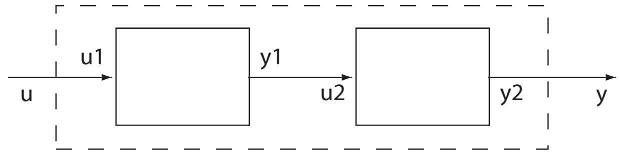
1. (2.2) Two systems in cascade (series). We assume for simplicity that $D_1 = 0$, $D_2 = 0$.



The connections here imply

$$u_1 = u,\ u_2 = y_1,\ \text{and } y = y_2.$$

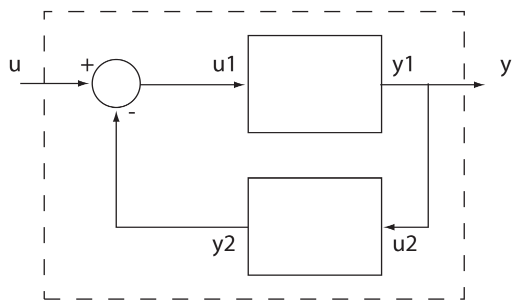
This gives the following equations

$$\begin{aligned} \dot{x}_1 &= A_1x_1 + B_1u_1 = A_1x_1 + B_1u \\ \dot{x}_2 &= A_2x_2 + B_2u_2 = A_2x_2 + B_2y_1 = A_2x_2 + B_2C_1x_1 = B_2C_1x_1 + A_2x_2 \\ y &= y_2 = C_2x_2 \end{aligned}$$

Putting these equations in "block matrix" notation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ B_2C_1 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(2.3) Two systems in feedback. We assume for simplicity that $D_1 = 0$, $D_2 = 0$.



The connections here imply

$$u_1 = u - y_2,\ y = y_1,\ \text{and } u_2 = y_1.$$

This gives the following equations

$$\begin{aligned} \dot{x}_1 &= A_1x_1 + B_1u_1 = A_1x_1 + B_1(u - y_2) \\ \dot{x}_2 &= A_2x_2 + B_2u_2 = A_2x_2 + B_2y_1 \\ y_1 &= C_1x_1 \\ y_2 &= C_2x_2 \\ y &= y_1 \end{aligned}$$

which imply

$$\begin{aligned} \dot{x}_1 &= A_1x_1 + B_1(u - C_2x_2) = A_1x_1 - B_1C_2x_2 + B_1u \\ \dot{x}_2 &= A_2x_2 + B_2C_1x_1 \\ y &= C_1x_1 \end{aligned}$$

[1]Please email Chunkai at ckgao@engr.ucsb.edu if you find any typos in the solutions. Thanks.

Putting these equations in "block matrix" notation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & -B_1C_2 \\ B_2C_1 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} C_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

2. (2.5) Servo with flexible shaft

   i) $\omega_2 = \dot{\theta}_2$
   $J\dot{\omega}_2 = K(\theta_1 - \theta_2)$ (Assuming no load torque)

   ii) $J_m\dot{\omega_m} = T_m - \frac{1}{N}K(\theta_1 - \theta_2)$
   Note the defined directions and the signs corresponding to the spring torques here.

   iii) Electrical subsystem

   $$L\frac{di}{dt} + Ri = v - K_m\omega_m$$

   Note also: $T_m = K_m i$ (Torque generated by motor is proportional to current)
   and $\omega_m = N\dot{\theta}_1 = N\omega_1$

   Now, defining $\Delta := \theta_1 - \theta_2$ and $\Omega := \dot{\Delta}$, and using $\theta_2,\ \Delta,\ \omega_2,\ \Omega,\ i$ as state variables and v as input

$$\begin{aligned}
\dot{\theta}_2 &= \omega_2 \\
\dot{\omega}_2 &= \frac{K}{J}(\theta_1 - \theta_2) = \frac{K}{J}\Delta \\
\dot{\Delta} &= \Omega \\
\dot{\Omega} &= \dot{\omega}_1 - \dot{\omega}_2 = \frac{1}{N}\dot{\omega}_m - \dot{\omega}_2 = \frac{1}{NJ_m}(K_m i - \frac{K}{N}\Delta) - \frac{K}{J}\Delta \\
\frac{d}{dt}i &= -\frac{R}{L}i + \frac{1}{L}v - \frac{K_m}{L}N\omega_1 = -\frac{R}{L}i + \frac{1}{L}v - \frac{K_m N}{L}(\Omega + \omega_2)
\end{aligned}$$

