

Control-oriented Modeling of the Dynamics of Stirling Engine Regenerators

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Abstract—We develop a first-principles model of the regenerator component of a generic Stirling engine. The model is based on the Euler equations of one dimensional gas dynamics coupled with its convective/conductive heat transfer with the embedded mesh material. We investigate various methods for deriving simpler and low order control-oriented models from this first principles model. The basic criterion being high fidelity representation of the dynamics of the regenerator when coupled to other dynamic components of the engine. We identify several non-dimensional parameters that potentially categorize different modes of operation, and investigate the corresponding time-scale separation. A hierarchy of singularly perturbed models are derived in which acoustic dynamics are eliminated, periodic mesh dynamics are averaged, and the shape of the distributed regenerator gas state is approximated respectively. In addition, since the reduced model is to be operated cyclically when connected to other parts of the engine, we develop such a feedback-aware model reduction algorithm based on a Proper Orthogonal Decomposition (POD) with a chirped signal input (chirp-POD). This algorithm yields reduced models that are accurate over a range of engine operating frequencies.

I. INTRODUCTION

Stirling devices are energy conversion devices which can be operated as heat engines or as heat pumps. In engine mode, they can operate using any heat source such as external combustion, waste heat, or solar thermal power. Although the basic Stirling engine design has been around for over two centuries, newer varieties continue to be periodically invented, with the Free Piston Stirling Engine [1] being the most widely-known modern example. Due to their ability to use any heat source, there has been a resurgent interest in using Stirling engines for utilization of Solar Thermal power [2]–[4], as well as in micro Combined Heat and Power (μ -CHP) applications [5]–[8] (with at least one recent commercial offering by Whispergen[®], which produces household-scale, Stirling engine based μ -CHP units). On the research side, there has also been a recent surge of interest in the dynamics and control of Stirling engines [5], [9]–[15]. Given the important role that electronic controls have had in improving the performance of the Internal Combustion Engine in the past few decades [16], it is arguable that control engineering can have a significant

impact on performance of analogously modern versions of the Stirling engine.

Of particular interest to the authors is the new concept of the actively controlled Stirling engine [9], [11] in which actuator control of the displacer piston allows for significant freedom in the design of the engine’s limit cycle. Potential improvements in both power density and efficiency can be substantial [9], although more reliable control designs will likely require higher fidelity models of the engine’s dynamics than the commonly used idealized Schmidt model. To this end we devote the present paper to control-oriented modeling of arguably the most important component of a Stirling engine, namely the regenerator. As a historical aside, Stirling’s original patent was not for the Air Engine (which was known at the time), but rather for the regenerator [17] (which he termed, the “economizer”).

The regenerator acts as a type of thermal capacitor that stores and retrieves thermal energy to the working gas during different phases of the engine’s cycle. A detailed description of its operation is provided in the next section. It is made up of a porous matrix of high thermal capacitance material through which the working gas flows. The working gas and the matrix interact via convective heat transfer. Thus a model of these dynamics involves the equations of compressible flow together with convective heat transfer through porous media. The partial differential equations modeling these processes are computationally intensive to simulate and difficult to use for control design. We therefore develop a hierarchy of reduced models in this paper which ultimately yield low order models with high fidelity for the typical frequencies of cyclic operation.

There is a significant challenge that arises in the present model reduction problem which is also relevant in a wider context. The regenerator operates in feedback with the dynamics of the remaining parts of the engine, which are assumed to be lumped. Thus the reduced model needs to approximate not only the regenerator’s open-loop behavior, but more importantly, the behavior of the overall coupled system. This is an instance of closed-loop (or “feedback-aware”) model reduction, which arises in the problem of controller reduction [18]–[21], but also in another contexts such as large-scale model reduction with the requirement to preserve certain properties such

as passivity [22]–[24]. In this paper, we argue that since the regenerator will ultimately operate periodically when coupled to the remainder of the engine, a chirp-POD technique (a Proper Orthogonal Decomposition (POD) technique with a chirp signal as the input) over the frequency range of potential operation provides an effective method for this nonlinear model reduction problem.

The paper is organized as follows. In section II we first introduce and summarize the basic operation of the regenerator in a Stirling engine. We then develop a one dimensional PDE model using the Euler equations of gas dynamics coupled with gas/mesh heat transfer. There are 4 fields in this coupled system of PDEs, 3 for the gas state (our choice is density, velocity and pressure) and an additional field for the mesh matrix temperature. In Section III these equations are non-dimensionalized to uncover four non-dimensional parameters that indicate potential time-scale separations. While one of those parameters is the well-known Mach number, the other three are lesser known and are specific to convective heat transfer between an oscillating gas and a mesh material. These parameters characterize various types of possible engines, and we pay special attention to this characterization. In Section IV we develop a hierarchy of singularly perturbed models by taking the respective small parameter limits, as well as averaging the mesh dynamics under the additional assumption of periodic operation. It turns out that the most useful and accurate model for typical operation is the one where the fast acoustic dynamics are removed, and we reduce that particular model further using a chirp-POD technique in Section V. We end with some conclusions about the utility of these model reduction techniques in active control design for Stirling engines.

II. MODELING

Figure 1 is a diagram a generic Stirling engine. A sealed working gas shuttles back and forth through the regenerator between the cold and hot sections, each section being in thermal contact with external cold and hot heat exchangers respectively. This movement of the gas is primarily caused by the displacer piston, and while the power piston's motion has some effect on the gas movement, it is a smaller effect than that due to the displacer, and can be neglected for a preliminary understanding of the engine's operation. The role of the regenerator will be explained below, but it can temporarily be thought of as simply a low-pressure-loss connection between the hot and cold sections.

A Stirling engine is a special type of air engine. The basic operating principle of an air engine is simple. At any one time, the hot and cold sections of the engine are at roughly the same pressure, and that pressure is dependent on the *average* temperature across those sections. As the working gas shuttles back and forth between the two sections, more or less of its volume is contained in either the hot or cold sections, and therefore the average temperature oscillates, which in turn causes the pressure to oscillate.

