1. (7.11)

   (a) Design a state feedback such that the closed-loop poles are located at -4, -5, -21.52, -150+150i, and -150-150i. We can use MATLAB to place the poles easily.

```
% Define constants
Km = 0.05; R = 1.2; L = 0.05; Jm = 8e-4; J = 0.02; N = 12; K = 500;
% Define A, B, C matrices
A = [0 0 1 0 0;
    0 0 0 1 0;
    -K*((1/J)+(1/(N^2*Jm))) 0 0 0 (Km/(Jm*N));
    K/J  0 0 0 0;
    0 0 -Km*N/L  -Km*N/L  -R/L];
B = [0; 0; 0; 0; 1/L];
C = eye(5); C(1,2) = 1; C(3,4) = 1;
D = zeros(5,1);

Pa=[-4 -5 -21.52 -150+150i -150-150i];
Ka = place(A,B,Pa)
```

   We get from MATLAB

```
Ka =

1.0e+004 *

-7.9292    0.0007    0.0240    0.0003    0.0015
```

   (b) repeat part (a) for the closed loop location 0 -2.47 -21.52 -150+150i -150-150i.

```
Pb=[0 -2.47 -21.52 -150+150i -150-150i];
Kb = place(A,B,Pb)
```

   We get from MATLAB

```
Kb =

1.0e+004 *

-8.0740   -0.0000    0.0219    0.0000    0.0015
```

   (c) The equilibrium is $\Delta^* = 0$, $\theta_2^* = \theta_d$, $\Omega^* = 0$, $\omega_2^* = 0$, $i^* = 0$ and $v^* = 0$.
   The control law is $v = -BK_a(x - x^*)$, where $x^* = \begin{pmatrix} 0 & \theta_d & 0 & 0 & 0 \end{pmatrix}^T$.
   The transfer function θ_2/θ_d is

$$H(s) = \frac{k}{(s+4)(s+5)(s+21.52)(s+150+150i)(s+150-150i)}$$
$$= \frac{k}{s^5 + 330.5s^4 + 5.437e004s^3 + 1.438e006s^2 + 9.745e006s + 1.937e007}$$

   where $k = (s^5 + 330.5s^4 + 5.437e004s^3 + 1.438e006s^2 + 9.745e006s + 1.937e007)|_{s=0} = 1.937e007$

   (d) The unit-step response is shown in Figure 1.

[1] Please email Chunkai at ckgao@engr.ucsb.edu if you find any typos in the solutions. Thanks.

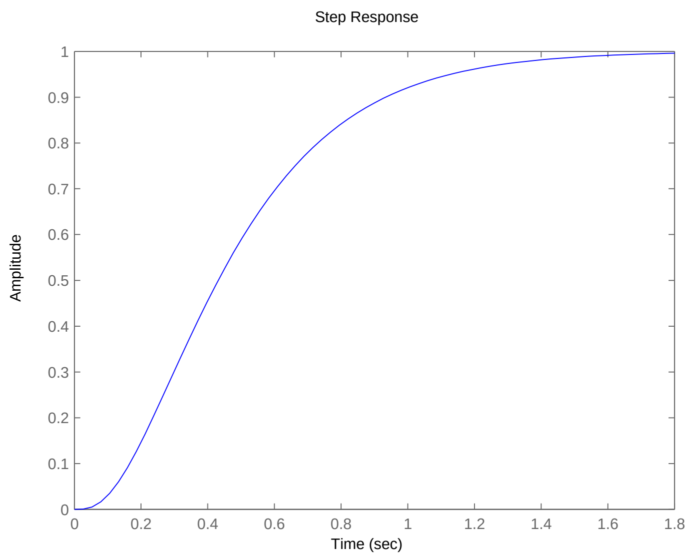

Figure 1: The unit-step response.

Problem 1. For the input $u^* = 1$, a) Find the associated equilibrium state x^* in the system below. b) Linearize the system at this equilibrium.

