ME 155B Fall 2008 Final Exam

Instructions:

1. This exam is open book and notes.
2. Write your name on this answer booklet.
3. Please, please, please write legibly.
4. *To receive full credit you* **must** *show your work and explain clearly what you are doing.*

NAME: SOLUTIONS

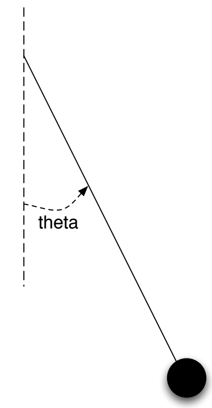
Problem 1 20%

1. Consider the following nonlinear system which is an approximate model for a damped pendulum

$$\ddot{\theta}(t) \; + \; \beta\dot{\theta}(t) \; + \; \sin(\theta(t)) \; = \; 0,$$

where the damping constant is such that $\beta > 0$.
Note that $\theta = 0$ corresponds to the downward position.

By making an appropriate choice of states, rewrite the dynamics in nonlinear state space form $\dot{x} \; = \; f(x)$.



Solution: As usual, we define positions and velocities as states

$$x_1 \; := \; \theta, \qquad x_2 \; := \; \dot{\theta}.$$

This gives the following state space model

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\sin(x_1) \; - \; \beta x_2 \end{aligned}$$

2. Find *all* equilibrium points of the damped pendulum.

Solution: Equilibrium conditions are:

$$
\begin{aligned}
0 &= x_2 \\
0 &= -\sin(x_1) - \beta x_2 = \sin(x_1) \quad \Rightarrow \quad x_1 = 0 + k\pi, \ k \text{ integer}
\end{aligned}
$$

Thus the equilibrium points of this system are

$$
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k\pi \\ 0 \end{bmatrix}, \ k \text{ integer}
$$

3. Classify all the equilibrium points you found as either stable or unstable by linearizing at each equilibrium and examining the stability of the linearized systems. Use your physical intuition as a guide, but you must provide this mathematical proof for stability/instability.

Useful fact: A second order polynomial $a\lambda^2 + b\lambda + c$ has both roots with negative real part if and only if $a > 0$, $b > 0$ and $c > 0$.

Solution: Each equilibrium is indexed by the integer k, and the corresponding linearization at that equilibrium is given by

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -cos(k\pi) & -\beta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -(-1)^k & -\beta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.
$$

The eigenvalues of each linearization are the roots of the following polynomial

$$
\lambda(\lambda + \beta) + (-1)^k = \lambda^2 + \beta\,\lambda + (-1)^k
$$

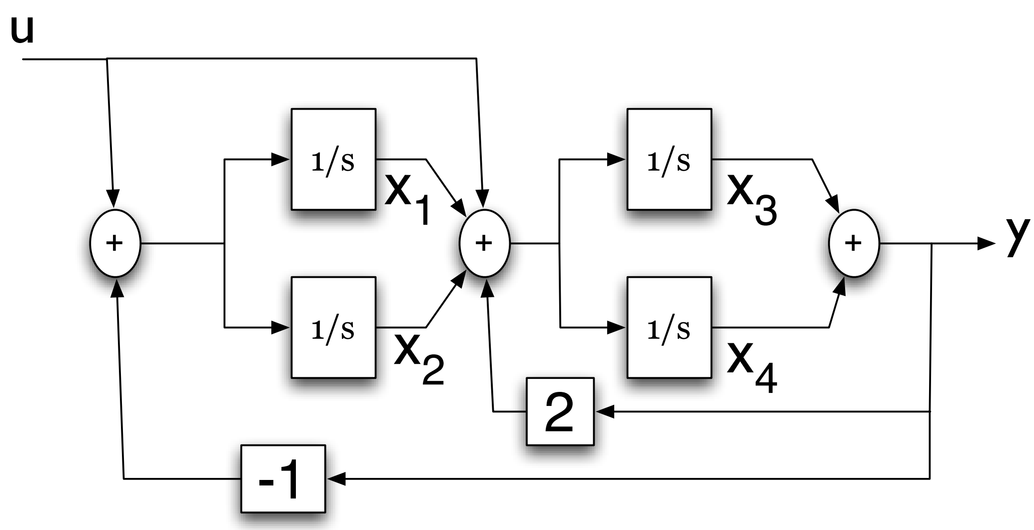
This is a second order polynomial and since $\beta > 0$, it has roots in the LHP if and only if $(-1)^k > 0$, which happens when k is even.
Thus the equilibrium points