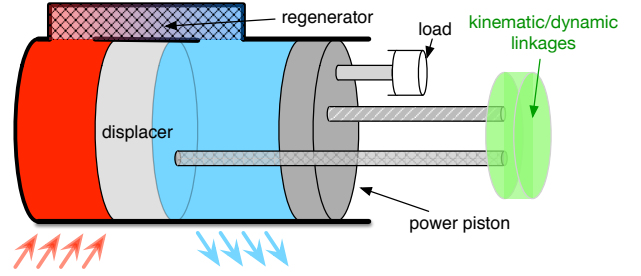


Fig. 1. A conceptual diagram of a generic Stirling engine. The displacer piston shuttles gas (through the regenerator) between the hot and cold sections. This causes the average temperature (and consequently the pressure) of the working gas to oscillate. These pressure oscillations drive the power piston which performs mechanical work on a load. Depending on the type of engine, kinematic or dynamic linkages use a small amount of that work to in turn drive the displacer piston, thus setting up a limit cycle. The geometry shown for linkages is conceptual.

These pressure oscillations then drive the working piston to perform mechanical work on a load.

The cycle described above is driven by the displacer's movement. In traditional Stirling engines displacer motion is induced through kinematic linkages from the power piston. Little power is required to drive the displacer since the working gases at either side of it are roughly at the same pressure. Linkages designed so that displacer and power pistons are approximately 90° out of phase will typically produce a stable oscillation [25]. Alternatively, in Free Piston Stirling Engines [1], the displacer and power piston are dynamically (rather than kinematically) linked through a gas spring (the so-called bounce space). Figure 1 shows these kinematic/dynamic linkages only conceptually since they differ from one type of Stirling engine to another.

A more recent concept is that of the Actively Controlled Stirling Engine [9], [11], where the displacer's motion is directly actuated by a control actuator whose motion can be designed to optimize the engine [9].

One of the motivations of the present work is that a more realistic active control design will probably require a better dynamical model than the commonly used idealized Schmidt model. An additional goal is to develop a modeling framework for a generic Stirling engine, i.e. largely independent of the presence or type of piston/displacer linkages. We therefore emphasize the most dynamically complex, and arguably the most important part of the engine, that is the regenerator.

The regenerator is an open connection between the cold and hot sections which is filled with a metal (or graphite) mesh of material. It acts like a porous channel for gas flow, but yet significant convective heat exchange occurs between the flowing gas and the mesh. The purpose of the regenerator is to act as a "thermal capacitor". For example, without the regenerator, as hot gas flows into the cold section, most its excess heat will be rejected to the cold exchanger. The regenerator retains some of that thermal energy in the mesh material, and the gas enters the cold

section colder than it would have otherwise been without a regenerator. In the other half of the cycle, the gas flowing back to the hot section retrieves some of the heat stored in the mesh, and arrives at a higher temperature than it would otherwise be at (without the regenerator), thus needing to extract less thermal energy from the hot exchanger to reach the hot side temperature. This concept of a thermal capacitor was the basis of Stirling's original patent [17] which referred to it as the "Economizer". Without the regenerator, an air engine would have very low efficiency and power output.

We begin this section with a distributed one dimensional Partial Differential Equation (PDE) model of the regenerator based on the Euler equations of gas dynamics together with mesh/gas heat exchange. We then describe a standard lumped model for the thermodynamics of each gas section, and finally the dynamics and possible kinematic linkages of the pistons are described. These three separate pieces of the model are schematically shown in Figure 2.

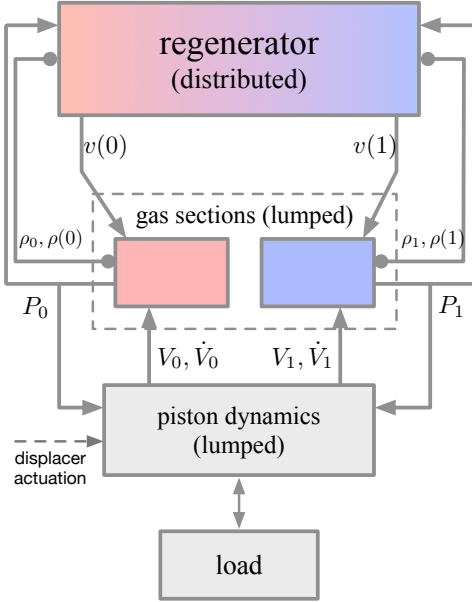


Fig. 2. A block diagram of the various components of our Stirling engine model. The displacer actuation input is only relevant to the case of the actively controlled engine. The piston dynamics block has the positions and velocities of the pistons as states (together with any kinematic constraints), while the gas sections blocks have the pressures and densities of each section as states. Lines with the port symbol (•-) indicate interactions whose directions switch depending on the sign of velocity at the boundaries. The only block with a distributed state is that of the regenerator, to which we apply the model reduction techniques described in this paper.

A. Distributed Regenerator Model

The regenerator consists of channel through which the working gas flows back and forth between the hot and cold gas section. This channel is typically filled with a mesh (a metal or graphite) material which runs lengthwise down the tube. The gas and the mesh material exchange heat as the working gas flows back and forth through the channel.

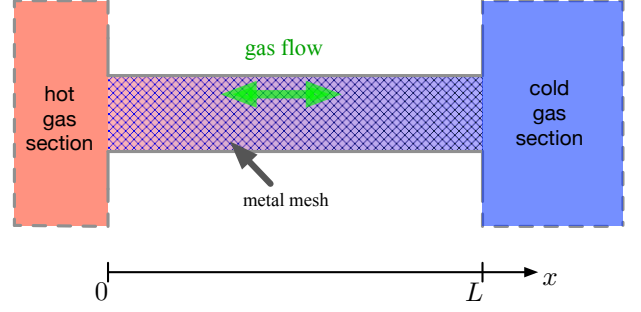


Fig. 3. A diagram of the first-principles model of the gas sections and regenerator interactions. Each gas section is considered as a well-mixed compartment with a lumped state. The regenerator is modeled using the Euler equations of one-dimensional compressible gas dynamics interacting through convective heat transfer with a spatially distributed metal mesh. The spatial coordinate axis is used for the distributed regenerator state only and not the lumped gas sections.

The main feature of our model of the regenerator is that due to gas flow being mainly in the axial direction, all variables are well approximated as being constant along axes perpendicular to the flow. For the gas, this leads to the Euler equations of one dimensional gas dynamics, while the mesh material's temperature can be modeled by the one dimensional heat equation. The thermal interaction between the mesh and the gas is captured by a simple model of convective heat transfer.

