Realizations

state space description

$$\begin{pmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} & & \\ & A & \\ & & \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} B \end{pmatrix} u$$

$$y = \begin{pmatrix} & C & \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

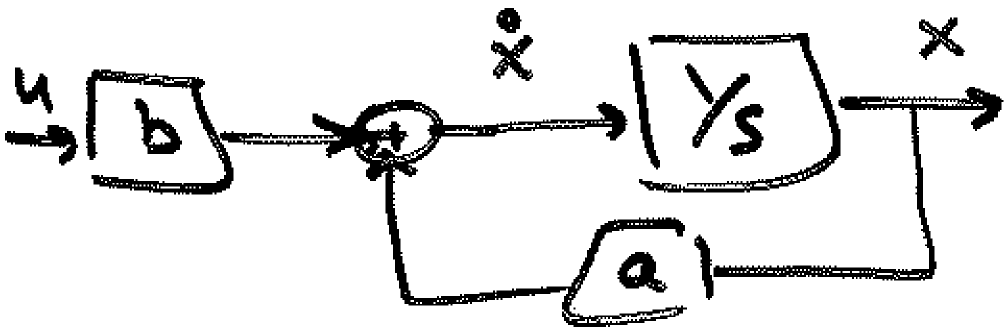
why is it called a "realization"?

we can "realize" or "build" this sytem from basic building block

build = with integrators, adders & gains

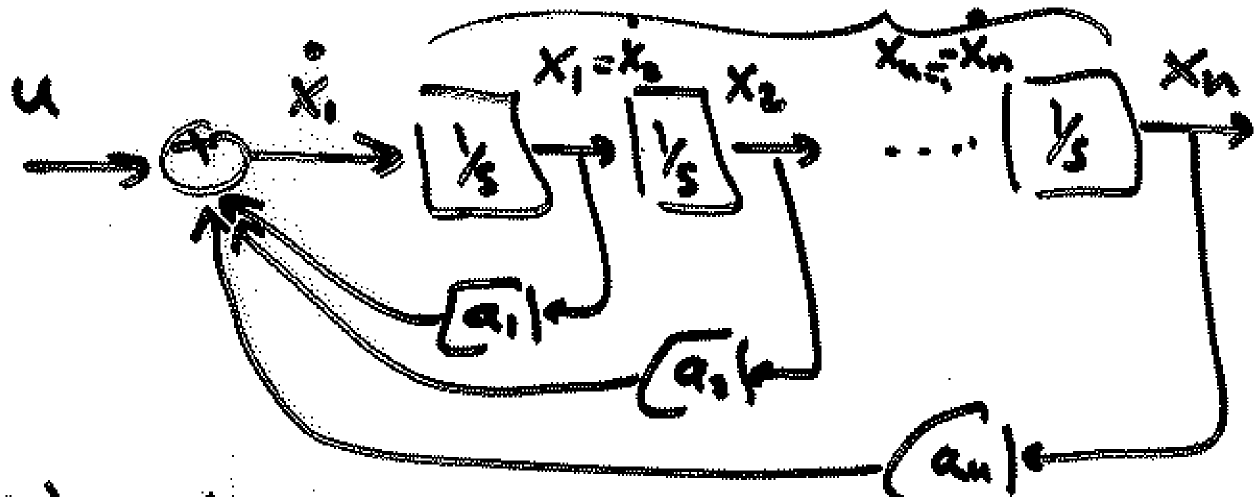
<u>Example:</u> let $\boxed{1/s}$ denote integ. block

1



build ~~[illegible]~~ $\dot{x} = ax + bu$

2 u



$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 \cdots & 0 \\ 0 & \ddots & 0 \; 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} u$$

