$ (or for $x^+ \in F(x)$) if it is locally stable and globally attractive.

An alternative definition of global asymptotic stability of $\mathcal{A}$ for $x^+ = f(x,w)$, $w \in \mathbb{W}$, if $\mathcal{A}$ is compact, is the existence of a $\mathcal{KL}$ function $\beta(\cdot)$ such that for each $x \in \mathbb{R}^n$, each $\phi \in S(x)$ satisfies $|\phi(i)|_{\mathcal{A}} \leq \beta(|x|_{\mathcal{A}}, i)$ for all $i \in \mathbb{I}_{\geq 0}$. To cope with disturbances we require a modified definition of a Lyapunov function.

**Definition B.22** (Lyapunov function (disturbances))**.** A function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is said to be a Lyapunov function for the system $x^+ = f(x,w)$, $w \in \mathbb{W}$ (or for $x^+ \in F(x)$) and set $\mathcal{A}$ if there exist functions $\alpha_i \in \mathcal{K}_\infty$, $i = 1,2$ and $\alpha_3 \in \mathcal{PD}$ such that for any $x \in \mathbb{R}^n$,

$$V(x) \geq \alpha_1(|x|_{\mathcal{A}}) \tag{B.8}$$

$$V(x) \leq \alpha_2(|x|_{\mathcal{A}}) \tag{B.9}$$

$$\sup_{z \in F(x)} V(z) - V(x) \leq -\alpha_3(|x|_{\mathcal{A}}) \tag{B.10}$$

Inequality B.10 ensures $V(f(x,w)) - V(x) \leq -\alpha_3(|x|_{\mathcal{A}})$ for all $w \in \mathbb{W}$. The existence of a Lyapunov function for the system $x^+ \in F(x)$ and set $\mathcal{A}$ is a sufficient condition for $\mathcal{A}$ to be globally asymptotically stable for $x^+ \in F(x)$ as shown in the next result.

**Theorem B.23** (Lyapunov function for GAS (disturbances))**.** *Suppose $V(\cdot)$ is a Lyapunov function for $x^+ = f(x,w)$, $w \in \mathbb{W}$ (or for $x^+ \in F(x)$) and set $\mathcal{A}$ with $\alpha_3(\cdot)$ a $\mathcal{K}_\infty$ function. Then $\mathcal{A}$ is globally asymptotically stable for $x^+ = f(x,w)$, $w \in \mathbb{W}$ (or for $x^+ \in F(x)$).*

*Proof.* (i) Local stability: Let $\varepsilon > 0$ be arbitrary and let $\delta := \alpha_2^{-1}(\alpha_1(\varepsilon))$. Suppose $|x|_{\mathcal{A}} < \delta$ so that, by (B.9), $V(x) \leq \alpha_2(\delta) = \alpha_1(\varepsilon)$. Let $\phi(\cdot)$ be

any solution in $S(x)$ so that $\phi(0) = x$. From (B.10), $\{V(\phi(i)) \mid i \in \mathbb{I}_{\geq 0}\}$ is a nonincreasing sequence so that, for all $i \in \mathbb{I}_{\geq 0}$, $V(\phi(i)) \leq V(x)$. From (B.8), $|\phi(i)|_{\mathcal{A}} \leq \alpha_1^{-1}(V(x)) \leq \alpha_1^{-1}(\alpha_1(\varepsilon)) = \varepsilon$ for all $i \in \mathbb{I}_{\geq 0}$. (ii) Global attractivity: Let $x \in \mathbb{R}^n$ be arbitrary. Let $\phi(\cdot)$ be any solution in $S(x)$ so that $\phi(0) = x$. From Equations B.8 and B.10, since $\phi(i+1) \in F(\phi(i))$, the sequence $\{V(\phi(i)) \mid i \in \mathbb{I}_{\geq 0}\}$ is nonincreasing and bounded from below by zero. Hence both $V(\phi(i))$ and $V(\phi(i+1))$ converge to $\bar{V} \geq 0$ as $i \to \infty$. But $\phi(i+1) \in F(\phi(i))$ so that, from (B.10), $\alpha_3(|\phi(i)|_{\mathcal{A}}) \to 0$ as $i \to \infty$. Since $|\phi(i)|_{\mathcal{A}} = \alpha_3^{-1}(\alpha_3(|\phi(i)|_{\mathcal{A}}))$ where $\alpha_3^{-1}(\cdot)$ is a $\mathcal{K}_\infty$ function, $|\phi(i)|_{\mathcal{A}} \to 0$ as $i \to \infty$. ■

## B.5 Control-Lyapunov Functions

A control-Lyapunov function is a useful generalization, due to Sontag (1998a, pp.218–233), of a Lyapunov function; while a Lyapunov function is relevant for a system $x^+ = f(x)$ and provides conditions for the (asymptotic) stability of a set for this system, a control-Lyapunov function is relevant for a control system $x^+ = f(x,u)$ and provides conditions for the existence of a controller $u = \kappa(x)$ that ensures (asymptotic) stability of a set for the controlled system $x^+ = f(x, \kappa(x))$. Consider the control system

$$x^+ = f(x,u)$$

where the control $u$ is subject to the constraint

$$u \in \mathbb{U}$$

Our standing assumptions in this section are that $f(\cdot)$ is continuous and $\mathbb{U}$ is compact.

**Definition B.24** (Global control-Lyapunov function (CLF))**.** A function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a global control-Lyapunov function for the system $x^+ = f(x,u)$ and set $\mathcal{A}$ if there exist $\mathcal{K}_\infty$-functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ and a $\mathcal{PD}$-function $\alpha_3(\cdot)$ satisfying for all $x \in \mathbb{R}^n$:

$$V(x) \geq \alpha_1(|x|_{\mathcal{A}})$$
$$V(x) \leq \alpha_2(|x|_{\mathcal{A}})$$
$$\inf_{u \in \mathbb{U}} V(f(x,u)) \leq V(x) - \alpha_3(|x|_{\mathcal{A}})$$

**Definition B.25** (Global stabilizability)**.** Let $\mathcal{A}$ be compact. The set $\mathcal{A}$ is globally stabilizable for the system $x^+ = f(x,u)$ if there exists a state-feedback function $\kappa : \mathbb{R}^n \to \mathbb{U}$ such that $\mathcal{A}$ is globally asymptotically stable for $x^+ = f(x, \kappa(x))$.

In a similar fashion one can extend the concept of control-Lyapunov functions to the case when the system is subject to disturbances. Consider the system

$$x^+ = f(x, u, w)$$

where the control $u$ is constrained to lie in $\mathbb{U}$ and the disturbance takes values in the set $\mathbb{W}$. We assume that $f(\cdot)$ is continuous and that $\mathbb{U}$ and $\mathbb{W}$ are compact. The system may be equivalently defined by

$$x^+ \in F(x, u)$$

where the set-valued function $F(\cdot)$ is defined by

$$F(x, u) := \{f(x, u, w) \mid w \in \mathbb{W}\}$$

We can now make the obvious generalizations of the definitions in Section B.4.2.

**Definition B.26** (Positive invariance (disturbance and control))**.** The set $\mathcal{A}$ is positive invariant for $x^+ = f(x, u, w)$, $w \in \mathbb{W}$ (or for $x^+ \in F(x, u)$) if for all $x \in \mathcal{A}$ there exists a $u \in \mathbb{U}$ such that $f(x, u, w) \in \mathcal{A}$ for all $w \in \mathbb{W}$ (or $F(x, u) \subseteq \mathcal{A}$).

**Definition B.27** (CLF (disturbance and control))**.** A function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is said to be a control-Lyapunov function for the system $x^+ = f(x, u, w)$, $u \in \mathbb{U}$, $w \in \mathbb{W}$ (or $x^+ \in F(x, u)$, $u \in \mathbb{U}$) and set $\mathcal{A}$ if there exist functions $\alpha_i \in \mathcal{K}_\infty$, $i = 1, 2$ and $\alpha_3 \in \mathcal{PD}$ such that for any $x \in \mathbb{R}^n$,

$$\begin{aligned} V(x) &\geq \alpha_1(|x|_{\mathcal{A}}) \\ V(x) &\leq \alpha_2(|x|_{\mathcal{A}}) \\ \inf_{u \in \mathbb{U}} \sup_{z \in F(x,u)} V(z) &\leq V(x) - \alpha_3(|x|_{\mathcal{A}}) \end{aligned} \tag{B.11}$$

**Remark B.28** (CLF implies control law)**.** Given a global control-Lyapunov function $V(\cdot)$, one can choose a control law $\kappa : \mathbb{R}^n \to \mathbb{U}$ satisfying

$$\sup_{z \in F(x, \kappa(x))} V(z) \leq V(x) - \alpha_3(|x|_{\mathcal{A}})/2$$

for all $x \in \mathbb{R}^n$. Since $\mathbb{U}$ is compact, $\kappa(\cdot)$ is locally bounded and, hence, so is $x \mapsto f(x, \kappa(x))$. Thus we may use Theorem B.23 to deduce that $\mathcal{A}$ is globally asymptotically stable for $x^+ = f(x, \kappa(x), w)$, $w \in \mathbb{W}$ (for $x^+ \in F(x, \kappa(x))$).

These results can be further extended to deal with the constrained case. First, we generalize the definitions of positive invariance of a set.

**Definition B.29** (Positive invariance (constrained))**.** The set $\mathcal{A}$ is control invariant for $x^+ = f(x,u)$, $u \in \mathbb{U}$ if, for all $x \in \mathcal{A}$, there exists a $u \in \mathbb{U}$ such that $f(x,u) \in \mathcal{A}$.

Suppose that the state $x$ is required to lie in the closed set $\mathbb{X} \subset \mathbb{R}^n$. In order to show that it is possible to ensure a decrease of a Lyapunov function, as in (B.11), in the presence of the state constraint $x \in \mathbb{X}$, we assume that there exists a control invariant set $X \subseteq \mathbb{X}$ for $x^+ = f(x,u,w)$, $u \in \mathbb{U}$, $w \in \mathbb{W}$. This enables us to obtain a control law that keeps the state in $X$ and, hence, in $\mathbb{X}$, and, under suitable conditions, to satisfy a variant of (B.11).

**Definition B.30** (CLF (constrained))**.** Suppose the sets $X$ and $\mathcal{A}$, $X \supset \mathcal{A}$, are control invariant for $x^+ = f(x,u)$, $u \in \mathbb{U}$. A function $V : X \to \mathbb{R}_{\geq 0}$ is said to be a control-Lyapunov function in $X$ for the system $x^+ = f(x,u)$, $u \in \mathbb{U}$, and set $\mathcal{A}$ in $X$ if there exist functions $\alpha_i \in \mathcal{K}_\infty$, $i = 1,2$ and $\alpha_3 \in \mathcal{PD}$, defined on $X$, such that for any $x \in X$,

$$
\begin{aligned}
V(x) &\geq \alpha_1(|x|_{\mathcal{A}}) \\
V(x) &\leq \alpha_2(|x|_{\mathcal{A}}) \\
\inf_{u \in \mathbb{U}} \{V(f(x,u)) \mid f(x,u) \in X\} &\leq V(x) - \alpha_3(|x|_{\mathcal{A}})
\end{aligned}
$$

Finally we consider the constrained case in the presence of disturbances. First we define control invariance in the presence of disturbances.

**Definition B.31** (Control invariance (disturbances, constrained))**.** The set $\mathcal{A}$ is control invariant for $x^+ = f(x,u,w)$, $u \in \mathbb{U}$, $w \in \mathbb{W}$ if, for all $x \in \mathcal{A}$, there exists a $u \in \mathbb{U}$ such that $f(x,u,w) \in \mathcal{A}$ for all $w \in \mathbb{W}$ (or $F(x,u) \subseteq \mathcal{A}$ where $F(x,u) := \{f(x,u,w) \mid w \in \mathbb{W}\}$.

Next, we define what we mean by a control-Lyapunov function in this context.

**Definition B.32** (CLF (disturbances, constrained))**.** Suppose the sets $X$ and $\mathcal{A}$, $X \supset \mathcal{A}$, are control invariant for $x^+ = f(x,u,w)$, $u \in \mathbb{U}$, $w \in \mathbb{W}$. A function $V : X \to \mathbb{R}_{\geq 0}$ is said to be a control-Lyapunov function in $X$ for the system $x^+ = f(x,u,w)$, $u \in \mathbb{U}$, $w \in \mathbb{W}$ and set $\mathcal{A}$ if there exist

functions $\alpha_i \in \mathcal{K}_\infty$, $i = 1, 2$ and $\alpha_3 \in \mathcal{PD}$, defined on $X$, such that for any $x \in X$,

$$V(x) \geq \alpha_1(|x|_{\mathcal{A}})$$
$$V(x) \leq \alpha_2(|x|_{\mathcal{A}})$$
$$\inf_{u \in \mathbb{U}} \sup_{z \in F(x,u)} \{V(z) \mid F(x,u) \subseteq X\} \leq V(x) - \alpha_3(|x|_{\mathcal{A}})$$

Suppose now that the state $x$ is required to lie in the closed set $\mathbb{X} \subset \mathbb{R}^n$. Again, in order to show that there exists a condition similar to (B.11), we assume that there exists a control invariant set $X \subseteq \mathbb{X}$ for $x^+ = f(x, u, w)$, $u \in \mathbb{U}$, $w \in \mathbb{W}$. This enables us to obtain a control law that keeps the state in $X$ and, hence, in $\mathbb{X}$, and, under suitable conditions, to satisfy a variant of (B.11).

## B.6 Input-to-State Stability

We consider, as in the previous section, the system

$$x^+ = f(x, w)$$

where the disturbance $w$ takes values in $\mathbb{R}^p$. In input-to-state stability (Sontag and Wang, 1995; Jiang and Wang, 2001) we seek a bound on the state in terms of a uniform bound on the disturbance sequence $\mathbf{w} := \{w(0), w(1), \ldots\}$. Let $\|\cdot\|$ denote the usual $\ell_\infty$ norm for sequences, i.e., $\|\mathbf{w}\| := \sup_{k \geq 0} |w(k)|$.

**Definition B.33** (Input-to-state stable (ISS))**.** The system $x^+ = f(x, w)$ is (globally) input-to-state stable (ISS) if there exists a $\mathcal{KL}$ function $\beta(\cdot)$ and a $\mathcal{K}$ function $\sigma(\cdot)$ such that, for each $x \in \mathbb{R}^n$, and each disturbance sequence $\mathbf{w} = \{w(0), w(1), \ldots\}$ in $\ell_\infty$

$$|\phi(i; x, \mathbf{w}_i)| \leq \beta(|x|, i) + \sigma(\|\mathbf{w}_i\|)$$

for all $i \in \mathbb{I}_{\geq 0}$, where $\phi(i; x, \mathbf{w}_i)$ is the solution, at time $i$, if the initial state is $x$ at time 0 and the input sequence is $\mathbf{w}_i := \{w(0), w(1), \ldots, w(i-1)\}$.

We note that this definition implies the origin is globally asymptotically stable if the input sequence is identically zero. Also, the norm of the state is asymptotically bounded by $\sigma(\|\mathbf{w}\|)$ where $\mathbf{w} := \{w(0), w(1), \ldots\}$. As before, we seek a Lyapunov function that ensures input-to-state stability.

**Definition B.34** (ISS-Lyapunov function)**.** A function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is an ISS-Lyapunov function for system $x^+ = f(x, w)$ if there exist $\mathcal{K}_\infty$ functions $\alpha_1(\cdot), \alpha_2(\cdot), \alpha_3(\cdot)$ and a $\mathcal{K}$ function $\sigma(\cdot)$ such that for all $x \in \mathbb{R}^n$, $w \in \mathbb{R}^p$

$$\begin{aligned} V(x) &\geq \alpha_1(|x|) \\ V(x) &\leq \alpha_2(|x|) \\ V(f(x,w)) - V(x) &\leq -\alpha_3(|x|) + \sigma(|w|) \end{aligned}$$

The following result appears in Jiang and Wang (2001), Lemma 3.5:

**Lemma B.35** (ISS-Lyapunov function implies ISS)**.** *Suppose $f(\cdot)$ is continuous and that there exists a continuous ISS-Lyapunov function for $x^+ = f(x, w)$. Then the system $x^+ = f(x, w)$ is ISS.*

The converse, i.e., input to state stability implies the existence of a smooth ISS-Lyapunov function for $x^+ = f(x, w)$ is also proved in Jiang and Wang (2002), Theorem 1. We now consider the case when the state satisfies the constraint $x \in \mathbb{X}$ where $\mathbb{X}$ is a closed subset of $\mathbb{R}^n$. Accordingly, we assume that the disturbance $w$ satisfies $w \in \mathbb{W}$ where $\mathbb{W}$ is a compact set containing the origin and that $X \subset \mathbb{X}$ is a closed robust positive invariant set for $x^+ = f(x, w)$, $w \in \mathbb{W}$ or, equivalently, for $x^+ \in F(x, u)$.

**Definition B.36** (ISS (constrained))**.** Suppose that $\mathbb{W}$ is a compact set containing the origin and that $X \subset \mathbb{X}$ is a closed robust positive invariant set for $x^+ = f(x, w)$, $w \in \mathbb{W}$. The system $x^+ = f(x, w)$, $w \in \mathbb{W}$ is ISS in $X$ if there exists a class $\mathcal{KL}$ function $\beta(\cdot)$ and a class $\mathcal{K}$ function $\sigma(\cdot)$ such that, for all $x \in X$, all $\mathbf{w} \in \mathcal{W}$ where $\mathcal{W}$ is the set of infinite sequences $\mathbf{w}$ satisfying $w(i) \in \mathbb{W}$ for all $i \in \mathbb{I}_{\geq 0}$

$$|\phi(i; x, \mathbf{w}_i)| \leq \beta(|x|, i) + \sigma(\|\mathbf{w}_i\|)$$

**Definition B.37** (ISS-Lyapunov function (constrained))**.** A function $V : X \to \mathbb{R}_{\geq 0}$ is an ISS-Lyapunov function in $X$ for system $x^+ = f(x, w)$ if there exist $\mathcal{K}_\infty$ functions $\alpha_1(\cdot), \alpha_2(\cdot), \alpha_3(\cdot)$ and a $\mathcal{K}$ function $\sigma(\cdot)$ such that for all $x \in X$, all $w \in \mathbb{W}$

$$\begin{aligned} V(x) &\geq \alpha_1(|x|) \\ V(x) &\leq \alpha_2(|x|) \\ V(f(x,w)) - V(x) &\leq -\alpha_3(|x|) + \sigma(|w|) \end{aligned}$$

The following result is a trivial generalization of Lemma 3.5 in Jiang and Wang (2001).