The geometry of the one dimensional model is depicted in Figure 3. The dynamics of the gas are given by the one dimensional Euler equations which reflect the conservation of mass, momentum and energy respectively

$$\rho_t = -(\rho v)_x, \quad (1)$$

$$(\rho v)_t = -(\rho v^2 + p)_x - \beta v, \quad (2)$$

$$E_t = -((E + p)v)_x + k_g(\Phi - T), \quad (3)$$

where ρ , v , p , E , T , Φ are the spatially distributed gas' density, velocity, pressure, energy, temperature, and mesh material temperature respectively. All of these fields are functions of space x and time t which are suppressed for notational simplicity. The terms in the last column represent the non-conservative effects of viscous friction and mesh-gas heat exchange respectively. The latter term represents a simple Fourier law of heat exchange between the gas and mesh with a heat transfer coefficient of k_g . It is a simplification which combines the effects of conductive and convective heat transfer in a single velocity-independent coefficient [12].

The variables ρ , p , T and E are not independent, but rather algebraically constrained by the following thermodynamic relations

$$p = \rho R T, \quad (4)$$

$$E = \frac{1}{2} \rho v^2 + \rho c_v T. \quad (5)$$

The first is the ideal gas law, and the second expresses total energy (per unit length) as the sum of kinetic and internal thermal energy for a "calorically perfect" gas with

specific heat capacity c_v . These two relations imply that the state of the gas can be described by several choices of 3 out of the 5 fields (ρ, v, p, T, E) . Common choices include either (ρ, v, p) or (ρ, v, T) , which means using either p or T to express the energy balance. These two choices are somewhat equivalent in terms of their utility. For brevity, we present here the choice of (ρ, v, p) as states, and rewrite the mass, momentum and energy conservation equations respectively as

$$\begin{bmatrix} \rho_t \\ v_t \\ p_t \end{bmatrix} = - \begin{bmatrix} v & \rho & 0 \\ 0 & v & \frac{1}{\rho} \\ 0 & \gamma p & v \end{bmatrix} \begin{bmatrix} \rho_x \\ v_x \\ p_x \end{bmatrix} - \begin{bmatrix} 0 \\ \beta \frac{v}{\rho} \\ \bar{\gamma} k_g \left(\frac{p}{R\rho} - \Phi \right) - \beta v^2 \end{bmatrix} \quad (6)$$

where $\gamma := 1 + \frac{R}{c_v}$ and $\bar{\gamma} := \frac{R}{c_v} = \gamma - 1$. Note that internal thermal energy is expressed in terms of the pressure by $E = \bar{\gamma} p$. The conversion from equations (1-3) to equations (6) using the relations (4-5) involves only algebraic manipulations and the chain rule, and is detailed in Appendix A. Note that these equations have the form

$$\Psi_t = F(\Psi, \Psi_x) + \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T \Phi,$$

where we've denoted the gas state $\Psi := [\rho \ v \ p]^T$. The mesh temperature field Φ can be regarded as a distributed input, while the gas temperature field T can be regarded as an output of this system using the ideal gas law $T = R \rho/p$. The gas dynamics are in feedback with the mesh temperature dynamics which are governed by the heat equation with the gas/mesh thermal exchange acting as a distributed input

$$c_m \rho_m \Phi_t = k_m \Phi_{xx} + k_g (T - \Phi), \quad (7)$$

where c_m , ρ_m and k_m are the (specific) heat capacity, density and heat conduction coefficient of the mesh material respectively. This coupling between the gas dynamics and the mesh thermal state is depicted in Figure 4.

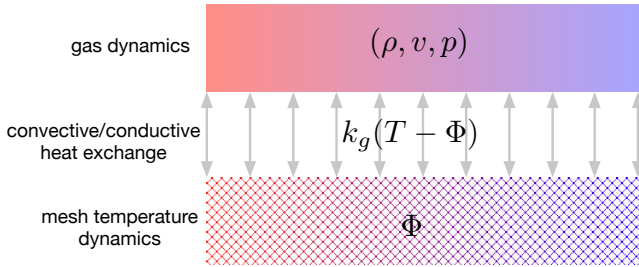


Fig. 4. A diagram depicting the coupling of one dimensional gas dynamics (eqs. (6)) with the distributed mesh temperature dynamics (eq. (7)) through spatially distributed convective and conductive heat exchange. Although the gas temperature T is not explicitly a state of the gas dynamics, it can be considered as an output using the ideal gas law $T = R \rho/p$.

Boundary Conditions and Numerical Methods for Gas Dynamics: One of the important features of Gas Dynamics is the switching nature of boundary conditions. The pressures at both ends of the regenerator are set equal to the pressure in the adjacent gas sections. However,

the density at either end is set equal to the density of the adjacent section only if the gas is flowing *into* the regenerator at that end (otherwise no boundary condition on density is enforced at that end). Thus the density boundary conditions turn on and off based on the sign of the velocity at the respective boundary. Formally the boundary conditions are

$$\begin{aligned} p(0, t) &= p_h(t), \quad p(1, t) = p_c(t), \\ \rho(0, t) &= \rho_h(t), \quad \text{if } v(0, t) \geq 0 \\ \rho(1, t) &= \rho_c(t), \quad \text{if } v(1, t) \leq 0. \end{aligned} \quad (8)$$

Thus there are 4 possible combinations of boundary conditions depending on whether the gas is flowing in or out of each end of the regenerator. This “state-dependent switching” of boundary conditions presents a special challenge in modeling and numerical approximations of this system.

The method we use to simulate such systems is the Essentially Nonoscillatory (ENO) scheme, which is commonly used to for compressible flow equations. The ENO scheme will not be covered here, but interested readers are referred to [26]–[28]. An important feature of the ENO scheme is that it is able to account for the state dependent switching of the boundary conditions. It does so by using an “upwinding” scheme, where the direction of upwinding switches with the sign of the velocity. Thus the ENO scheme is able to capture this state-dependent switching automatically.

B. Lumped Gas Sections Models

Each of the two gas sections are modeled as a well-mixed lumped system with a spatially uniform temperature and pressure. The thermodynamic variables describing the i 'th compartment ($i = 0, 1$, for the hot and cold sections respectively) are its volume V_i , density ρ_i , temperature T_i , pressure P_i , and internal thermal energy E_i . The kinetic energy due to gas velocity is neglected in this lumped model as it is much smaller than variations in thermal energy. As mentioned earlier in discussing gas dynamics, these time varying thermodynamic variables are not independent, but are related by

$$\begin{aligned} P_i &= \rho_i R T_i, \\ E_i &= c_v T_i \rho_i V_i, \end{aligned} \quad i = \begin{cases} 0, & \text{hot section,} \\ 1, & \text{cold section.} \end{cases} \quad (9)$$

The first is the ideal gas law, and the second is the expression for the gas internal thermal energy.

