Today

(1) Lagrangian Mechanics (crash course)

(2) Linearizations

(3) State space equations for Composite systems.

Modeling w/ Lagrange's eqs.

For any mechanical system

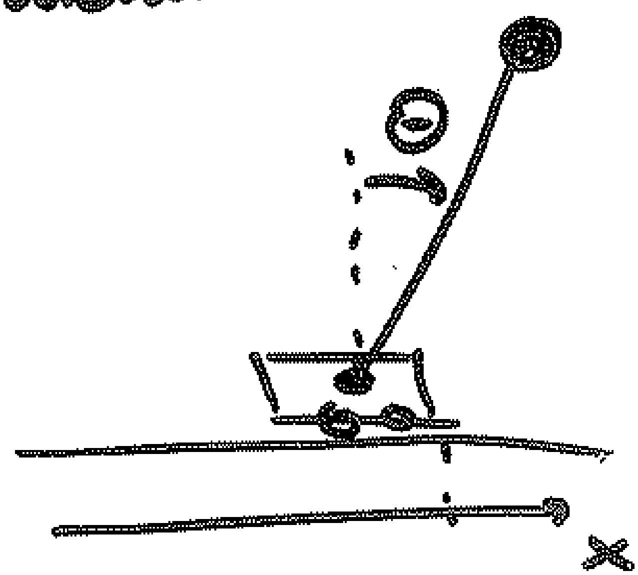
q_i : generalized coordinates

$\dot{q}_i$: generalized velocities

Ex: Inverted pendulum

x : cart position

θ : arm angle



The Lagrangian L is defined by

$$L := T - V$$

③

T = kinetic energy

$T(q,\dot{q})$ function of positions & velocities

V = Potential Energy

$V(q)$ function of position

Equations of Motion:

Euler - Lagrange eqs.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = F_i$$

$i = 1, \dots, n$

F_i : generalized forces

account work done by non-conservative or external forces recal that

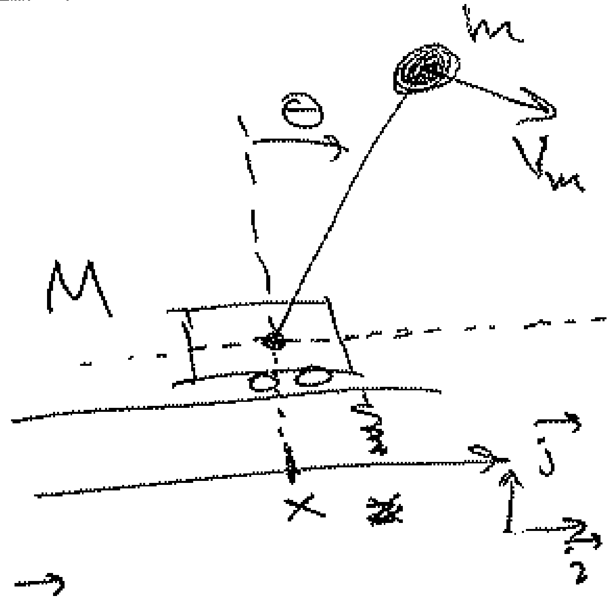
$$L(q_1,\dot q_1,\dots,q_n,\dot q_n) = T(q_1,\dot q_1,\dots,q_n,\dot q_n) - V(q_1,\dots,q_n)$$

<u>Pendulum Example:</u>

$v = \dot x$

$\omega = \dot\theta$

$$T = \tfrac{1}{2} M v^2 + \tfrac{1}{2} m V_m^2$$



$$\vec V_m = \frac{d}{dt}(x + l\sin\theta)\,\vec i + \frac{d}{dt}(l\cos\theta)\,\vec j$$

$$= (\dot x + l\dot\theta\cos\theta)\,\vec i - l\dot\theta\sin\theta\,\vec j$$

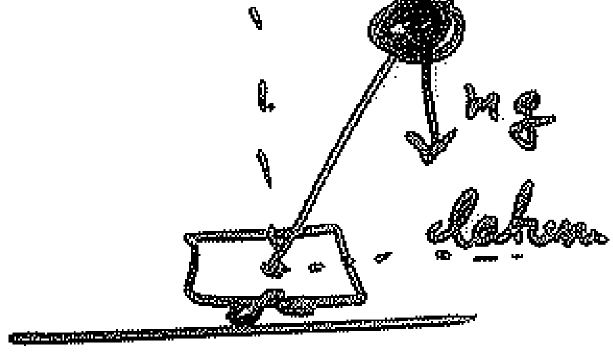
$$V_m^2 = v^2 + l^2\omega^2\cos^2\theta + 2vl\omega\cos\theta + l^2\omega^2\sin^2\theta$$

$$= \underline{v}^2 + l^2\underline{\omega}^2 + 2\underline{v}\,l\,\underline{\omega}\cos\underline{\theta}$$

