State Space Representations

Linear State space equations

<u>in detail:</u>

State eq. $\longrightarrow$ $$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} & & \\ & B & \\ & & \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_p \end{bmatrix}$$

output eq. $\longrightarrow$ $$\begin{bmatrix} y_1 \\ \vdots \\ y_q \end{bmatrix} = \begin{bmatrix} & C & \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} & D & \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_p \end{bmatrix}$$

State variables: determined by underlying dynamics

outputs & inputs : determined by problem set up

n: states
p : inputs
q : outputs

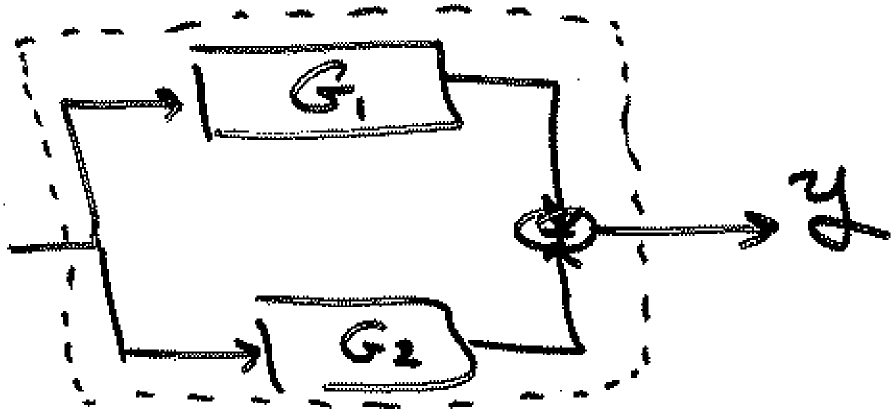
typically we write in matrix-vector form as

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

<u>Note:</u> The <u>state space realization</u> is given by the matrices (A, B, C, D)

<u>State Space Realizations of "composite" systems</u>

<u>Ex:</u>



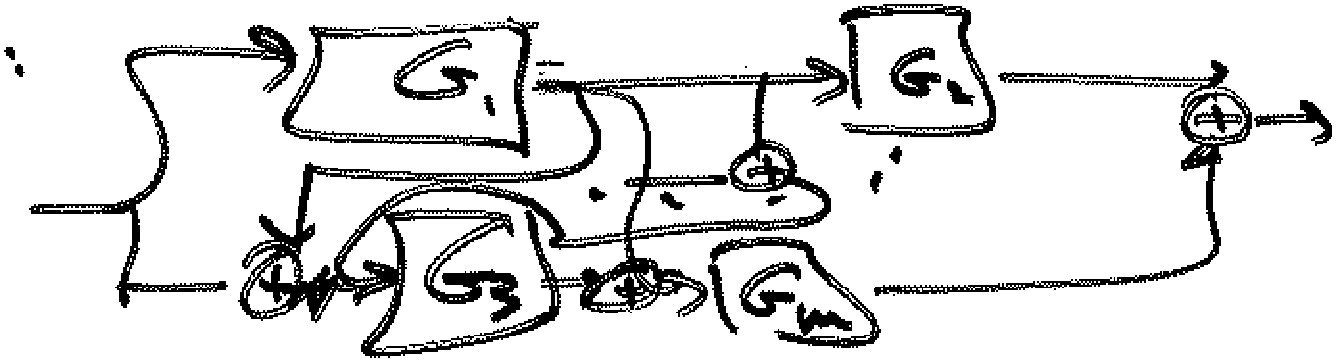
given realization for G_1 & G_2, find realization for G

Universal rule:

Given m systems $G_1, \dots, G_m$ with internal state vectors $x_1, \dots, x_m$

Any interconnection of these systems

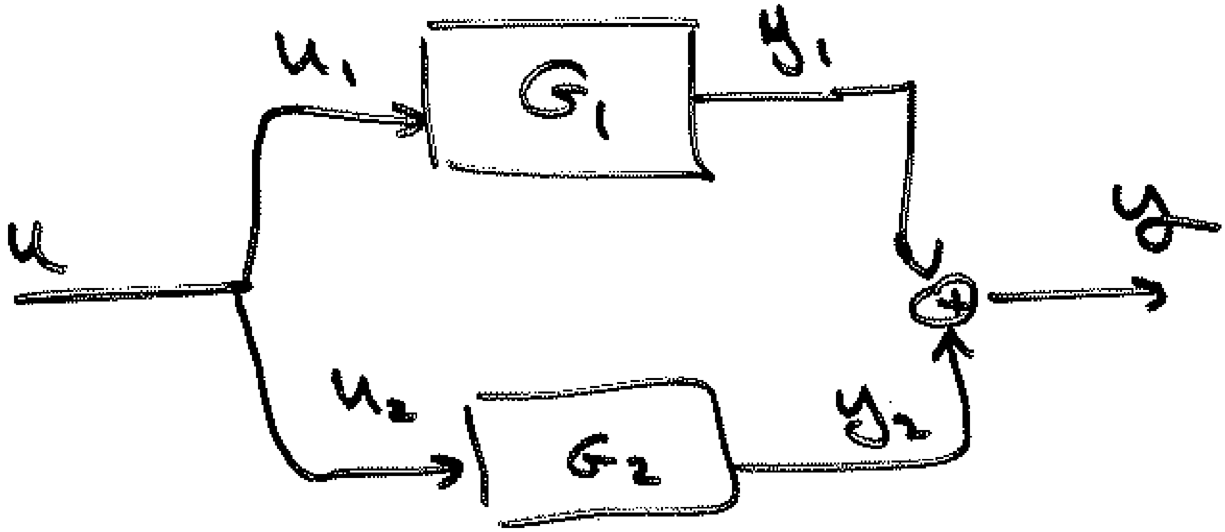
Ex.:



the "state" of the composite system is the vector

$$\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$$

Parallel Connection Example:



Systems Equations:

G_1 :

$$\dot{x}_1 = A_1 x_1 + B_1 u_1$$
$$y_1 = C_1 x_1$$

G_2 :

$$\dot{x}_2 = A_2 x_2 + B_2 u_2$$
$$y_2 = C_2 x_2$$

Interconnections:

$$y = y_1 + y_2$$

$$u_1 = u$$
$$u_2 = u$$

Realization of Composite system has state $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\boxed{\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \\ y &= \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}}$$

$$\dot{x}_1 = A_1 x_1 + B_1 u_1 \qquad u = u_1$$
$$= \underline{A_1 x_1 + B_1 u}$$

$$\dot{x}_2 = A_2 x_2 + B_2 u_2 \qquad u = u_2$$
$$= A_2 x_2 + B_2 u$$

$$y = y_1 + y_2 = C_1 x_1 + C_2 x_2$$
$$= \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Non linear State Space Models

in vector form

state equations $\longrightarrow$ $\dot{x} = f(x, u)$

output equations $\longrightarrow$ $y = h(x, u)$

short hand form:

$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} f_1(x_1, \dots, x_n, u_1, \dots, u_p) \\ \vdots \\ f_n(x_1, \dots, x_n, u_1, \dots, u_p) \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ \vdots \\ y_q \end{bmatrix} = \begin{bmatrix} h_1(x_1, \dots, u_p) \\ \vdots \\ h_q(x_1, \dots, u_p) \end{bmatrix}$$

When $f(x,u)$ is linear in x & u

then $f(x,u) = Ax + Bu$

Similarly when $h(x,u)$ is linear in x & u

Linearization of State Equations:

- Equilibrium

given $\dot{x} = f(x, u)$ --- (*)

Def: the point $(\bar{x}, \bar{u})$ is an equilibrium of (*) if

$$f(\bar{x}, \bar{u}) = 0$$

$\bar{x}, \bar{u}$ are fixed vectors (not functions of time)

Linearization around equilibria

Formaly let $x = \bar{x} + \tilde{x}$

fixed equilibrium — small pert.

$u = \bar{u} + \tilde{u}$

let $x(t) = \bar{x} + \tilde{x}(t)$

$u(t) = \bar{u} + \tilde{u}(t)$

plug into $\dot{x}(t) = f(x(t), u(t))$

then derive a diff equations for

$\tilde{x}(t)$, $\tilde{u}(t)$

use Taylor series form for $f(\cdot,\cdot)$ and retain only linear terms to get linear equations for

$\tilde{x}$, $\tilde{u}$

Aside: Multivariable Taylor Series around $(\bar{x},\bar{u})$

$$f(x,u) = f(\bar{x},\bar{u}) \leftarrow \text{constant term}$$

$$+ \underbrace{\overbrace{\boxed{\left.\frac{\partial f}{\partial x}\right|_{(\bar{x},\bar{u})}}}^{n \times n}}_{A} \tilde{x} + \underbrace{\overbrace{\boxed{\left.\frac{\partial f}{\partial u}\right|_{(\bar{x},\bar{u})}}}^{n \times p}}_{B} \tilde{u}$$

$\leftarrow$ linear term

$$+ \text{H.O.T.}$$

diff eq. $\quad \dot{x} = f(x,u)$

$$\tilde{x} = x - \bar{x}$$

$$\cancel{\dot{\tilde{x}} = f(\bar{x},\bar{u})\,\tilde{x}}$$

$$\dot{\tilde{x}} = \dot{x} - \cancel{\dot{\bar{x}}}^{\,0} = f(x,u)$$

$$= \cancel{f(\bar{x},\bar{u})}^{\,0} + A\tilde{x} + B\tilde{u} + \text{HOT}$$

Dynamics derived from E-L eqs.

$$L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n)$$

E-L: $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = F_i$$

$\updownarrow$ re written

$$\dot{x} = f(x, u) \quad \left[\text{state space form}\right]$$

Short cut: to insure that f is linear, insure that L is at most quadratic in q_i's, $\dot{q}_i$'s

Single Pendulum:

$$L = \ldots v^2 \ldots \quad \sin\theta \quad \ldots \quad \cos\theta$$

$$\sin\theta \to \theta + 0 + O(\theta^3) \qquad \cos\theta \to 1 - \frac{\theta^2}{2} + O(\theta^4)$$

In summary:

given $\dot{x} = f(x, u)$

equilibrium $\bar{x}, \bar{u} \iff f(\bar{x}, \bar{u}) = 0$

then $\tilde{x} = x - \bar{x}$, $\tilde{u} = u - \bar{u}$

has linearized dynamics

$$\dot{\tilde{x}} = A\,\tilde{x} + B\,\tilde{u}$$

$$A = \left.\frac{\partial f}{\partial x}\right|_{(\bar{x}, \bar{u})} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}\right|_{(\bar{x}, \bar{u})}$$

$$B = \left.\frac{\partial f}{\partial u}\right|_{(\bar{x}, \bar{u})} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_p} \\ & & \\ \frac{\partial f_p}{\partial u_p} & \cdots & \frac{\partial f_p}{\partial u_p} \end{bmatrix}$$

Assignment:

- Go to Help section in Matlab

- Go to "Control System Toolbox"

"Getting Started"

~~[illegible]~~ "System Models"

"Simulink"