The dynamics of each lumped section as it interacts with its boundaries can be derived from the conservation of mass and energy. First, note that the total mass in each section is $V_i \rho_i$, and therefore

$$\frac{d}{dt} (V_0 \rho_0) = \begin{cases} -a_0 \rho_0 v(0) & \text{if } v(0) > 0 \\ -a_0 \rho(0) v(0) & \text{if } v(0) < 0 \end{cases} \quad (10)$$

$$\frac{d}{dt} (V_1 \rho_1) = \begin{cases} a_1 \rho(L) v(L) & \text{if } v(L) > 0 \\ a_1 \rho_1 v(L) & \text{if } v(L) < 0 \end{cases}, \quad (11)$$

where a_0, a_1 are the cross sectional areas of the boundaries between the hot and cold sections respectively and the regenerator. $v(\cdot)$ and $\rho(\cdot)$ are the velocity and density fields

as determined by the gas dynamics model of the regenerator. Note the switching nature of the above equations. If the velocity $v(0)$ at the boundary between the hot section and the regenerator is positive, then the hot section is losing mass of density ρ_0 , the density of the mass already in the hot section. If $v(0)$ is negative, then it gains mass of density $\rho(0)$, i.e. the density of the gas at the left boundary of the regenerator.

To make the subsequent notation significantly more compact, the following “switching selection” functions are defined

$$\begin{aligned} \mathcal{S}_0(a, b; c) &:= \begin{cases} a, & \text{if } c > 0 \\ b, & \text{if } c < 0 \end{cases} \\ \mathcal{S}_1(a, b; c) &:= \begin{cases} b, & \text{if } c > 0 \\ a, & \text{if } c < 0 \end{cases} \end{aligned}$$

and assume the regenerator to have length $L = 1$ (this choice is made without loss of generality as we will non-dimensionalize the model’s variables in the next section). Equations (10)-(11) can now be written more compactly

$$\frac{d}{dt}(V_i \rho_i) = (-1)^{i+1} a_i v(i) \mathcal{S}_i(\rho_i, \rho(i); v(i)), \quad i = 0, 1.$$

The energy dynamics can also be compactly written using the switching function. The rate of change of energy for each section is

$$\begin{aligned} \dot{E}_i &= (-1)^{i+1} a_i v(i) \\ &\mathcal{S}_i \left(c_v T_i \rho_i + \rho_i \frac{v^2(i)}{2}, c_v T(i) \rho(i) + \rho(i) \frac{v^2(i)}{2}; v(i) \right) \\ &+ (-1)^{i+1} a_i P_i v(i) - P_i \dot{V}_i + K_i (H_i - T_i), \quad i = 0, 1. \end{aligned} \quad (12)$$

which is equal to the rate at which (thermal+kinetic) energy is flowing in/out of the section, plus the work rate of pressure forces, minus the mechanical work done by the gas on its surroundings due to volume changes, and the last term represents the conductive heat transfer (with coefficient K_i) between the heat exchanger at the walls of the section which are assumed kept at constant temperatures H_i . Note that $T(i)$, $\rho(i)$ and $v(i)$ are the boundary values of fields determined by the distributed regenerator model described in the previous section.

We finally note the number of state variables and inputs. Both volume and volume change rate V_i and $\dot{V}_i$ can be regarded as inputs imposed by the pistons’ positions and velocities. The boundary velocities $v(i)$ can also be regarded as inputs imposed by the regenerator model. The remaining variables T_i , ρ_i , P_i and E_i are dynamic state variables that are constrained by 2 (for each section) static algebraic relations (9), and therefore one can choose only 2 state variables for each lumped section. The pressure of each section can then be considered as an output effecting both the pistons’ as well as the regenerator dynamics by imposing the pressure boundary conditions on the latter. This is summarized in Figure 2.

A particularly convenient choice of the sections’ state variables are densities and pressures. Since pressure boundary conditions do not undergo switching, the final

form of these equations takes the following form which is simpler than that written with temperatures instead of pressure s

$$\begin{aligned} V_i \dot{\rho}_i &= -\dot{V}_i \rho_i + (-1)^{i+1} a_i v(i) \mathcal{S}_i(\rho_i, \rho(i); v(i)) \\ V_i \dot{P}_i &= (-1)^{i+1} a_i \frac{\bar{\gamma}}{2} v^3(i) \mathcal{S}_i(\rho_i, \rho(i); v(i)) \\ &+ (-1)^{i+1} \gamma a_i v(i) P_i - \gamma P_i \dot{V}_i + \bar{\gamma} K_i \left(H_i - \frac{P_i}{\rho_i R} \right) \end{aligned}$$

Note that these equations are of the form

$$\begin{bmatrix} \dot{\rho}_i \\ \dot{P}_i \end{bmatrix} = \mathcal{F}_i \left(\begin{bmatrix} \rho_i \\ P_i \end{bmatrix}, v(i), \rho(i), V_i, \dot{V}_i \right),$$

where $v(i)$, $\rho(i)$, V_i and $\dot{V}_i$ are regarded as inputs.

C. Piston Dynamics and Kinematics

The case of the actively controlled Stirling engine corresponds to having no kinematic linkages in Figure 1, and the displacer is driven directly by an external input u . In this case, the pistons’ equations of motion are

$$\begin{aligned} m_p \ddot{x}_p &= a_p [P_1 - P_{ex}] - c_p \dot{x}_p, \\ \dot{x}_d &= u \\ V_0 &= \bar{V}_0 + a_d x_d, \\ V_1 &= \bar{V}_1 + a_p x_p - a_d x_d, \end{aligned} \quad (13)$$

where x_p and m_p are the position and mass of the power piston respectively, $P_1 - P_{ex}$ is the pressure difference between the cold section and the external side of the power piston, $c_p \dot{x}_p$ a viscous dissipation term which models the power dissipated in the load, x_d is the displacer’s position whose velocity is assumed directly assignable by an external control input u . The time-varying section volumes V_0 and V_1 and their derivatives $\dot{V}_0$ and $\dot{V}_1$ can be considered as outputs of this dynamical system as shown above (the equations for $\dot{V}_0$, $\dot{V}_1$ are not shown) with a_d , a_p the pistons’ cross sectional areas, and $\bar{V}_0$, $\bar{V}_1$ as the respective volumes when $x_p = 0$, $x_d = 0$.