   Putting all the equations in matrix-vector notation:

$$\frac{d}{dt}\begin{bmatrix} \Delta \\ \theta_2 \\ \Omega \\ \omega_2 \\ i \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -K(\frac{1}{J} + \frac{1}{N^2J_m}) & 0 & 0 & 0 & \frac{K_m}{J_m N} \\ \frac{K}{J} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{K_m N}{L} & -\frac{K_m N}{L} & \frac{-R}{L} \end{bmatrix} \begin{bmatrix} \Delta \\ \theta_2 \\ \Omega \\ \omega_2 \\ i \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{L} \end{bmatrix} v$$

   If we choose as outputs $\theta_1,\ \theta_2,\ \omega_1, \omega_2, i$, the output equation is

$$\begin{bmatrix} \theta_1 \\ \theta_2 \\ \omega_1 \\ \omega_2 \\ i \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta \\ \theta_2 \\ \Omega \\ \omega_2 \\ i \end{bmatrix}$$

Below is the MATLAB code used to do this problem.

```
% Define constants
Km = 0.05; R = 1.2; L = 0.05; Jm = 8e-4; J = 0.02;
N = 12; K = 500;

% Define A, B, C matrices
A = [0 0 1 0 0;
    0 0 0 1 0;
    -K*((1/J)+(1/(N^2*Jm))) 0 0 0 (Km/(Jm*N));
    K/J  0 0 0 0;
    0 0 -Km*N/L  -Km*N/L  -R/L];

B = [0; 0; 0; 0; 1/L];

C = eye(5); C(1,2) = 1; C(3,4) = 1;
D = zeros(5,1);

% Construct input using STEPFUN function (see below)
T = 0:0.01:4;
V = 3 * ones(size(T));
V = V - 6 * stepfun(T,2);

% Simulate
SYS = ss(A,B,C,D);  X0 = zeros(5,1);
Y = lsim(SYS,V,T,X0);

plot(T,Y)


% >> help stepfun
%
%  STEPFUN  Unit step function.
%
%     STEPFUN(T,T0), where T is a monotonically increasing vector,
%     returns a vector the same length as T with zeros where T < T0
%     and ones where T >= T0.
```

The result of this simulation is shown in figure 1.

Note how the trajectories of θ_1 and θ_2 are almost identical. The same holds for ω_1 and ω_2. The reason for this is that the torsional spring constant of $K = 500$ is a very large constant (compared with the other parameters in this system). This represents a very inflexible shaft. The natural frequency of oscillation of the shaft is very high, and the corresponding amplitude is very small. This can be seen if we zoom in on a part of the plot for ω_1 and ω_2 as shown in figure 1.

If instead we consider a more flexible shaft (representing one that is longer, or made of a more compliant material), say with $K = 50$, then the oscillations are more pronounced. Such a simulation is shown in figure 2. Note how the oscillations have larger amplitude and a lower frequency.

