10/16 (1)

* Check the web site regularly

* HW is now due Wednesdays @

Last time:

Solutions of $\dot{x}(t) = A\,x(t)$; $x(0) = x_0$

$$x(t) = e^{At} x_0$$

Now consider systems with inputs:

$$\dot{x}(t) = A\,x(t) + B\,u(t)$$

$$x(0) = x_0$$

$$x(t) = \underbrace{e^{At} x_0}_{x_{zi}} + \underbrace{\int_0^t e^{A(t-\tau)} B\,u(\tau)\,d\tau}_{x_{zs}}$$

Let's prove that it is indeed a solution of $\dot{x}(t)=Ax(t)+Bu(t)$

assume $X(0)=0$

∴ proposed sol $X(t)=\int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$

Compute

$$\frac{dX(t)}{dt} = \frac{d}{dt}\left(\int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau\right)$$

$$=\frac{d}{dt}\left(\int_0^t e^{+At}e^{-A\tau}Bu(\tau)\,d\tau\right)$$

$$=\frac{d}{dt}\left(\underline{e^{At}}\;\underline{\int_0^t e^{-A\tau}Bu(\tau)\,d\tau}\right)$$

$$=Ae^{At}\int_0^t e^{-A\tau}Bu(\tau)\,d\tau + e^{At}e^{-At}Bu(t)$$

$$=A\underbrace{\int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau}_{X(t)} + Bu(t)$$

recap:

$$\dot{x} = Ax + Bu \quad ; \; x(0) = x_0$$

$$x(t) = e^{At} x_0 + \int_0^t e^{A(t-\tau)} B \, u(\tau) \, d\tau$$

$$\begin{bmatrix} x(t) \end{bmatrix} = \begin{bmatrix} e^{At} \end{bmatrix} \begin{bmatrix} x_0 \end{bmatrix} + \int_0^t \begin{bmatrix} e^{A(t-\tau)} \end{bmatrix} \begin{bmatrix} B \end{bmatrix} u(\tau) \, d\tau$$

single input system

~~[illegible]~~

with outputs:

$$\dot{x}(t) = A\,x(t) + B\,u(t)$$

$$y(t) = C\,x(t) + D\,u(t)$$

assume for now ; $x(0) = 0$

solution

$$x(t) = \int_0^t e^{A(t-\tau)} B\,u(\tau)\,d\tau$$

$$y(t) = C \int_0^t e^{A(t-\tau)} B\,u(\tau)\,d\tau$$

$$+ \underbrace{D\,u(t)}_{}$$

direct feedthrough term

From Realizations to Transfer Functions

① $\dot{x}(t) = A\,x(t) + B\,u(t)$; $x(0) = 0$

② $y(t) = C\,x(t) + D\,u(t)$

What is the transfer function $H(s)$ from u to y?

i.e.

$$Y(s) = \boxed{?\,H(s)}\;U(s)$$

for SISO systems, $H(s) = \dfrac{Y(s)}{U(s)}$

take L.T. of ① & ②

$$\dot{x} = Ax + Bu \longrightarrow sX(s) = AX(s) + BU(s)$$

$$y = Cx + Du \longrightarrow Y(s) = CX(s) + DU(s)$$

$$(sI - A)\,X(s) = B\,U(s)$$

$$X(s) = (sI-A)^{-1} B \; U(s)$$

$$Y(s) = \underbrace{\left[C (sI-A)^{-1} B + D \right]}_{H(s)} U(s)$$

$$\therefore \quad H(s) = C (sI-A)^{-1} B + D$$

<u>example:</u>

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

<u>Compute $H(s)$,</u>

$$(sI-A) = s \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} s & -1 \\ 1 & s+2 \end{pmatrix}$$

$$\begin{pmatrix} s & -1 \\ 1 & s+2 \end{pmatrix}^{-1} = \frac{1}{s(s+2)+1}\begin{pmatrix} s+2 & 1 \\ -1 & s \end{pmatrix}$$

$$= \frac{1}{s^2+2s+1}\begin{pmatrix} s+2 & 1 \\ -1 & s \end{pmatrix}$$

$$(sI-A)^{-1} = \begin{pmatrix} \frac{s+2}{s^2+2s+1} & \frac{1}{s^2+2s+1} \\ \frac{-1}{s^2+2s+1} & \frac{s}{s^2+2s+1} \end{pmatrix}$$

$$(sI-A)^{-1} = \frac{\operatorname{adj}(sI-A)}{\det(sI-A)}$$

<u>Aside:</u> $\det(sI-A)$ = characteristic poly. of A.

roots of $\det(sI-A)$ = eigenvalues of A

put it all together

$$H(s) = C\,(sI - A)^{-1}\,B$$

$$= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} s+2 & 1 \\ -1 & s \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

$$\times \frac{1}{s^2+2s+1}$$

$$= \begin{pmatrix} s+2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \frac{1}{s^2+2s+1}$$

$$= \begin{bmatrix} (s+4) & 1 \end{bmatrix} \frac{1}{s^2+2s+1}$$

$$= \begin{bmatrix} \frac{(s+4)}{s^2+2s+1} & \frac{1}{s^2+2s+1} \end{bmatrix}$$

$$Y(s) = \begin{bmatrix} \frac{s+4}{s^2+2s+1} & \frac{1}{s^2+2s+1} \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix}$$

meaning

$$Y(s) = \frac{s+4}{s^2+2s+1} U_1(s) + \frac{1}{s^2+2s+1} U_2(s)$$