Today: <u>Observability & Cont.</u>
<u>& the requisite Linear Algebra</u>

Singular Value Decomposition

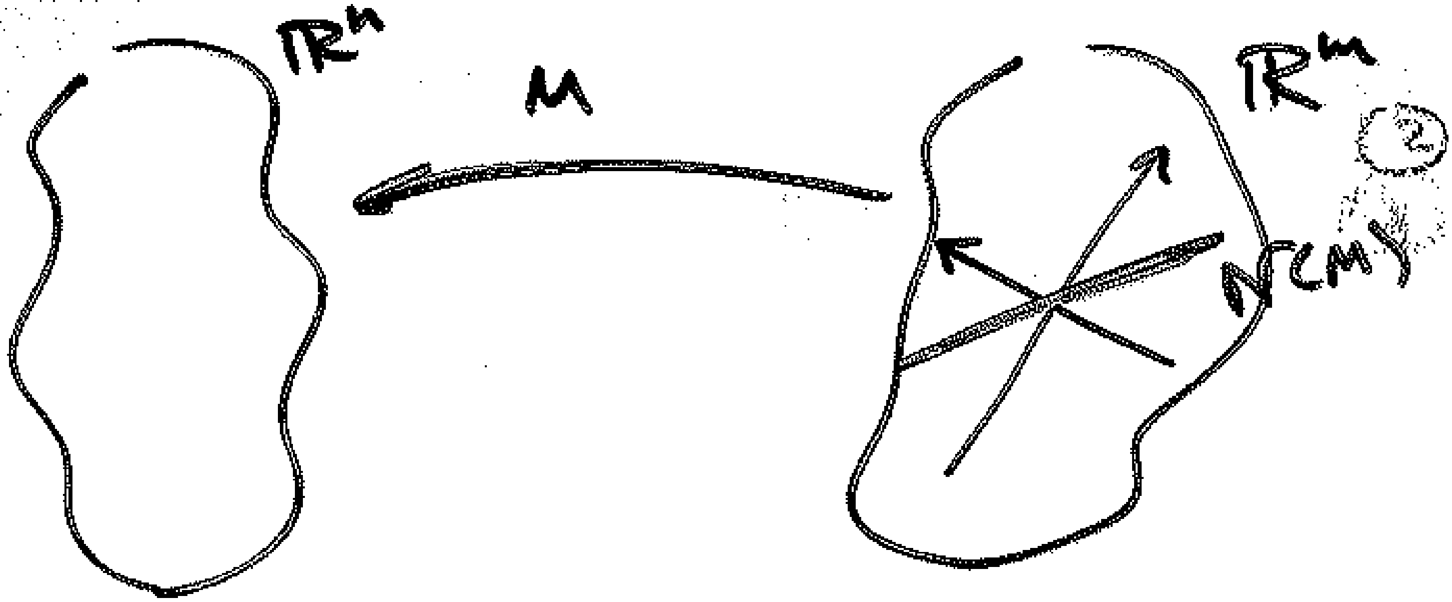
Concepts: (1) Null Space
(2) Range Space (column span)
(3) Rank

Geometric View: An $n \times m$ matrix maps $\mathbb{R}^m$ to $\mathbb{R}^n$

$$\underset{n\text{-vector}}{\begin{bmatrix} w \end{bmatrix}} = \underset{n \times m}{\begin{bmatrix} \quad M \quad \end{bmatrix}} \underset{m\text{-vector}}{\begin{bmatrix} v \end{bmatrix}}$$