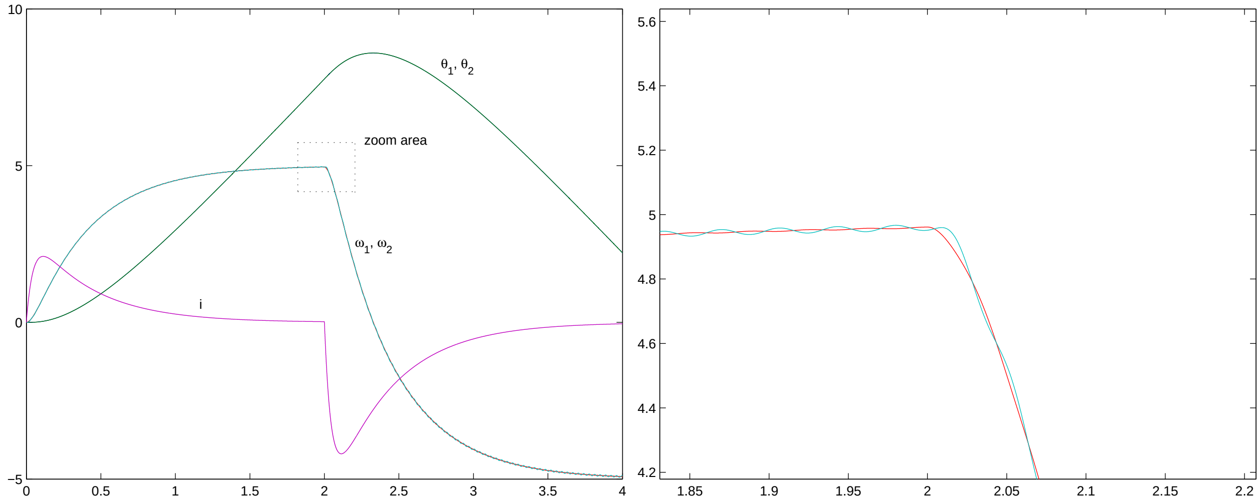


Figure 1: Simulation with $K = 500$

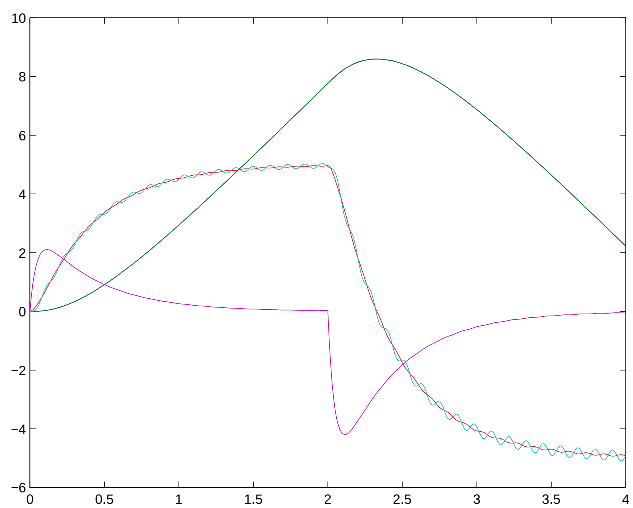


Figure 2: Simulation with $K = 50$

3. (2.19)

(a) **First approach: through away the higher order terms at the very beginning.**

Following example 2.9, we write the approximate Lagrangian for the two pendula as the real Lagrangian minus terms that are higher than second order. This means that we throw out terms of order three or higher from the series expressions for the sin and cos functions. This yields

$$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}+l_1\dot{\theta}_1)^2 + \frac{1}{2}m(\dot{x}+l_2\dot{\theta}_2)^2 - (V_o + mgl_1(1-\frac{1}{2}\theta_1^2)) - (V_o + mgl_2(1-\frac{1}{2}\theta_2^2))$$

Now we evaluate the six partial derivatives required to form the Euler Lagrange equations

$$\begin{array}{lcl} \frac{\partial L}{\partial \dot{x}} &=& M\dot{x} + m(\dot{x}+l_1\dot{\theta}_1) + m(\dot{x}+l_2\dot{\theta}_2) \\ \frac{\partial L}{\partial \dot{\theta}_1} &=& ml_1(\dot{x}+l_1\dot{\theta}_1) \\ \frac{\partial L}{\partial \dot{\theta}_2} &=& ml_2(\dot{x}+l_2\dot{\theta}_2) \end{array} \quad , \quad \begin{array}{lcl} \frac{\partial L}{\partial x} &=& 0 \\ \frac{\partial L}{\partial \theta_1} &=& mgl_1\theta_1 \\ \frac{\partial L}{\partial \theta_2} &=& mgl_2\theta_2 \end{array} .$$

