* HW will be due Friday, 1 additional problem

Midterm statistics:

mean : 82
median : 87

The Linear Quadratic Regulator (LQR)

* An "optimally designed" state feedback gain matrix K

For us:
(1) specify a good performance criterion, ~~...~~ in terms of closed loop response
(2) LQR algorithm returns optimal K

Mathematical Setup of LQR problem

System Dynamics $\dot{x} = Ax + Bu$

control input

Problem: Design K, so that

$$u(t) = K\, x(t)$$

Performance Objective

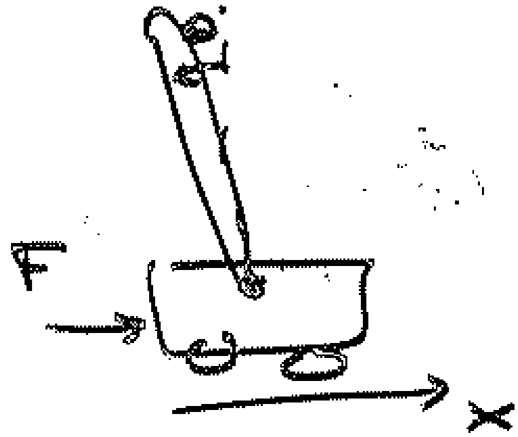
$$J = \int_0^\infty \Big[\underbrace{x^T(t)\, Q\, x(t)}_{\text{quadratic function of } x} + \underbrace{u^T(t)\, R\, u(t)}_{\text{quadratic function of } u}\Big]\, dt$$

LQR: given A, B, Q, R

produce K that gives the smallest possible objective J

Meaning of quadratic objectives:

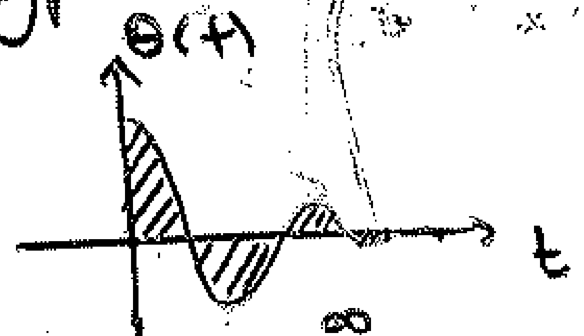
Ex: Inverted Pendulum

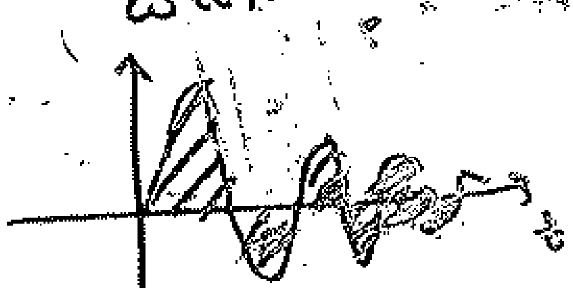


$$\frac{d}{dt}\begin{bmatrix}\theta\\ x\\ \omega\\ v\end{bmatrix} = \begin{bmatrix} & & & \\ & A & & \\ & & & \end{bmatrix}\begin{bmatrix}\theta\\ x\\ \omega\\ v\end{bmatrix} + \begin{bmatrix} \\ B \\ \\ \end{bmatrix} F$$

state feedback $\Rightarrow$ $F(t) = \begin{bmatrix} & K & \end{bmatrix}\begin{bmatrix}\theta\\ x\\ \omega\\ v\end{bmatrix}$

typical closed response





measure of performance $\int_0^\infty \theta^2(t)\,dt$ $\qquad \int_0^\infty \omega^2(t)\,dt$

choose the objective

$$J = q_1 \int_0^\infty \Theta^2(t)\,dt + q_2 \int_0^\infty \omega^2(t)\,dt + q_3 \int_0^\infty x^2(t)\,dt$$

$$+ q_4 \int_0^\infty v^2(t)\,dt + r \int_0^\infty F^2(t)\,dt$$

Aside: q_i reflects the importance of $\int_0^\infty x_i^2(t)\,dt$ in the optimization

q_i is the penalty on x_i in J

$$J = \int_0^\infty \left(\{\Theta \; \omega \; x \; v\} \begin{bmatrix} q_1 & & & 0 \\ & q_2 & & \\ & & q_3 & \\ 0 & & & q_4 \end{bmatrix} \begin{bmatrix} \Theta \\ \omega \\ x \\ v \end{bmatrix} + F r F \right) dt$$

general form $\int_0^\infty (x^T Q x + u^T R u)\, dt$

5

Terminology:

$$J = \int_0^\infty \underbrace{x^T Q x}_{\text{state regulation}} + \underbrace{u^T R u}_{\text{control effort}} \, dt$$

typically: state regulation and control effort are competing objectives

LQR gives a systematic method to trade off regulation vs. control effort

Rule of thumb for picking weights (Q, R are called the LQR weights)

$$J = \int_0^\infty \left(\frac{x}{x_{max}}\right)^2 + \left(\frac{\theta}{\theta_{max}}\right)^2 + \left(\frac{F}{F_{max}}\right)^2 dt$$

these are called Bryson's Rules

Q, R diagonal

$$q_i = \frac{1}{\text{max. allowable deviation in } x_i^2}$$

This is a rough guideline, not a precise mathematical constraint.