**Lemma B.38** (ISS-Lyapunov function implies ISS (constrained))**.** *Suppose that $\mathbb{W}$ is a compact set containing the origin and that $X \subset \mathbb{X}$ is a closed robust positive invariant set for $x^+ = f(x,w)$, $w \in \mathbb{W}$. If $f(\cdot)$ is continuous and there exists a continuous ISS-Lyapunov function in $X$ for the system $x^+ = f(x,w)$, $w \in \mathbb{W}$, then the system $x^+ = f(x,w)$, $w \in \mathbb{W}$ is ISS in $X$.*

## B.7 Output-to-State Stability and Detectability

We present some definitions and results that are discrete time versions of results due to Sontag and Wang (1997) and Krichman, Sontag, and Wang (2001). The output-to-state (OSS) property corresponds, informally, to the statement that "no matter what the initial state is, if the observed outputs are small, then the state must eventually be small". It is therefore a natural candidate for the concept of nonlinear (zero-state) detectability. We consider first the autonomous system

$$x^+ = f(x) \qquad y = h(x) \tag{B.12}$$

where $f(\cdot) : \mathbb{X} \to \mathbb{X}$ is locally Lipschitz continuous and $h(\cdot)$ is continuously differentiable where $\mathbb{X} = \mathbb{R}^n$ for some $n$. We assume $x = 0$ is an equilibrium state, i.e., $f(0) = 0$. We also assume $h(0) = 0$. We use $\phi(k; x_0)$ to denote the solution of (B.12) with initial state $x_0$, and $y(k; x_0)$ to denote $h(\phi(k; x_0))$. The function $y_{x_0}(\cdot)$ is defined by

$$y_{x_0}(k) := y(k; x_0)$$

We use $|\cdot|$ and $\|\cdot\|$ to denote, respectively the Euclidean norm of a vector and the sup norm of a sequence; $\|\cdot\|_{0:k}$ denotes the max norm of a sequence restricted to the interval $[0,k]$. For conciseness, $\mathbf{u}$, $\mathbf{y}$ denote, respectively, the sequences $\{u(j)\}$, $\{y(j)\}$.

**Definition B.39** (Output-to-state stable (OSS))**.** The system (B.12) is output-to-state stable (OSS) if there exist functions $\beta(\cdot) \in \mathcal{KL}$ and $\gamma(\cdot) \in \mathcal{K}$ such that for all $x_0 \in \mathbb{R}^n$ and all $k \geq 0$

$$|x(k; x_0)| \leq \max\left\{\beta(|x_0|, k),\ \gamma(\|\mathbf{y}\|_{0:k})\right\}$$

**Definition B.40** (OSS-Lyapunov function)**.** An OSS-Lyapunov function for system (B.12) is any function $V(\cdot)$ with the following properties

(a) There exist $\mathcal{K}_\infty$-functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ such that

$$\alpha_1(|x|) \le V(x) \le \alpha_2(|x|)$$

for all $x$ in $\mathbb{R}^n$.

(b) There exist $\mathcal{K}_\infty$-functions $\alpha(\cdot)$ and $\sigma(\cdot)$ such that for all $x \in \mathbb{R}^n$ either

$$V(x^+) \le V(x) - \alpha(|x|) + \sigma(|y|)$$

or

$$V(x^+) \le \rho V(x) + \sigma(|y|) \tag{B.13}$$

with $x^+ = f(x)$, $y = h(x)$, and $\rho \in (0,1)$.

Inequality (B.13) corresponds to an exponential-decay OSS-Lyapunov function.

**Theorem B.41** (OSS and OSS-Lyapunov function)**.** *The following properties are equivalent for system* (B.12)*:*

*(a) The system is OSS.*

*(b) The system admits an OSS-Lyapunov function.*

*(c) The system admits an exponential-decay OSS-Lyapunov function.*

## B.8 Input/Output-to-State Stability

Consider now a system with both inputs and outputs

$$x^+ = f(x,u) \qquad y = h(x) \tag{B.14}$$

Input/output-to-state stability corresponds, roughly, to the statement that, no matter what the initial state is, if the input and the output converge to zero, so does the state. We assume $f(\cdot)$ is continuous and locally Lipschitz in $x$ on bounded $u$ and that $h(\cdot)$ is continuously differentiable. We also assume $f(0,0) = 0$ and $h(0) = 0$. Let $x(\cdot, x_0, \mathbf{u})$ denote the solution of (B.14) which results from initial state $x_0$ and control $\mathbf{u} = \{u(j)\}$ and let $y_{x_0,\mathbf{u}}(k) = y(k; x_0, \mathbf{u})$ denote $h(x(k; x_0, \mathbf{u}))$.

**Definition B.42** (Input/output-to-state stable (IOSS))**.** The system (B.14) is input/output-to-state stable (IOSS) if there exist functions $\beta(\cdot) \in \mathcal{KL}$ and $\gamma_1(\cdot)$, $\gamma_2(\cdot) \in \mathcal{K}$ such that

$$|x(k; x_0)| \le \max\left\{\beta(|x_0|, k), \gamma_1\left(\|\mathbf{u}\|_{0:k-1}\right), \gamma_2\left(\|\mathbf{y}\|_{0:k}\right)\right\}$$

for every initial state $x_0 \in \mathbb{R}^n$, every control sequence $\mathbf{u} = \{u(j)\}$, and all $k \ge 0$.

**Definition B.43** (IOSS-Lyapunov function)**.** An IOSS-Lyapunov function for system (B.14) is any function $V(\cdot)$ with the following properties:
(a) There exist $\mathcal{K}_\infty$-functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ such that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$$

for all $x \in \mathbb{R}^n$.

(b) There exist $\mathcal{K}_\infty$-functions $\alpha(\cdot)$, $\sigma_1(\cdot)$, and $\sigma_2(\cdot)$ such that for every $x$ and $u$ either

$$V(x^+) \leq V(x) - \alpha(|x|) + \sigma_1(|u|) + \sigma_2(|y|)$$

or

$$V(x^+) \leq \rho V(x) + \sigma_1(|u|) + \sigma_2(|y|)$$

with $x^+ = f(x,u)$, $y = h(x)$, and $\rho \in (0,1)$.

**Conjecture B.44** (IOSS and IOSS-Lyapunov function)**.** *The following properties are equivalent for system* (B.14)*:*

*(a) The system is IOSS.*

*(b) The system admits a smooth IOSS-Lyapunov function.*

*(c) The system admits an exponential-decay IOSS-Lyapunov function.*

## B.9 Incremental-Input/Output-to-State Stability

**Definition B.45** (Incremental input/output-to-state stable)**.** The system (B.14) is incrementally input/output-to-state stable (i-IOSS) if there exists some $\beta(\cdot) \in \mathcal{KL}$ and $\gamma_1(\cdot)$, $\gamma_2(\cdot) \in \mathcal{K}$ such that, for every two initial states $z_1$ and $z_2$ and any two control sequences $\mathbf{u}_1 = \{u_1(j)\}$ and $\mathbf{u}_2 = \{u_2(j)\}$

$$|x(k;z_1,\mathbf{u}_1) - x(k;z_2,\mathbf{u}_2)| \leq$$
$$\max\left\{\beta(|z_1 - z_2|,k),\ \gamma_1(\|\mathbf{u}_1 - \mathbf{u}_2\|_{0:k-1}),\ \gamma_2(\|\mathbf{y}_{z_1,\mathbf{u}_1} - \mathbf{y}_{z_2,\mathbf{u}_2}\|_{0:k})\right\}$$

## B.10 Observability

**Definition B.46** (Observability)**.** The system (B.14) is (uniformly) observable if there exists a positive integer $N$ and an $\alpha(\cdot) \in \mathcal{K}$ such that

$$\sum_{j=0}^{k-1} |h(x(j;x,u)) - h(x(j;z,u))| \geq \alpha(|x-z|) \qquad \text{(B.15)}$$

for all $x, z$, all $k \geq N$ and all control sequences $u$; here $x(j; z, u) = \phi(j; z, u)$, the solution of (B.14) when the initial state is $z$ at time 0 and the control sequence is $u$.

When the system is linear, i.e., $f(x, u) = Ax + Bu$ and $h(x) = Cx$, this assumption is equivalent to assuming the observability Gramian $\sum_{j=0}^{n-1} CA^j(A^j)'C'$ is positive definite. Consider the system described by

$$z^+ = f(z, u) + w \qquad y + v = h(z) \tag{B.16}$$

with output $y_w = y + v$. Let $z(k; z, u, w)$ denote the solution, at time $k$ of (B.16) if the state at time 0 is $z$, the control sequence is $u$ and the disturbance sequence is $w$. We assume, in the sequel, that

**Assumption B.47** (Lipschitz continuity of model)**.**

(a) The function $f(\cdot)$ is globally Lipschitz continuous in $\mathbb{R}^n \times \mathbf{U}$ with Lipschitz constant $c$.

(b) The function $h(\cdot)$ is globally Lipschitz continuous in $\mathbb{R}^n$ with Lipschitz constant $c$.

**Lemma B.48** (Lipschitz continuity and state difference bound)**.** *Suppose Assumption B.47 is satisfied (with Lipschitz constant c). Then,*

$$|x(k; x, u) - z(k; z, u, w)| \leq c^k |x - z| + \sum_{i=0}^{k-1} c^{k-i-1} |w(i)|$$

*Proof.* Let $\delta(k) := |x(k; x, u) - z(k; z, u, w)|$. Then

$$\begin{aligned} \delta(k+1) &= \left| f(x(k; x, u), u(k)) - f(z(k; z, u, w), u(k)) - w(k) \right| \\ &\leq c\,|\delta(k)| + |w(k)| \end{aligned}$$

Iterating this equation yields the desired result. ∎

**Theorem B.49** (Observability and convergence of state)**.** *Suppose* (B.14) *is (uniformly) observable and that Assumption B.47 is satisfied. Then, $w(k) \to 0$ and $v(k) \to 0$ as $k \to \infty$ imply $|x(k; x, u) - z(k; z, u, w)| \to 0$ as $k \to \infty$.*

*Proof.* Let $x(k)$ and $z(k)$ denote $x(k; x, u)$ and $z(k; z, u, w)$, respectively, in the sequel. Since (B.14) is observable, there exists an integer

$N$ satisfying (B.15). Consider the sum

$$\begin{aligned} S(k) = \sum_{j=k}^{k+N} v(k) &= \sum_{j=k}^{k+N} \left| h(x(j;x,u)) - h(z(j;z,u,w)) \right| \\ &\geq \sum_{j=k}^{k+N} \left| h(x(j;x(k),u)) - h(x(j;z(k),u)) \right| \\ &\quad - \sum_{j=k}^{k+N} \left| h(x(j;z(k),u)) - h(z(j;z(k),u,w)) \right| \end{aligned} \tag{B.17}$$

where we have used the fact that $|a+b| \geq |a| - |b|$. By the assumption of observability

$$\sum_{j=k}^{k+N} \left| h(x(j;x(k),u)) - h(x(j;z(k),u)) \right| \geq \alpha(|x(k) - z(k)|)$$

for all $k$. From Lemma B.48 and the Lipschitz assumption on $h(\cdot)$

$$\begin{aligned} &\left| h(x(j;z(k),u)) - h(z(j;z(k),u,w)) \right| \leq \\ &\qquad c \left| x(j;z(k),u) - z(j;z(k),u,w) \right| \leq c \sum_{i=k}^{j-1} c^{j-1-i} |w(i)| \end{aligned}$$

for all $j$ in $\{k+1, k+2, \ldots k+N\}$. Hence there exists a $d \in (0,\infty)$ such that the last term in (B.17) satisfies

$$\sum_{j=k}^{k+N} \left| h(x(j;x(k),u)) - h(x(j;z(k),u)) \right| \leq d \, \|w\|_{k-N:k}$$

Hence, (B.17) becomes

$$\alpha(|x(k) - z(k)|) \leq N \, \|v\|_{k-N:k} + d \, \|w\|_{k-N:k}$$

Since, by assumption, $w(k) \to 0$ and $v(k) \to 0$ as $k \to \infty$, and $\alpha(\cdot) \in \mathcal{K}$, it follows that $|x(k) - z(k)| \to 0$ as $k \to \infty$. ■

## B.11 Exercises

**Exercise B.1: Lyapunov equation and linear systems**

Establish the equivalence of (a) and (b) in Lemma B.15.

### Exercise B.2: Lyapunov function for exponential stability

Let $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a Lyapunov function for the system $x^+ = f(x)$ with the following properties. For all $x \in \mathbb{R}^n$

$$a_1 |x|^\sigma \leq V(x) \leq a_2 |x|^\sigma$$
$$V(f(x)) - V(x) \leq -a_3 |x|^\sigma$$

in which $a_1, a_2, a_3, \sigma > 0$. Show that the origin of the system $x^+ = f(x)$ is globally exponentially stable.

### Exercise B.3: A converse theorem for exponential stability

(a) Assume that the origin is globally exponentially stable (GES) for the system

$$x^+ = f(x)$$

in which $f(\cdot)$ is continuous. Show that there exists a continuous Lyapunov function $V(\cdot)$ for the system satisfying for all $x \in \mathbb{R}^n$

$$a_1 |x|^\sigma \leq V(x) \leq a_2 |x|^\sigma$$
$$V(f(x)) - V(x) \leq -a_3 |x|^\sigma$$

in which $a_1, a_2, a_3, \sigma > 0$.

Hint: Consider summing the solution $|\phi(i;x)|^\sigma$ on $i$ as a candidate Lyapunov function $V(x)$.

(b) Establish that in the Lyapunov function defined above, any $\sigma > 0$ is valid, and also that the constant $a_3$ can be chosen as large as one wishes.

### Exercise B.4: A converse theorem for asymptotic stability

Show that if the origin is globally asymptotically stable (GAS) for the system

$$x^+ = f(x)$$

Then there exists a Lyapunov function $V(\cdot)$ for the system satisfying for all $x \in \mathbb{R}^n$

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$$
$$V(f(x)) - V(x) \leq -\alpha_3(|x|)$$

in which $\alpha_1(\cdot), \alpha_2(\cdot), \alpha_3(\cdot) \in \mathcal{K}_\infty$.

Hint: use the following result due to Sontag (1998b, Proposition 7) and the approach of Exercise B.3.

**Proposition B.50** (Improving convergence (Sontag (1998b))). *Assume that $\beta(\cdot) \in \mathcal{KL}$. Then there exists $\theta_1(\cdot), \theta_2(\cdot) \in \mathcal{K}_\infty$ so that*

$$\beta(s,t) \leq \theta_1(\theta_2(s)e^{-t}) \qquad \forall s \geq 0, \quad \forall t \geq 0 \tag{B.18}$$

### Exercise B.5: Revisit Lemma 1.3 in Chapter 1

Establish Lemma 1.3 in Chapter 1 using the Lyapunov function tools established in this appendix. Strengthen the conclusion and establish that the closed-loop system is globally exponentially stable.

### Exercise B.6: Continuity of Lyapunov function for asymptotic stability

Let $X$ be a compact subset of $\mathbb{R}^n$ containing the origin in its interior that is positive invariant for the system $x^+ = f(x)$. If $f(\cdot)$ is continuous on $X$ and the origin is asymptotically stable with a region of attraction $X$, show that the Lyapunov function suggested in Exercise B.4 is continuous on $X$.

### Exercise B.7: A Lipschitz continuous converse theorem for exponential stability

Consider the system $x^+ = f(x)$, $f(0) = 0$, with function $f : D \to \mathbb{R}^n$ Lipschitz continuous on compact set $D \subset \mathbb{R}^n$ containing the origin in its interior. Choose $R > 0$ such that $B_R \subseteq D$. Assume that there exist scalars $c > 0$ and $\lambda \in (0,1)$ such that

$$|\phi(k;x)| \leq c\,|x|\,\lambda^k \qquad \text{for all } |x| \leq r, \quad k \geq 0$$

with $r := R/c$.

Show that there exists a *Lipschitz continuous* Lyapunov function $V(\cdot)$ satisfying for all $x \in B_r$

$$a_1\,|x|^2 \leq V(x) \leq a_2\,|x|^2$$
$$V(f(x)) - V(x) \leq -a_3\,|x|^2$$

with $a_1, a_2, a_3 > 0$.

Hint: Use the proposed Lyapunov function of Exercise B.3 with $\sigma = 2$. See also (Khalil, 2002, Exercise 4.68).

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# C
# Optimization

## C.1 Dynamic Programming

The name *dynamic programming* dates from the 1950s when it was coined by Richard Bellman for a technique for solving dynamic optimization problems, i.e., optimization problems associated with deterministic or stochastic systems whose behavior is governed by differential or difference equations. Here we review some of the basic ideas behind dynamic programming (DP) Bellman (1957); Bertsekas, Nedic, and Ozdaglar (2001).

To introduce the topic in its simplest form, consider the simple routing problem illustrated in Figure C.1. To maintain connection with optimal control, each node in the graph can be regarded as a point $(x,t)$ in a subset $S$ of $X \times T$ where both the state space $X = \{a,b,c,\ldots,g\}$ and the set of times $T = \{0,1,2,3\}$ are discrete. The set of permissible control actions is $\mathbb{U} = \{U,D\}$, i.e., to go "up" or "down." The control problem is to choose the lowest cost path from event $(d,0)$ (state $d$ at $t = 0$) to any of the states at $t = 3$; the cost of going from one event to the next is indicated on the graph. This problem is equivalent to choosing an open-loop control, i.e., a sequence $\{u(0),u(1),u(2)\}$ of admissible control actions. There are $2^N$ controls where $N$ is the number of stages, 3 in this example. The cost of each control can, in this simple example, be evaluated and is given in Table C.1.

There are two different *open-loop* optimal controls, namely {U,D,U}

| control | UUU | UUD | UDU | UDD | DUU | DUD | DDU | DDD |
|---|---|---|---|---|---|---|---|---|
| cost | 20 | 24 | 16 | 24 | 24 | 32 | 20 | 16 |

**Table C.1:** Control Cost.

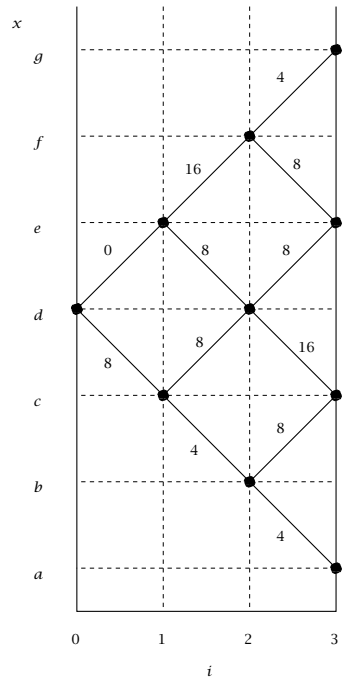



**Figure C.1:** Routing problem.

and {D,D,D}, each incurring a cost of 16. The corresponding state trajectories are $\{d, e, d, e\}$ and $\{d, c, b, a\}$.

In discrete problems of this kind, DP replaces the $N$-stage problem by M single stage problems, where M is the total number of nodes, i.e., the number of elements in $S \subset X \times T$. The first set of optimization problems deals with the states $b, d, f$ at time $N - 1 = 2$. The optimal decision at event $(f, 2)$, i.e., state $f$ at time 2, is the control $U$ and gives rise to a cost of 4. The optimal cost and control for node $(f, 2)$ are recorded; see Table C.2. The procedure is then repeated for states $d$ and $b$ at time $t = 2$ (nodes $(d, 2)$ and $(b, 2)$) and recorded as shown in Table C.2. Attention is next focused on the states $e$ and $c$ at $t = 1$ (nodes $(e, 1)$ and $(c, 1)$). The lowest cost that can be achieved at node $(e, 1)$ *if* control $U$ is chosen, is $16 + 4$, the sum of the path cost 16 associated with the control $U$, and the *optimal* cost 4 associated with the node $(f, 2)$ that results from using control $U$ at node $(e, 1)$.

| $t$ | 0 | 1 | | 2 | | |
|---|---|---|---|---|---|---|
| state | $d$ | $e$ | $c$ | $f$ | $d$ | $b$ |
| control | U or D | D | D | U | U | D |
| optimal cost | 16 | 16 | 8 | 4 | 8 | 4 |

**Table C.2:** Optimal Cost and Control

Similarly the lowest possible cost, if control $D$ is chosen, is $8+8$. Hence the optimal control and cost for node $(e,1)$ are, respectively, $D$ and 16. The procedure is repeated for the remaining state $d$ at $t = 1$ (node $(d,1)$). A similar calculation for the state $d$ at $t = 0$ (node $(d,0)$), where the optimal control is $U$ or $D$, completes this backward recursion; this backward recursion provides the optimal cost and control for each $(x,t)$, as recorded in Table C.2. The procedure therefore yields an optimal *feedback* control that is a function of $(x,t) \in S$. To obtain the optimal open-loop control for the initial node $(d,0)$, the feedback law is obeyed, leading to control U or D at $t = 0$; if U is chosen, the resultant state at $t = 1$ is $e$. From Table C.2, the optimal control at $(e,1)$ is D, so that the successor node is $(d,2)$. The optimal control at node $(d,2)$ is $U$. Thus the optimal open-loop control sequence $\{U,D,U\}$ is re-obtained. On the other hand, if the decision at $(d,0)$ is chosen to be $D$, the optimal sequence $\{D,D,D\}$ is obtained. This simple example illustrates the main features of DP that we will now examine in the context of discrete time optimal control.