The E-L equations are then

$$\frac{d}{dt}\begin{bmatrix} M\dot{x} + m(\dot{x}+l_1\dot{\theta}_1) + m(\dot{x}+l_2\dot{\theta}_2) \\ ml_1(\dot{x}+l_1\dot{\theta}_1) \\ ml_2(\dot{x}+l_2\dot{\theta}_2) \end{bmatrix} - \begin{bmatrix} 0 \\ mgl_1\theta_1 \\ mgl_2\theta_2 \end{bmatrix} = \begin{bmatrix} F \\ 0 \\ 0 \end{bmatrix},$$

which give

$$\begin{bmatrix} M\ddot{x} + m(\ddot{x}+l_1\ddot{\theta}_1) + m(\ddot{x}+l_2\ddot{\theta}_2) \\ ml_1(\ddot{x}+l_1\ddot{\theta}_1) \\ ml_2(\ddot{x}+l_2\ddot{\theta}_2) \end{bmatrix} - \begin{bmatrix} 0 \\ mgl_1\theta_1 \\ mgl_2\theta_2 \end{bmatrix} = \begin{bmatrix} F \\ 0 \\ 0 \end{bmatrix}.$$

To rewrite these equations in state space form, we need to identify the state variables. As in any mechanical system, the states are the positions and velocities of all masses. In this case, they are x, θ_1, θ_2, $v := \dot{x}$, $\omega_1 := \dot{\theta}_1$ and $\omega_2 := \dot{\theta}_2$. The first three equations in the state space description will be the definitions of velocities, and the last three equations will be expressions for $\ddot{x}$, $\ddot{\theta}_1$ and $\ddot{\theta}_2$ in terms of the state variables. The E-L equations above are not quite in that form yet. In order to convert to that form we need to solve for $\ddot{x}$, $\ddot{\theta}_1$ and $\ddot{\theta}_2$ in terms of state variables (and input). To do this, we rewrite the E-L equations as

$$\begin{bmatrix} M+2m & ml_1 & ml_2 \\ ml_1 & ml_1^2 & 0 \\ ml_2 & 0 & ml_2^2 \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & mgl_1 & 0 \\ 0 & 0 & mgl_2 \end{bmatrix}\begin{bmatrix} F \\ \theta_1 \\ \theta_2 \end{bmatrix}$$

These equations can be solved for $\ddot{x}$, $\ddot{\theta}_1$ and $\ddot{\theta}_2$ either by hand or using some symbolic mathematics package like Mathematica or Maple. MATLAB has a symbolic math package which is basically an interface to Maple. I used it to solve these equations. Below is the MATLAB code I used (it inverts the matrix on the left hand side of the equation above and multiplies the result by the matrix on the right) :

```
syms M m l1 l2 g    % Declare symbolic variables

% Solve for second derivatives of x, theta 1 & 2 in terms of
% state and input variables
H = [(M+2*m) m*l1 m*l2; m*l1 m*l1^2 0; m*l2 0 m*l2^2];
G = inv(H) * [1 0 0; 0 m*g*l1 0; 0 0 m*g*l2];
pretty(G)
```

The last command simply prints out a symbolic expression in a more readable form. The result is

```
[                 m g          m g  ]
[ 1/M           - ---        - ---  ]
[                  M            M   ]
[                                   ]
[   1       (M + m) g        m g    ]
[- ----     ---------        ----   ]
[  l1 M        l1 M          l1 M   ]
[                                   ]
[   1          m g        (M + m) g]
[- ----        ----       ---------]
[  l2 M        l2 M          l2 M   ]
```

Combining all of the above, we write the state space model as

$$\frac{d}{dt}\begin{bmatrix} x \\ \theta_1 \\ \theta_2 \\ v \\ \omega_2 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -\frac{mg}{M} & -\frac{mg}{M} & 0 & 0 & 0 \\ 0 & \frac{(M+m)g}{l_1 M} & \frac{mg}{l_1 M} & 0 & 0 & 0 \\ 0 & \frac{mg}{l_2 M} & \frac{(M+m)g}{l_2 M} & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ \theta_1 \\ \theta_2 \\ v \\ \omega_2 \\ \omega_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{M} \\ -\frac{1}{Ml_1} \\ -\frac{1}{Ml_2} \end{bmatrix} F$$