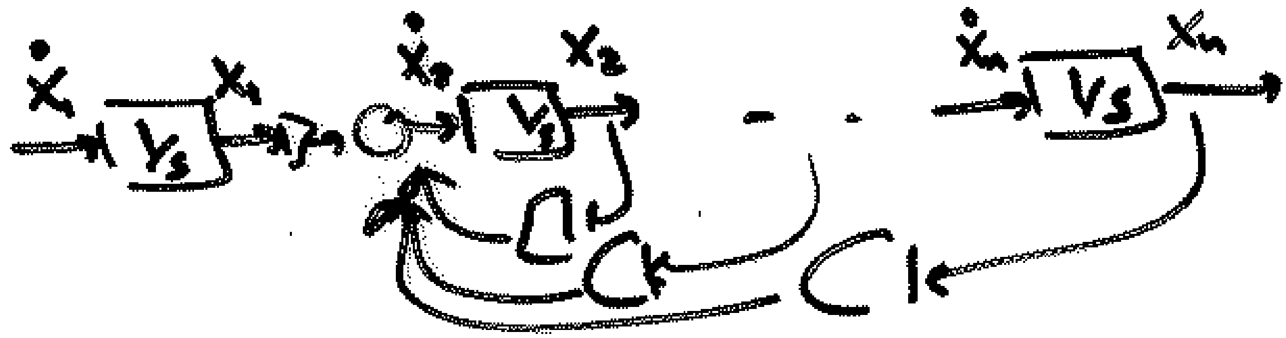
previous process:

Diagram with Integrators, adders, gains $\Longrightarrow$ $\dot{x} = Ax + Bu$
$y = Cx$

