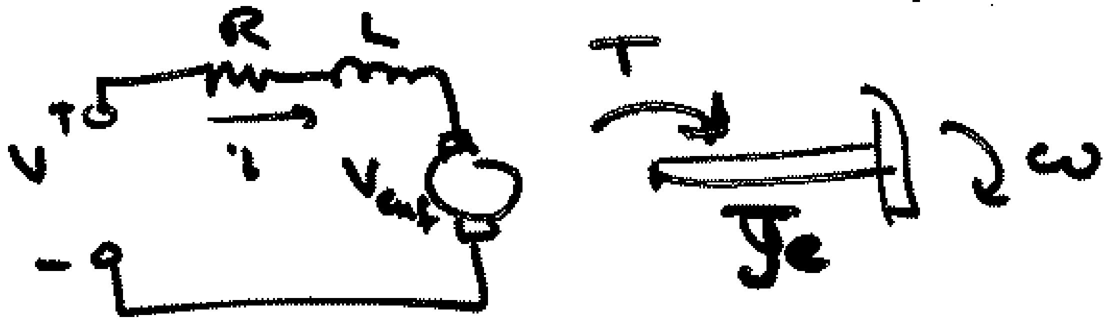
Example 2.1 (prob. 3.28)

$$\frac{d}{dt}\begin{pmatrix}\theta\\ \omega\\ i\end{pmatrix} = \begin{pmatrix}0&1&0\\0&0&\frac{Nk_m}{J_e}\\0&-\frac{Nk_m}{L}&-R/L\end{pmatrix}\begin{pmatrix}\theta\\ \omega\\ i\end{pmatrix} + \begin{pmatrix}0&0\\0&-1/J_e\\1/L&0\end{pmatrix}\begin{pmatrix}V\\ T_L\end{pmatrix}$$



if $L \approx 0$ look at $\frac{di}{dt}$ equation

$$\cancel{L\frac{di}{dt}}^{\to 0} = -Nk_m\,\omega - Ri + V$$

left with an algebraic relation

System Models:

time domain
Diff. eq.

LaPlace domain
"Frequency domain"
Transfer
Functions

$$y^{(n)} + \cdots + a_0 y = b_{n-1} u^{(n-1)} + \cdots + b_0 u$$

$$H(s) = \frac{Y(s)}{U(s)} = \frac{b_n s^{n-1} + \cdots + b_0}{s^n + \cdots + a_0}$$

(I)

(II)

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

(I) $H(s) = C(sI - A)^{-1} B$

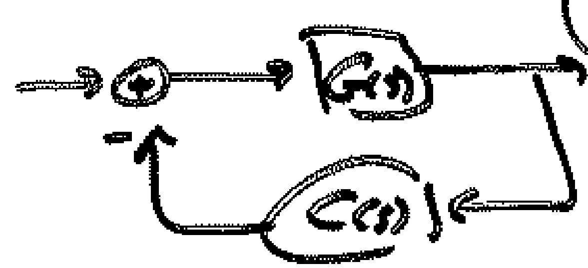
(II) many ways (non unique)

Controllable Canonical form

Observable " "

there are many others

in ISSA :



Designed T.F. of controller C(s)

Given T.F. of Plant G(s)

in ISSB: Design S.S. realization of Controller, Given S.S. realization of Plant

State Feedback Design

Given $\dot{x} = Ax + Bu$

Assume controller can measure $x(t)$

A "state feedback" controller has the form

$$u(t) = K\,x(t)$$

$$= \begin{bmatrix} k_1 & \cdots & k_n \end{bmatrix} \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

$$= k_1 x_1(t) + \cdots + k_n x_n(t)$$

$k_1, \ldots, k_n$ are called "feedback gains"

K is called the feedback gain

Given $\dot{x} = Ax + Bu$

Controller $u = kx$

closed loop system is

$$\dot{x} = Ax + Bkx = (A+Bk)x$$

$$\underline{\underline{\text{c.l.}}} \quad \dot{x} = \underbrace{(A+Bk)}_{\text{closed loop "A" matrix}} x$$

<u>Design question</u>: choose k such that $A+Bk$ has some desired properties

Pole Placement:

Given $\dot{x} = Ax + Bu$, find K s.t. the closed loop system has eigenvalues at some prescribed locations, $\bar{\lambda}_1, \ldots, \bar{\lambda}_n$

This is really a linear algebra prob.
Given (A, B) find K s.t.
$(A+BK)$ has eigenvalues $\bar{\lambda}_1, \ldots, \bar{\lambda}_n$

Answer: This can be done if and only if (A, B) is controllable

i.e. $[B \; AB \ldots A^{n-1}B]$ is full rank

First, to appreciate this

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \begin{bmatrix} k_1 & \cdots & k_n \end{bmatrix}$$

$$A + Bk = \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} + \begin{bmatrix} b_1 k_1 & \cdots & b_1 k_n \\ \vdots & & \\ b_n k_1 & \cdots & b_n k_n \end{bmatrix}$$

very special structure

Special cases:

Example:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

find k s.t. c.l. poles are at -1 & -2.

c.l. system is $\dot{x} = \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [k_1 \; k_2] \right) x$

$$A + BK = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ k_1 & k_2 \end{bmatrix}$$

$$\det(sI - A) = \left| \begin{pmatrix} s & -1 \\ -k_1 & s - k_2 \end{pmatrix} \right| = s(s - k_2) - k_1$$

$$= s^2 - k_2 s - k_1$$

need

$$(S+1)(s+2) = S^2 - k_2 s - k_1$$

$$S^2 + 3S + 2 = S^2 - k_2 s - k_1$$

$$\therefore \boxed{\begin{aligned} k_1 &= -2 \\ k_2 &= -3 \end{aligned}}$$

Given $A = \begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ -a_0 & & & -a_{n-1} \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

$$A+Bk = \begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ -a_0 & & & -a_{n-1} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} [k_1 \cdots k_n]$$

$$= \begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ (k_1-a_0) & - & - & (k_n-a_{n-1}) \end{bmatrix}$$

$\therefore$ need

$$(s-\bar{\lambda}_1)\ldots(s-\bar{\lambda}_n)$$

$$= s^n + (k_n - a_{n-1})s^{n-1} + \ldots + (k_1 - a_0)$$

Can choose k_i's to satisfy this

So if A_c, B_c are in controllable canonical form, can find K_c s.t. eigenvalues of $(A_c + B_c K_c)$ can be placed arbitrarily.

Fact

If (A,B) is controllable,

can find state transformation T

$z = Tx$ $\qquad \dot{z} = T\dot{x} = T(Ax+Bu)$

s.t. $\dot{z} = \underbrace{TAT^{-1}}_{A_c} z + \underbrace{TB}_{B_c} u$

A_c, B_c are in controllable canonical form

Can design $u = K_c z$ to place eigenvalues

Find the feedback $u = Kx$ from that.

Think about it & Do it!