### C.1.1 Optimal Control Problem

The discrete time system we consider is described by

$$x^+ = f(x,u) \tag{C.1}$$

where $f(\cdot)$ is continuous. The system is subject to the mixed state-control constraint

$$(x,u) \in \mathbb{Z}$$

where $\mathbb{Z}$ is a closed subset of $\mathbb{R}^n \times \mathbb{R}^m$ and $\mathcal{P}_u(\mathbb{Z})$ is compact where $\mathcal{P}_u$ is the projection operator $(x,u) \mapsto u$. Often $\mathbb{Z} = \mathbb{X} \times \mathbb{U}$ in which case the constraint $(x,u) \in \mathbb{Z}$ becomes $x \in \mathbb{X}$ and $u \in \mathbb{U}$ and $\mathcal{P}_u(\mathbb{Z}) = \mathbb{U}$ so that $\mathbb{U}$ is compact. In addition there is a constraint on the terminal state $x(N)$:

$$x(N) \in \mathbb{X}_f$$

where $\mathbb{X}_f$ is closed. In this section we find it easier to express the value function and the optimal control in terms of the current state and current time $i$ rather than using time-to-go $k$. Hence we replace time-to-go $k$ by time $i$ where $k = N - i$, replace $V_k^0(x)$ (the optimal cost at state $x$ when the time-to-go is $k$) by $V^0(x, i)$ (the optimal cost at state $x$, time $i$) and replace $X_k$ by $X(i)$ where $X(i)$ is the domain of $V^0(\cdot, i)$).

The cost associated with an initial state $x$ at time 0 and a control sequence $\mathbf{u} := \{u(0), u(1), \ldots, u(N-1)\}$ is

$$V(x, 0, \mathbf{u}) = V_f(x(N)) + \sum_{i=1}^{N-1} \ell(x(i), u(i)) \tag{C.2}$$

where $\ell(\cdot)$ and $V_f(\cdot)$ are continuous and, for each $i$, $x(i) = \phi(i; (x, 0), \mathbf{u})$ is the solution at time $i$ of (C.1) if the initial state is $x$ at time 0 and the control sequence is $\mathbf{u}$. The optimal control problem $\mathbb{P}(x, 0)$ is defined by

$$V^0(x, 0) = \min_{\mathbf{u}} V(x, 0, \mathbf{u}) \tag{C.3}$$

subject to the constraints $(x(i), u(i)) \in \mathbb{Z}$, $i = 0, 1, \ldots, N-1$ and $x(N) \in \mathbb{X}_f$. Equation (C.3) may be rewritten in the form

$$V^0(x, 0) = \min_{\mathbf{u}} \{V(x, 0, \mathbf{u}) \mid \mathbf{u} \in \mathcal{U}(x, 0)\} \tag{C.4}$$

where $\mathbf{u} := \{u(0), u(1), \ldots, u(N-1)\}$,

$$\mathcal{U}(x, 0) := \{\mathbf{u} \in \mathbb{R}^{Nm} \mid (x(i), u(i)) \in \mathbb{Z}, i = 0, 1, \ldots, N-1; x(N) \in \mathbb{X}_f\}$$

and $x(i) := \phi(i; (x, 0), \mathbf{u})$. Thus $\mathcal{U}(x, 0)$ is the set of admissible control sequences[1] if the initial state is $x$ at time 0. It follows from the continuity of $f(\cdot)$ that for all $i \in \{0, 1, \ldots, N-1\}$ and all $x \in \mathbb{R}^n$, $\mathbf{u} \mapsto \phi(i; (x, 0), \mathbf{u})$ is continuous, $\mathbf{u} \mapsto V(x, 0, \mathbf{u})$ is continuous and $\mathcal{U}(x, 0)$ is compact. Hence the minimum in (C.4) exists at all $x \in \{x \in \mathbb{R}^n \mid \mathcal{U}(x, 0) \neq \emptyset\}$.

DP embeds problem $\mathbb{P}(x, 0)$ for a given state $x$ in a whole family of problems $P(x, i)$ where, for each $(x, i)$, problem $\mathbb{P}(x, i)$ is defined by

$$V^0(x, i) = \min_{\mathbf{u}^i} \{V(x, i, \mathbf{u}^i) \mid \mathbf{u}^i \in \mathcal{U}(x, i)\}$$

where

$$\mathbf{u}^i := \{u(i), u(i+1), \ldots, u(N-1)\}$$

[1] An admissible control sequence satisfies all constraints.

$$V(x,i,\mathbf{u}^i) := V_f(x(N)) + \sum_{j=i}^{N-1} \ell(x(j),u(j)) \tag{C.5}$$

and

$$\mathcal{U}(x,i) := \{\mathbf{u}^i \in \mathbb{R}^{(N-i)m} \mid (x(j),u(j)) \in \mathbb{Z}, j = i, i+1, \ldots, N-1 \\ x(N) \in \mathbb{X}_f\} \tag{C.6}$$

In (C.5) and (C.6), $x(j) = \phi(j;(x,i),\mathbf{u}^i)$, the solution at time $j$ of (C.1) if the initial state is $x$ at time $i$ and the control sequence is $\mathbf{u}^i$. For each $i$, $X(i)$ denotes the domain of $V^0(\cdot,i)$ and $\mathcal{U}(\cdot,i)$ so that

$$X(i) = \{x \in \mathbb{R}^n \mid \mathcal{U}(x,i) \neq \emptyset\}. \tag{C.7}$$

### C.1.2 Dynamic Programming

One way to approach DP for discrete time control problems is the simple observation that for all $(x,i)$

$$\begin{aligned} V^0(x,i) &= \min_{\mathbf{u}^i}\{V(x,i,\mathbf{u}^i) \mid \mathbf{u}^i \in \mathcal{U}(x,i)\} \\ &= \min_{u}\{\ell(x,u) + \min_{\mathbf{u}^{i+1}} V(f(x,u),i+1,\mathbf{u}^{i+1}) \mid \\ &\qquad \{u,\mathbf{u}^{i+1}\} \in \mathcal{U}(x,i)\} \end{aligned} \tag{C.8}$$

where $\mathbf{u}^i = \{u, u(i+1), \ldots, u(N-1)\} = \{u, \mathbf{u}^{i+1}\}$. We now make use of the fact that $\{u,\mathbf{u}^{i+1}\} \in \mathcal{U}(x,i)$ if and only if $(x,u) \in \mathbb{Z}$, $f(x,u) \in X(i+1)$, and $\mathbf{u}^{i+1} \in \mathcal{U}(f(x,u),i+1)$ since $f(x,u) = x(i+1)$. Hence we may rewrite (C.8) as

$$\begin{aligned} V^0(x,i) = \min_{u}\{&\ell(x,u) + V^0(f(x,u),i+1) \mid \\ &(x,u) \in \mathbb{Z}, f(x,u) \in X(i+1)\} \end{aligned} \tag{C.9}$$

for all $x \in X(i)$ where

$$X(i) = \{x \in \mathbb{R}^n \mid \exists u \text{ such that } (x,u) \in \mathbb{Z} \text{ and } f(x,u) \in X(i+1)\} \tag{C.10}$$

Equations (C.9) and (C.10), together with the boundary condition

$$V^0(x,N) = V_f(x) \ \forall x \in X(N), \quad X(N) = \mathbb{X}_f$$

constitute the DP recursion for constrained discrete time optimal control problems. If there are no state constraints, i.e., if $\mathbb{Z} = \mathbb{R}^n \times \mathbb{U}$ where

$\mathbb{U} \subset \mathbb{R}^m$ is compact, then $X(i) = \mathbb{R}^n$ for all $i \in \{0, 1, \ldots, N\}$ and the DP equations revert to the familiar DP recursion:

$$V^0(x, i) = \min_u \{\ell(x, u) + V^0(f(x, u), i + 1)\} \ \forall x \in \mathbb{R}^n$$

with boundary condition

$$V^0(x, N) = V_f \ \forall x \in \mathbb{R}^n$$

We now prove some basic facts; the first is the well known *principle of optimality*.

**Lemma C.1** (Principle of optimality)**.** *Let $x \in X_N$ be arbitrary, let $\mathbf{u} := \{u(0), u(1), \ldots, u(N-1)\} \in \mathcal{U}(x, 0)$ denote the solution of $\mathbb{P}(x, 0)$ and let $\{x, x(1), x(2), \ldots, x(N)\}$ denote the corresponding optimal state trajectory so that for each $i$, $x(i) = \phi(i; (x, 0), \mathbf{u})$. Then, for any $i \in \{0, 1, \ldots, N-1\}$, the control sequence $\mathbf{u}^i := \{u(i), u(i+1), \ldots, u(N-1)\}$ is optimal for $\mathbb{P}(x(i), i)$ (any portion of an optimal trajectory is optimal).*

*Proof.* Since $\mathbf{u} \in \mathcal{U}(x, 0)$, the control sequence $\mathbf{u}^i \in \mathcal{U}(x(i), i)$. If $\mathbf{u}^i = \{u(i), u(i+1), \ldots, u(N-1)\}$ is not optimal for $\mathbb{P}(x(i), i)$, there exists a control sequence $\mathbf{u}' = \{u'(i), u'(i+1), \ldots, u(N-1)'\} \in \mathcal{U}(x(i), i)$ such that $V(x(i), i, \mathbf{u}') < V(x(i), \mathbf{u})$. Consider now the control sequence $\tilde{\mathbf{u}} := \{u(0), u(1), \ldots, u(i-1), u'(i), u'(i+1), \ldots, u(N-1)'\}$. It follows that $\tilde{\mathbf{u}} \in \mathcal{U}(x, 0)$ and $V(x, 0, \tilde{\mathbf{u}}) < V(x, 0, \mathbf{u}) = V^0(x, 0)$, a contradiction. Hence $\mathbf{u}(x(i), i)$ is optimal for $\mathbb{P}(x(i), i)$. ∎

The most important feature of DP is the fact that the DP recursion yields the optimal value $V^0(x, i)$ and the optimal control $\kappa(x, i) = \arg\min_u \{\ell(x, u) + V^0(f(x, u), i + 1) \mid (x, u) \in \mathbb{Z}, f(x, u) \in X(i + 1)\}$ for each $(x, i) \in X(i) \times \{0, 1, \ldots, N - 1\}$.

**Theorem C.2** (Optimal value function and control law from DP)**.** *Suppose that the function $\Psi : \mathbb{R}^n \times \{0, 1, \ldots, N\} \to \mathbb{R}$, satisfies, for all $i \in \{1, 2, \ldots, N - 1\}$, all $x \in X(i)$, the DP recursion*

$$\Psi(x, i) = \min\{\ell(x, u) + \Psi(f(x, u), i + 1) \mid (x, u) \in \mathbb{Z}, f(x, u) \in X(i + 1)\}$$
$$X(i) = \{x \in \mathbb{R}^n \mid \exists u \in \mathbb{R}^m \text{ such that } (x, u) \in \mathbb{Z},\ f(x, u) \in X(i + 1)\}$$

*with boundary conditions*

$$\Psi(x, N) = V_f(x) \ \forall x \in \mathbb{X}_f, \quad X(N) = \mathbb{X}_f$$

*Then $\Psi(x, i) = V^0(x, i)$ for all $(x, i) \in X(i) \times \{0, 1, 2, \ldots, N\}$; the DP recursion yields the optimal value function and the optimal control law.*

*Proof.* Let $(x,i) \in X(i) \times \{0,1,\ldots,N\}$ be arbitrary. Let $\mathbf{u} = \{u(i), u(i+1), \ldots, u(N-1)\}$ be an arbitrary control sequence in $\mathcal{U}(x,i)$ and let $\mathbf{x} = \{x, x(i+1), \ldots, x(N)\}$ denote the corresponding trajectory starting at $(x,i)$ so that for each $j \in \{i, i+1, \ldots, N\}$, $x(j) = \phi(j; x, i, \mathbf{u})$. For each $j \in \{i, i+1, \ldots, N-1\}$, let $\mathbf{u}^j := \{u(j), u(j+1), \ldots, u(N-1)\}$; clearly $\mathbf{u}^j \in \mathcal{U}(x(j), j)$. The cost due to initial event $(x(j), j)$ and control sequence $\mathbf{u}^j$ is $\Phi(x(j), j)$ defined by

$$\Phi(x(j), j) := V(x(j), j, \mathbf{u}^j)$$

Showing that $\Psi(x,i) \leq \Phi(x,i)$ proves that $\Psi(x,i) = V^0(x,i)$ since $\mathbf{u}$ is an arbitrary sequence in $\mathcal{U}(x,i)$; because $(x,i) \in X(i) \times \{0,1,\ldots,N\}$ is arbitrary, that fact that $\Psi(x,i) = V^0(x,i)$ proves that DP yields the optimal value function.

To prove that $\Psi(x,i) \leq \Phi(x,i)$, we compare $\Psi(x(j), j)$ and $\Phi(x(j), j)$ for each $j \in \{i, i+1, \ldots, N\}$, i.e., we compare the costs yielded by the DP recursion and by the arbitrary control $\mathbf{u}$ along the corresponding trajectory $\mathbf{x}$. By definition, $\Psi(x(j), j)$ satisfies for each $j$

$$\Psi(x(j), j) = \min_u \{\ell(x(j), u) + \Psi(f(x(j), u), j+1) \mid (x(j), u) \in \mathbb{Z}, f(x(j), u) \in X(j+1)\} \quad \text{(C.11)}$$

To obtain $\Phi(x(j), j)$ for each $j$ we solve the following recursive equation

$$\Phi(x(j), j) = \ell(x(j), u(j)) + \Phi(f(x(j), u(j)), j+1) \quad \text{(C.12)}$$

The boundary conditions are

$$\Psi(x(N), N) = \Phi(x(N), N) = V_f(x(N)) \quad \text{(C.13)}$$

Since $u(j)$ satisfies $(x(j), u(j)) \in \mathbb{Z}$ and $f(x(j), u(j)) \in X(j+1)$ but is not necessarily a minimizer in (C.11), we deduce that

$$\Psi(x(j), j) \leq \ell(x(j), u(j)) + \Psi(f(x(j), u(j)), j+1) \quad \text{(C.14)}$$

For each $j$, let $E(j)$ be defined by

$$E(j) := \Psi(x(j), j) - \Phi(x(j), j)$$

Subtracting (C.12) from (C.14) and replacing $f(x(j), u(j))$ by $x(j+1)$ yields

$$E(j) \leq E(j+1) \ \forall j \in \{i, i+1, \ldots N\}$$

Since $E(N) = 0$ by virtue of (C.13), we deduce that $E(j) \leq 0$ for all $j \in \{i, i+1, \ldots, N\}$; in particular, $E(i) \leq 0$ so that

$$\Psi(x, i) \leq \Phi(x, i) = V(x, i, \mathbf{u})$$

for all $\mathbf{u} \in \mathcal{U}(x, i)$. Hence $\Psi(x, i) = V^0(x, i)$ for all $(x, i) \in X(i) \times \{0, 1, \ldots, N\}$. ■

**Example C.3: DP applied to linear quadratic regulator**

A much used example is the familiar linear quadratic regulator problem. The system is defined by

$$x^+ = Ax + Bu$$

There are no constraints. The cost function is defined by (C.2) where

$$\ell(x, u) := (1/2)x'Qx + (1/2)u'Ru$$

and $V_f(x) = 0$ for all $x$; the horizon length is $N$. We assume that $Q$ is symmetric and positive semidefinite and that $R$ is symmetric and positive definite. The DP recursion is

$$V^0(x, i) = \min_u \{\ell(x, u) + V^0(Ax + Bu, i + 1)\} \ \forall x \in \mathbb{R}^n$$

with terminal condition

$$V^0(x, N) = 0 \ \forall x \in \mathbb{R}^n$$

Assume that $V^0(\cdot, i + 1)$ is quadratic and positive semidefinite and, therefore, has the form

$$V^0(x, i + 1) = (1/2)x'P(i + 1)x$$

where $P(i + 1)$ is symmetric and positive semidefinite. Then

$$V^0(x, i) = (1/2) \min_u \{x'Qx + u'Ru + (Ax + Bu)'P(i + 1)(Ax + Bu)\}$$

The right-hand side of the last equation is a positive definite function of $u$ for all $x$, so that it has a unique minimizer given by

$$\kappa(x, i) = K(i)x \qquad K(i) := -(B'P(i + 1)B + R)^{-1}B'P(i + 1)$$

Substituting $u = K(i)x$ in the expression for $V^0(x, i)$ yields

$$V^0(x, i) = (1/2)x'P(i)x$$

where $P(i)$ is given by:

$$P(i) = Q + K(i)'RK(i) - A'P(i+1)B(B'P(i+1)B + R)^{-1}B'P(i+1)A$$

Hence $V^0(\cdot, i)$ is quadratic and positive semidefinite if $V^0(\cdot, i+1)$ is. But $V^0(\cdot, N)$, defined by

$$V^0(x, N) := (1/2)x'P(N)x = 0 \qquad P(N) := 0$$

is symmetric and positive semidefinite. By induction $V^0(\cdot, i)$ is quadratic and positive semidefinite (and $P(i)$ is symmetric and positive semidefinite) for all $i \in \{0, 1, \ldots, N\}$. Substituting $K(i) = -(B'P(i+1)B + R)^{-1}B'P(i+1)A$ in the expression for $P(i)$ yields the more familiar matrix Riccati equation

$$P(i) = Q + A'P(i+1)A - A'P(i+1)B(B'P(i+1)B + R)^{-1}BP(i+1)A$$

□

## C.2 Optimality Conditions

In this section we obtain optimality conditions for problems of the form

$$f^0 = \inf_u \{f(u) \mid u \in U\}$$

In these problems, $u \in \mathbb{R}^m$ is the *decision* variable, $f(u)$ the cost to be minimized by appropriate choice of $u$and $U \subset \mathbb{R}^m$ the constraint set. The value of the problem is $f^0$. Some readers may wish to read only Section C.2.2, which deals with convex optimization problems and Section C.2.3 which deals with convex optimization problems in which the constraint set $U$ is polyhedral. These sections require some knowledge of tangent and normal cones discussed in Section C.2.1; Proposition C.7 in particular derives the normal cone for the case when $U$ is convex.