In beta-type kinematically linked Stirling engines, there are further algebraic constraints. The displacer and power piston are connected to a flywheel which provides the feedback necessary for a stable limit cycle to form. The kinematics for the beta engine can be expressed using the geometrical relations from Figure 5 as follows

$$m_p \ddot{x}_p = a_p [P_1 - P_{ex}] - c_p \dot{x}_p - F_p, \quad (14)$$

$$I \ddot{\theta} = F_p r_p \sin(\theta - \phi) - a_d r_d (P_1 - P_0) \sin(\theta), \quad (15)$$

$$x_d = -r_d \cos(\theta), \quad (16)$$

$$x_p = -r_p \cos(\theta - \phi), \quad (17)$$

$$V_0 = \bar{V}_0 + a_d x_d, \quad (18)$$

$$V_1 = \bar{V}_1 + a_p x_p - a_d x_d, \quad (19)$$

where I and θ are the moment of inertia and angular position of the flywheel respectively, F_p is the reaction force between the power piston and the flywheel, ϕ is the phase difference between the two pistons, r_p and r_d are the radial attachment locations of the pistons on the flywheel. The equations above assume that the displacer and arms

connecting the pistons to the flywheel are massless. The latter are also assumed to be sufficiently long so that the forces they exert on the flywheel and pistons are essentially horizontal.

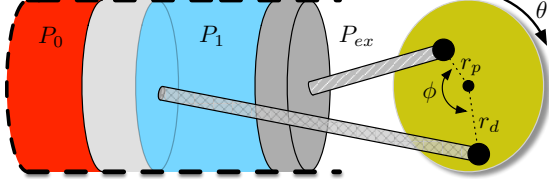


Fig. 5. Conceptual diagram of a beta-type Stirling engine with flywheel kinematic connections. P_0 and P_1 are the pressures in the hot and cold sections respectively, while P_{ex} is the pressure on the external side of the power piston.

Note that although there are 4 states in (14)-(15), the two algebraic constraints (16)-(17) reduce the number of state variables in this model to 2. We choose θ and $\dot{\theta}$ as coordinates for these states.

The parameters used in the sequel for numerical examples for both the mechanical model and the section models are based on the work of [29], where a martini type Stirling engine is modeled. The engine modeled here is a simplified version of that engine. The choice of wall temperatures, helium as the working gas, approximate dimensions, and nominal pressure are all taken from this source.

III. TIME SCALE SEPARATIONS

The Euler equations of gas dynamics contain several phenomena and time scales. The fastest scale corresponds to acoustic waves. Other scales are induced by the periodic oscillation and interaction with the lumped gas sections, as well as thermal exchange with the mesh material. Various types of engines will have differing time scale separations. The systematic method to uncover potential time scale separation is to first rewrite the equations in non-dimensional form. Both time and space, as well as the dynamic fields are normalized as follows

$$\begin{aligned} \tilde{t} &:= \frac{t}{T_b} = \omega_b t, & \tilde{x} &:= \frac{x}{L}, \\ \tilde{\rho} &:= \frac{\rho}{\bar{\rho}}, & \tilde{v} &:= \frac{v}{\bar{v}}, & \tilde{p} &:= \frac{p}{\bar{p}}, & \tilde{\Phi} &:= \frac{\Phi}{\bar{\Phi}}, & \tilde{T} &:= \frac{T}{\bar{T}}, \end{aligned} \quad (20)$$

where $\bar{\rho}$, $\bar{p}$, are nominal gas density and pressure, $\bar{\Phi}$ is nominal mesh temperature, ω_b and $\bar{v}$ are the engine's frequency (T_b is the engine's period) and nominal advection speed. The latter two parameters are typically not known in advance, but can be estimated based on other parameters and the engine's geometry. Note that in the nondimensional variables, the ideal gas law becomes simply $\tilde{p} = \tilde{\rho}\tilde{T}$.

With the nondimensional variables and the use of the chain rule and some algebraic manipulations, the gas and

mesh thermal dynamics can be rewritten as

$$\frac{\omega_b}{\bar{v}/L} \frac{\partial \tilde{\rho}}{\partial \tilde{t}} = - \frac{\partial}{\partial \tilde{x}} (\tilde{\rho} \tilde{v}), \quad (21)$$

$$\frac{\omega_b \bar{v}}{c^2/L} \tilde{\rho} \frac{\partial \tilde{v}}{\partial \tilde{t}} = - \frac{\tilde{v}^2}{c^2} \tilde{\rho} \tilde{v} \frac{\partial \tilde{v}}{\partial \tilde{x}} - \frac{1}{\gamma} \frac{\partial \tilde{p}}{\partial \tilde{x}} - \frac{1}{\gamma} \frac{\beta L \bar{v}}{\bar{p}} \tilde{v}, \quad (22)$$

$$\frac{\omega_b}{\bar{v}/L} \frac{\partial \tilde{p}}{\partial \tilde{t}} = - \tilde{v} \frac{\partial \tilde{p}}{\partial \tilde{x}} - \gamma \tilde{p} \frac{\partial \tilde{v}}{\partial \tilde{x}} \quad (23)$$

$$\begin{aligned} &+ \frac{\bar{\gamma} L k_g}{R \bar{\rho} \bar{v}} \left(\tilde{\Phi} - \frac{\tilde{p}}{\bar{\rho}} \right) + \bar{\gamma} \frac{\beta L \bar{v}}{\bar{p}} \tilde{v}^2, \quad (24) \\ c_p \rho_m \omega_b \frac{\partial \tilde{\Phi}}{\partial \tilde{t}} &= \frac{k_m}{L^2} \frac{\partial^2 \tilde{\Phi}}{\partial \tilde{x}^2} + k_g \left(\frac{\tilde{p}}{\bar{\rho}} - \tilde{\Phi} \right) \end{aligned}$$

where $c = \sqrt{\gamma \frac{\bar{p}}{\bar{\rho}}}$ is the speed of sound. The following potentially small parameters are immediately recognizable in the equations above

$$\begin{aligned} \epsilon_1 &:= \frac{\omega_b}{\bar{v}/L} = \frac{L/\bar{v}}{T_b} = \frac{\text{advection time through regenerator}}{\text{engine oscillation period}} \\ \epsilon_2 &:= \frac{\bar{v}}{c} = \text{Mach number} \\ \epsilon_f &:= \frac{\beta L \bar{v}}{\bar{p}} = \frac{\text{head loss through regenerator}}{\text{nominal pressure}} \\ \epsilon_m &:= \frac{k_g}{c_p \rho_m \omega_b} = \frac{T_b}{c_p \rho_m / k_g} = \frac{\text{engine oscillation period}}{\text{mesh thermal time constant}} \end{aligned}$$