Using the values $M = m = 1$ gives a slightly simpler form

$$\frac{d}{dt}\begin{bmatrix} x \\ \theta_1 \\ \theta_2 \\ v \\ \omega_2 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -g & -g & 0 & 0 & 0 \\ 0 & \frac{2g}{l_1} & \frac{g}{l_1} & 0 & 0 & 0 \\ 0 & \frac{g}{l_2} & \frac{2g}{l_2} & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ \theta_1 \\ \theta_2 \\ v \\ \omega_2 \\ \omega_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -\frac{1}{l_1} \\ -\frac{1}{l_2} \end{bmatrix} F$$

This state space model is already in linear form (around the upright position for both arms), we do not need further linearization.

(b) **Second approach: derive the nonlinear system first and then linearize it.**
The total kinetic energy is

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m[(\dot{x} + l_1\dot{\theta}_1\cos\theta_1)^2 + (l_1\dot{\theta}_1\sin\theta_1)^2] + \frac{1}{2}m[(\dot{x} + l_2\dot{\theta}_2\cos\theta_2)^2 + (l_2\dot{\theta}_2\sin\theta_2)^2]$$

Suppose V_0 is the potential energy of the two bodies for $\theta_1 = \theta_2 = \frac{\pi}{2}$, then

$$V = V_0 + mgl_1\cos\theta_1 + mgl_2\cos\theta_2.$$

Thus,

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m[(\dot{x} + l_1\dot{\theta}_1\cos\theta_1)^2 + (l_1\dot{\theta}_1\sin\theta_1)^2 + (\dot{x} + l_2\dot{\theta}_2\cos\theta_2)^2 + (l_2\dot{\theta}_2\sin\theta_2)^2] - V_0 - mg(l_1\cos\theta_1 + l_2\cos\theta_2). \end{aligned}$$

The only nonconservative force if F. We are ready to write Lagrange's equations:

$$\begin{aligned} \frac{\partial L}{\partial \dot{x}} &= M\dot{x} + m[2\dot{x} + l_1\dot{\theta}_1\cos\theta_1 + l_2\dot{\theta}_2\cos\theta_2] \\ \frac{\partial L}{\partial x} &= 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) &= M\ddot{x} + m[2\ddot{x} + l_1\ddot{\theta}_1\cos\theta_1 - l_1(\dot{\theta}_1)^2\sin\theta_1 + l_2\ddot{\theta}_2\cos\theta_2 - l_2(\dot{\theta}_2)^2\sin\theta_2] \end{aligned}$$

We have, consequently

$$(M+2m)\ddot{x} + m[l_1\ddot{\theta}_1\cos\theta_1 - l_1(\dot{\theta}_1)^2\sin\theta_1 + l_2\ddot{\theta}_2\cos\theta_2 - l_2(\dot{\theta}_2)^2\sin\theta_2] = F \tag{1}$$

For θ_1, we have

$$\begin{aligned}
\frac{\partial L}{\partial\dot{\theta}_1} &= ml_1\dot{x}\cos\theta_1 + ml_1^2\dot{\theta}_1 \\
\frac{\partial L}{\partial\theta_1} &= -ml_1\dot{\theta}_1\dot{x}\sin\theta_1 + mgl_1\sin\theta_1 \\
\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_1}\right) &= ml_1\ddot{x}\cos\theta_1 - ml_1\dot{\theta}_1\dot{x}\sin\theta_1 + ml_1^2\ddot{\theta}_1
\end{aligned}$$