$\dot{x} = xu - 1$

$y = xu^2$

Sol: Equilibrium states are fixed $\Rightarrow \dot{x}^* = 0$

$x^* u^* - 1 = 0$, $u^* = 1 \Rightarrow x^* = 1$

The linearized equations are:

$$\Delta\dot{\mathbf{x}}(t) = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_* \Delta\mathbf{x}^*(t) + \left.\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right|_* \Delta\mathbf{u}^*(t)$$

$$\Delta\mathbf{y}(t) = \left.\frac{\partial \mathbf{h}}{\partial \mathbf{x}}\right|_* \Delta\mathbf{x}^*(t) + \left.\frac{\partial \mathbf{h}}{\partial \mathbf{u}}\right|_* \Delta\mathbf{u}^*(t)$$

And here: $f(x, u) = xu - 1$ and $h(x, u) = xu^2$.

$$\left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_* = u^* = 1$$

$$\left.\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right|_* = x^* = 1$$

$$\left.\frac{\partial \mathbf{h}}{\partial \mathbf{x}}\right|_* = u^{*2} = 1$$

$$\left.\frac{\partial \mathbf{h}}{\partial \mathbf{u}}\right|_* = 2x^* u^* = 2$$

The result linearized equations:

$\Delta\dot{\mathbf{x}}(t) = \Delta\mathbf{x}^*(t) + \Delta\mathbf{u}^*(t)$

$\Delta\mathbf{y}(t) = \Delta\mathbf{x}^*(t) + 2\Delta\mathbf{u}^*(t)$

Problem 2. Consider the system $\dot{x} = f(x,u)$; $y = h(x,u)$; $x(0) = 0$. For the input $u = 1$ the output is $y(t) = e^{-t}$, and for $u = -1$, $y(t) = -e^{-t}$. Is the system linear?

Sol: The system may or may not be linear.

Additivity is a necessary condition for linearity not a sufficient one. Besides, this property should hold for every two state-input-output solutions. But here it holds only for one pair. For example following system satisfies the condition, but it is not linear.

$\dot{x}(t) = x(t) = 0$

$y(t) = ue^{-t} + u^2 - 1$

Problem 3. In the frequency domain, let $y(s) = u(s)v(s)$. What is $y(t)$?

Sol: According to Convolution Integral:

$y(t) = \mathcal{L}^{-1}(u(s)v(s)) = u(t) * v(t) = \int_0^t u(t-\tau)v(\tau)d\tau$

Problem 4. For the system [A,B,C,D], derive the relationship

$y(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)\,d\tau + Du(t)$

Sol:

$\dot{x} = Ax + Bu$

$y = Cx + Du$

By Laplace transform:

$sx(s) - x(0) = Ax(s) + Bu(s)$

$y(s) = Cx(s) + Du(s)$

$\Rightarrow (sI - A)x(s) = x(0) + Bu(s)$

$\Rightarrow x(s) = (sI - A)^{-1}(x(0) + Bu(s))$

$\Rightarrow y(s) = C(sI - A)^{-1}x(0) + C(sI - A)^{-1}Bu(s) + Du(s)$

By taking the inverse Laplace transform of both sides we have:

$y(t) = \mathcal{L}^{-1}[C(sI - A)^{-1}x(0)] + \mathcal{L}^{-1}[C(sI - A)^{-1}Bu(s)] + Du(t)$

x(0),A,B,C, and D are time invariant and by linearity, the inverse Laplace transform of constant matrices multiplied by a function is inverse Laplace of the function times the constant and for the second term according to Convolution Integral we can write:

$y(t) = C\mathcal{L}^{-1}[(sI - A)^{-1}]x(0) + C\mathcal{L}^{-1}[(sI - A)^{-1}] * (Bu(t)) + Du(t)$
$= Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)\, d\tau + Du(t)$

Problem 5, Sol: Refer to page 69.

Problem 6, Sol:

Eigenvalues of A are $\lambda_{1,2} = 1, 2$ and the corresponding eigenvectors are $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and

$v_2 = \begin{bmatrix} 1 \\ \\ 2 \end{bmatrix}$. For checking the observability as it was mentioned before:

$$\mathcal{O} = \begin{bmatrix} 1 & -1 \\ \\ 2 & -2 \end{bmatrix} \rightarrow \mathcal{O}v_1 = \begin{bmatrix} 0 \\ \\ 0 \end{bmatrix}$$

$\Rightarrow$ the set of λ_1 and v_1, is the unobservable mode.

$$\mathcal{O}v_2 \neq \begin{bmatrix} 0 \\ \\ 0 \end{bmatrix}$$