ϵ_1 is the ratio of the advection time (average time it takes the gas to advect from one end of the regenerator to the other) to the engine's period. This ratio is less than one for a well designed engine, otherwise motion reversal in the regenerator occurs before gas has shuttled from one section to the other. The Mach number ϵ_2 is the smallest of the above parameters and is typically much less than 1. A reasonably efficient engine would also have a much smaller head loss (through the regenerator) than the nominal operating pressure, and therefore ϵ_f is typically much less than 1. Finally, the mesh material is designed to have a high thermal capacity, and therefore the mesh thermal time constant $c_p \rho_m / k_g$ is typically much longer than the engine's period T_b .

Using the definitions of the small parameters above, and simplifying subsequent notation by dropping the $\sim$'s on all variables, the equations become

$$\begin{aligned} \epsilon_1 \rho_t &= -(\rho v)_x, \\ \epsilon_1 \epsilon_2^2 \rho v_t &= -\epsilon_2^2 \rho v v_x - \frac{1}{\gamma} p_x - \frac{1}{\gamma} \epsilon_f v, \\ \epsilon_1 p_t &= -v p_x - \gamma p v_x + \frac{\bar{\gamma} L k_g}{R \bar{\rho} \bar{v}} \left(\Phi - \frac{p}{\rho} \right) + \bar{\gamma} \epsilon_f v^2. \\ \Phi_t &= \epsilon_m \left(\frac{k_m}{k_g L^2} \Phi_{xx} + \left(\frac{p}{\rho} - \Phi \right) \right) \end{aligned} \quad (25)$$

It is now clear that the first three equations are candidates for a singular perturbation type of model reduction, while the last equation may be simplified using an averaging technique. These simplifications are detailed in Section IV. We close this section with a further discussion on which properties of a particular engine determine how small each of the parameters are.

Estimating the Small Parameters

In order to justify that assumption that the ϵ 's are small, they need to be expressed in terms of easily measurable engine parameters. This allows them to be quickly approximated for a given engine design. Estimating the flow speed is arguably the most difficult and will be tackled first. Conservation of mass for the hot section yields

$$\dot{M}_h = \dot{\rho}_h V_h + \rho_h \dot{V}_h = \rho_h v_{hr} A_{hr}, \quad (26)$$

here M_h is the mass in the hot section, ρ_h is the density, V_h is the volume, v_{hr} is the velocity of the flow in or out of the section, and A_{hr} is the cross-sectional area of the regenerator void volume. Differentiating the ideal gas law produces

$$\dot{P}_h = \dot{\rho}_h R T_h + \rho_h R \dot{T}_h. \quad (27)$$

If the gas is assumed to be in perfect thermal contact with the wall, as is done in the popular Schmidt analysis of Stirling engines, then $\dot{T}_h$ can be assumed to be zero. Solving for $\dot{\rho}_h$ and substituting it into the mass equation results in

$$\frac{\dot{P}_h}{P} V_h + \dot{V}_h = v_{hr} A_{hr}. \quad (28)$$

Assuming that the volume and the pressure in the hot section both vary sinusoidally and are in phase then they can be expressed as

$$P_h(t) = P_a \sin(\omega t) + P_0, \quad (29)$$

$$V_h(t) = A_d R_d \sin(\omega t) + V_{h_0}, \quad (30)$$

where A_d is the cross sectional area of the displacer, and R_d is the amplitude of the motion of the displacer. Plugging these in and simplifying results in

$$\left[\frac{\sin(\omega t) + \frac{V_{h_0}}{A_d R_d}}{\sin(\omega t) + \frac{P_0}{P_a}} + 1 \right] \cos(\omega t) \frac{A_d R_d}{A_{hr}} \omega = v_{hr}(t). \quad (31)$$

For most engines, all of these parameters are generally known or easily estimated. Keeping in mind that $\omega = 2\pi\omega_b$, it should be simple to get an approximation for the maximum value of $v_{hr}(t)$ for a given engine. If further simplification is desired, one can assume that the dead volume in the hot section is close to zero which makes $\frac{V_{h_0}}{A_d R_d}$ close to one. The maximum possible value for P_a is P_0 and the minimum value for P_a is zero, this implies that the maximum possible value for the velocity at the regenerator and hot section boundary can be approximated as

$$\bar{v}_{hr} = k \frac{A_d R_d 2\pi\omega_b}{A_v}, \quad k \in [1, 2]. \quad (32)$$

If the velocity throughout the regenerator is approximately uniform, then this maximum velocity can be used as an estimate for the maximum velocity in the regenerator, $\bar{v} \approx \bar{v}_{hr}$. The small parameters ϵ_2 and ϵ_1 can now be expressed in terms of engine parameters as

$$\epsilon_2 = \frac{\bar{v}}{\bar{c}} \approx \frac{2\pi k A_d R_d \omega_b}{A_{zr} \sqrt{\gamma \frac{P}{\rho}}}, \quad (33)$$

and

$$\epsilon_1 = \frac{L/\bar{v}}{T_b} \approx \frac{L A_{zr}}{2\pi k A_d R_d} = \frac{V_r}{\pi k V_s}, \quad (34)$$

where V_r is the void volume in the regenerator, and V_s is the volume displaced by the displacer during one stroke of the engine.