$$T = \frac{1}{2}(M+m)V^2 + \frac{1}{2}ml^2\omega^2 + mVl\omega\cos\theta$$

Potential Energy

$$V = mgl\cos\theta$$



$$L = T - V$$

$$= \frac{1}{2}(M+m)V^2 + \frac{1}{2}ml^2\omega^2 + mVl\omega\cos\theta - mgl\cos\theta$$

$$\frac{\partial L}{\partial q_i}, \frac{\partial L}{\partial \dot{q}_i} =: \frac{\partial L}{\partial x}, \frac{\partial L}{\partial \theta}, \frac{\partial L}{\partial V}, \frac{\partial L}{\partial \omega}$$

$$\frac{\partial L}{\partial x} = 0$$

$$\frac{\partial L}{\partial \theta} = -mVl\omega\sin\theta + mgl\sin\theta$$

$$\frac{\partial L}{\partial V} = (M+m)V + ml\omega\cos\theta$$

$$\frac{\partial L}{\partial \omega} = ml^2\omega + mvl\cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \omega}\right) = ml^2\dot{\omega} + m\dot{v}l\cos\theta - mvl\omega\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v}\right) = (M+m)\dot{v} + ml\dot{\omega}\cos\theta - ml\omega^2\sin\theta$$

temporarily
assume external forces are zero

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v}\right) - \frac{\partial L}{\partial x} : (M+m)\dot{v} + ml(\dot{\omega}\cos\theta - \omega^2\sin\theta)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \omega}\right) - \frac{\partial L}{\partial \theta} : ml^2\dot{\omega} + m\dot{v}l\cos\theta - mlg$$

in matrix form

$$\begin{bmatrix} M+m & ml\cos\theta \\ \cos\theta & l \end{bmatrix}\begin{bmatrix} \dot{v} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} \omega^2\sin\theta\, ml \\ g \end{bmatrix}$$

invert
to get

$$\dot{v} = f_1(x, \theta, v, \omega)$$
$$\dot{\omega} = f_2(x, \theta, v, \omega)$$

<u>Observe</u>: The state variables are generalised coordinates
& " velocities

state: $\begin{bmatrix} x \\ \theta \\ v \\ \omega \end{bmatrix}$

state equation:

$$\frac{d}{dt}\begin{bmatrix} x \\ \theta \\ v \\ \omega \end{bmatrix} = \begin{bmatrix} v \\ \omega \\ f_1(x,\theta,v,\omega) \\ f_2(x,\theta,v,\omega) \end{bmatrix}$$

Generalized Forces:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial q_i}\right) - \frac{\partial L}{\partial q_i} = F_i$$

dW_i = total work done by all non-conservative forces when q_i changes by dq_i ~~and~~ and all others fixed

$$dW_i = F_i \, dq_i$$

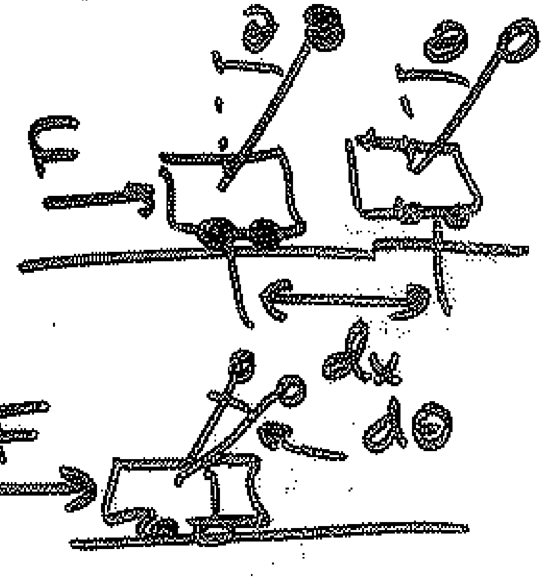
definition of F_i

$dW_x = F \, dx$

$\therefore \; F_x = F$

$dW_\theta = 0 = F_\theta \, d\theta$

$\therefore \; F_\theta = 0$



Linearization

General nonlinear state space system:

$$\dot{x} = f(x)$$

$$\Updownarrow$$

$$\dot{x}_1 = f_1(x_1, \dots, x_n)$$

$$\vdots$$

$$\dot{x}_n = f_n(x_1, \dots, x_n)$$

if $\bar{x}$ is such that $f(\bar{x}) = 0$

$$\Updownarrow$$

$$f_1(\bar{x}_1, \dots, \bar{x}_n) = 0$$

$$\vdots$$

$$f_n(\bar{x}_1, \dots, \bar{x}_n) = 0$$

then $\bar{x}$ is called an equilibrium point if $\bar{x}$ is s.t. $f(\bar{x}) = 0$ 10

then consider

$$\dot{x} = f(x)$$

$$\dot{x}(t) = f(x(t)) \quad ; \ x(0) = \bar{x}$$

$\therefore$ $x(t) = \bar{x}$ is a solution

for $t \geq 0$

We can linearize the dynamics around equilibria