$$
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k\pi \\ 0 \end{bmatrix},
$$

are stable when k is even and unstable when k is odd.
Note that k even means that $\theta = 0$, which is the downward position, and k odd means that $\theta = \pi$, which is the upright position.

Problem 2 20%

1. Write down a state space realization of the following system



Solution: With the choice of states shown above, the realization can be derived as

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

2. Find all unobservable states of this system.

Hint: You can do this in either of two ways (a) by using the observability matrix, or (b) by examining the diagram and discovering non-zero initial conditions that produce zero output. Note that each integrator can have a different initial condition.

You should do both (a) and (b) as a cross check.

Solution: (a) The observability matrix is computed to be

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 2 & 2 & 4 & 4 \\ 8 & 8 & 12 & 12 \\ 24 & 24 & 32 & 32 \end{bmatrix}$$

By inspection, we can immediately conclude that this matrix has rank 2 since the first two columns and the last two columns are identical respectively.

The unbobservable states are the states nulled by $\mathcal{O}$. We can find these by inspection. Observe that

$$\mathcal{O} \begin{bmatrix} \alpha \\ -\alpha \\ 0 \\ 0 \end{bmatrix} = 0, \qquad \mathcal{O} \begin{bmatrix} 0 \\ 0 \\ \beta \\ -\beta \end{bmatrix} = 0,$$

and any linear combination of those two vectors will also be nulled by $\mathcal{O}$. We thus conclude that the null space of $\mathcal{O}$ (the "unobservable" subspace) is

$$\text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

Another way to say this is that any initial state such that

$$x_1(0) = -x_2(0), \quad \text{and} \quad x_3(0) = -x_4(0),$$

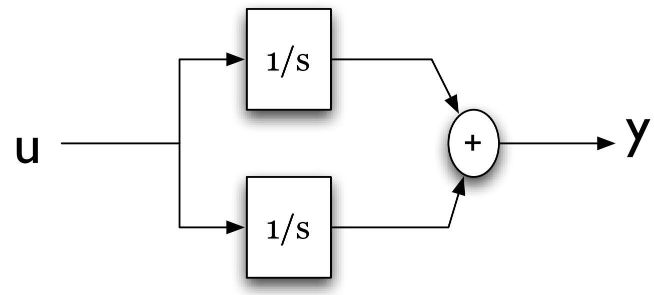
will produce zero output.

(a) We could reach this conclusion by analyzing the block diagram. Imagine a scenario where the initial condition on the x_1 integrator is the negative of the initial condition on the x_2 integrator (and the initial conditions on the other integrators is zero).

The outputs of those two integrators are added, which then produces a zero signal. With zero initial conditions on the remaining integrators and the absence of an input, all other signals in the system are then zero, and the output is zero for all time. A similar scenario shows that initial condtions $x_3(0) = -x_4(0)$, $x_1(0) = x_2(0) = 0$, also produce zero output.

Problem 3 10%

Is the following system controllable?



Give both a mathematical reasoning using the controllability test, as well as an intuitive reasoning using the diagram above (e.g. starting from zero initial conditions, what target states can or cannot be reached regardless of the choice of input u?).

Solution: A state space realization for this system is given by

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \end{aligned}$$

The controllability matrix is

$$\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix},$$

which clearly has rank 1, and thus the system is not controllable.