### C.2.1 Tangent and Normal Cones

In determining conditions of optimality, it is often convenient to employ approximations to the cost function $f(\cdot)$ and the constraint set $U$. Thus the cost function $f(\cdot)$ may be approximated, in the neighborhood of a point $\bar{u}$, by the first order expansion $f(\bar{u}) + \langle \nabla f(\bar{u}), (u - \bar{u}) \rangle$ or by the second order expansion $f(\bar{u}) + \langle \nabla f(\bar{u}), (u - \bar{u}) \rangle + (1/2)((u - \bar{u})'\nabla^2 f(\bar{x})(u - \bar{u}))$ if the necessary derivatives exist. Thus we see that

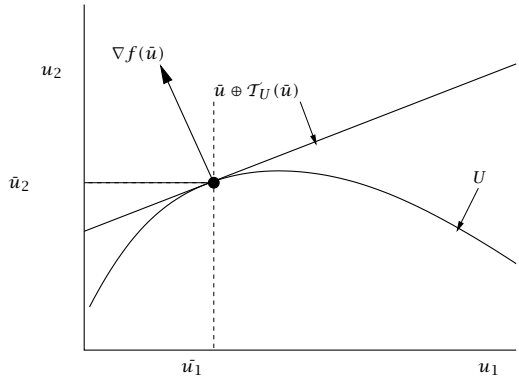



**Figure C.2:** Approximation of the set $U$.

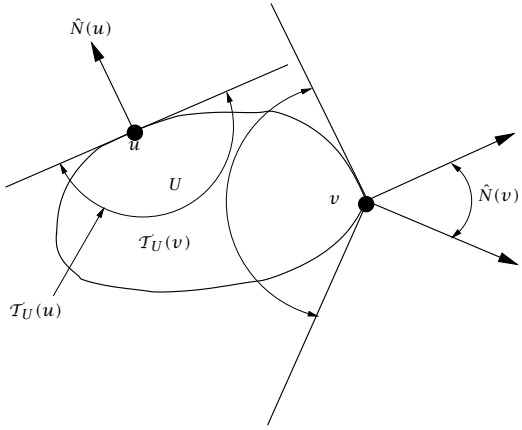



**Figure C.3:** Tangent cones.

in the unconstrained case, a necessary condition for the optimality of $\bar{u}$ is $\nabla f(\bar{u}) = 0$. To obtain necessary conditions of optimality for constrained optimization problems, we need to approximate the constraint set as well; this is more difficult. An example of $U$ and its approximation is shown in Figure C.2; here the set $U = \{u \in \mathbb{R}^2 \mid g(u) = 0\}$ where $g : \mathbb{R} \to \mathbb{R}$ is approximated in the neighborhood of a point $\bar{u}$ satisfying $g(\bar{u}) = 0$ by the set $\bar{u} \oplus \mathcal{T}_U(\bar{u})$ where[2] the tangent cone $\mathcal{T}_U(\bar{u}) := \{h \in \mathbb{R}^2 \mid \nabla g(\bar{u}), u - \bar{u}\rangle = 0\}$. In general, a set $U$ is approx-

[2]If $A$ and $B$ are two subsets of $\mathbb{R}^n$, say, then $A \oplus B := \{a + b \mid a \in A, b \in B\}$ and $a \oplus B := \{a + b \mid b \in B\}$.

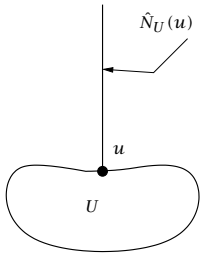



**Figure C.4:** Normal at $u$.

imated, near a point $\bar{u}$, by $\bar{u} \oplus \mathcal{T}_U(\bar{u})$ where its *tangent cone* $\mathcal{T}_U(\bar{u})$ is defined below. Following Rockafellar and Wets (1998), we use $u^\nu \underset{U}{\longrightarrow} v$ to denote that the sequence $\{u^\nu \mid \nu \in \mathbb{I}_{\geq 0}\}$ converges to $v$ as $\nu \to \infty$ while satisfying $u^\nu \in U$ for all $\nu \in \mathbb{I}_{\geq 0}$.

**Definition C.4** (Tangent vector)**.** A vector $h \in \mathbb{R}^m$ is tangent to the set $U$ at $\bar{u}$ if there exist sequences $u^\nu \underset{U}{\longrightarrow} \bar{u}$ and $\lambda^\nu \searrow 0$ such that

$$[u^\nu - \bar{u}]/\lambda^\nu \to h$$

$\mathcal{T}_U(u)$ is the set of all tangent vectors.

Equivalently, a vector $h \in \mathbb{R}^m$ is tangent to the set $U$ at $\bar{u}$ if there exist sequences $h^\nu \to h$ and $\lambda^\nu \searrow 0$ such that $\bar{u} + \lambda^\nu h^\nu \in U$ for all $\nu \in \mathbb{I}_{\geq 0}$. This equivalence can be seen by identifying $u^\nu$ with $\bar{u} + \lambda^\nu h^\nu$.

**Proposition C.5** (Tangent vectors are closed cone)**.** *The set $\mathcal{T}_U(u)$ of all tangent vectors to $U$ at any point $u \in U$ is a closed cone.*

See Rockafellar and Wets (1998), Proposition 6.2. That $\mathcal{T}_U(\bar{u})$ is a cone may be seen from its definition; if $h$ is a tangent, so is $\alpha h$ for any $\alpha \geq 0$. Two examples of a tangent cone are illustrated in Figure C.3.

Associated with each tangent cone $\mathcal{T}_U(u)$ is a normal cone $\hat{N}(u)$ defined as follows Rockafellar and Wets (1998):

**Definition C.6** (Regular normal)**.** A vector $g \in \mathbb{R}^m$ is a regular normal to a set $U \subset \mathbb{R}^m$ at $\bar{u} \in U$ if

$$\langle g, u - \bar{u} \rangle \leq o(|u - \bar{u}|) \ \forall u \in U \tag{C.15}$$

where $o(\cdot)$ has the property that $o(|u - \bar{u}|)/|u - \bar{u}| \to 0$ as $u \underset{U}{\longrightarrow} \bar{u}$ with $u \neq \bar{u}$; $\hat{N}_U(u)$ is the set of all regular normal vectors.

Some examples of normal cones are illustrated in Figure C.3; here the set $\hat{N}_U(u) = \{\lambda g \mid \lambda \geq 0\}$ is a cone generated by a single vector $g$, say, while $\hat{N}_U(v) = \{\lambda_1 g_1 + \lambda_2 g_2 \mid \lambda_1 \geq 0, \lambda_2 \geq 0\}$ is a cone generated by two vectors $g_1$ and $g_2$, say. The term $o(|u - \bar{u}|)$ may be replaced by 0 if $U$ is convex as shown in Proposition C.7(b) below but is needed in general since $U$ may not be locally convex at $\bar{u}$ as illustrated in Figure C.4.

The tangent cone $\mathcal{T}_U(\bar{u})$ and the normal cone $\hat{N}_U(\bar{u})$ at a point $\bar{u} \in U$ are related as follows.

**Proposition C.7** (Relation of normal and tangent cones)**.**
*(a) At any point $\bar{u} \in U \subset \mathbb{R}^m$,*

$$\hat{N}_U(\bar{u}) = \mathcal{T}_U(\bar{u})^* := \{g \mid \langle g, h\rangle \leq 0 \ \forall h \in \mathcal{T}_U(\bar{u})\}$$

*where, for any cone $V$, $V^* := \{g \mid \langle g, h\rangle \leq 0 \ \forall h \in V\}$ denotes the polar cone of $V$.*

*(b) If $U$ is convex, then, at any point $\bar{u} \in U$*

$$\hat{N}_U(\bar{u}) = \{g \mid \langle g, u - \bar{u}\rangle \leq 0 \ \forall u \in U\} \tag{C.16}$$

*Proof.* (a) To prove $\hat{N}_U(\bar{u}) \subset \mathcal{T}_U(\bar{u})^*$, we take an arbitrary point $g$ in $\hat{N}_U(\bar{u})$ and show that $\langle g, h\rangle \leq 0$ for all $h \in \mathcal{T}(\bar{u})$ implying that $g \in \mathcal{T}_U^*(\bar{u})$. For, if $h$ is tangent to $U$ at $\bar{u}$, there exist, by definition, sequences $u^\nu \xrightarrow[U]{} \bar{u}$ and $\lambda^\nu \searrow 0$ such that

$$h^\nu := (u^\nu - \bar{u})/\lambda^\nu \to h$$

Since $g \in \hat{N}_U(\bar{u})$, it follows from (C.15) that $\langle g, h^\nu\rangle \leq o(|(u^\nu - \bar{u})|) = o(\lambda^\nu |h^\nu|)$; the limit as $\nu \to \infty$ yields $\langle g, h\rangle \leq 0$, so that $g \in \mathcal{T}_U^*(\bar{u})$. Hence $\hat{N}_U(\bar{u}) \subset \mathcal{T}_U(\bar{u})^*$. The proof of this result, and the more subtle proof of the converse, that $\mathcal{T}_U(\bar{u})^* \subset \hat{N}_U(\bar{u})$, are given in Rockafellar and Wets (1998), Proposition 6.5.

(b) This part of the proposition is proved in (Rockafellar and Wets, 1998, Theorem 6.9).

■

We wish to derive optimality conditions for problems of the form $\mathbb{P} : \inf_u \{f(u) \mid u \in U\}$. The *value* of the problem is defined to be

$$f^0 := \inf_u \{f(u) \mid u \in U\}$$

There may not exist a $u \in U$ such that $f(u) = f^0$. If, however, $f(\cdot)$ is continuous and $U$ is compact, there exists a minimizing $u$ in $U$, i.e.,

$$f^0 = \inf_u \{f(u) \mid u \in U\} = \min_u \{f(u) \mid u \in U\}$$

The minimizing $u$, if it exists, may not be unique so

$$u^0 := \arg\min_u \{f(u) \mid u \in U\}$$

may be a set. We say $u$ is feasible if $u \in U$. A point $u$ is *globally optimal* for problem $\mathbb{P}$ if $u$ is feasible and $f(v) \geq f(u)$ for all $v \in U$. A point $u$ is *locally optimal* for problem $\mathbb{P}$ if $u$ is feasible and there exists a $\varepsilon > 0$ such that $f(v) \geq f(u)$ for all $v$ in $(u \oplus \varepsilon\mathcal{B}) \cap U$ where $\mathcal{B}$ is the closed unit ball $\{u \mid \min|u| \leq 1\}$.

### C.2.2 Convex Optimization Problems

The optimization problem $\mathbb{P}$ is convex if the function $f : \mathbb{R}^m \to \mathbb{R}$ and the set $U \subset \mathbb{R}^m$ are convex. In convex optimization problems, $U$ often takes the form $\{u \mid g_j(u) \leq 0, j \in \mathcal{J}\}$ where $\mathcal{J} := \{1, 2, \ldots, J\}$ and each function $g_j(\cdot)$ is convex. A useful feature of convex optimization problems is the following result:

**Proposition C.8** (Global optimality for convex problems)**.** *Suppose the function $f(\cdot)$ is convex and differentiable and the set $U$ is convex. Any locally optimal point of the convex optimization problem* $\inf_u \{f(u) \mid u \in U\}$ *is globally optimal.*

*Proof.* Suppose $u$ is locally optimal so that there exists an $\varepsilon > 0$ such that $f(v) \geq f(u)$ for all $v \in (u \oplus \varepsilon\mathcal{B}) \cap U$. If, contrary to what we wish to prove, $u$ is *not* globally optimal, there exists a $w \in U$ such that $f(w) < f(u)$. For any $\lambda \in [0, 1]$, the point $w_\lambda := \lambda w + (1 - \lambda)u$ lies in $[u, w]$ (the line joining $u$ and $w$). Then $w_\lambda \in U$ (because $U$ is convex) and $f(w_\lambda) \leq \lambda f(w) + (1 - \lambda)f(u) < f(u)$ for all $\lambda \in (0, 1]$ (because $f(\cdot)$ is convex and $f(w) < f(u)$). We can choose $\lambda > 0$ so that $w_\lambda \in (u \oplus \varepsilon\mathcal{B}) \cap U$ and $f(w_\lambda) < f(u)$. This contradicts the local optimality of $u$. Hence $u$ is globally optimal. ∎

On the assumption that $f(\cdot)$ is differentiable, we can obtain a simple necessary and sufficient condition for the (global) optimality of a point $u$.

**Proposition C.9** (Optimality conditions – normal cone)**.** *Suppose the function $f(\cdot)$ is convex and differentiable and the set $U$ is convex. The point $u$ is optimal for problem $\mathbb{P}$ if and only if $u \in U$ and*

$$df(u; v-u) = \langle \nabla f(u), v-u \rangle \geq 0 \ \forall v \in U \tag{C.17}$$

*or, equivalently*

$$-\nabla f(u) \in \hat{N}_U(u) \tag{C.18}$$

*Proof.* Because $f(\cdot)$ is convex, it follows from Theorem 7 in Appendix A1 that

$$f(v) \geq f(u) + \langle \nabla f(u), v-u \rangle \tag{C.19}$$

for all $u, v$ in $U$. To prove sufficiency, suppose $u \in U$ and that the condition in (C.17) is satisfied. It then follows from (C.19) that $f(v) \geq f(u)$ for all $v \in U$ so that $u$ is globally optimal. To prove necessity, suppose that $u$ is globally optimal but that, contrary to what we wish to prove, the condition on the right-hand side of (C.17) is not satisfied so that there exists a $v \in U$ such that

$$df(u; h) = \langle \nabla f(u), v-u \rangle = -\delta < 0$$

where $h := v - u$. For all $\lambda \in [0,1]$, let $v_\lambda := \lambda v + (1-\lambda)u = u + \lambda h$; because $U$ is convex, each $v_\lambda$ lies in $U$. Since

$$df(u; h) = \lim_{\lambda \searrow 0} \frac{f(u+\lambda h) - f(u)}{\lambda} = \lim_{\lambda \searrow 0} \frac{f(v_\lambda) - f(u)}{\lambda} = -\delta$$

there exists a $\lambda \in (0,1]$ such that $f(v_\lambda) - f(u) \leq -\lambda\delta/2 < 0$ which contradicts the optimality of $u$. Hence the condition in (C.17) must be satisfied. That (C.17) is equivalent to (C.18) follows from Proposition C.7 (ii).

■

It is an interesting fact that $U$ in Proposition C.9 may be replaced by its approximation $u \oplus \mathcal{T}_U(u)$ at $u$ yielding

**Proposition C.10** (Optimality conditions – tangent cone)**.** *Suppose the function $f(\cdot)$ is convex and differentiable and the set $U$ is convex. The point $u$ is optimal for problem $\mathbb{P}$ if and only if $u \in U$ and*

$$df(u; v-u) = \langle \nabla f(u), h \rangle \geq 0 \ \forall h \in \mathcal{T}_U(u)$$

*or, equivalently*

$$-\nabla f(u) \in \hat{N}_U(u) = \mathcal{T}_U^*(u).$$

*Proof.* It follows from Proposition C.9 that $u$ is optimal for problem $\mathbb{P}$ if and only if $u \in U$ and $-\nabla f(u) \in \hat{N}_U(u)$. But, by Proposition C.7, $\hat{N}_U(u) = \{g \mid \langle g, h\rangle \leq 0 \ \forall h \in \mathcal{T}_U(u)\}$ so that $-\nabla f(u) \in \hat{N}_U(u)$ is equivalent to $\langle \nabla f(u), h\rangle \geq 0$ for all $h \in \mathcal{T}_U(u)$. ∎

### C.2.3 Convex Problems: Polyhedral Constraint Set

The definitions of tangent and normal cones given above may appear complex but this complexity is necessary for proper treatment of the general case when $U$ is not necessarily convex. When $U$ is polyhedral, i.e., when $U$ is defined by a set of linear inequalities

$$U := \{u \in \mathbb{R}^m \mid Au \leq b\}$$

where $A \in \mathbb{R}^{p\times m}$ and $b \in \mathbb{R}^p$, $\mathcal{I} := \{1, 2, \ldots, p\}$, then the normal and tangent cones are relatively simple. We first note that $U$ is equivalently defined by

$$U := \{u \in \mathbb{R}^m \mid \langle a_i, u\rangle \leq b_i, \ i \in \mathcal{I}\}$$

where $a_i$ is the $i$th row of $A$ and $b_i$ is the $i$th element of $b$. For each $u \in U$, let

$$\mathcal{I}^0(u) := \{i \in \mathcal{I} \mid \langle a_i, u\rangle = b_i\}$$

denote the index set of constraints *active* at $u$. Clearly $\mathcal{I}^0(u) = \emptyset$ if $u$ lies in the interior of $U$. An example of a polyhedral constraint set is shown in Figure C.5. The next result shows that in this case, the tangent cone is the set of $h$ in $\mathbb{R}^m$ that satisfy $\langle a_i, h\rangle \leq 0$ for all $i$ in $\mathcal{I}^0(u)$ and the normal cone is the cone generated by the vectors $a_i$, $i \in \mathcal{I}^0(u)$; each normal $h$ in the normal cone may be expressed as $\sum_{i\in\mathcal{I}^0(u)} \mu_i a_i$ where each $\mu_i \geq 0$.

**Proposition C.11** (Representation of tangent and normal cones)**.** *Let* $U := \{u \in \mathbb{R}^m \mid \langle a_i, u\rangle \leq b_i, \ i \in \mathcal{I}\}$. *Then, for any* $u \in U$*:*

$$\mathcal{T}_U(u) = \{h \mid \langle a_i, h\rangle \leq 0, \ i \in \mathcal{I}^0(u)\}$$
$$\hat{N}_U(u) = \mathcal{T}_U^*(u) = \text{cone}\{a_i \mid i \in \mathcal{I}^0(u)\}$$

*Proof.* (i) Suppose $h$ is any vector in $\{h \mid \langle a_i, h\rangle \leq 0, \ i \in \mathcal{I}^0(u)\}$. Let the sequences $u^\nu$ and $\lambda^\nu$ satisfy $u^\nu = u + \lambda^\nu h$ and $\lambda^\nu \searrow 0$ with $\lambda^0$, the first element in the sequence $\lambda^\nu$, satisfying $u + \lambda^0 h \in U$. It follows that $[u^\nu - u]/\lambda^\nu \equiv h$ so that from Definition C.4, $h$ is tangent to $U$ at $u$. Hence $\{h \mid \langle a_i, h\rangle \leq 0, \ i \in \mathcal{I}^0(u)\} \subset \mathcal{T}_U(u)$. (ii) Conversely, if $h \in \mathcal{T}_U(u)$, then there exist sequences $\lambda^\nu \searrow 0$ and $h^\nu \to h$ such that

$\langle a_i, u + \lambda^\nu h^\nu \rangle \le b_i$ for all $i \in \mathcal{I}$, all $\nu \in \mathbb{I}_{\ge 0}$. Since $\langle a_i, u \rangle = b_i$ for all $i \in \mathcal{I}^0(u)$, it follows that $\langle a_i, h^\nu \rangle \le 0$ for all $i \in \mathcal{I}^0(u)$, all $\nu \in \mathbb{I}_{\ge 0}$; taking the limit yields $\langle a_i, h \rangle \le 0$ for all $i \in \mathcal{I}^0(u)$ so that $h \in \{h \mid \langle a_i, h \rangle \le 0,\ i \in \mathcal{I}^0(u)\}$ which proves $\mathcal{T}_U(u) \subset \{h \mid \langle a_i, h \rangle \le 0,\ i \in \mathcal{I}^0(u)\}$. We conclude from (i) and (ii) that $\mathcal{T}_U(u) = \{h \mid \langle a_i, h \rangle \le 0,\ i \in \mathcal{I}^0(u)\}$. That $\hat{N}_U(u) = \mathcal{T}_U^*(u) = \text{cone}\{a_i \mid i \in \mathcal{I}^0(u)\}$ then follows from Proposition C.7 above and Proposition 9 in Appendix A1. ■

The next result follows from Proposition C.5 and Proposition C.7.