Next is $\epsilon_f = \frac{\beta L \bar{v}}{\bar{p}}$. It can be shown that β is given by

$$\beta = \frac{8}{R_r^2} \mu, \quad (35)$$

where R_r is the pore radius in the regenerator and μ is the dynamics viscosity of the gas. Since $\bar{v}$ has already been estimated, ϵ_f can be expressed as

$$\epsilon_f = \frac{\beta L \bar{v}}{\bar{p}} = k \frac{16\pi \mu A_d R_d L \omega_b}{\bar{p} R_r^2 A_{zr}}. \quad (36)$$

Given these approximations, it is possible to justify the assumption that all of the epsilons are small. If the flow speed in the regenerator does not become supersonic, which should be the case for most Stirling engines, then $\epsilon_2 = \frac{\bar{v}}{\bar{c}}$ will be less than one. If the void volume (V_r) is less than the stroke volume (V_s) then $\epsilon_1 = \frac{V_r}{\pi k V_s}$ must be less than one. This should also be the case for most Stirling engines as a well designed Stirling engine minimizes the dead volume (volume that is not part of the expansion or compression process) throughout the engine, which the regenerator volume is considered to be. Because of the large thermal inertia of the matrix material, the engine frequency is much faster than the matrix temperature dynamics. Therefore, $\frac{T_b}{c_p \rho_m / k_g}$ will be small. The final small parameter $\epsilon_f = \frac{\beta L \bar{v}}{\bar{p}}$ should also be less than one since $\beta L \bar{v} = \Delta p$ where Δp is pressure difference across the regenerator. Most Stirling engine assumptions, including the common Schmidt assumptions, assume the pressure difference across the regenerator is negligible compared to the nominal pressure. Thus, it is safe to assume that for most Stirling engines these values will all be less than or much less than one.

IV. SINGULARLY PERTURBED MODELS

In this section we develop a hierarchy of singularly perturbed models in progressing order of coarseness. The first model (42) is concerned with only the gas' states, but eliminates acoustic dynamics. This is applicable to low Mach number conditions, which is typical for Stirling engines. The second model involves (Bogoliubov) averaging of the mesh matrix dynamics motivated by its relatively large thermal inertia and periodic operation. This leads to the conclusion that the solid mesh temperature profile is linear (50). We also present a Quasi-Steady State (QSS) model that parametrizes the temperature, pressure and density distributions across the regenerator with a small number of parameters that are algebraically related to the boundary conditions.

A. Model without Acoustic Dynamics

As defined in the previous section, ϵ_2 is the Mach number, which is low in typical Stirling engines. Furthermore, ϵ_2 appears squared in equation (25) which means that ϵ_2^2 it is by far the smallest parameter. Setting $\epsilon_2 = 0$ in equation (25) gives the following

$$\epsilon_1 \rho_t = -(\rho v)_x, \quad (37)$$

$$0 = -p_x - \epsilon_f v, \quad (38)$$

$$\epsilon_1 p_t = -v p_x - \gamma p v_x + \frac{\bar{\gamma} L k_g}{R \bar{\rho} \bar{v}} \left(\Phi - \frac{p}{\rho} \right) + \bar{\gamma} \epsilon_f v^2. \quad (39)$$

The second equation implies that v is simply proportional to the pressure gradient $v = -\frac{1}{\epsilon_f} p_x$. This can be used to eliminate v from the other two equations to arrive at

$$\epsilon_1 \epsilon_f \rho_t = (\rho p_x)_x, \quad (40)$$

$$\epsilon_1 \epsilon_f p_t = \gamma (p p_x)_x + \epsilon_f \frac{\bar{\gamma} L k_g}{R \bar{\rho} \bar{v}} \left(\Phi - \frac{p}{\rho} \right) \quad (41)$$

The above equations represent flow, compression and thermal exchange with the mesh at the advective time scale. The much faster acoustic dynamics have been removed from this model.

To understand the above model better we rewrite the equations in the following equivalent form

$$\begin{aligned} \epsilon_1 \epsilon_f \rho_t &= p_x \rho_x + p_{xx} \rho, \\ \epsilon_1 \epsilon_f p_t &= \frac{\gamma}{2} (p^2)_{xx} + \epsilon_f \frac{\bar{\gamma} L k_g}{R \bar{\rho} \bar{v}} \left(\Phi - \frac{p}{\rho} \right), \end{aligned} \quad (42)$$

which follows from $2(pp_x)_x = (p^2)_{xx}$. We note that the dynamics of p are primarily diffusive with the temperature difference $(\Phi - T)$ as a distributed source term. The dynamics of ρ on the other hand are advective (with the advection velocity $-\frac{1}{\epsilon_f} p_x = v$, and a regenerative term proportional to p_{xx}).

Boundary Conditions and Numerical Method for the No-Acoustics Model: The pressure equation in (42) is of the diffusive type with boundary conditions given at both ends. It can therefore be easily discretized using a central difference scheme. The density equation however is of the advection type with switching boundary conditions according to (8) (note that $v(x, t) = -p_x(x, t)/\epsilon_f$, and therefore the switching boundary conditions on ρ depend on p in the no-acoustics model). We therefore use a standard first-order upwinding scheme to discretize ρ_x , where at each grid point, the direction of upwinding is based on the local velocity $-p_x(x, t)/\epsilon_f$.

B. Averaging the Mesh-Matrix Dynamics

Recall equation (25) for the mesh thermal dynamics (and substitute $T = \frac{p}{\rho}$ for the gas temperature)

$$\Phi_t = \epsilon_m \left(\frac{k_m}{k_g L^2} \Phi_{xx} + (T - \Phi) \right). \quad (43)$$

Averaging analysis [30] tells us that the difference between the solution to the T periodic system $\dot{x} = \epsilon f(t, x, \epsilon)$ and the solution to $\dot{x}_{av} = \epsilon f_{av}(x_{av})$ (where $f_{av} = \frac{1}{T} \int_0^T f(t, x, 0) dt$) is of order ϵ . Assuming that the engine

has reached steady state, averaging equation (43) and the pressure dynamics in (42) over one period of the limit cycle ($\bar{T}_b$) results in

$$\begin{aligned} 0 &= \frac{1}{\bar{T}_b} \int_0^{\bar{T}_b} \left[\frac{k_m}{k_g L^2} \Phi_{xx} + (T - \Phi) \right] dt, \\ 0 &= \frac{1}{\bar{T}_b} \int_0^{\bar{T}_b} \left[\frac{\gamma}{2} (p^2)_{xx} + \epsilon_f \frac{\bar{\gamma} L k_g}{R \bar{\rho} \bar{v}} (\Phi - T) \right] dt. \end{aligned} \quad (44)$$