The reachable subspace is the column span of $\mathcal{C}$, which is just the span $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

Another way to obtain the set of reachable states is to consider the block diagram with zero initial conditions on all integrators. Both integrators have exactly the same input signal, and therefore, their outputs will be equal for all time. In other words, regardless of the input, for zero initial conditions, the state vector will have equal components for all time

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} f(t) \\ f(t) \end{bmatrix} = f(t) \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad t \geq 0$$

where the function f will depend on the input chosen.

Problem 4 20%

Consider the single state unstable system

$$\dot{x}(t) = 2\,x(t) + u(t).$$

We would like to design a state feedback controller to stabilize it, and minimizes the following performance objective function

$$J = \int_0^\infty \left(x^2(t) + \alpha^2\, u^2(t)\right)\, dt,$$

where the parameter $\alpha^2 > 0$ represents the "weight" of the control effort in the objective function.

1. Find the optimal state feedback gain, and the corresponding closed loop system eigenvalue as a function of the parameter α.

Solution: This is a particularly simple LQR problem, where the Ricatti equation can be easily solved since it is a scalar equation.

$$A^*P + PA - PBR^{-1}B^*P + Q = 0,$$

becomes

$$4p - \frac{1}{\alpha^2}p^2 + 1 = 0,$$

in this case.
Using the quadratic formula to find the two possible solutions

$$p = \frac{-4 \pm \sqrt{16 + 4/\alpha^2}}{-2/\alpha^2} = \frac{-4 \pm \sqrt{(4 - 2/\alpha)^2}}{-2/\alpha^2} = \frac{-4 \pm |4 - 2/\alpha|}{-2/\alpha^2}$$

$$= 2\alpha^2 \mp |2\alpha^2 - \alpha| = \begin{cases} 2\alpha^2 - |(2\alpha-1)\alpha| = \begin{cases} 2\alpha^2 - (2\alpha-1)\alpha & = \alpha, & \alpha \geq 1/2 \\ 2\alpha^2 - (-2\alpha+1)\alpha & = 4\alpha^2 - \alpha, & \alpha \leq 1/2 \end{cases} \\ \\ 2\alpha^2 + |(2\alpha-1)\alpha| = \begin{cases} 2\alpha^2 + (2\alpha-1)\alpha & = 4\alpha^2 - \alpha, & \alpha \geq 1/2 \\ 2\alpha^2 + (-2\alpha+1)\alpha & = \alpha, & \alpha \leq 1/2 \end{cases} \end{cases}.$$

To ascertain which of the two choices give the stabilizing solution, we form the closed loop "A" matrix $A_{cl} = A + BK = A + B\left(-R^{-1}B^*P\right)$. For $\alpha \leq 1/2$, the choices are

$$A_{cl} = \begin{cases} 2 - \frac{1}{\alpha^2}(4\alpha^2 - \alpha) & = -2 + \frac{1}{\alpha^2} \\ \\ 2 - \frac{1}{\alpha^2}\alpha & = 2 - \frac{1}{\alpha} \end{cases},$$

and the second choice is the stabilizing one. For $\alpha \geq 1/2$, the choices are reversed. We conclude that

for $\alpha \leq \frac{1}{2}$, $\quad p = \alpha, \quad k = -\frac{1}{\alpha}, \quad A_{cl} = 2 - \frac{1}{\alpha^2}$

for $\alpha \geq \frac{1}{2}$, $\quad p = 4\alpha^2 - \alpha, \quad k = -4 + \frac{1}{\alpha}, \quad A_{cl} = -2 - \frac{1}{\alpha}$

2. (a) The case when $\alpha \to 0$ is called the "cheap control" case. Explain this terminology, and examine the location of the closed loop system eigenvalue in this case.

Solution: When α is very small, the control signal size contributes very little to J, and thus large controls are "tolerated" by the objective in the sense that they do not contribute as much as the regulation term.
As $\alpha \to 0$, we expect the control gain to be large, and indeed since

$$k = -\frac{1}{\alpha},$$

we have a high gain controller in this case. On the other hand, the closed loop eigenvalue $A_{cl} = 2 - \frac{1}{\alpha^2}$, and thus

$$A_{cl} \to -\infty.$$

This is consistent with state regulation being heavily weighted, implying that the state will decay rapidly towards zero.