**Proposition C.12** (Optimality conditions — linear inequalities)**.** *Suppose the function $f(\cdot)$ is convex and differentiable and $U$ is the convex set $\{u \mid Au \le b\}$. Then $u$ is optimal for $\mathbb{P} : \min_u \{f(u) \mid u \in U\}$ if and only if $u \in U$ and*

$$-\nabla f(u) \in \hat{N}_U(u) = \text{cone}\{a_i \mid i \in \mathcal{I}^0(u)\}$$

**Corollary C.13** (Optimality conditions — linear inequalities)**.** *Suppose the function $f(\cdot)$ is convex and differentiable and $U = \{u \mid Au \le b\}$. Then $u$ is optimal for $\mathbb{P} : \min_u \{f(u) \mid u \in U\}$ if and only if $Au \le b$ and there exist multipliers $\mu_i \ge 0, i \in \mathcal{I}^0(u)$ satisfying*

$$\nabla f(u) + \sum_{i \in \mathcal{I}^0(u)} \mu_i \nabla g_i(u) = 0 \tag{C.20}$$

*where, for each $i$, $g_i(u) := \langle a_i, u \rangle - b_i$ so that $g_i(u) \le 0$ is the constraint $\langle a_i, u \rangle \le b_i$ and $\nabla g_i(u) = a_i$.*

*Proof.* Since any point $g \in \text{cone}\{a_i \mid i \in \mathcal{I}^0(u)\}$ may be expressed as $g = \sum_{i \in \mathcal{I}^0(u)} \mu_i a_i$ where, for each $i$, $\mu_i \ge 0$, the condition $-\nabla f(u) \in \text{cone}\{a_i \mid i \in \mathcal{I}^0(u)\}$ is equivalent to the existence of multipliers $\mu_i \ge 0, i \in \mathcal{I}^0(u)$ satisfying (C.20). ■

The above results are easily extended if $U$ is defined by linear equality and inequality constraints, i.e., if

$$U := \{\langle a_i, u \rangle \le b_i,\ i \in \mathcal{I},\ \langle c_i, u \rangle = d_i,\ i \in \mathcal{E}\}$$

In this case, at any point $u \in U$, the tangent cone is

$$\mathcal{T}_U(u) = \{h \mid \langle a_i, h \rangle \le 0,\ i \in \mathcal{I}^0(u),\ \langle c_i, h \rangle = 0,\ i \in \mathcal{E}\}$$

and the normal cone is

$$\hat{N}_U(u) = \{ \sum_{i \in \mathcal{I}^0(u)} \lambda_i a_i + \sum_{i \in \mathcal{E}} \mu_i c_i \mid \lambda_i \ge 0\ \forall i \in \mathcal{I}^0(u), \mu_i \in \mathbb{R}\ \forall i \in \mathcal{E}\}$$

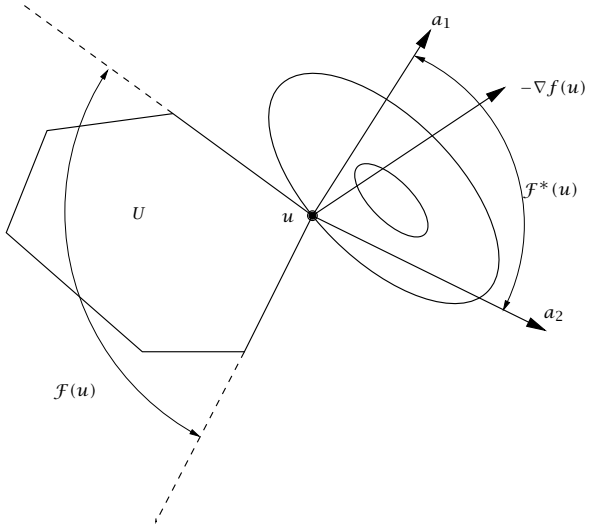



**Figure C.5:** Condition of optimality.

With $U$ defined this way, $u$ is optimal for $\min_u\{f(u) \mid u \in U\}$ where $f(\cdot)$ is convex and differentiable if and only if

$$-\nabla f(u) \in \hat{N}_U(u)$$

For each $i \in \mathcal{I}$ let $g_i(u) := \langle a_i, u\rangle - b_i$ and for each $i \in \mathcal{E}$, let $h_i(u) := \langle c_i, u\rangle - d_i$ so that $\nabla g(u_i) = a_i$ and $\nabla h_i = c_i$. It follows from the characterization of $\hat{N}_U(u)$ that $u$ is optimal for $\min_u\{f(u) \mid u \in U\}$ if and only if there exist multipliers $\lambda_i \geq 0,\ i \in \mathcal{I}^0(u)$ and $\mu_i \in \mathbb{R},\ i \in \mathcal{E}$ such that

$$\nabla f(u) + \sum_{i\in\mathcal{I}^0(u)} \mu_i \nabla g_i(u) + \sum_{i\in\mathcal{E}} h_i(u) = 0 \tag{C.21}$$

### C.2.4 Nonconvex Problems

We first obtain a necessary condition of optimality for the problem $\min\{f(u) \mid u \in U\}$ where $f(\cdot)$ is differentiable but not necessarily convex and $U \subset \mathbb{R}^m$ is not necessarily convex; this result generalizes the necessary condition of optimality in Proposition C.9.

**Proposition C.14** (Necessary condition for nonconvex problem)**.** *A necessary condition for $u$ to be locally optimal for the problem of minimizing a differentiable function $f(\cdot)$ over the set $U$ is*

$$df(u;h) = \langle \nabla f(u), h\rangle \geq 0,\ \forall h \in \mathcal{T}_U(u)$$

*which is equivalent to the condition*

$$-\nabla f(u) \in \hat{N}_U(u)$$

*Proof.* Suppose, contrary to what we wish to prove, that there exists a $h \in \mathcal{T}_U(u)$ and a $\delta > 0$ such that $\langle \nabla f(u), h \rangle = -\delta < 0$. Because $h \in \mathcal{T}_U(u)$, there exist sequences $h^\nu \xrightarrow[U]{} h$ and $\lambda^\nu \searrow 0$ such that $u^\nu := u + \lambda^\nu h^\nu$ converges to $u$ and satisfies $u^\nu \in U$ for all $\nu \in \mathbb{I}_{\geq 0}$. Then

$$f(u^\nu) - f(u) = \langle \nabla f(u), \lambda^\nu h^\nu \rangle + o(\lambda^\nu |h^\nu|)$$

Hence

$$[f(u^\nu) - f(u)]/\lambda^\nu = \langle \nabla f(u), h^\nu \rangle + o(\lambda^\nu)/\lambda^\nu$$

where we make use of the fact that $|h^\nu|$ is bounded for $\nu$ sufficiently large. It follows that

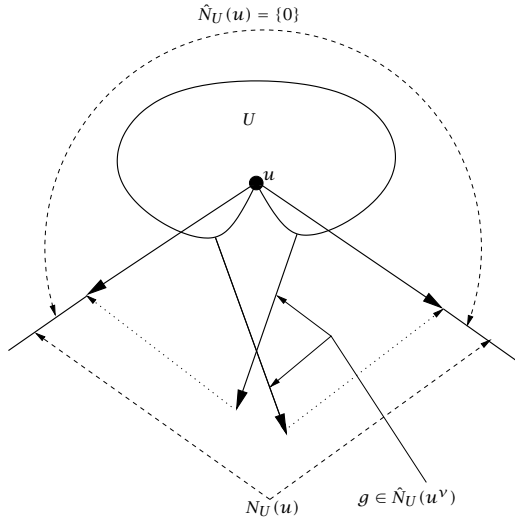
$$[f(u^\nu) - f(u)]/\lambda^\nu \to \langle \nabla f(u), h \rangle = -\delta$$

so that there exists a finite integer $j$ such that $f(u^j) - f(u) \leq -\lambda^j \delta/2 < 0$ which contradicts the local optimality of $u$. Hence $\langle \nabla f(u), h \rangle \geq 0$ for all $h \in \mathcal{T}_U(u)$. That $-\nabla f(u) \in \hat{N}_U(u)$ follows from Proposition C.7. ∎

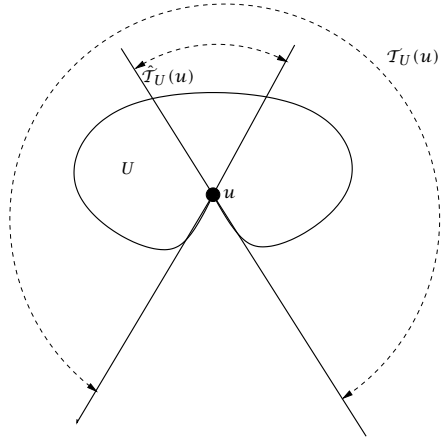
A more concise proof proceeds as follows Rockafellar and Wets (1998). Since $f(v) - f(u) = \langle \nabla f(u), v - u \rangle + o(|v - u|)$ it follows that $\langle -\nabla f(u), v - u \rangle = o(|v - u|) - (f(v) - f(u))$. Because $u$ is locally optimal, $f(v) - f(u) \geq 0$ for all $v$ in the neighborhood of $u$ so that $\langle -\nabla f(u), v - u \rangle \leq o(|v - u|)$ which, by (C.15), is the definition of a normal vector. Hence $-\nabla f(u) \in \hat{N}_U(u)$.

### C.2.5 Tangent and Normal Cones

The material in this section is *not* required for Chapters 1-7; it is presented merely to show that alternative definitions of tangent and normal cones are useful in more complex situations than those considered above. Thus, the normal and tangent cones defined in C.2.1 have some limitations when $U$ is not convex or, at least, not similar to the constraint set illustrated in Figure C.4. Figure C.6 illustrates the type of difficulty that may occur. Here the tangent cone $\mathcal{T}_U(u)$ is not convex, as shown in Figure C.6(b), so that the associated normal cone $\hat{N}_U(u) = \mathcal{T}_U(u)^* = \{0\}$. Hence the necessary condition of optimality of $u$ for the problem of minimizing a differentiable function $f(\cdot)$




(a) Normal cones.




(b) Tangent cones.

**Figure C.6:** Tangent and normal cones.

over $U$ is $\nabla f(u) = 0$; the only way a *differentiable* function $f(\cdot)$ can achieve a minimum over $U$ at $u$ is for the condition $\nabla f(u) = 0$ to be satisfied. Alternative definitions of normality and tangency are sometimes necessary. In Rockafellar and Wets (1998), a vector $g \in \hat{N}_U(u)$ is normal in the *regular* sense; a normal in the *general* sense is then defined by:

**Definition C.15** (General normal)**.** A vector $g$ is normal to $U$ at $u$ in the general sense if there exist sequences $u^\nu \underset{U}{\longrightarrow} u$ and $g^\nu \to g$ where $g^\nu \in \hat{N}_U(u^\nu)$ for all $\nu$; $N_U(u)$ is the set of all general normal vectors.

The cone $N_U(u)$ of general normal vectors is illustrated in Figure C.6(a); here the cone $N_U(u)$ is the union of two distinct cones each having form $\{\alpha g \mid \alpha \geq 0\}$. Also shown in Figure C.6(a) are single elements of two sequences $g^\nu$ in $\hat{N}_U(u^\nu)$ converging to $N_U(u)$. Counter intuitively, the general normal vectors in this case point into the interior of $U$. Associated with $N_U(u)$ is the set $\hat{\mathcal{T}}_U(u)$ of regular tangents to $U$ at $u$ defined, when $U$ is locally closed,[3] in (Rockafellar and Wets, 1998, Theorem 6.26) by:

**Definition C.16** (General tangent)**.** Suppose $U$ is locally closed at $u$. A vector $h$ is tangent to $U$ at $u$ in the regular sense if, for all sequences $u^\nu \underset{U}{\longrightarrow} u$, there exists a sequence $h^\nu \to h$ that satisfies $h^\nu \in \mathcal{T}_u(u^\nu)$ for all $\nu$; $\hat{\mathcal{T}}_U(u)$ is the set of all regular tangent vectors to $U$ at $u$.

Alternatively, a vector $h$ is tangent to $U$ at $u$ in the regular sense if, for all sequences $u^\nu \underset{U}{\longrightarrow} u$ and $\lambda^\nu \searrow 0$, there exists a sequence $h^\nu \to h$ satisfying $u^\nu + \lambda^\nu h^\nu \in U$ for all $\nu \in \mathbb{I}_{\geq 0}$. The cone of regular tangent vectors for the example immediately above is shown in Figure C.6(b). The following result is proved in Rockafellar and Wets (1998), Theorem 6.26:

**Proposition C.17** (Set of regular tangents is closed convex cone)**.** *At any $u \in U$, the set $\hat{\mathcal{T}}_U(u)$ of regular tangents to $U$ at $u$ is a closed convex cone with $\hat{\mathcal{T}}_U(u) \subset \mathcal{T}_U(u)$. Moreover, if $U$ is locally closed at $u$, then $\hat{\mathcal{T}}_U(u) = N_U(u)^*$.*

Figure C.7 illustrates some of these results. In Figure C.7, the constant cost contour $\{v \mid f(v) = f(u)\}$ of a *nondifferentiable* cost function $f(\cdot)$ is shown together with a sublevel set $D$ passing through the

[3] A set $U$ is locally closed at a point $u$ if there exists a closed neighborhood $\mathcal{N}$ of $u$ such that $U \cap \mathcal{N}$ is closed; $U$ is locally closed if it is locally closed at all $u$.

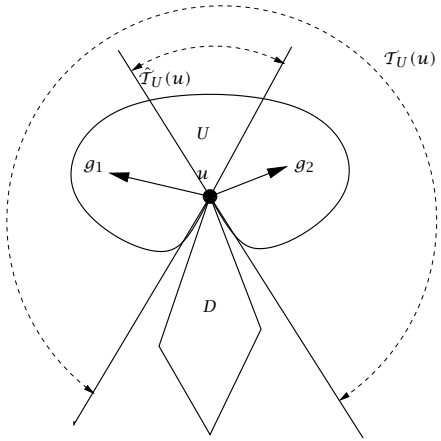



**Figure C.7:** Condition of optimality.

point $u$: $f(v) \leq f(u)$ for all $v \in D$. For this example, $df(u;h) = \max\{\langle g_1, h\rangle, \langle g_2, h\rangle\}$ where $g_1$ and $g_2$ are normals to the level set of $f(\cdot)$ at $u$ so that $df(u;h) \geq 0$ for all $h \in \hat{\mathcal{T}}_U(u)$, a necessary condition of optimality; on the other hand, there exist $h \in \mathcal{T}_U(u)$ such that $df(u;h) < 0$. The situation is simpler if the constraint set $U$ is *regular* at $u$.

**Definition C.18** (Regular set)**.** A set $U$ is regular at a point $u \in U$ in the sense of Clarke if it is locally closed at $u$ and if $N_U(u) = \hat{N}_U(u)$ (all normal vectors at $u$ are regular).

The following consequences of Clarke regularity are established in Rockafellar and Wets (1998), Corollary 6.29:

**Proposition C.19** (Conditions for regular set)**.** *Suppose $U$ is locally closed at $u \in U$. Then $U$ is regular at $u$ is equivalent to each of the following.*

*(a)* $N_U(u) = \hat{N}_U(u)$ *(all normal vectors at $u$ are regular).*

*(b)* $\mathcal{T}_U(u) = \hat{\mathcal{T}}_U(u)$ *(all tangent vectors at $u$ are regular).*

*(c)* $N_U(u) = \mathcal{T}_U(u)^*$.

*(d)* $\mathcal{T}_U(u) = N_U(u)^*$.

*(e)* $\langle g, h\rangle \leq 0$ *for all* $h \in \mathcal{T}_U(u)$, *all* $g \in N_U(u)$.

It is shown in Rockafellar and Wets (1998) that if $U$ is regular at $u$ and a constraint qualification is satisfied, then a necessary condition

of optimality, similar to (C.21), may be obtained. To obtain this result, we pursue a slightly different route in Sections C.2.6 and C.2.7.

### C.2.6 Constraint Set Defined by Inequalities

We now consider the case when the set $U$ is specified by a set of differentiable inequalities:

$$U := \{u \mid g_i(u) \leq 0 \ \forall i \in \mathcal{I}\} \tag{C.22}$$

where, for each $i \in \mathcal{I}$, the function $g_i : \mathbb{R}^m \to \mathbb{R}$ is differentiable. For each $u \in U$

$$\mathcal{I}^0(u) := \{i \in \mathcal{I} \mid g_i(u) = 0\}$$

is the index set of active constraints. For each $u \in U$, the set $\mathcal{F}_U(u)$ of feasible variations for the *linearized* set of inequalities; $\mathcal{F}_U(u)$ is defined by

$$\mathcal{F}_U(u) := \{h \mid \langle \nabla g_i(u), h\rangle \leq 0 \ \forall i \in \mathcal{I}^0(u)\} \tag{C.23}$$

The set $\mathcal{F}_U(u)$ is a closed, convex cone and is called a cone of first order feasible variations in Bertsekas (1999) because $h$ is a descent direction for $g_i(u)$ for all $i \in \mathcal{I}^0(u)$, i.e., $g_i(u+\lambda h) \leq 0$ for all $\lambda$ sufficiently small. When $U$ is polyhedral, the case discussed in C.2.3, $g_i(u) = \langle a_i, u\rangle - b_i$ and $\nabla g_i(u) = a_i$ so that $\mathcal{F}_U(u) = \{h \mid \langle a_i, h\rangle \leq 0 \ \forall i \in \mathcal{I}^0(u)\}$ which was shown in Proposition C.11 to be the tangent cone $\mathcal{T}_U(u)$. An important question whether $\mathcal{F}_U(u)$ is the tangent cone $\mathcal{T}_U(u)$ for a wider class of problems because, if $\mathcal{F}_U(u) = \mathcal{T}_U(u)$, a condition of optimality of the form in (C.20) may be obtained. In the example in Figure C.8, $\mathcal{F}_U(u)$ is the horizontal axis $\{h \in \mathbb{R}^2 \mid h_2 = 0\}$ whereas $\mathcal{T}_U(u)$ is the half-line $\{h \in \mathbb{R}^2 \mid h_1 \geq 0,\ h_2 = 0\}$ so that in this case, $\mathcal{F}_U(u) \neq \mathcal{T}_U(u)$. While $\mathcal{F}_U(u)$ is always convex, being the intersection of a set of half-spaces, the tangent cone $\mathcal{T}_U(u)$ is not necessarily convex as Figure C.6b shows. The set $U$ is said to be *quasiregular* at $u \in U$ if $\mathcal{F}_U(u) = \mathcal{T}_U(u)$ is which case $u$ is said to be a quasiregular point Bertsekas (1999). The next result, due to Bertsekas (1999), shows that $\mathcal{F}_U(u) = \mathcal{T}_U(u)$, i.e., $U$ is quasiregular at $u$, when a certain constraint qualification is satisfied.

**Proposition C.20** (Quasiregular set)**.** *Suppose* $U := \{u \mid g_i(u) \leq 0 \ \forall i \in \mathcal{I}\}$ *where, for each* $i \in \mathcal{I}$*, the function* $g_i : \mathbb{R}^m \to \mathbb{R}$ *is differentiable. Suppose also that* $u \in U$ *and that there exists a vector* $\bar{h} \in \mathcal{F}_U(u)$ *such that*

$$\langle \nabla g_i(u), \bar{h}\rangle < 0, \ \forall \ i \in \mathcal{I}^0(u) \tag{C.24}$$

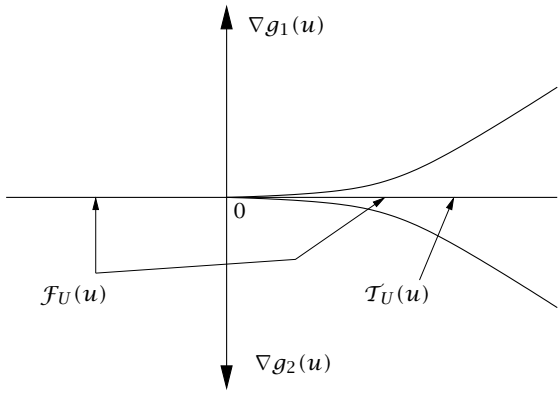



**Figure C.8:** $\mathcal{F}_U(u) \neq \mathcal{T}_U(u)$.

*Then*

$$\mathcal{T}_U(u) = \mathcal{F}_U(u)$$

*i.e., $U$ is quasiregular at $u$.*

Equation (C.24) is the constraint qualification; it can be seen that it precludes the situation shown in Figure C.8.