$$W \xleftarrow{\quad M \quad} V$$

Linear Mapping



Null space of M

$$\mathcal{N}(M) := \{ v \in \mathbb{R}^m ;\ Mv = 0 \}$$

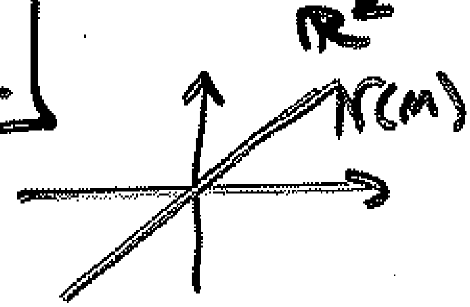
i.e. vectors "nulled" by M

it is a linear ~~sp~~ subspace

$$M(v_1 + v_2) = Mv_1 + Mv_2 \quad ✓$$

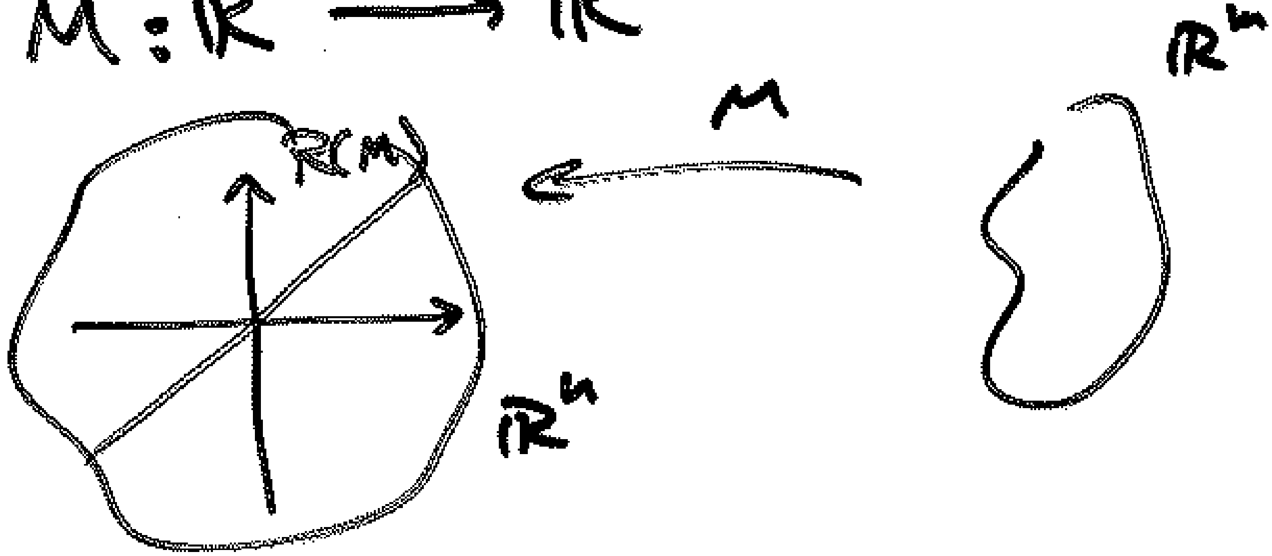
Example $M = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}$

$$M \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



2 Range Space = Column Space

$$M : \mathbb{R}^m \longrightarrow \mathbb{R}^n$$



$$R(M) := \{ w \in \mathbb{R}^n ; \quad w = Mv, \ v \in \mathbb{R}^m \}$$

__Col. span:__

$$\begin{bmatrix} \\ w \\ \\ \end{bmatrix} = \begin{bmatrix} & & \\ & M & \\ & & \end{bmatrix} \begin{bmatrix} \\ v \\ \\ \end{bmatrix}$$

$$= \begin{bmatrix} | & & | \\ m_1 & \cdots & m_m \\ | & & | \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} = \begin{bmatrix} \\ m_1 \\ \\ \end{bmatrix} v_1 + \cdots \begin{bmatrix} \\ m_m \\ \\ \end{bmatrix} v_m$$