The equation pertaining to θ_1 is

$$ml_1\ddot{x}\cos\theta_1 - ml_1\dot{\theta}_1\dot{x}\sin\theta_1 + ml_1^2\ddot{\theta}_1 + ml_1\dot{\theta}_1\dot{x}\sin\theta_1 - mgl_1\sin\theta_1 = 0$$

or equivalently

$$\ddot{x}\cos\theta_1 + l_1\ddot{\theta}_1 - g\sin\theta_1 = 0 \tag{2}$$

Similarly, the equation pertaining to θ_2 is

$$\ddot{x}\cos\theta_2 + l_2\ddot{\theta}_2 - g\sin\theta_2 = 0 \tag{3}$$

Define the following relations

$$v = \dot{x} \tag{4}$$

$$\omega_1 = \dot{\theta}_1 \tag{5}$$

$$\omega_2 = \dot{\theta}_2 \tag{6}$$

Putting together Equations (1)-(6), we have the following

$$\begin{bmatrix} 0 & 0 & 0 & M+2m & ml_1\cos\theta_1 & ml_2\cos\theta_2 \\ 0 & 0 & 0 & \cos\theta_1 & l_1 & 0 \\ 0 & 0 & 0 & \cos\theta_2 & 0 & l_2 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{v} \\ \dot{\omega}_1 \\ \dot{\omega}_2 \end{bmatrix} = \begin{bmatrix} F + ml_1\omega_1^2\sin\theta_1 + ml_2\omega_2^2\sin\theta_2 \\ g\sin\theta_1 \\ g\sin\theta_2 \\ v \\ \omega_1 \\ \omega_2 \end{bmatrix} \tag{7}$$

Invert the 6 by 6 matrix to get

$$\begin{bmatrix} \dot{x} \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{v} \\ \dot{\omega}_1 \\ \dot{\omega}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -\frac{1}{W} & \frac{m\cos\theta_1}{W} & \frac{m\cos\theta_2}{W} & 0 & 0 & 0 \\ \frac{\cos\theta_1}{Wl_1} & \frac{-M-2m+m(\cos\theta_2)^2}{Wl_1} & -\frac{m\cos\theta_1\cos\theta_2}{Wl_1} & 0 & 0 & 0 \\ \frac{\cos\theta_2}{Wl_2} & -\frac{m\cos\theta_1\cos\theta_2}{Wl_2} & \frac{-M-2m+m(\cos\theta_1)^2}{Wl_2} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} F + ml_1\omega_1^2\sin\theta_1 + ml_2\omega_2^2\sin\theta_2 \\ g\sin\theta_1 \\ g\sin\theta_2 \\ v \\ \omega_1 \\ \omega_2 \end{bmatrix} \tag{8}$$

where $W = -M - 2m + m(\cos\theta_1)^2 + m(\cos\theta_2)^2$. This is the nonlinear state-space model of the two-pendula.

To linearize system described by Equation (8) around the equilibrium, we substitute in Equation (8) the approximations $\cos\theta_1 = \cos\theta_2 = 1$, $\sin\theta_1 = \theta_1$, $\sin\theta_2 = \theta_2$, $\omega_1^2 = \omega_2^2 = 0$. We get

$$\begin{bmatrix} \dot{x} \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{v} \\ \dot{\omega}_1 \\ \dot{\omega}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{1}{M} & -\frac{m}{M} & -\frac{m}{M} & 0 & 0 & 0 \\ -\frac{1}{Ml_1} & \frac{M+m}{Ml_1} & \frac{m}{Ml_1} & 0 & 0 & 0 \\ -\frac{1}{Ml_2} & \frac{m}{Ml_2} & \frac{M+m}{Ml_2} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} F \\ g\theta_1 \\ g\theta_2 \\ v \\ \omega_1 \\ \omega_2 \end{bmatrix} \tag{9}$$

Write Equation (9) in standard state-space model format

$$\begin{bmatrix} \dot{x} \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{v} \\ \dot{\omega}_1 \\ \dot{\omega}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -\frac{mg}{M} & -\frac{mg}{M} & 0 & 0 & 0 \\ 0 & \frac{(M+m)g}{Ml_1} & \frac{mg}{Ml_1} & 0 & 0 & 0 \\ 0 & \frac{mg}{Ml_2} & \frac{(M+m)g}{Ml_2} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta_1 \\ \theta_2 \\ v \\ \omega_1 \\ \omega_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{M} \\ -\frac{1}{Ml_1} \\ -\frac{1}{Ml_2} \end{bmatrix} F \tag{10}$$

This is the linearized model of the two-pendula.