*Proof.* It follows from the definition (C.23) of $\mathcal{F}_U(u)$ and the constraint qualification (C.24) that:

$$\langle \nabla g_i(u), h + \alpha(\bar{h} - h) \rangle < 0, \ \forall h \in \mathcal{F}_U(u), \alpha \in (0,1], i \in \mathcal{I}^0(u)$$

Hence, for all $h \in \mathcal{F}_U(u)$, all $\alpha \in (0,1]$, there exists a vector $h_\alpha := h + \alpha(\bar{h} - h)$, in $\mathcal{F}_U(u)$ satisfying $\langle \nabla g_i(u), h_\alpha \rangle < 0$ for all $i \in \mathcal{I}^0(u)$. Assuming for the moment that $h_\alpha \in \mathcal{T}_U(u)$ for all $\alpha \in (0,1]$, it follows, since $h_\alpha \to h$ as $\alpha \to 0$ and $\mathcal{T}_U(u)$ is closed, that $h \in \mathcal{T}_U(u)$, thus proving $\mathcal{F}_U(u) \subset \mathcal{T}_U(u)$. It remains to show that $h_\alpha$ is tangent to $U$ at $u$. Consider the sequences $h^\nu$ and $\lambda^\nu \searrow 0$ where $h^\nu := h_\alpha$ for all $\nu \in \mathbb{I}_{\geq 0}$. There exists a $\delta > 0$ such that $\langle \nabla g_i(u), h_\alpha \rangle \leq -\delta$ for all $i \in \mathcal{I}^0(u)$ and $g_i(u) \leq -\delta$ for all $i \in \mathcal{I} \setminus \mathcal{I}^0(u)$. Since

$$g_i(u + \lambda^\nu h^\nu) = g_i(u) + \lambda^\nu \langle \nabla g_i(u), h_\alpha \rangle + o(\lambda^\nu) \leq -\lambda^\nu \delta + o(\lambda^\nu)$$

for all $i \in \mathcal{I}^0(u)$, it follows that there exists a finite integer $N$ such that $g_i(u + \lambda^\nu h^\nu) \leq 0$ for *all* $i \in \mathcal{I}$, all $\nu \geq N$. Since the sequences $\{h^\nu\}$ and $\{\lambda^\nu\}$ for all $\nu \geq N$ satisfy $h^\nu \to h_\alpha$, $\lambda^\nu \searrow 0$ and $u + \lambda^\nu h^\nu \in U$ for

all $i \in \mathcal{I}$, it follows that $h_\alpha \in \mathcal{T}_U(u)$, thus completing the proof that $\mathcal{F}_U(u) \subset \mathcal{T}_U(u)$.

Suppose now that $h \in \mathcal{T}_U(u)$. There exist sequences $h^\nu \to h$ and $\lambda^\nu \to 0$ such that $u + \lambda^\nu h^\nu \in U$ so that $g(u + \lambda^\nu h^\nu) \le 0$ for all $\nu \in \mathbb{I}_{\ge 0}$. Since $g(u + \lambda^\nu h^\nu) = g(u) + \langle \nabla g_j(u), \lambda^\nu h^\nu \rangle + o(\lambda^\nu |h^\nu|) \le 0$, it follows that $\langle \nabla g_j(u), \lambda^\nu h^\nu \rangle + o(\lambda^\nu) \le 0$ for all $j \in \mathcal{I}^0(u)$, all $\nu \in \mathbb{I}_{\ge 0}$. Hence $\langle \nabla g_j(u), h^\nu \rangle + o(\lambda^\nu)/\lambda^\nu \le 0$ for all $j \in \mathcal{I}^0(u)$, all $\nu \in \mathbb{I}_{\ge 0}$. Taking the limit yields $\langle \nabla g_j(u), h^\nu \rangle \le 0$ for all $j \in \mathcal{I}^0(u)$ so that $h \in \mathcal{F}_U(u)$ which proves $\mathcal{T}_U(u) \subset \mathcal{F}_U(u)$. Hence $\mathcal{T}_U(u) = \mathcal{F}_U(u)$. ∎

The existence of a $\bar{h}$ satisfying (C.24) is, as we have noted above, a constraint qualification. If $u$ is locally optimal for the inequality constrained optimization problem of minimizing a differentiable function $f(\cdot)$ over the set $U$ defined in (C.22) and, if (C.24) is satisfied thereby ensuring that $\mathcal{T}_U(u) = \mathcal{F}_U(u)$, then a condition of optimality of the form (C.20) may be easily obtained as shown in the next result.

**Proposition C.21** (Optimality conditions nonconvex problem)**.** *Suppose $u$ is locally optimal for the problem of minimizing a differentiable function $f(\cdot)$ over the set $U$ defined in* (C.22) *and that $\mathcal{T}_U(u) = \mathcal{F}_U(u)$. Then*

$$-\nabla f(u) \in \text{cone}\{\nabla g_i(u) \mid i \in \mathcal{I}^0(u)\}$$

*and there exist multipliers $\mu_i \ge 0$, $i \in \mathcal{I}^0(u)$ satisfying*

$$\nabla f(u) + \sum_{i \in \mathcal{I}^0(u)} \mu_i \nabla g_i(u) = 0 \tag{C.25}$$

*Proof.* It follows from Proposition C.14 that $-\nabla f(u) \in \hat{\mathcal{N}}_U(u)$ and from Proposition C.7 that $\hat{\mathcal{N}}_U(u) = \mathcal{T}_U^*(u)$. But, by hypothesis, $\mathcal{T}_U(u) = \mathcal{F}_U(u)$ so that $\hat{\mathcal{N}}_U(u) = \mathcal{F}_U^*(u)$, the polar cone of $\mathcal{F}_U(u)$. It follows from (C.23) and the definition of a polar cone, given in Appendix A1, that

$$\mathcal{F}_U^*(u) = \text{cone}\{\nabla g_i(u) \mid i \in| I^0(u)\}$$

Hence

$$-\nabla f(u) \in \text{cone}\{\nabla g_i(u) \mid i \in \mathcal{I}^0(u)\}$$

The existence of multipliers $\mu_i$ satisfying (C.25) follows from the definition of a cone generated by $\{\nabla g_i(u) \mid i \in \mathcal{I}^0(u)\}$. ∎

### C.2.7 Constraint Set Defined by Equalities and Inequalities

Finally, we consider the case when the set $U$ is specified by a set of differentiable equalities *and* inequalities:

$$U := \{u \mid g_i(u) \leq 0 \ \forall i \in \mathcal{I},\ h_i(u) = 0 \ \forall i \in \mathcal{E}\}$$

where, for each $i \in \mathcal{I}$, the function $g_i : \mathbb{R}^m \to \mathbb{R}$ is differentiable and for each $i \in \mathcal{E}$, the function $h_i : \mathbb{R}^m \to \mathbb{R}$ is differentiable. For each $u \in U$

$$\mathcal{I}^0(u) := \{i \in \mathcal{I} \mid g_i(u) = 0\}$$

the index set of active inequality constraints is defined as before. We wish to obtain necessary conditions for the problem of minimizing a differentiable function $f(\cdot)$ over the set $U$. The presence of equality constraints makes this objective more difficult than for the case when $U$ is defined merely by differentiable inequalities. The result we wish to prove is a natural extension of Proposition C.21 in which the equality constraints are included in the set of active constraints:

**Proposition C.22** (Fritz-John necessary conditions)**.** *Suppose $u$ is a local minimizer for the problem of minimizing $f(u)$ subject to the constraint $u \in U$ where $U$ is defined in* (C.22)*. Then there exist multipliers $\mu_0$, $\mu_i,\ i \in \mathcal{I}$ and $\lambda_i,\ i \in \mathcal{E}$, not all zero, such that*

$$\mu_0 \nabla f(u) + \sum_{i \in \mathcal{I}} \mu_i \nabla g_i(u) + \sum_{j \in \mathcal{E}} \lambda_j \nabla h_j(u) = 0 \tag{C.26}$$

*and*

$$\mu_i g_i(u) = 0 \ \forall i \in \mathcal{I}$$

*where $\mu_0 \geq 0$ and $\mu_i \geq 0$ for all $i \in \mathcal{I}^0$.*

The condition $\mu_i g_i(u) = 0$ for all $i \in \mathcal{I}$ is known as the *complementarity* conditions and implies $\mu_i = 0$ for all $i \in \mathcal{I}$ such that $g_i(u) < 0$. If $\mu_0 > 0$, then (C.26) may be normalized by dividing each term by $\mu_0$ yielding the more familiar expression

$$\nabla f(u) + \sum_{i \in \mathcal{I}} \mu_i \nabla g_i(u) + \sum_{j \in \mathcal{E}} \nabla h_j(u) = 0$$

We return to this point later. Perhaps the simplest method for proving Proposition C.22 is the penalty approach adopted by Bertsekas (1999), Proposition 3.3.5. We merely give an outline of the proof. The constrained problem of minimizing $f(v)$ over $U$ is approximated, for each

$k \in \mathbb{I}_{\geq 0}$ by a penalized problem defined below; as $k$ increases the penalized problem becomes a closer approximation to the constrained problem. For each $i \in \mathcal{I}$, we define

$$g_i^+(v) := \max\{g_i(v), 0\}$$

For each $k$, the penalized problem $\mathbb{P}^k$ is then defined as the problem of minimizing $F^k(v)$ defined by

$$F^k(v) := f(v) + (k/2)\sum_{i\in\mathcal{I}}(g_i^+(v))^2 + (k/2)\sum_{j\in\mathcal{E}}(h_j(v))^2 + (1/2)|v-u|^2$$

subject to the constraint

$$S := \{v \mid |v-u| \leq \varepsilon\}$$

where $\epsilon > 0$ is such that $f(u) \leq f(v)$ for all $v$ in $S \cap U$. Let $v^k$ denote the solution of $\mathbb{P}^k$. Bertsekas shows that $v^k \to u$ as $k \to \infty$ so that for all $k$ sufficiently large, $v^k$ lies in the interior of $S$ and is, therefore, the unconstrained minimizer of $F^k(v)$. Hence for each $k$ sufficiently large, $v^k$ satisfies $\nabla F^k(v^k) = 0$, or

$$\nabla f(v^k) + \sum_{i\in\mathcal{I}} \bar{\mu}_i^k \nabla g(v^k) + \sum_{i\in\mathcal{E}} \bar{\lambda}_i^k \nabla h(v^k) = 0 \tag{C.27}$$

where

$$\bar{\mu}_i^k := k g_i^+(v^k), \quad \bar{\lambda}_i^k := k h_i(v^k)$$

Let $\mu^k$ denote the vector with elements $\mu_i^k$, $i \in \mathcal{I}$ and $\lambda^k$ the vector with elements $\lambda_i^k$, $k \in \mathcal{E}$. Dividing (C.27) by $\delta^k$ defined by

$$\delta^k := [1 + |\mu^k|^2 + |\lambda^k|^2]^1/2$$

yields

$$\mu_0^k \nabla f(v^k) + \sum_{i\in\mathcal{I}} \mu_i^k \nabla g(v^k) + \sum_{j\in\mathcal{E}} \lambda_j^k \nabla h(v^k) = 0$$

where

$$\mu_0^k := \bar{\mu}_i^k/\delta^k, \quad \mu_i^k := \bar{\mu}_i^k/\delta^k, \quad \lambda_j^k := \bar{\lambda}_i^k/\delta^k$$

and

$$(\mu_0^k)^2 + |\mu^k|^2 + |\lambda^k|^2 = 1$$

Because of the last equation, the sequence $\{\mu_0^k, \mu^k, \lambda^k\}$ lies in a compact set, and therefore has a subsequence, indexed by $K \subset \mathbb{I}_{\geq 0}$, converging to some limit $\{\mu_0, \mu, \lambda\}$ where $\mu$ and $\lambda$ are vectors whose elements are,

respectively, $\mu_i,\ i \in \mathcal{I}$ and $\lambda_j,\ j \in \mathcal{E}$. Because $v^k \to u$ as $k \in K$ tends to infinity, it follows from (C.27) that

$$\mu_0 \nabla f(u) + \sum_{i \in \mathcal{I}} \mu_i \nabla g_i(u) + \sum_{j \in \mathcal{E}} \lambda_j \nabla h_j(u) = 0$$

To prove the complementarity condition, suppose, contrary to what we wish to prove, that there exists a $i \in \mathcal{I}$ such that $g_i(u) < 0$ but $\mu_i > 0$. Since $\mu_i^k \to \mu_i > 0$ and $g_i(v^k) \to g_i(u)$ as $k \to \infty$, $k \in K$, it follows that $\mu_i \mu_i^k > 0$ for all $k \in K$ sufficiently large. But $\mu_i^k = \bar{\mu}_i^k / \delta^k = k g_i^+(v^k)/\delta^k$ so that $\mu_i \mu_i^k > 0$ implies $\mu_i g_i^+(v^k) > 0$ which in turn implies $g_i^+(v^k) = g_i(v^k) > 0$ for all $k \in K$ sufficiently large. This contradicts the fact that $g_i(v^k) \to g_i(u) < 0$ as $k \to \infty$, $k \in K$. Hence we must have $g_i(u) = 0$ for all $i \in \mathcal{I}$ such that $\mu_i > 0$.

The Fritz-John condition in Proposition C.22 is known as the Karush-Kuhn-Tucker (KKT) condition if $\mu_0 > 0$; if this is the case, $\mu_0$ may be normalized to $\mu_0 = 1$. A constraint qualification is required for the Karush-Kuhn-Tucker condition to be a necessary condition of optimality for the optimization problem considered in this section. A simple constraint qualification is linear independence of $\{\nabla g_i(u),\ i \in \mathcal{I}^0(u),\ \nabla h_j(u),\ j \in \mathcal{E}\}$ at a local minimizer $u$. For, if $u$ is a local minimizer and $\mu_0 = 0$, then the Fritz-John condition implies that $\sum_{i \in \mathcal{I}^0(u)} \mu_i \nabla g_i(u) + \sum_{j \in \mathcal{E}} \lambda_j \nabla h_j(u) = 0$ which contradicts the linear independence of $\{\nabla g_i(u),\ i \in \mathcal{I}^0(u),\ \nabla h_j(u),\ j \in \mathcal{E}\}$ since not all the multipliers are zero. Another constraint qualification, used in Propositions C.20 and C.21 for an optimization problem in which the constraint set is $U := \{u \mid g_i(u) \le 0,\ i \in \mathcal{I}\}$, is the existence of a vector $\bar{h}(u) \in \mathcal{F}_U(u)$ such that $\langle \nabla g_i(u), \bar{h} \rangle < 0$ for all $i \in \mathcal{I}^0(u)$; this condition ensures $\mu_0 = 1$ in C.25. Many other constraint qualifications exist; see, for example, Bertsekas (1999), Chapter 3.

## C.3 Set-Valued Functions and Continuity of Value Function

A set-valued function $U(\cdot)$ is one for which, for each value of $x$, $U(x)$ is a set; these functions are encountered in parametric programming. For example, in the problem $\mathbb{P}(x) : \inf_u \{f(x,u) \mid u \in U(x)\}$ (which has the same form as an optimal control problem in which $x$ is the state and $u$ is a control sequence), the constraint set $U$ is a set-valued function of the state. The solution to the problem $\mathbb{P}(x)$ (the value of $u$ that achieves the minimum) can also be set-valued. It is important to

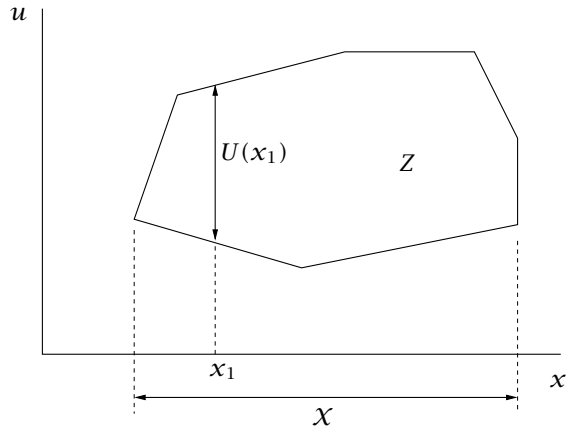
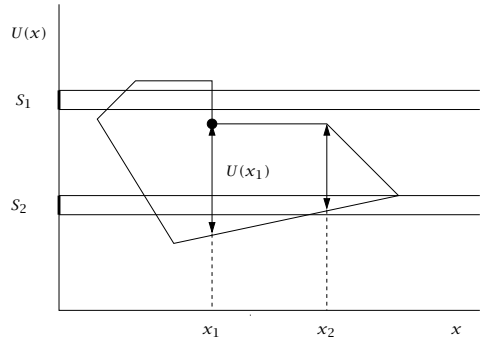


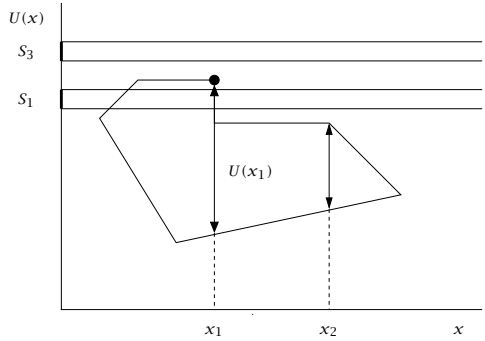

**Figure C.9:** Graph of set-valued function $U(\cdot)$.

know how smoothly these set-valued functions vary with the parameter $x$. In particular, we are interested in the continuity properties of the value function $x \mapsto f^0(x) = \inf_u \{f(x,u) \mid u \in U(x)\}$ since, in optimal control problems we employ the value function as a Lyapunov function and robustness depends, as we have discussed earlier, on the continuity of the Lyapunov function. Continuity of the value function depends, in turn, on continuity of the set-valued constraint set $U(\cdot)$. We use the notation $U : \mathbb{R}^n \rightsquigarrow \mathbb{R}^m$ to denote the fact that $U(\cdot)$ maps points in $\mathbb{R}^n$ into subsets of $\mathbb{R}^m$.

The *graph* of a set-valued functions is often a useful tool. The graph of $U : \mathbb{R}^n \rightsquigarrow \mathbb{R}^m$ is defined to be the set $Z := \{(x,u) \in \mathbb{R}^n \times \mathbb{R}^m \mid u \in U(x)\}$; the *domain* of the set-valued function $U$ is the set $\mathcal{X} := \{x \in \mathbb{R}^n \mid U(x) \neq \emptyset\} = \{x \in \mathbb{R}^n \mid \exists u \in \mathbb{R}^m \text{ such that } (x,u) \in Z\}$; clearly $\mathcal{X} \subset \mathbb{R}^n$. Also $\mathcal{X}$ is the *projection* of the set $Z \subset \mathbb{R}^n \times \mathbb{R}^m$ onto $\mathbb{R}^n$, i.e., $(x,u) \in Z$ implies $x \in \mathcal{X}$. An example is shown in Figure C.9. In this example, $U(x)$ varies continuously with $x$. Examples in which $U(\cdot)$ is discontinuous are shown in Figure C.10. In Figure C.10(a), the set $U(x)$ varies continuously if $x$ increases from its initial value of $x_1$, but jumps to a much larger set if $x$ decreases an infinitesimal amount (from its initial value of $x_1$); this is an example of a set-valued function that is inner semicontinuous at $x_1$. In Figure C.10(b), the set $U(x)$ varies continuously if $x$ decreases from its initial value of $x_1$, but jumps to a much smaller set if $x$ increases an infinitesimal amount (from its initial value of $x_1$); this is an example of a set-valued function that is




(a) Inner semicontinuous set-valued function.