<u>Example:</u> 7.6 (DC servo)

states

$$J = \int_0^\infty \left(Q_{11} (\theta - \theta_d)^2 + Q_{22}\omega^2 + v^2 \right) dt$$

$$= \int_0^\infty [\Delta\theta \;\; \omega \;\; i] \begin{bmatrix} Q_{11} & 0 & 0 \\ 0 & Q_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\theta \\ \omega \\ i \end{bmatrix} + v^2$$

Design K using LQR for varying values of Q

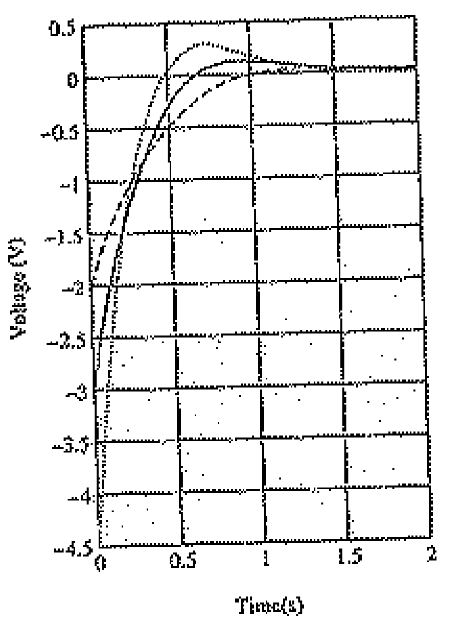


Figure 7.8 Results of LQ design for the dc servo: input voltage and angle error for $Q11 = 4$ (dashed), 9 (solid), and 20 (dotted)

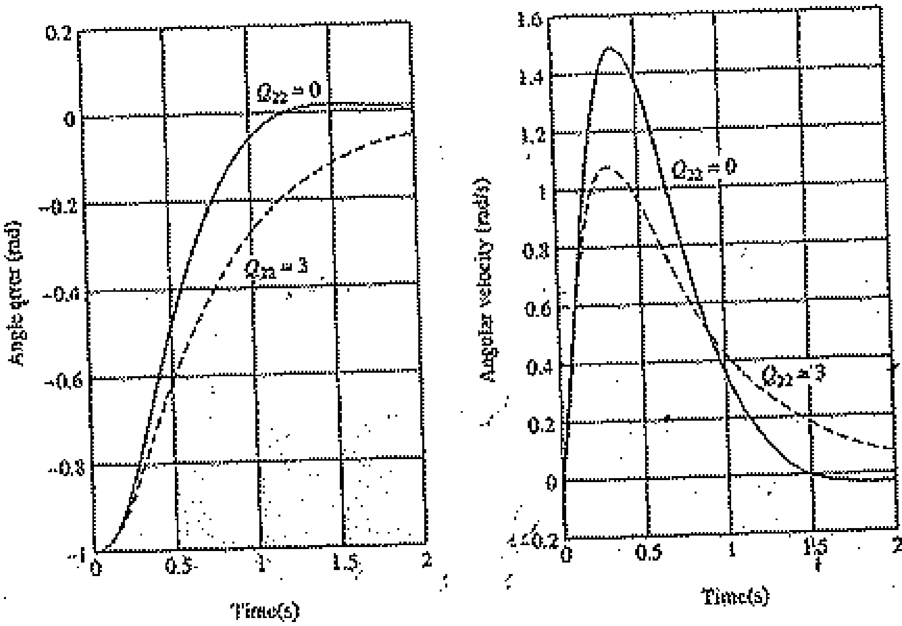


Figure 7.9 Results of LQ design for the dc servo: effect of weighting the angular velocity

Optimal Observers:

- Observation model

$$\dot{x} = Ax + Bu + w$$

$$y = Cx + v$$

process noise

measurement noise

matrices V & W represent the variances of these noise processes

think diagonal V & W

the diagonal elements of V & W are the variances of corresponding components of random vectors v & w

optimal observer design gives a procedure to produce optimal observer gain given

$$A, C, \bar{W}, \bar{V} \longrightarrow \text{optimal } L$$

(book calls L a G)

via solving Algebraic Ricatti Equation

(in matlab "care")