1. (3.36)

   (a) Show that the system is controllable from the input F if and only if $l_1 \neq l_2$.
   We know from HW 1 that the linearized system is

$$\frac{d}{dt}\begin{bmatrix} x \\ \theta_1 \\ \theta_2 \\ v \\ \omega_2 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -g & -g & 0 & 0 & 0 \\ 0 & \frac{2g}{l_1} & \frac{g}{l_1} & 0 & 0 & 0 \\ 0 & \frac{g}{l_2} & \frac{2g}{l_2} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta_1 \\ \theta_2 \\ v \\ \omega_2 \\ \omega_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -\frac{1}{l_1} \\ -\frac{1}{l_2} \end{bmatrix} F$$

   We then calculate the controllability matrix Ctl using MATLAB as following

```
syms L1 L2 g
A=[0      0       0      1 0 0
   0      0       0      0 1 0
   0      0       0      0 0 1
   0     -g      -g      0 0 0
   0   2*g/L1   g/L1     0 0 0
   0    g/L2    2*g/L2   0 0 0];
B=[0;  0;  0;  1;  -1/L1; -1/L2];
Ctl=[B, A*B, A^2*B, A^3*B, A^4*B, A^5*B];
determ=det(Ctl)
pretty(simple(determ))
```

   Where we get $det(Ctl) = \frac{g^6(l_1-l_2)^2}{l_1^6 l_2^6}$, and the arguments continue as following

$$\text{System controllable} \iff \text{rank}(Ctl) = 6 \iff det(Ctl) \neq 0 \iff l_1 \neq l_2.$$

   (b) Do part (a) numerically as follows: plot the minimum singular value of the controllability matrix for a fixed value of l_1 and as l_2 ranges near l_1. For our simulation we set $l_1 = 1$, and the plot is shown in Figure 1. We see from the plot that the system is uncontrollable if $l_1 = l_2$. Here is the MATLAB code

```
g=9.8; L1=1;L2=1; SingularValues=[];
for L2=L1-0.2:0.01:L1+0.3
   A=[0      0       0      1 0 0
   0      0       0      0 1 0
   0      0       0      0 0 1
   0     -g      -g      0 0 0
   0   2*g/L1   g/L1     0 0 0
   0    g/L2    2*g/L2   0 0 0];
B=[0;  0;  0;  1;  -1/L1; -1/L2];
Ctl=[B, A*B, A^2*B, A^3*B, A^4*B, A^5*B];
[u,s,v]=svd(Ctl);
SingularValues=[SingularValues, min(diag(s))];
end %for
L2=L1-0.2:0.01:L1+0.3;
plot(L2,SingularValues);xlabel('l2');ylabel('minimum singular value');
```

[1] Please email Chunkai at ckgao@engr.ucsb.edu if you find any typos in the solutions. Thanks.

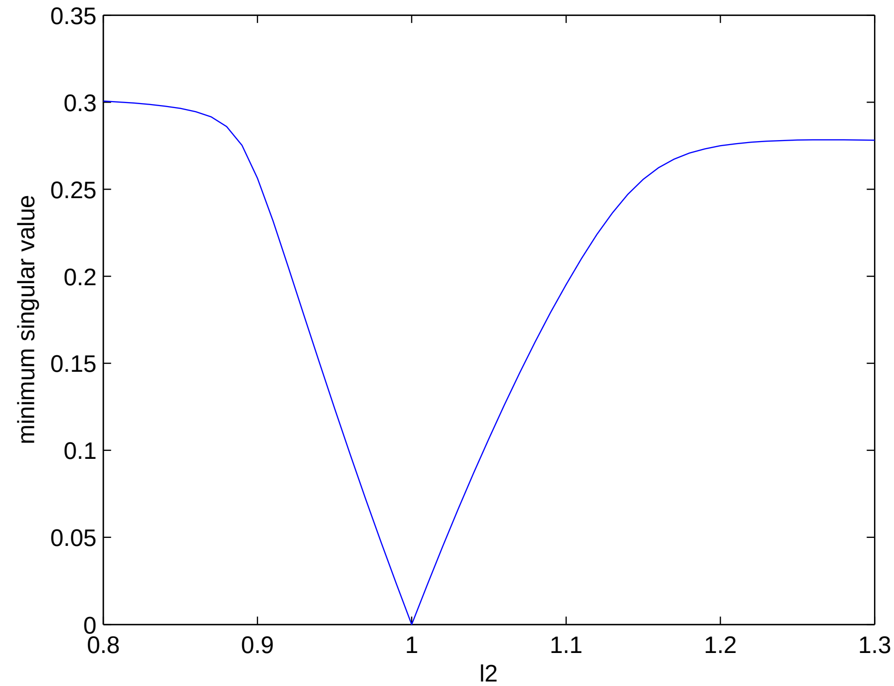

Figure 1: The minimum singular value of the controllability matrix for fixed value of $l_1 = 1$ and l_2 ranges near l_1.

(c) Assess obervability

```
g=9.8; L1=1;L2=0.8;
A=[0      0       0      1  0  0
   0      0       0      0  1  0
   0      0       0      0  0  1
   0     -g      -g      0  0  0
   0   2*g/L1   g/L1     0  0  0
   0    g/L2    2*g/L2   0  0  0];
B=[0;  0;  0;  1;  -1/L1; -1/L2];
C1=[1 0 0 0 0 0]
C2=[0 1 1 0 0 0]
C3=[1 1 1 0 0 0]

Ob1 = obsv(A,C1); r1=rank(Ob1)
Ob2 = obsv(A,C2); r2=rank(Ob2)
Ob3 = obsv(A,C3); r3=rank(Ob3)
```

We get $r_1 = 6$, $r_2 = 4$ and $r_3 = 6$. So (i) observable; (ii) unobservable; (iii) observable.

2. (3.43)

The following is one possible answer:

$$\dot{z} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} z + \begin{bmatrix} 0 \\ 0 \\ -2.2361 \\ -1.1541 \\ 2.2361 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & -1.0831 & -0.2236 & -0.6498 & 0.2236 \end{bmatrix} z$$

It is possible to get different answers by permuting the diagonal elements of the new $\mathbb{A}$ matrix and by switching the positive and negative signs of corresponding eigenvectors.