(b) Outer semicontinuous set-valued function.

**Figure C.10:** Graphs of discontinuous set-valued functions.

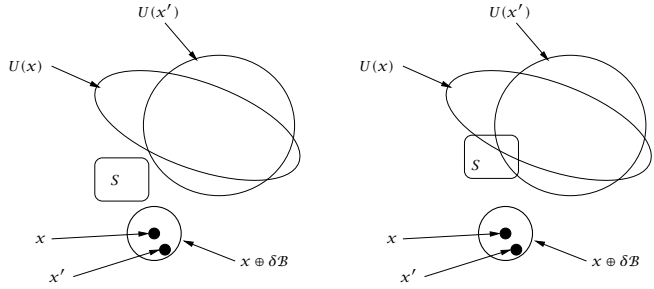
outer semicontinuous at $x_1$. The set-valued function is continuous at $x_2$ where it is both outer and inner semicontinuous.

We can now give precise definitions of inner and outer semicontinuity.

### C.3.1 Outer and Inner Semicontinuity

The concepts of inner and outer semicontinuity were introduced by Rockafellar and Wets (1998, p. 144) to replace earlier definitions of lower and upper semicontinuity of set-valued functions. This section is based on the useful summary provided by Polak (1997, pp. 676-682).

**Definition C.23** (Outer semicontinuous function)**.** A set-valued function $U : \mathbb{R}^n \rightsquigarrow \mathbb{R}^m$ is said to be outer semicontinuous (osc) at $x$ if $U(x)$




(a) Outer semicontinuity. (b) Inner semicontinuity.

**Figure C.11:** Outer and inner semicontinuity of $U(\cdot)$.

is closed and if, for every compact set $S$ such that $U(x) \cap S = \emptyset$, there exists a $\delta > 0$ such that $U(x') \cap S = \emptyset$ for all $x' \in x \oplus \delta\mathcal{B}$.[4] The set-valued function $U : \mathbb{R}^n \rightsquigarrow \mathbb{R}^m$ is outer semicontinuous if it is outer semicontinuous at each $x \in \mathbb{R}^n$.

**Definition C.24** (Inner semicontinuous function)**.** A set-valued function $U : \mathbb{R}^n \rightsquigarrow \mathbb{R}^m$ is said to be inner semicontinuous (isc) at $x$ if, for every open set $S$ such that $U(x) \cap S \neq \emptyset$, there exists a $\delta > 0$ such that $U(x') \cap S \neq \emptyset$ for all $x' \in x \oplus \delta\mathcal{B}$. The set-valued function $U : \mathbb{R}^n \rightsquigarrow \mathbb{R}^m$ is inner semicontinuous if it is inner semicontinuous at each $x \in \mathbb{R}^n$.

These definitions are illustrated in Figure C.11. Roughly speaking, a set-valued function that is outer semicontinuous at $x$ cannot explode as $x$ changes to $x'$ arbitrarily close to $x$; similarly, a set-valued function that is inner semicontinuous at $x$ cannot collapse as $x$ changes to $x'$ arbitrarily close to $x$.

**Definition C.25** (Continuous function)**.** A set-valued function is continuous (at $x$) if it is both outer and inner continuous (at $x$).

If we return to Figure C.10(a) we see that $S_1 \cap U(x_1) = \emptyset$ but $S_1 \cap U(x) \neq \emptyset$ for $x$ infinitesimally less than $x_1$ so that $U(\cdot)$ is not outer semicontinuous at $x_1$. For all $S_2$ such that $S_2 \cap U(x_1) \neq \emptyset$, however, $S_2 \cap U(x) \neq \emptyset$ for all $x$ in a sufficiently small neighborhood of $x_1$ so that $U(\cdot)$ is inner semicontinuous at $x_1$. If we turn to Figure C.10(b) we see that $S_1 \cap U(x_1) \neq \emptyset$ but $S_1 \cap U(x) = \emptyset$ for $x$ infinitesimally greater than $x_1$ so that in this case $U(\cdot)$ is not inner semicontinuous at $x_1$. For all $S_3$ such that $S_3 \cap U(x_1) = \emptyset$, however, $S_3 \cap U(x) = \emptyset$ for

[4] Recall that $\mathcal{B} := \{x \mid |x| \leq 1\}$ is the closed unit ball in $\mathbb{R}^n$.

all $x$ in a sufficiently small neighborhood of $x_1$ so that $U(\cdot)$ is outer semicontinuous at $x_1$.

The definitions of outer and inner semicontinuity may be interpreted in terms of infinite sequences (Rockafellar and Wets, 1998, p. 152), (Polak, 1997, pp. 677-678).

**Theorem C.26** (Equivalent conditions for outer and inner semicontinuity)**.**

*(a) A set-valued function $U : \mathbb{R}^n \rightsquigarrow \mathbb{R}^m$ is outer semicontinuous at $x$ if and only if for every infinite sequence $\{x_i\}$ converging to $x$, any accumulation point*[5] *$u$ of any sequence $\{u_i\}$, satisfying $u_i \in U(x_i)$ for all $i$, lies in $U(x)$ ($u \in U(x)$).*

*(b) A set-valued function $U : \mathbb{R}^n \rightsquigarrow \mathbb{R}^m$ is inner semicontinuous at $x$ if and only if for every $u \in U(x)$ and for every infinite sequence $\{x_i\}$ converging to $x$, there exists an infinite sequence $\{u_i\}$, satisfying $u_i \in U(x_i)$ for all $i$, that converges to $u$.*

Proofs of these results may be found in Rockafellar and Wets (1998); Polak (1997). Another result that we employ is:

**Proposition C.27** (Outer semicontinuity and closed graph)**.** *A set-valued function $U : \mathbb{R}^n \rightsquigarrow \mathbb{R}^m$ is outer semicontinuous in its domain if and only if its graph $Z$ is closed in $\mathbb{R}^n \times \mathbb{R}^m$.*

*Proof.* Since $(x, u) \in Z$ is equivalent to $u \in U(x)$, this result is a direct consequence of the Theorem C.26. ■

In the above discussion we have assumed, as in Polak (1997), that $U(x)$ is defined everywhere in $\mathbb{R}^n$; in constrained parametric optimization problems, however, $U(x)$ is defined on $\mathcal{X}$, a closed subset of $\mathbb{R}^n$; see Figure C.9. Only minor modifications of the above definitions are then required. In definitions C.23 and C.24 we replace the closed set $\delta\mathcal{B}$ by $\delta\mathcal{B} \cap \mathcal{X}$ and in Theorem C.26 we replace "every infinite sequence (in $\mathbb{R}^n$)" by "every infinite sequence in $\mathcal{X}$." In effect, we are replacing the topology of $\mathbb{R}^n$ by its topology relative to $\mathcal{X}$.

### C.3.2 Continuity of the Value Function

Our main reason for introducing set-valued functions is to provide us with tools for analyzing the continuity properties of the value function and optimal control law in constrained optimal control problems.

[5] Recall, $u$ is the limit of $\{u_i\}$ if $u_i \to u$ as $i \to \infty$; $u$ is an accumulation point of $\{u_i\}$ if it is the limit of a subsequence of $\{u_i\}$.

These problems have the form

$$V^0(x) = \min\{V(x,u) \mid u \in U(x)\} \tag{C.28}$$

$$u^0(x) = \arg\min\{V(x,u) \mid u \in U(x)\} \tag{C.29}$$

where $U : \mathbb{R}^n \rightsquigarrow \mathbb{R}^m$ is a set-valued function and $V : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is continuous; in optimal control problems arising from MPC, $u$ should be replaced by $\mathbf{u} = \{u(0), u(1), \ldots, u(N-1)\}$ and $m$ by $Nm$. We are interested in the continuity properties of the value function $V^0 : \mathbb{R}^n \to \mathbb{R}$ and the control law $u^0 : \mathbb{R}^n \to \mathbb{R}^m$; the latter may be set-valued (if the minimizer in (C.28) is not unique).

The following max problem has been extensively studied in the literature

$$\phi^0(x) = \max\{\phi(x,u) \mid u \in U(x)\}$$

$$\mu^0(x) = \arg\max\{\phi(x,u) \mid u \in U(x)\}$$

If we define $\phi(\cdot)$ by $\phi(x,u) := -V(x,u)$, we see that $\phi^0(x) = -V^0(x)$ and $\mu^0(x) = u^0(x)$ so that we can obtain the continuity properties of $V^0(\cdot)$ and $u^0(\cdot)$ from those of $\phi^0(\cdot)$ and $\mu^0(\cdot)$ respectively. Using this transcription and Corollary 5.4.2 and Theorem 5.4.3 in Polak (1997) we obtain the following result:

**Theorem C.28** (Minimum theorem). *Suppose that $V : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is continuous, that $U : \mathbb{R}^n \rightsquigarrow \mathbb{R}^m$ is continuous, compact-valued and satisfies $U(x) \subset \mathbb{U}$ for all $x \in \mathcal{X}$ where $\mathbb{U}$ is compact. Then $V^0(\cdot)$ is continuous and $u^0(\cdot)$ is outer semicontinuous. If, in addition, $u^0(x) = \{\mu^0(x)\}$ (there is a unique minimizer $\mu^0(x)$), then $\mu^0(\cdot)$ is continuous.*

It is unfortunately the case, however, that due to state constraints, $U(\cdot)$ is often not continuous in constrained optimal control problems. If $U(\cdot)$ is constant, which is the case in optimal control problem if state or mixed control-state constraints are absent, then, from the above results, the value function $V^0(\cdot)$ is continuous. Indeed, under slightly stronger assumptions, the value function is Lipschitz continuous.

**Lipschitz continuity of the value function.** If we assume that $V(\cdot)$ is Lipschitz continuous and that $U(x) \equiv U$, we can establish Lipschitz continuity of $V^0(\cdot)$. Interestingly the result does not require, nor does it imply, Lipschitz continuity of the minimizer $u^0(\cdot)$.

**Theorem C.29** (Lipschitz continuity of value function)**.** *Suppose that* $V : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ *is Lipschitz continuous on bounded sets*[6] *and that* $U(x) \equiv U$ *where* $U$ *is a compact subset of* $\mathbb{R}^m$*. Then* $V^0(\cdot)$ *is Lipschitz continuous on bounded sets.*

*Proof.* Let $S$ be an arbitrary bounded set in $\mathcal{X}$, the domain of the value function $V^0(\cdot)$, and let $R := S \times \mathbb{U}$; $R$ is a bounded subset of $\mathcal{Z}$. Since $R$ is bounded, there exists a Lipschitz constant $L_S$ such that

$$\left| V(x', u) - V(x'', u) \right| \leq L_S |x' - x''|$$

for all $x', x'' \in S$, all $u \in U$. Hence,

$$V^0(x') - V^0(x'') \leq V(x', u'') - V(x'', u'') \leq L_S |x' - x''|$$

for all $x', x'' \in S$, any $u'' \in u^0(x'')$. Interchanging $x'$ and $x''$ in the above derivation yields

$$V^0(x'') - V^0(x') \leq V(x'', u') - V(x', u') \leq L_S |x'' - x'|$$

for all $x', x'' \in S$, any $u' \in u^0(x')$. Hence $V^0(\cdot)$ is Lipschitz continuous on bounded sets. ■

We now specialize to the case where $U(x) = \{u \in \mathbb{R}^m \mid (x, u) \in \mathcal{Z}\}$ where $\mathcal{Z}$ is a polyhedron in $\mathbb{R}^n \times \mathbb{R}^m$; for each $x$, $U(x)$ is a polytope. This type of constraint arises in constrained optimal control problems when the system is linear and the state and control constraints are polyhedral. What we show in the sequel is that, in this special case, $U(\cdot)$ is continuous and so, therefore, is $V^0(\cdot)$. An alternative proof, which many readers may prefer, is given in Chapter 7 where we exploit the fact that if $V(\cdot)$ is strictly convex and quadratic and $\mathcal{Z}$ polyhedral, then $V^0(\cdot)$ is piecewise quadratic and continuous. Our first concern is to obtain a bound on $d(u, U(x'))$, the distance of any $u \in U(x)$ from the constraint set $U(x')$.

**A bound on $d(u, U(x'))$, $u \in U(x)$.** The bound we require is given by a special case of a theorem due to Clarke, Ledyaev, Stern, and Wolenski (1998, Theorem 3.1, page 126). To motivate this result, consider a differentiable convex function $f : \mathbb{R} \to \mathbb{R}$ so that $f(u) \geq f(v) + \langle \nabla f(v), u - v \rangle$ for any two points $u$ and $v$ in $\mathbb{R}$. Suppose also that there exists a nonempty interval $U = [a, b] \subset \mathbb{R}$ such that $f(u) \leq 0$

[6] A function $V(\cdot)$ is Lipschitz continuous on bounded sets if, for any bounded set $S$, there exists a constant $L_S \in [0, \infty)$ such that $|V(z') - V(z)| \leq L_S |z - z'|$ for all $z, z' \in S$.

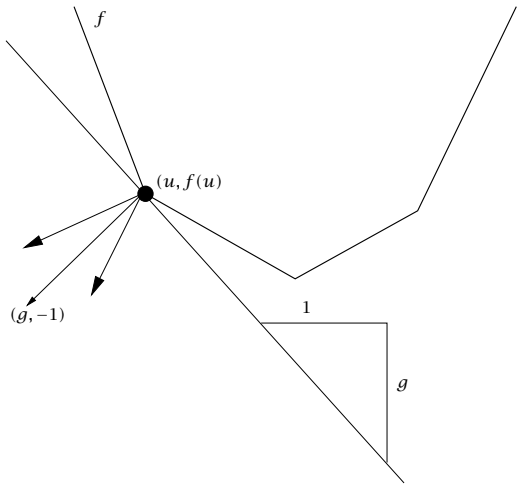



**Figure C.12:** Subgradient of $f(\cdot)$.

for all $u \in U$ and that there exists a $\delta > 0$ such that $\Delta f(u) > \delta$ for all $u \in \mathbb{R}$. Let $u > b$ and let $v = b$ be the closest point in $U$ to $u$. Then $f(u) \geq f(v) + \langle \nabla f(v), u - v \rangle \geq \delta |v - u|$ so that $d(u, U) \leq f(u)/\delta$. The theorem of Clarke et al. (1998) extends this result to the case when $f(\cdot)$ is not necessarily differentiable but requires the concept of a subgradient of a convex function

**Definition C.30** (Subgradient of convex function)**.** Suppose $f : \mathbb{R}^m \to \mathbb{R}$ is convex. Then the subgradient $\delta f(u)$ of $f(\cdot)$ at $u$ is defined by

$$\delta f(u) := \{g \mid f(v) \geq f(u) + \langle g, v - u \rangle \ \forall v \in \mathbb{R}^m\}$$

Figure C.12 illustrates a subgradient. In the figure, $g$ is one element of the subgradient because $f(v) \geq f(u) + \langle g, v - u \rangle$ for all $v$; $g$ is the slope of the line passing through the point $(u, f(u))$. To obtain a bound on $d(u, U(x))$ we require the following result which is a special case of the much more general result of the theorem of Clarke *et al.*:

**Theorem C.31** (Clarke et al. (1998))**.** *Take a nonnegative valued, convex function $\psi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$. Let $U(x) := \{u \in \mathbb{R}^m \mid \psi(x, u) = 0\}$ and $\mathcal{X} := \{x \in \mathbb{R}^n \mid U(x) \neq \emptyset\}$. Assume there exists a $\delta > 0$ such that*

$$u \in \mathbb{R}^m,\ x \in \mathcal{X}, \psi(x, u) > 0 \text{ and } g \in \partial_u \psi(x, u) \implies |g| > \delta$$

*where $\partial_u \psi(x, u)$ denotes the convex subgradient of $\psi$ with respect to*

*the variable $u$. Then, for each $x \in \mathcal{X}$, $d(u, U(x)) \leq \psi(x,u)/\delta$ for all $u \in \mathbb{R}^m$.*

The proof of this result is given in the reference cited above. We next use this result to bound the distance of $u$ from $U(x)$ where, for each $x$, $U(x)$ is polyhedral.

**Corollary C.32** (A bound on $d(u, U(x'))$ for $u \in U(x)$)**.** [7] *Suppose $\mathcal{Z}$ is a polyhedron in $\mathbb{R}^n \times \mathbb{R}^m$ and let $\mathcal{X}$ denote its projection on $\mathbb{R}^n$ ($\mathcal{X} = \{x \mid \exists u \in \mathbb{R}^m$ such that $(x,u) \in \mathcal{Z}\}$). Let $\mathcal{U}(x) := \{u \mid (x,u) \in \mathcal{Z}\}$. Then there exists a $K > 0$ such that for all $x, x' \in \mathcal{X}$, $d(u, U(x')) \leq K|x' - x|$ for all $u \in U(x)$ (or, for all $x, x' \in \mathcal{X}$, all $u \in U(x)$, there exists a $u' \in U(x')$ such that $|u' - u| \leq K|x' - x|$).*

*Proof.* The polyhedron $\mathcal{Z}$ admits the representation $\mathcal{Z} = \{(x,u) \mid \langle m^j, u\rangle - \langle n^j, x\rangle - p^j \leq 0,\ j \in \mathcal{J}\}$ for some $m^j \in \mathbb{R}^m$, $n^j \in \mathbb{R}^n$ and $p^j \in \mathbb{R}$, $j \in \mathcal{J} := \{1, \ldots, J\}$. Define $\mathcal{D}$ to be the collection of all index sets $I \subseteq \mathcal{J}$ such that $\sum_{j\in I} \lambda^j m^j \neq 0,\ \forall \lambda \in \Lambda_I$ in which, for a particular index set $I$, $\Lambda_I$ is defined to be $\Lambda_I := \{\lambda \mid \lambda^j \geq 0,\ \sum_{j\in I} \lambda^j = 1\}$. Because $\mathcal{D}$ is a finite set, there exists a $\delta > 0$ such that for all $I \in \mathcal{D}$, all $\lambda \in \Lambda_I$, $|\sum_{j\in I} \lambda^j m^j| > \delta$. Let $\psi(\cdot)$ be defined by $\psi(x,u) := \max\{\langle m^j, u\rangle - \langle n^j, x\rangle - p^j, 0 \mid j \in \mathcal{J}\}$ so that $(x,u) \in \mathcal{Z}$ (or $u \in \mathcal{U}(x)$) if and only if $\psi(x,u) = 0$. We now claim that, for every $(x,u) \in \mathcal{X} \times \mathbb{R}^m$ such that $\psi(x,u) > 0$ and every $g \in \partial_u \psi(x,u)$, the subgradient of $\psi$ with respect to $u$ at $(x,u)$, we have $|g| > \delta$. Assuming for the moment that the claim is true, the proof of the Corollary may be completed with the aid of Theorem C.31. Assume, as stated in the Corollary, that $x, x' \in \mathcal{X}$ and $u \in \mathcal{U}(x)$; the theorem asserts

$$d(u, \mathcal{U}(x')) \leq (1/\delta)\psi(x', u),\ \forall x' \in \mathcal{X}$$

But $\psi(x,u) = 0$ (since $u \in \mathcal{U}(x)$) so that

$$d(u, \mathcal{U}(x')) \leq (1/\delta)[\psi(x', u) - \psi(x,u)] \leq (c/\delta)|x' - x|$$

where $c$ is the Lipschitz constant for $x \mapsto \psi(x,u)$ ($\psi(\cdot)$ is piecewise affine and continuous). This proves the Corollary with $K = c/\delta$.

