Matrix Exponential:

$$\exp(M) = \sum_{k=0}^{\infty} M^k / k!$$

$$e^{At} = \begin{bmatrix} M_{11}(t) & \cdots \\ \vdots & \\ M_{n1}(t) & \cdots \end{bmatrix}$$

Special Case:

Diagonal Matrices

$$A = \begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix}$$

$$A^2 = \begin{bmatrix} a_1^2 & & \\ & \ddots & \\ & & a_n^2 \end{bmatrix}$$

$$\vdots$$

$$A^k = \begin{bmatrix} a_1^n & & 0 \\ & \ddots & \\ 0 & & a_n^k \end{bmatrix}$$

$\therefore$ if A is diagonal then 2

$$e^{A} = \sum_{k=0}^{\infty} A^{k}/k! = \cancel{\sum \left[a_1^k \right.}$$

$$= \sum_{k=0}^{\infty} \begin{bmatrix} a_1^k/k! & & 0 \\ & \ddots & \\ 0 & & a_n^k/k! \end{bmatrix}$$

$$= \begin{bmatrix} \sum_k a_1^k/k! & & \\ & \ddots & \\ & & \sum_k a_n^k/k! \end{bmatrix}$$

$$= \begin{bmatrix} e^{a_1} & & \\ & \ddots & \\ & & e^{a_n} \end{bmatrix}$$

$$e^{\begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix} t} = \begin{bmatrix} e^{a_1 t} & & \\ & \ddots & \\ & & e^{a_n t} \end{bmatrix}$$

Suppose we have the simplest of all Realizations where "A" is diagonal

$$\dot{x} = A x$$

$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\Updownarrow$$

$$\dot{x}_1(t) = a_1 \; x_1(t)$$

$$\vdots$$

$$\dot{x}_n(t) = a_n \, x_n(t)$$

scalar equations

diagonal $A \Longleftrightarrow$ equations are decoupled

Solutions: $x_1(t) = e^{a_1 t}\, x_1(0)$

$\vdots$

$x_n(t) = e^{a_n t}\, x_n(0)$

or

$$\begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} = \begin{bmatrix} e^{a_1 t} & & \\ & \ddots & \\ & & e^{a_n t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ \vdots \\ x_n(0) \end{bmatrix}$$

or

$$\boxed{x(t) = e^{At}\, x(0)}$$

true for any A

special case when A is diagonal

One (of many) way to compute

Matrix exponential:

Diagonalization:

Given a matrix A that is diagonalizable, i.e. there exists T

s.t.

$$A = T \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix} T^{-1}$$

Conditions on A s.t. it is diagonalizable.

1. A has n distinct eigenvalues (no repeated eigenvalues)
2. A symmetric
3. A has n lin. indp. eigenvectors

meaning of $A = T \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} T^{-1}$

$$\Updownarrow$$

$$A\,T = T \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}$$

decompose $T = \left[v_1 \middle| \cdots \middle| v_n \right]$

$$\Big[\; A \;\Big] \left[v_1 \middle| \cdots \middle| v_n \right] = \left[v_1 \middle| \cdots \middle| v_n \right] \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix}$$

$$= \left[v_1 a_1 \middle| \cdots \middle| v_n a_n \right]$$

$$\Updownarrow$$

$$\Big[\; A \;\Big] \big[v_i \big] = a_i \big[v_i \big] \qquad i = 1, \ldots, n$$

Note $A = T \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix} T^{-1}$

$$T^{-1} A T = \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix}$$

<u>Back To differential equations</u>

$\dot{x} = Ax$

assume A diagonalizable : $T^{-1} A T = \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

define new state variables

$$\begin{bmatrix} z_1(t) \\ \vdots \\ z_n(t) \end{bmatrix} = \begin{bmatrix} & T & \end{bmatrix} \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

②

what diff. eq. do $z(t)$'s satisfy?

$$\boxed{z(t) = T\, x(t)} \; ; \quad \dot{x} = Ax$$

$$\boxed{T^{-1} z = x}$$

$$\dot{z}(t) = T\, \dot{x}(t)$$

$$= T\, \big(A\, x(t)\big)$$

$$\dot{z}(t) = \left(T A T^{-1}\right) z(t)$$

$$\dot{z}(t) = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} z(t)$$

Solution:

$$z(t) = \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} z(0)$$

$$= \begin{bmatrix} e^{\lambda_1 t} z_1(0) \\ \vdots \\ e^{\lambda_n (t)} z_n(0) \end{bmatrix}$$

back to $x(t)$:

$$x(t) = T^{-1} z(t)$$

$$= \begin{bmatrix} & & \\ & T^{-1} & \\ & & \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} z_1(0) \\ \vdots \\ e^{\lambda_n t} z_n(0) \end{bmatrix}$$

<u>Main Observation:</u>

$$\boxed{\begin{array}{l} \text{the solution for any of the state variables is a linear} \\ \text{combination of terms of the form} \\ \qquad e^{\lambda_1 t}, \ldots, e^{\lambda_n t} \\ \text{where } \lambda_1, \ldots, \lambda_n \text{ are eigenvalues of } A \text{ !} \end{array}}$$

Asymptotic Stability:

a system $\dot{x} = Ax$ is asymptotically stable if for any initial condition, $X(t) \xrightarrow{t \to \infty} 0$

$\left(X(t) \longrightarrow 0 \iff \text{each component } x_i(t) \to 0 \right)$

Since each component looks like

$$x_i(t) = \alpha_1 e^{\lambda_1 t} + \cdots + \alpha_n e^{\lambda_n t}$$

α's depend on i.c.'s

recall $e^{(\sigma + j\omega)t} = e^{\sigma t} \underline{\underline{e^{j\omega t}}}$

decays $\iff \sigma < 0$

Results:

1. $\dot{x} = Ax$ asymptotically stable iff all eigenvalues of A have negative real part

2. $\dot{x} = Ax$ has oscillatory solution if some eigenvalue of A has non-zero imaginary part

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The eigenvalues of A are called the modes of

$\dot{x} = Ax$

λ_i

$e^{\lambda_i t}$