building (realizing) it

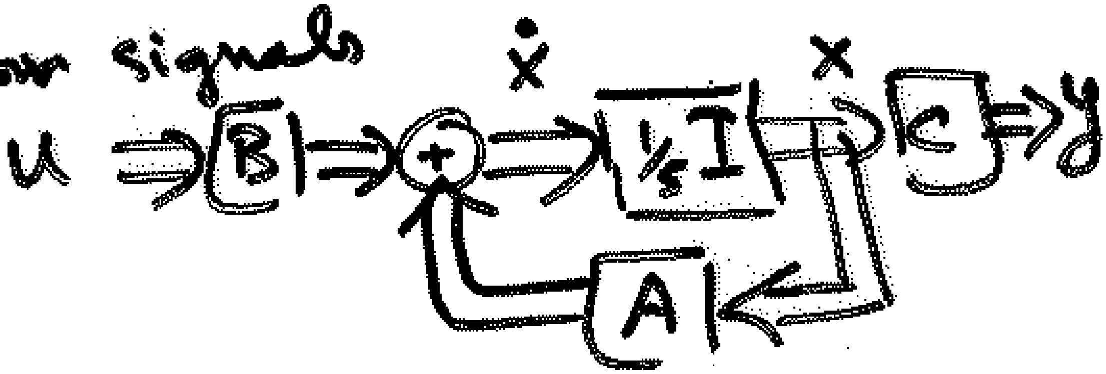
<u>going the other way</u>

start with $\dot{x} = Ax + Bu$



$$\dot{x}_2 = a_{21}x_1 + \cdots + a_{2n}x_n + b_2 u$$

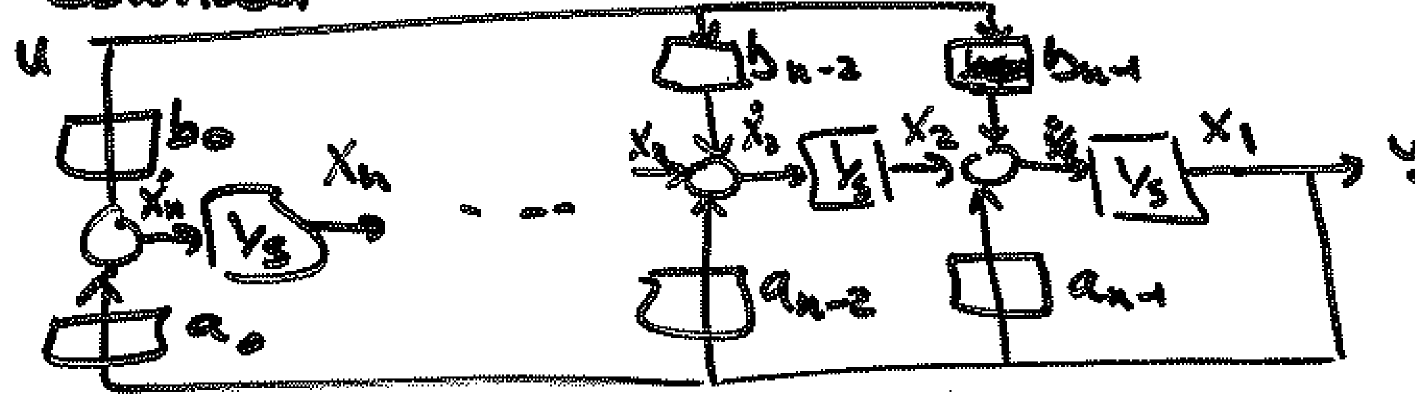
with vector signals



Observable Canonical form realization

let $\dfrac{Y(s)}{U(s)} = H(s) = \dfrac{b_{n-1}s^{n-1} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}$

Consider



$$\begin{pmatrix} \\ \dot{x} \\ \\ \end{pmatrix} = \begin{pmatrix} & & \\ & A & \\ & & \end{pmatrix} \begin{pmatrix} \\ x \\ \\ \end{pmatrix} + \begin{pmatrix} \\ B \\ \\ \end{pmatrix} u$$

$$y = \begin{pmatrix} & C & \end{pmatrix} \begin{pmatrix} \\ x \\ \\ \end{pmatrix}$$

This is indeed "a realization" of the above transfer function

Silly example:

given $H(s) = \frac{Y(s)}{U(s)}$

find a realization

$$\boxed{\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}}$$

Observe that

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} C & 0 \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix}$$

realizes $H(s)$ as well

$$\begin{pmatrix} C & 0 \end{pmatrix} \left(sI - \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \right)^{-1} \begin{pmatrix} B \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} C & 0 \end{pmatrix} \begin{bmatrix} (sI-A)^{-1} & 0 \\ 0 & (sI)^{-1} \end{bmatrix} \begin{bmatrix} B \\ 0 \end{bmatrix} = C(sI-A)^{-1}B$$

Realizations are not unique

Suppose $\dot{x} = Ax + Bu$
$y = Cx$

The T.F. $H(s) = C(sI-A)^{-1}B$

note $z := Tx$ (state transformat

$$\dot{z} = T\dot{x} = T(Ax+Bu)$$
$$\dot{z} = TAT^{-1}z + TBu$$
$$y = Cx = CT^{-1}z$$

one realization | another

$\dot{x} = Ax + Bu$
$y = Cx$

$\dot{z} = \underline{TAT^{-1}}\, z + \underline{\underline{TB}}\, u$
$y = \underline{\underline{CT^{-1}}}\, z$

$\dot{z} = \hat{A} z + \hat{B} u$
$y = \hat{C} z$

check $\hat{H}(s) = \hat{C}(sI-\hat{A})^{-1}\hat{B}$

$= CT^{-1}(sI-TAT^{-1})^{-1}TB$

$= C(sI-A)^{-1}B$ ✓

Controllable Canonical Form

given $H(s) = \dfrac{b_{n-1}s^{n-1} + \cdots + b_0}{s^n + \cdots + a_0}$

the following is a realization

$$\begin{pmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -a_0 & \cdots & & -a_{n-1} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} b_0 & \cdots & b_{n-1} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

matrices of the form

$$\begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & 0 \\ -a_0 & \cdots & & a_{n-1} \end{pmatrix}$$

are called "Companion" matrices

their characteristic polynomial is precisely

$$\det(sI-A) = s^n + a_{n-1}s^{n-1} + \cdots + a_0$$