(b) The case when $\alpha \to \infty$ is called the "expensive control" case. Explain this terminology, and examine the location of the closed loop system eigenvalue in this case. How does it compare with the open loop eigenvalue?

Solution: When α is very large, the control signal is heavily weighted in J, and and we don't expect the controls to be large. Let's investigate. As $\alpha \to \infty$,

$$k = -4 + \frac{1}{\alpha},$$

and $k \to -4$. On the other hand, $A_{cl} = -2 - \frac{1}{\alpha}$, and thus

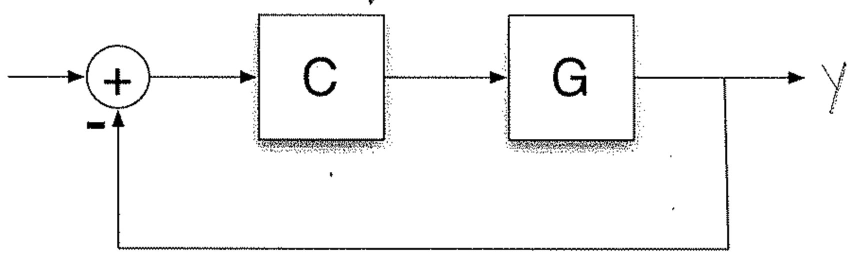
$$A_{cl} \to -2.$$

This shows something interesting when the open loop system is unstable. Even when control is very expensive, an optimal LQR controller must at least stabilize the system, and $k = 0$ will not do. The gain $k = -4$ is the least expensive gain (in the LQR sense) required to stabilize the system.
It's interesting to observe also that the closed loop eigenvalue (at -2) in this case is the "reflection" of the unstable open loop eigenvalue at $+2$. This is actually a special case of a more general theorem: For any unstable system, the expensive control LQR limit produces a closed loop system that has the same stable eigenvalues as the open loop system, but also has the reflection (across the imaginary axis) of the unstable open loop eigenvalues.

The controller can only see y;
It has to include an observer. who can estimate the states.

Problem 5 30%

In the following feedback control system



the transfer function of the plant is given as

$$G(s) = \frac{s+1}{s^2 - 2s - 1}.$$

Design a second order controller C such that the poles of the closed loop system are at $\{-1, -1, -1 \pm i\}$. As a final answer, find the transfer function of the controller $C(s)$.
You must use an observer-based controller design for this problem, and explain every step of your procedure. You will be graded based on the individual steps in this procedure. As a final check, you should calculate the closed loop system formed from the given $G(s)$ and your calculated $C(s)$, and check that the poles of the closed loop system are where they should be.

The Procedure

1. Take $G(s)$ and find a state space realization of G

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

✓

2. Find K such that $A+BK$ is stable ✓
3. Find L such that $A-LC$ is stable ✓
4. Construct an observerbaised controller ✓

1

$$G(s) = \frac{b_1 s + b_0}{s^2 + a_1 s + a_0} = \frac{s+1}{s^2 - 2s - 1}$$

We put it into the controllable canonical form:

$$A = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} b_0 & b_1 \end{bmatrix}$$

Inserting values:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \dot{x} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} x$$

(2) Now we have a system with $2 \overset{+}{\cdot} 2 = 4$ states $\begin{bmatrix} x \\ \hat{x} \end{bmatrix}$ the real state and its estimate, we have 4 poles.
The first two are "connected to" the state feedback gain.

