Instructions:

1. This exam is open book and notes.
2. Write your name on this answer booklet.
3. Please, please, please write legibly.
4. *To receive full credit you* **must** *show your work and explain clearly what you are doing.*

NAME: SOLUTIONS

Problem 1

1. Consider the following nonlinear system

$$\begin{aligned}\dot{x}_1 &= x_2, \\ \dot{x}_2 &= -x_1^2 + u.\end{aligned}$$

Find *all* equilibrium states when the input is $u(t) = 1$, for all $t \geq 0$.

Solution: Equilibrium for $u = 1$ means that the right hand sides of the equations above are all zero when $u = 1$, i.e.

$$\begin{aligned}0 &= x_2, \\ 0 &= -x_1^2 + 1\end{aligned}$$

Which means that $x_2 = 0$, and $x_1^2 = 1$, or $x_1 = \pm 1$.
Therefore the two possible equilibrium states are

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

2. Consider the nonlinear system

$$\begin{aligned}\dot{x}_1 &= \sin(x_2),\\ \dot{x}_2 &= \cos(x_1) \;-\; u.\end{aligned}$$

Verify that the input $u = 1$ makes the state $(x_1, x_2) = (0, 0)$ into an equilibrium.

Find the linearization of this system about that equilibrium condition.

Solution: Indeed

$$\begin{aligned}0 &= \sin(x_2) &\Rightarrow\quad & x_2 = 0 + \pi k, \quad k \text{ integer}\\ 0 &= \cos(x_1) \;-\; 1 &\Rightarrow\quad & x_2 = 0 + 2\pi k, \quad k \text{ integer}\end{aligned}$$

Thus $(x_1, x_2) = (0, 0)$ is one of the equilibrium points for $u = 1$.
The easiest way to find the linearization is to use the taylor series expansions of $\sin$ and $\cos$ around 0

$$\begin{aligned}\sin(\tilde{x}_2) &= 0 + \tilde{x}_2 + 0 \times \tilde{x}_2^2 \;+\; \cdots\\ \cos(\tilde{x}_1) &= 1 + 0 \times \tilde{x}_1 \;-\; \frac{1}{2}\tilde{x}_1^2 \;+\; \cdots\end{aligned}$$

Now substituting $x_1 = 0 + \tilde{x}_1$, $x_2 = 0 + \tilde{x}_2$ and $u = 1 + \tilde{u}$ in the state equations and removing second order and higher terms gives

$$\begin{aligned}\dot{\tilde{x}}_1 &= \tilde{x}_2,\\ \dot{\tilde{x}}_2 &= -\tilde{u}.\end{aligned}$$

Alternatively, we can use the formulae that involve the Jacobians evaluated at the equilibrium

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2}\\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2}\end{bmatrix} = \begin{bmatrix} \frac{\partial \sin(x_2)}{\partial x_1} & \frac{\partial \sin(x_2)}{\partial x_2}\\ \frac{\partial(\cos(x_1)-u)}{\partial x_1} & \frac{\partial(\cos(x_1)-u)}{\partial x_2}\end{bmatrix} = \begin{bmatrix} 0 & \cos(x_2)\\ \sin(x_1) & 0\end{bmatrix} = \begin{bmatrix} 0 & 1\\ 0 & 0\end{bmatrix}.$$

$$B = \begin{bmatrix} \frac{\partial f_1}{\partial u}\\ \frac{\partial f_2}{\partial u}\end{bmatrix} = \begin{bmatrix} \frac{\partial \sin(x_2)}{\partial u}\\ \frac{\partial(\cos(x_1)-u)}{\partial u}\end{bmatrix} = \begin{bmatrix} 0\\ -1\end{bmatrix}.$$

This gives the same result as obtained with the first method.

Problem 2

Consider the system two-input two-ouput system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 0.1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$
$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

1. Is the system stable? Explain.

Solution: NO. The A matrix is diagonal, so its eigenvalues are just the diagonal entries -1 and 0.1. Since one of them is in the right half of the complex plane, the system is unstable.

2. Is the system controllable from the input u_1? Is it controllable from u_2? Explain.

Solution: Considering u_1 as the input, the pair $\left(\begin{bmatrix} -1 & 0 \\ 0 & 0.1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$ has the controllability matrix

$$\mathcal{C}_1 = \begin{bmatrix} 1 & -1 \\ -1 & -0.1 \end{bmatrix},$$

which has rank 2, and thus the system is controllable from u_1.

Now considering u_2 as the input, the pair $\left(\begin{bmatrix} -1 & 0 \\ 0 & 0.1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}\right)$ has the controllability matrix

$$\mathcal{C}_2 = \begin{bmatrix} 0 & 0 \\ 2 & 0.2 \end{bmatrix},$$

which has rank 1, and thus the system is NOT controllable from u_2.

3. Is the system observable from the output y_2? Explain.

Solution: Considering y_1 as the output, the pair $\left(\begin{bmatrix} 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 0.1 \end{bmatrix}\right)$ has the observability matrix

$$\mathcal{O}_1 = \begin{bmatrix} 1 & 1 \\ -1 & 0.1 \end{bmatrix},$$

which has rank 2, and thus the system is observable from y_1.