It remains to confirm the claim. Take any $(x,u) \in \mathcal{X} \times \mathbb{R}^m$ such that $\psi(x,u) > 0$. Then $\max_j\{\langle m^j, u\rangle - \langle n^j, x\rangle - p^j, 0 \mid j \in \mathcal{J}\} > 0$. Let

[7] The authors wish to thank Richard Vinter and Francis Clarke for providing this result.

$I^0(x,u)$ denote the active constraint set (the set of those constraints at which the maximum is achieved). Then

$$\langle m^j, u\rangle - \langle n^j, x\rangle - p^j > 0,\ \forall j \in I^0(x,u)$$

Since $x \in \mathcal{X}$, there exists a $\bar{u} \in \mathcal{U}(x)$ so that

$$\langle m^j, \bar{u}\rangle - \langle n^j, x\rangle - p^j \leq 0,\ \forall j \in I^0(x,u)$$

Subtracting these two inequalities yields

$$\langle m^j, u - \bar{u}\rangle > 0,\ \forall j \in I^0(x,u)$$

But then, for all $\lambda \in \Lambda_{I^0(x,u)}$, it follows that $|\sum_{j\in I^0(x,u)} \lambda^j m^j(u-\bar{u})| > 0$, so that

$$\sum_{j\in I^0(x,u)} \lambda^j m^j \neq 0$$

It follows that $I^0(x,u) \in \mathcal{D}$. Hence

$$\Big|\sum_{j\in I^0(x,u)} \lambda^j m^j\Big| > \delta,\ \forall \lambda \in \Lambda_{I^0(x,u)}$$

Now take any $g \in \partial_u f(x,u) = \text{co}\{m^j \mid j \in I^0(x,u)\}$ (co denotes "convex hull"). There exists a $\lambda \in \Lambda_{I^0(x,u)}$ such that $g = \sum_{j\in I^0(x,u)} \lambda^j m_j$. But then $|g| > \delta$ by the inequality above. This proves the claim and, hence, completes the proof of the Corollary. ■

**Continuity of the value function when $U(x) = \{u \mid (x,u) \in \mathcal{Z}\}$.** In this section we investigate continuity of the value function for the constrained linear quadratic optimal control problem $\mathbb{P}(x)$; in fact we establish continuity of the value function for the more general problem where the cost is continuous rather than quadratic. We showed in Chapter 2 that the optimal control problem of interest takes the form

$$V^0(x) = \min_u \{V(x,u) \mid (x,u) \in \mathcal{Z}\}$$

where $\mathcal{Z}$ is a polyhedron in $\mathbb{R}^n \times \mathbb{U}$ where $\mathbb{U} \subset \mathbb{R}^m$ is a polytope and, hence, is compact and convex; in MPC problems we replace the control $u$ by the sequence of controls $\mathbf{u}$ and $m$ by $Nm$. Let $u^0 : \mathbb{R}^n \rightsquigarrow \mathbb{R}^m$ be defined by

$$u^0(x) := \arg\min_u \{V(x,u) \mid (x,u) \in \mathcal{Z}\}$$

and let $\mathcal{X}$ be defined by

$$\mathcal{X} := \{x \mid \exists u \text{ such that } (x,u) \in \mathcal{Z}\}$$

so that $\mathcal{X}$ is the projection of $\mathcal{Z} \subset \mathbb{R}^n \times \mathbb{R}^m$ onto $\mathbb{R}^n$. Let the set-valued function $U : \mathbb{R}^n \rightsquigarrow \mathbb{R}^m$ be defined by

$$U(x) := \{u \in \mathbb{R}^m \mid (x, u) \in Z\}$$

The domain of $V^0(\cdot)$ and of $U(\cdot)$ is $\mathcal{X}$. The optimization problem may be expressed as $V^0(x) = \min_u \{V(x, u) \mid u \in U(x)\}$. Our first task is establish the continuity of $U : \mathbb{R}^n \rightsquigarrow \mathbb{R}^m$.

**Theorem C.33** (Continuity of $U(\cdot)$)**.** *Suppose $\mathcal{Z}$ is a polyhedron in $\mathbb{R}^n \times \mathbb{U}$ where $\mathbb{U} \subset \mathbb{R}^m$ is a polytope. Then the set-valued function $U : \mathbb{R}^n \rightsquigarrow \mathbb{R}^m$ defined above is continuous in $\mathcal{X}$.*

*Proof.* By Proposition C.27, the set-valued map $U(\cdot)$ is outer semicontinuous in $\mathcal{X}$ because its graph, $\mathcal{Z}$, is closed. We establish inner semicontinuity using Corollary C.32 above. Let $x, x'$ be arbitrary points in $\mathcal{X}$ and $U(x)$ and $U(x')$ the associated control constraint sets. Let $S$ be any open set such that $U(x) \cap S \neq \emptyset$ and let $u$ be an arbitrary point in $U(x) \cap S$. Because $S$ is open, there exist an $\varepsilon > 0$ such that $u \oplus \varepsilon\mathcal{B} \subset S$. Let $\varepsilon' := \varepsilon/K$ where $K$ is defined in Corollary 1. From Corollary C.32, there exists a $u' \in U(x')$ such that $|u' - u| \leq K|x' - x|$ which implies $|u' - u| \leq \varepsilon$ ($u' \in u \oplus \varepsilon\mathcal{B}$) for all $x' \in \mathcal{X}$ such that $|x' - x| \leq \varepsilon'$ ($x' \in (x \oplus \varepsilon'\mathcal{B}) \cap \mathcal{X}$). This implies $u \in \mathcal{U}(x') \cap S$ for all $x' \in \mathcal{X}$ such that $|x' - x| \leq \varepsilon'$ ($x' \in (x \oplus \varepsilon'\mathcal{B}) \cap \mathcal{X}$). Hence $\mathcal{U}(x') \cap S \neq \emptyset$ for all $x' \in (x \oplus \varepsilon'\mathcal{B}) \cap \mathcal{X}$, so that $\mathcal{U}(\cdot)$ is inner semicontinuous in $\mathcal{X}$. Since $U(\cdot)$ is both outer and inner semicontinuous in $\mathcal{X}$, it is continuous in $\mathcal{X}$. ∎

We can now establish continuity of the value function.

**Theorem C.34** (Continuity of the value function)**.** *Suppose that $V : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is continuous and that $\mathcal{Z}$ is a polyhedron in $\mathbb{R}^n \times \mathbb{U}$ where $\mathbb{U} \subset \mathbb{R}^m$ is a polytope. Then $V^0 : \mathbb{R}^n \to \mathbb{R}$ is continuous and $u^0 : \mathbb{R}^n \rightsquigarrow \mathbb{R}^m$ is outer semicontinuous in $\mathcal{X}$. Moreover, if $u^0(x)$ is unique (not set-valued) at each $x \in \mathcal{X}$, then $u^0 : \mathbb{R}^n \to \mathbb{R}^m$ is continuous in $\mathcal{X}$.*

*Proof.* Since the real-valued function $V(\cdot)$ is continuous (by assumption) and since the set-valued function $U(\cdot)$ is continuous in $\mathcal{X}$ (by Theorem C.33), it follows from Theorem C.28 that $V^0 : \mathbb{R}^n \to \mathbb{R}$ is continuous and $u^0 : \mathbb{R}^n \rightsquigarrow \mathbb{R}^m$ is outer semicontinuous in $\mathcal{X}$; it also follows that if $u^0(x)$ is unique (not set-valued) at each $x \in \mathcal{X}$, then $u^0 : \mathbb{R}^n \to \mathbb{R}^m$ is continuous in $\mathcal{X}$. ∎

**Lipschitz continuity when $U(x) = \{u \mid (x,u) \in \mathcal{Z}\}$.** Here we establish that $V^0(\cdot)$ is Lipschitz continuous if $V(\cdot)$ is Lipschitz continuous and $U(x) := \{u \in \mathbb{R}^m \mid (x,u) \in \mathcal{Z}\}$; this result is more general than Theorem C.29 where it is assumed that $U$ is constant.

**Theorem C.35** (Lipschitz continuity of the value function)**.** *Suppose that $V : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is continuous, that $\mathcal{Z}$ is a polyhedron in $\mathbb{R}^n \times \mathbb{U}$ where $\mathbb{U} \subset \mathbb{R}^m$ is a polytope. Suppose, in addition, that $V : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is Lipschitz continuous on bounded sets.*[8] *Then $V^0(\cdot)$ is Lipschitz continuous on bounded sets.*

*Proof.* Let $S$ be an arbitrary bounded set in $\mathcal{X}$, the domain of the value function $V^0(\cdot)$, and let $R := S \times \mathbb{U}$; $R$ is a bounded subset of $\mathcal{Z}$. Let $x, x'$ be two arbitrary points in $S$. Then

$$V^0(x) = V(x, \kappa(x))$$
$$V^0(x') = V(x', \kappa(x'))$$

where $V(\cdot)$ is the cost function, assumed to be Lipschitz continuous on bounded sets, and $\kappa(\cdot)$, the optimal control law, satisfies $\kappa(x) \in U(x) \subset \mathbb{U}$ and $\kappa(x') \in U(x') \subset \mathbb{U}$. It follows from Corollary C.32 that there exists a $K > 0$ such that for all $x, x' \in \mathcal{X}$, there exists a $u' \in U(x') \subset \mathbb{U}$ such that $|u' - \kappa(x)| \leq K|x' - x|$. Since $\kappa(x)$ is optimal for the problem $\mathbb{P}(x)$, and since $(x, \kappa(x))$ and $(x', u')$ both lie in $R = S \times \mathbb{U}$, there exists a constant $L_R$ such that

$$\begin{aligned} V^0(x') - V^0(x) &\leq V(x', u') - V(x, \kappa(x)) \\ &\leq L_R(|(x', u') - (x, \kappa(x))|) \\ &\leq L_R|x' - x| + L_R K|x' - x| \\ &\leq M_S|x' - x|, \quad M_S := L_R(1 + K) \end{aligned}$$

Reversing the role of $x$ and $x'$ we obtain the existence of a $u \in \mathcal{U}(x)$ such that $|u - \kappa(x')| \leq K|x - x'|$; it follows from the optimality of $\kappa(x')$ that

$$\begin{aligned} V^0(x) - V^0(x') &\leq V(x, u) - V(x', \kappa(x')) \\ &\leq M_S|x - x'| \end{aligned}$$

where, now, $u \in U(x)$ and $\kappa(x') \in U(x')$. Hence $|V^0(x') - V^0(x)| \leq M_S|x - x'|$ for all $x, x'$ in $S$. Since $S$ is an arbitrary bounded set in $\mathcal{X}$, $V^0(\cdot)$ is Lipschitz continuous on bounded sets. ∎

[8] A function $V(\cdot)$ is Lipschitz continuous on bounded sets if, for any bounded set $S$, there exists a constant $L_S \in [0, \infty)$ such that $|V(z') - V(z)| \leq L_S|z - z'|$ for all $z, z' \in S$.

## C.4 Exercises

### Exercise C.1: Nested optimization and switching order of optimization

Consider the optimization problem in two variables

$$\min_{(x,y)\in\mathbb{Z}} V(x,y)$$

in which $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, and $V : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$. Assume this problem has a solution. This assumption is satisfied, for example, if $V$ is continuous and $\mathbb{Z}$ is compact, but, in general, we do not require either of these conditions.

Define the following four sets

$$\mathbb{X}(y) = \{x \mid (x,y) \in \mathbb{Z}\} \qquad \mathbb{Y}(x) = \{y \mid (x,y) \in \mathbb{Z}\}$$
$$\mathbb{B} = \{y \mid \mathbb{X}(y) \neq \emptyset\} \qquad \mathbb{A} = \{x \mid \mathbb{Y}(x) \neq \emptyset\}$$

Note that $\mathbb{A}$ and $\mathbb{B}$ are the projections of $\mathbb{Z}$ onto $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Projection is defined in Section C.3. Show the solutions of the following two nested optimization problems exist and are equal to the solution of the original problem

$$\min_{x\in\mathbb{A}} \left( \min_{y\in\mathbb{Y}(x)} V(x,y) \right)$$
$$\min_{y\in\mathbb{B}} \left( \min_{x\in\mathbb{X}(y)} V(x,y) \right)$$

### Exercise C.2: DP nesting

Prove the assertion made in Section C.1.2 that $\mathbf{u}^i = \{u, \mathbf{u}^{i+1}\} \in \mathcal{U}(x,i)$ if and only if $(x,u) \in \mathbb{Z}$, $f(x,u) \in X(i+1)$, and $\mathbf{u}^{i+1} \in \mathcal{U}(f(x,u), i+1)$.

### Exercise C.3: Recursive feasibility

Prove the assertion in the proof of Theorem C.2 that $(x(j), u(j)) \in \mathbb{Z}$ and that $f(x(j), u(j)) \in X(j+1)$.

### Exercise C.4: Basic minmax result

Consider the following two minmax optimization problems in two variables

$$\inf_{x\in\mathbb{X}} \sup_{y\in\mathbb{Y}} V(x,y) \qquad \sup_{y\in\mathbb{Y}} \inf_{x\in\mathbb{X}} V(x,y)$$

in which $x \in \mathbb{X} \subseteq \mathbb{R}^n$, $y \in \mathbb{Y} \subseteq \mathbb{R}^m$, and $V : \mathbb{X} \times \mathbb{Y} \to \mathbb{R}$.

(a) Show that the values are ordered as follows

$$\inf_{x\in\mathbb{X}} \sup_{y\in\mathbb{Y}} V(x,y) \geq \sup_{y\in\mathbb{Y}} \inf_{x\in\mathbb{X}} V(x,y)$$

or, if the solutions to the problems exist,

$$\min_{x\in\mathbb{X}} \max_{y\in\mathbb{Y}} V(x,y) \geq \max_{y\in\mathbb{Y}} \min_{x\in\mathbb{X}} V(x,y)$$

A handy mnemonic for this result is that the player who goes first (inner problem) has the advantage.[9]

[9] Note that different conventions are in use. Boyd and Vandenberghe (2004, p. 240) say that the player who "goes" *second* has the advantage, meaning that the inner problem is optimized *after* the outer problem has selected a value for its variable. We say that since the inner optimization is solved first, this player "goes" first.

(b) Use your results to order these three problems

$$\sup_{x\in\mathbb{X}}\inf_{y\in\mathbb{Y}}\sup_{z\in\mathbb{Z}} V(x,y,z) \qquad \inf_{y\in\mathbb{Y}}\sup_{z\in\mathbb{Z}}\sup_{x\in\mathbb{X}} V(x,y,z) \qquad \sup_{z\in\mathbb{Z}}\sup_{x\in\mathbb{X}}\inf_{y\in\mathbb{Y}} V(x,y,z)$$

### Exercise C.5: Lagrange multipliers and minmax

Consider the constrained optimization problem

$$\min_{x\in\mathbb{R}^n} V(x) \qquad \text{subject to } g(x) = 0 \tag{C.30}$$

in which $V : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^m$. Introduce the Lagrange multiplier $\lambda \in \mathbb{R}^m$ and Lagrangian function $L(x,\lambda) = V(x) - \lambda' g(x)$ and consider the following minmax problem

$$\min_{x\in\mathbb{R}^n}\max_{\lambda\in\mathbb{R}^m} L(x,\lambda)$$

Show that if $(x_0,\lambda_0)$ is a solution to this problem with finite $L(x_0,\lambda_0)$, then $x_0$ is also a solution to the original constrained optimization (C.30).

### Exercise C.6: Dual problems and duality gap

Consider again the constrained optimization problem of Exercise C.5

$$\min_{x\in\mathbb{R}^n} V(x) \qquad \text{subject to } g(x) = 0$$

and its equivalent minmax formulation

$$\min_{x\in\mathbb{R}^n}\max_{\lambda\in\mathbb{R}^m} L(x,\lambda)$$

Switching the order of optimization gives the maxmin version of this problem

$$\max_{\lambda\in\mathbb{R}^m}\min_{x\in\mathbb{R}^n} L(x,\lambda)$$

Next define a new (dual) objective function $q : \mathbb{R}^m \to \mathbb{R}$ as the inner optimization

$$q(\lambda) = \min_{x\in\mathbb{R}^n} L(x,\lambda)$$

Then the maxmin problem can be stated as

$$\max_{\lambda\in\mathbb{R}^m} q(\lambda) \tag{C.31}$$

Problem (C.31) is known as the *dual* of the original problem (C.30), and the original problem (C.30) is then denoted as the *primal* problem in this context (Nocedal and Wright, 2006, p. 343–345), (Boyd and Vandenberghe, 2004, p. 223).

(a) Show that the solution to the dual problem is a lower bound for the solution to the primal problem

$$\max_{\lambda\in\mathbb{R}^m} q(\lambda) \le \min_{x\in\mathbb{R}^n} V(x) \qquad \text{subject to } g(x) = 0 \tag{C.32}$$

This property is known as *weak duality* (Nocedal and Wright, 2006, p. 345), (Boyd and Vandenberghe, 2004, p. 225).

(b) The difference between the dual and the primal solutions is known as the duality gap. *Strong duality* is defined as the property that equality is achieved in (C.32) and the duality gap is zero (Boyd and Vandenberghe, 2004, p. 225).

$$\max_{\lambda \in \mathbb{R}^m} q(\lambda) = \min_{x \in \mathbb{R}^n} V(x) \qquad \text{subject to } g(x) = 0 \tag{C.33}$$

Show that strong duality is equivalent to the existence of $\lambda_0$ such that

$$\min_{x \in \mathbb{R}^n} V(x) - \lambda_0' g(x) = \min_{x \in \mathbb{R}^n} V(x) \qquad \text{subject to } g(x) = 0 \tag{C.34}$$

Characterize the set of all $\lambda_0$ that satisfy this equation.

### Exercise C.7: Example with duality gap

Consider the following function and sets (Peressini, Sullivan, and Uhl, Jr., 1988, p. 34)

$$V(x, y) = (y - x^2)(y - 2x^2) \qquad \mathbb{X} = [-1, 1] \qquad \mathbb{Y} = [-1, 1]$$

Make a contour plot of $V(\cdot)$ on $\mathbb{X} \times \mathbb{Y}$ and answer the following question. Which of the following two minmax problems has a nonzero duality gap?

$$\min_{y \in \mathbb{Y}} \max_{x \in \mathbb{X}} V(x, y)$$

$$\min_{x \in \mathbb{X}} \max_{y \in \mathbb{Y}} V(x, y)$$

Notice that the two problems are different because the first one minimizes over $y$ and maximizes over $x$, and the second one does the reverse.

### Exercise C.8: The Heaviside function and inner and outer semicontinuity

Consider the (set-valued) function

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}$$

and you are charged with deciding how to define $H(0)$.

(a) Characterize the choices of set $H(0)$ that make $H$ outer semicontinuous. Justify your answer.

(b) Characterize the choices of set $H(0)$ that make $H$ inner semicontinuous. Justify your answer.

(c) Can you define $H(0)$ so that $H$ is both outer and inner semicontinuous? Explain why or why not.

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