1. (3.28)

   (a) We first derive the model when L is negligible. The derivation is identical as that of the Ex 2.1 in the text up to

$$\dot{\omega} = \frac{NK_m}{J_e} i - \frac{T_L}{J_e}.$$

Then, we have

$$i = \frac{v - K_m \omega_m}{R} = \frac{v}{R} - \frac{NK_m \omega}{R}.$$

So,

$$\dot{\omega} = -\frac{N^2 K_m^2}{RJ_e}\omega + \frac{NK_m}{RJ_e} v - \frac{T_L}{J_e}.$$

The state equations are

$$\frac{d}{dt}\begin{bmatrix} \theta \\ \omega \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{N^2K_m^2}{RJ_e} \end{bmatrix}\begin{bmatrix} \theta \\ \omega \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{NK_m}{RJ_e} & -\frac{1}{J_e} \end{bmatrix}\begin{bmatrix} v \\ T_L \end{bmatrix}$$

When we have only control input v, then the controllability matrix $\mathbb{C}$ is

$$\mathbb{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & \frac{NK_m}{RJ_e} \\ \frac{NK_m}{RJ_e} & -\frac{N^3K_m^3}{R^2J_e^2} \end{bmatrix}.$$

It is clear that rank($\mathbb{C}$) = 2, so the system is controllable from input v.

   (b) Assess observability

      i. Output is θ, then $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and the observability matrix is

$$\mathbb{O} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

rank($\mathbb{O}$) = 2, so observable.

      ii. Output is ω, then $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$ and the observability matrix is

$$\mathbb{O} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{N^2K_m^2}{RJ_e} \end{bmatrix}$$

rank($\mathbb{O}$) = 1, so unobservable.

In this case vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is unobservable, this vector is the eigenvector of A corresponding to the eigenvalue 0, so mode 0 is unobservable.

2. (3.29) The rank of the controllability matrix is 5, so controllable. With θ_2 as output, the observability matrix has full rank, so observable. With θ_1 as output, the observability matrix has full rank too, so observable as well. Clearly, also observable with both θ_1 and θ_2 as output.

   From Problem 3.14, we know the denominators of both transfer functions are identical with order 5. Since the dimension of A is 5, this means that the state space models are minimal realization of corresponding transfer functions, so it much be both controllable and observable.

[1] Please email Chunkai at `ckgao@engr.ucsb.edu` if you find any typos in the solutions. Thanks.

3. (3.55)

(a) Give realizations in controllable canonical form for $\mathbb{S}_1$

$$\dot{x_1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x_1$$

for $\mathbb{S}_2$

$$\dot{x_2} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} -2 & 2 \end{bmatrix} x_2$$

(b) A realization for the composite system

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 0 & -2 & 2 \end{bmatrix} x$$

(c) Not minimal, since H_1 and H_2 cancels $(s-1)$ term. The observability matrix has rank=3, so unobservable. The mode $s = 1$ is the unobservable mode.

4. (3.59)

(a) The controllability matrix is

$$\mathbb{C} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$$

Since the sum of the columns is zero, $\mathbb{C}$ is singular, so uncontrollable. The vector $x^* = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$ is an uncontrollable state. x^* should be an eigenvector of A^T:

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

So the uncontrollable mode is $s = 1$. Since $s = 1 > 0$ is the mode, the system is not internally stable.

(b) Calculate the transfer function y/u

$$sI - A = \begin{pmatrix} s & -1 & 0 \\ 0 & s & -1 \\ -1 & 0 & s \end{pmatrix}$$
$$(sI - A)^{-1} = \frac{1}{s^3 - 1} \begin{pmatrix} s^2 & * & 1 \\ 1 & * & s \\ s & * & s^2 \end{pmatrix}$$
$$H(s) = C(sI - A)^{-1}B = \frac{1}{s^3 - 1} \begin{pmatrix} 1 & -0.5 & -1 \end{pmatrix} \begin{pmatrix} s^2 - 1 \\ 1 - s \\ s - s^2 \end{pmatrix}$$
$$= \frac{2s^2 - 0.5s - 1.5}{s^3 - 1} = \frac{(s-1)(2s + 1.5)}{(s-1)(s^2 + s + 1)} = \frac{2s + 1.5}{s^2 + s + 1}.$$

All LHP poles, so input-output stable.

(c) With $B = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$,

$$\begin{aligned} H(s) &= C(sI - A)^{-1}B = \frac{1}{s^3 - 1} \begin{pmatrix} 1 & -0.5 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ s \\ s^2 \end{pmatrix} \\ &= \frac{-s^2 - 0.5s + 1}{(s-1)(s^2 + s + 1)}. \end{aligned}$$

Since we have pole $s = 1$ on the RHP, the transfer function y/w is not input-output stable.

5. (7.10) NEXT PAGE

Prob. 7.10

$$\frac{d}{dt}\begin{bmatrix}\theta \\ \omega\end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -2.219\end{bmatrix}\begin{bmatrix}\theta \\ \omega\end{bmatrix} + \begin{bmatrix} 0 \\ 3.65\end{bmatrix} v$$

a)

$$sI - A + \underline{b}\,\underline{k}^T = \begin{bmatrix} s & -1 \\ 0 & s+2.219\end{bmatrix} + \begin{bmatrix} 0 \\ 3.65\end{bmatrix}[k_1 \; k_2]$$

$$= \begin{bmatrix} s & -1 \\ +3.65k_1 & s+2.219+3.65k_2\end{bmatrix}$$

$$\det = s^2 + s(2.219+3.65k_2) + 3.65k_1$$

$$= s^2 + 1.414\,\omega_0 s + \omega_0^2$$

$$k_1 = \frac{\omega_0^2}{3.65}$$

$$k_2 = \frac{1.414\omega_0 - 2.219}{3.65}$$

b) Steady state: $\omega^* = 0$, $v^* = 0$, $\theta^* = \theta_d$.

Control law: $v = -k_1(\theta - \theta_d) - k_2\omega$

Closed-loop transfer function $\dfrac{\theta}{\theta_d} = \dfrac{\omega_0^2}{s^2 + 1.414\omega_0 s + \omega_0^2}$

(No zeros, 1 at dc.)

c)

$$v(s) = -k_1\left(\frac{\omega_0^2}{s^2+1.414\omega_0 s+\omega_0^2} - 1\right)\theta_d - k_2\frac{s\,\omega_0^2\,\theta_d}{s^2+1.414\omega_0 s+\omega_0^2}$$

$$\frac{v}{\theta_d} = \frac{-k_1 s(s+1.414\omega_0) - k_2 s\omega_0^2}{s^2+1.414\omega_0 s+\omega_0^2}$$

$$= \frac{-k_1 s^2 - s(1.414\omega_0 k_1 + k_2\omega_0^2)}{s^2+1.414\omega_0 s+\omega_0^2}$$

Larger with increasing ω_0, since k_1, k_2 increase