4. Calculate the transfer function from u_1 to y_1.

Solution: Using the formula $C(sI - A)^{-1}B$ where C and B are the output and input matrices to y_1 and from u_1 respectively

$$\begin{aligned}
H(s) &= \begin{bmatrix} 1 & 1 \end{bmatrix} \left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & 0.1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} s+1 & 0 \\ 0 & s-0.1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{1}{s-0.1} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s-0.1} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\
&= \frac{1}{s+1} - \frac{1}{s-0.1} = \frac{-1.1}{(s+1)(s-0.1)}
\end{aligned}$$

Problem 3

Consider the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

1. By using the series definition of the matrix exponential show that

$$\exp\left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} t \right\} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}.$$

Solution:
$$\exp\left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} t \right\} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} t + \frac{1}{2} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^2 t^2 + \cdots$$

Observer that $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, and consequently all higher powers of that matrix are also zero. Thus the series expression is made up of only the first two terms, and we have the result above.

2. Assume the initial conditions are zero. Using the formula for the solution $x(t)$ in the presence of an input, derive a formula for $x_1(t)$ that involves an integral in in the input u.

Solution: For zero initial conditions

$$x(t) = \int_0^t \exp A(t-\tau) B \; u(t) \; d\tau,$$

which gives

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \int_0^t \begin{bmatrix} 1 & t-\tau \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(\tau) \; d\tau = \int_0^t \begin{bmatrix} t-\tau \\ 1 \end{bmatrix} u(\tau) \; d\tau,$$

specifically

$$x_1(t) = \int_0^t (t-\tau) \; u(\tau) \; d\tau = t \int_0^t u(\tau) \; d\tau - \int_0^t \tau \; u(\tau) \; d\tau$$

3. (FOR A LITTLE EXTRA CREDIT ONLY)

Using the formula derived in the pervious part, and a similar formula for $x_2(t)$, give an algorithm to generate an input such that the state is driven from zero to any specified target state $(\bar{x}_1, \bar{x}_2)$. Your method should generate the input function given the two numbers $\bar{x}_1$ and $\bar{x}_2$. The time at which the target state is achieved is not specified, and can be chosen by your algorithm.

Hint: Try constant inputs, and use the extra freedom in selecting the final time

Solution: From the previous part

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \int_0^t \begin{bmatrix} t-\tau \\ 1 \end{bmatrix} u(\tau)\ d\tau$$

Using constant inputs $u(t) = \bar{u}$, we get

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \int_0^t \begin{bmatrix} t-\tau \\ 1 \end{bmatrix} d\tau\ \bar{u} = \begin{bmatrix} t\int_0^t d\tau - \int_0^t \tau d\tau \\ \int_0^t d\tau \end{bmatrix} \bar{u} = \begin{bmatrix} t^2/2 \\ t \end{bmatrix} \bar{u}.$$

Now to reach the target state, we need to select t and $\bar{u}$ so that the two equations

$$\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} t^2/2 \\ t \end{bmatrix} \bar{u}$$

are satisfied.
One way to do this is as follows: let

$$t = \bar{x}_2/\bar{u},$$

so

$$\bar{x}_1 = \frac{1}{2}\left(\bar{x}_2/\bar{u}\right)^2 \bar{u} = \frac{\bar{x}_2^2}{2\bar{u}}.$$

Thus, if we select

$$\bar{u} = \frac{\bar{x}_2^2}{2\bar{x}_1},$$

and

$$t = \frac{\bar{x}_2}{\frac{\bar{x}_2^2}{2\bar{x}_1}} = 2\frac{\bar{x}_1}{\bar{x}_2},$$

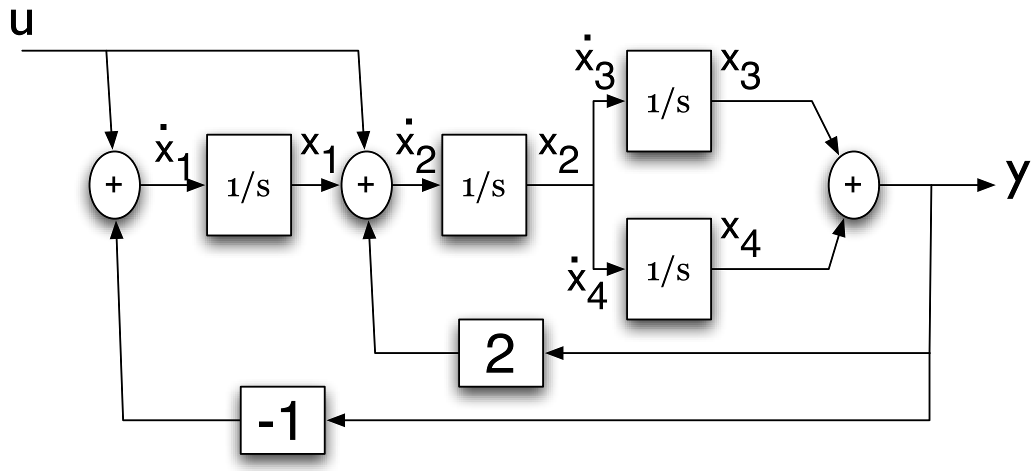
we can reach the specified target state.

Problem 4

Consider the block diagram shown below. It is a realization made up of integrator blocks (labeled **1/s**), adders and gains (the blocks with numbers representing the value of the gain).
Label all the proper choices of the state variables on the diagram, and write down the state space equations in the standard form, i.e. in the form

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx \end{aligned}$$

by finding the matrices A, B and C.



Solution:

x

Identifying the output of each integrator as a state, we label the states as shown in the diagram above.
After noting that $y = x_3 + x_4$, we can write down the state equations by inspection as follows

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

The answer you give may differ from the above by a permutation of the state vector components depending on how you chose to label the states.