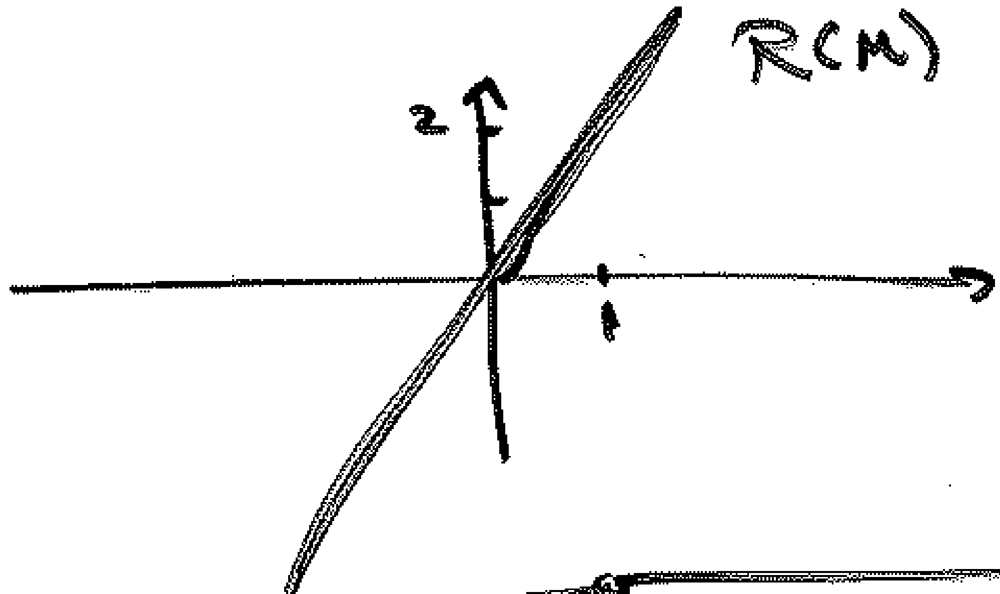
$= \text{col. span}$

$$M = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}$$

Compute $\mathcal{R}(M)$

$$w = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 - v_2 \\ 2(v_1 - v_2) \end{bmatrix}$$

$$= (v_1 - v_2) \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$



Solvability of Linear equations

① for a given b, $\exists$ a sol. $\begin{bmatrix} & M & \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} b \end{bmatrix}$ iff $b \in \mathcal{R}(M)$

② sol. are unique $\iff$ $\mathcal{N}(M) = 0$

Singular Value Decomposition:

Given any matrix M it can be written as

$$\begin{bmatrix} M \end{bmatrix} = \begin{bmatrix} U \end{bmatrix} \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \; 0 \cdots 0 \end{bmatrix} \begin{bmatrix} V^T \end{bmatrix}$$

U, V are square orthonormal matrices
(i.e. cols. & rows are mutually orthogonal & have length $= 1$)

σ's = singular values of M

can be done so that

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$$

special case : Square M

n singular values

$$\sigma_1, \ldots, \sigma_n$$

r of them $\sigma_1, \ldots, \sigma_r, 0, \ldots, 0$ are non-zero

$$M = \begin{bmatrix} & & \\ & U & \\ & & \end{bmatrix} \begin{bmatrix} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_r & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{bmatrix} \begin{bmatrix} & & \\ & V^T & \\ & & \end{bmatrix}$$

$$= \begin{bmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{bmatrix} \begin{bmatrix} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_r & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{bmatrix} \begin{bmatrix} - & v_1^T & - \\ & \vdots & \\ - & v_n^T & - \end{bmatrix}$$

let this act on vectors

$$\begin{bmatrix} v_1^T v \\ \vdots \\ v_n^T v \end{bmatrix} = \begin{bmatrix} - & v_1^T & - \\ & \vdots & \\ - & v_n^T & - \end{bmatrix} \begin{bmatrix} v \end{bmatrix}$$

Conclusion:

$$\mathcal{R}(M) = \text{span}\{u_1, \dots, u_r\}$$

$$\mathcal{N}(M) = \text{span}\{v_{r+1}, \dots, v_n\}$$

$$\dim(\mathcal{R}(M)) = r = \boxed{\text{rank of } M}$$

$$\dim(\mathcal{N}(M)) = n-r = \text{nullity of } M$$

For any square matrix M
with singular values

$$\sigma_{max} \geq \sigma_2 \geq \cdots \geq \sigma_{n-1} \geq \sigma_{min}$$

M is singular $\iff$ $\sigma_{min} = 0$

$\sigma_{min} \neq 0$ measures how close
a matrix is to being
singular

Observability

$$\dot{x} = Ax$$
$$y = Cx$$

Def 1: System is observable if no non-zero i.C. gives output $y(t) = 0$, $0 \le t \le T$

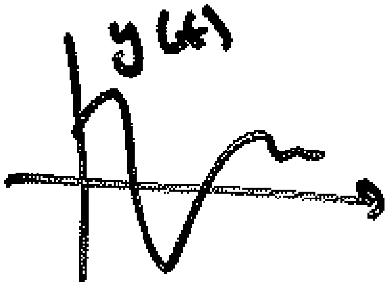
Def 2: System is observable if no two initial conditions $x_1(0) \neq x_2(0)$ give the same output over $0 \le t \le T$

$$y_1(t) = e^{At} x_1(0)$$

$$y_2(t) = e^{At} x_2(0)$$

if $y_1(t) = y_2(t)$

$$0 = y_1(t) - y_2(t) = e^{At} x_1(0) - e^{At} x_2(0)$$
$$= e^{At}(x_1(0) - x_2(0))$$



$$y(t) = e^{At} x(0) \longleftarrow \begin{bmatrix} x(0) \end{bmatrix}$$

__Aside:__ set of unobservable states is precisely the null space of

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$