Since our system is in the controllable canonical form we make use of the shortcut. The desired poles are -1 and -1.

$$\Rightarrow (s+1)(s+1) = s^2 - K_2 s - k_1$$

$$s^2 + \underline{2}s + \underline{1} = s^2 - \underline{K_2} s - \underline{k_1}$$

Comparing the coefficients we get $K_2 = -2$ and $K_1 = -1$

$$K = [-1 \quad -2]$$

(2) Find L such that $(A-LC)$ is stable

$$A - LC = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} l_1 & l_1 \\ l_2 & l_2 \end{bmatrix} = \begin{bmatrix} -l_1 & 1-l_1 \\ 1-l_2 & 2-l_2 \end{bmatrix}$$

$$sI - (A-LC) = \begin{bmatrix} s+l_1 & -1+l_1 \\ -1+l_2 & s-2+l_2 \end{bmatrix}$$

$$\det(sI-(A-LC)) = (s+l_1)(s-2+l_2) - (-1+l_2)(-1+l_1) = 0$$

$$s^2 - 2s + l_2 s + l_1 s - 2l_1 + \cancel{l_1 l_2} - (1 - l_1 - l_2 + \cancel{l_1 l_2}) = 0$$

$$s^2 + s(l_1 + l_2 - 2) + (l_2 - l_1 - 1) = 0$$

Desired poles: $(s-1-i)(s-1+i) = s^2 - s + \cancel{si} - s + 1 - \cancel{i} - \cancel{si} + \cancel{i} - \underset{-1}{\underset{\uparrow}{i^2}}$

$$= s^2 - 2s + 2$$

Compare: $l_1 + l_2 - 2 = -2 \iff l_1 = -l_2$

$-l_1 + l_2 - 1 = 2 \implies 2l_2 = 3 \iff l_2 = \frac{3}{2} \quad l_1 = -\frac{3}{2}$

$$L = \begin{bmatrix} -\frac{3}{2} \\ \frac{3}{2} \end{bmatrix}$$

③ The Combined system with The observerbased controller:

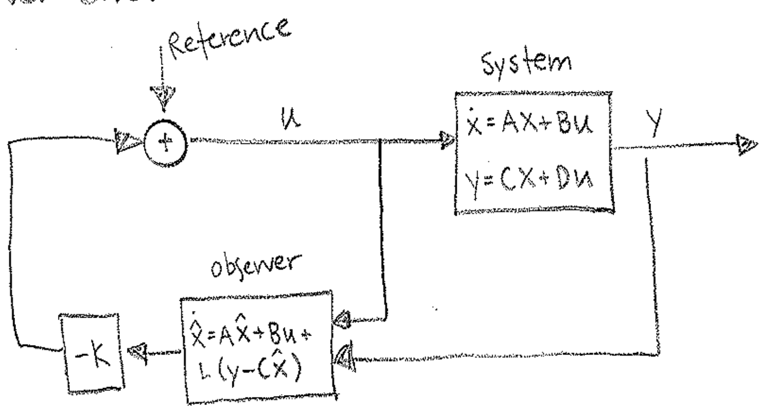
$$\left.\begin{aligned} &\dot{x} = Ax + Bu \\ &\text{where } u = -k\hat{x} \end{aligned}\right\} \dot{x} = Ax - BK\hat{x}$$

$\hat{x}$ is the state estimate: $\hat{x} = x - e$ (e is an error)

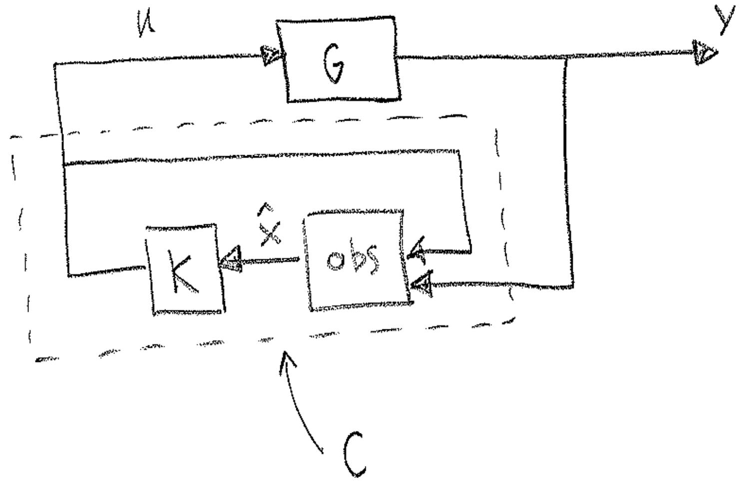
$$\implies \dot{x} = Ax - BKx + BKe$$

$\dot{e} = (A - LC)e$ ← The observer error

$$\implies \begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A-BK & BK \\ 0 & A-LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}$$



Simplified:



$$\dot{\hat{x}} = (A + BK - LC)\hat{x} + Ly$$
$$u = -K\hat{x}$$

We need to find $\dfrac{u}{y}$

$$s\hat{X} = (A + BK - LC)\hat{X} + LY \iff (sI - A - BK + LC)\hat{X} = LY$$

$$\iff \hat{X} = (sI - A - BK + LC)^{-1} LY$$

$$u = \underbrace{-K(sI - A - BK + LC)^{-1}L}_{C(s)}Y$$

$$-\begin{bmatrix}-1 & -2\end{bmatrix}\left(\begin{bmatrix} s & 0 \\ 0 & s\end{bmatrix} + \begin{bmatrix}-1.5 & -2.5 \\ 1.5 & 1.5\end{bmatrix}\right)^{-1}\begin{bmatrix}-1.5 \\ 1.5\end{bmatrix} = -\begin{bmatrix}-1 & -2\end{bmatrix}\left(\underbrace{\begin{bmatrix} s-1.5 & -2.5 \\ 1.5 & s+1.5\end{bmatrix}^{-1}}\right)\begin{bmatrix}-1.5 \\ 1.5\end{bmatrix}$$

$$= \frac{1}{(s-1.5)(s+1.5)+3.75}\underset{1\times 2}{\begin{bmatrix}1 & 2\end{bmatrix}}\begin{bmatrix} s+1.5 & 2.5 \\ -1.5 & s-1.5\end{bmatrix}\begin{bmatrix}-1.5 \\ 1.5\end{bmatrix} = \frac{1}{\underbrace{(s-1.5)(s+1.5)+3.75}_{\substack{s^2+1.5s-1.5s-2.25+3.75 \\ = s^2+1.5}}}\begin{bmatrix}s-1.5 & 2s-0.5\end{bmatrix}\begin{bmatrix}-1.5 \\ 1.5\end{bmatrix}$$

$$C(s) = \frac{1}{s^2+1.5}\left(-1.5s + 2.25 + 3s - 0.75\right) = \frac{1.5s+1.5}{s^2+1.5}$$

Answer: $C(s) = \dfrac{1.5s+1.5}{s^2+1.5}$ ✓

Control:

Check the poles of: $\boxed{\dfrac{Y}{U} = \dfrac{G}{1-GC}}$

SR

$$1 - GC = 1 - \frac{1.5s+1.5}{s^2+1.5} \cdot \frac{s+1}{s^2-2s-1} = 1 - \frac{1.5s^2+1.5s+1.5s+1.5}{s^4-2s^3-s^2+1.5s^2-3s-1.5} =$$

$$= 1 - \frac{1.5s^2+3s+1.5}{s^4-2s^3+0.5s^2-3s-1.5} = \frac{s^4-2s^3+0.5s^2-3s-1.5-1.5s^2-3s-1.5}{s^4-2s^3+0.5s^2-3s-1.5} =$$

$$= \frac{s^4-2s^3-s^2-6s-3}{s^4-2s^3+0.5s^2-3s-1.5}$$

must be a sign error in the ~~feedback~~

$$\frac{G}{1-GC} = \frac{(s^4-2s^3+0.5s^2-3s-1.5)(1+s)}{(s^2-2s-1)(s^4-2s^3-s^2-6s-3)}$$

The poles -1 and -1 are at least where they should be!