The gas and mesh temperature during steady state can be decomposed as

$$\begin{aligned} T(x, t) &= T_p(x, t) + T_{av}(x), \\ \Phi(x, t) &= \Phi_p(x, t) + \Phi_{av}(x), \end{aligned} \quad (45)$$

where $T_p(x, t)$ and $\Phi_p(x, t)$ are periodic in t and have zero mean over one cycle. Using these substitutions yields

$$\begin{aligned} 0 &= \frac{k_m}{k_g L^2} \Phi_{avxx} + (T_{av} - \Phi_{av}), \\ 0 &= \frac{1}{\bar{T}_b} \int_0^{\bar{T}_b} \left[\frac{\gamma}{2} (p^2)_{xx} \right] dt + \epsilon_f \frac{\bar{\gamma} L k_g}{R \bar{\rho} \bar{v}} (\Phi_{av} - T_{av}). \end{aligned} \quad (46)$$

Solving for $(T_{av} - \Phi_{av})$ in one equation substituting it into the other results in the relation

$$\epsilon_f \Phi_{avxx} = -\frac{c_1}{\bar{T}_b} \int_0^{\bar{T}_b} [(p^2)_{xx}] dt, \quad (47)$$

where $c_1 = \frac{\gamma R \bar{\rho} \bar{v} L}{2 \bar{\gamma} k_m}$. Expressing the pressure profile as $p(x, t) = p_0(t) + \epsilon_f p_\Delta(x, t)$ equation (47) then becomes

$$\Phi_{avxx} = -\frac{c_1}{\bar{T}_b} \int_0^{\bar{T}_b} [2p_0(t)p_\Delta(x, t) + \epsilon_f p_\Delta^2(x, t)]_{xx} dt. \quad (48)$$

Given that $p_{xx} = p_{\Delta xx}$, and that equation (38) implies that $p_{xx} = -\epsilon_f v_x$, the integrand above can be expressed as

$$-2\epsilon_f [p_0(t)v_x(x, t) + \epsilon_f p_\Delta(x, t)v_x(x, t) + \epsilon_f^2 v^2(x, t)]. \quad (49)$$

Setting ϵ_2 equal to zero earlier removed the fast acoustic phenomena from the system, which includes the ability for shocks to form. This implies that $v(x, t)$ will be relatively smooth and $v_x(x, t)$ will be reasonable in size. Thus, as ϵ_f becomes very small so does the integrand in equation (48) and the average mesh profile can be assumed to be

$$\Phi_{av}(x) = \Phi_0 + \Phi_1 x, \quad (50)$$

where Φ_0 and Φ_1 are constants of integration determined by the boundary conditions. In our case the ends of the mesh are assumed to be in thermal contact with the section walls and as such, the boundary conditions are that the temperature at the ends of the mesh must be equal to the temperature of the section walls. As was mentioned at the start of this section, averaging analysis indicates that the difference between this and the true limit cycle is of order ϵ_m . Therefore, the matrix material will be assumed to be a fixed linear profile which interpolates between the two wall temperatures.

As an aside, we note that the linear temperature profile (50) would have also been a consequence of assuming that the thermal conductivity of the mesh (k_m) is much greater than the combined conduction and convection coefficient between the gas and the mesh (k_g). However, a

large k_m/k_g essentially implies a “thermal short” between the hot and cold exchangers through the regenerator, which leads to low efficiency. Therefore, a well-designed engine would not have a large k_m/k_g ratio, and we did not invoke this assumption here.

C. Quasi-Steady State (QSS) Model

In engines where the product $\epsilon_1 \epsilon_f$ can be considered small (e.g. low head loss through regenerator and/or slower engine frequency than advection time through regenerator), a significant simplification occurs in the equations. It turns out that the regenerator states depend algebraically on the boundary conditions. We term this the Quasi-Steady State (QSS) model since it represents a situation where the regenerator dynamics are much faster than the remaining engine’s dynamics.

Starting from the no-acoustics model (42), we set the product $\epsilon_1 \epsilon_f$ equal to zero (we only assume $\epsilon_1 = 0$) and obtain

$$\begin{aligned} 0 &= (\rho p_x)_x, \\ 0 &= \gamma (pp_x)_x + \epsilon_f \frac{\bar{\gamma} L k_g}{R \bar{\rho} \bar{v}} \left(\Phi - \frac{p}{\rho} \right). \end{aligned} \quad (51)$$

There are two immediate consequences of this simplification. The first equation implies that ρp_x is constant in x , so we define it as the scalar variable

$$\alpha(t) := \rho(x, t) p_x(x, t) = -\epsilon_f \rho(x, t) v(x, t), \quad (52)$$

where the second equality follows from (38).

The second simplification is in the reduction of the number of possible combinations of boundary conditions (8) on ρ . Observe that since ρ is always positive, equation (52) implies that $v(x, t)$ has the same sign for all x at any one time t . This means that velocity throughout the regenerator can only be either positive or negative at any one time. This reduces the number of possible boundary conditions on ρ to the two mutually exclusive possibilities

$$\rho(0, t) = \rho_0(t) \quad \text{if } (\alpha(t) < 0 \Leftrightarrow v(x, t) > 0), \quad (53)$$

$$\rho(1, t) = \rho_1(t) \quad \text{if } (\alpha(t) > 0 \Leftrightarrow v(x, t) < 0), \quad (54)$$

$$p(0, t) = p_0(t),$$

$$p(1, t) = p_1(t),$$

where we have also listed the pressure boundary conditions as well.

We now come to the second equation in (51) which we rewrite (using the definition of α) as

$$0 = \gamma (pp_x)_x + \epsilon_f \frac{\bar{\gamma} L k_g}{R \bar{\rho} \bar{v}} \left(\Phi - \frac{pp_x}{\alpha} \right).$$

Utilizing the identities $(p^2)_x = 2pp_x$ and $(p^2)_{xx} = 2(pp_x)_x$, this equation can be further rewritten as

$$0 = \gamma \alpha(t) (p^2)_{xx} + \epsilon_f \frac{\bar{\gamma} L k_g}{R \bar{\rho} \bar{v}} (2\alpha(t)\Phi(x) - (p^2)_x). \quad (55)$$

Note that at each t , this is a second order linear differential equation (in x) for p^2 with Φ as a forcing function. It is a two point boundary value problem given the values of

the pressure at each end. Thus for each t , it can be solved in terms of an integral (in x) of Φ . For the special case in this paper where the averaging analysis has shown that Φ can be well approximated (50) as an affine function of x , an analytical solution to equation (55) can be given as a linear combination of the boundary conditions p_0, p_1, H_0, H_1 and the parameter α as follows

$$\begin{aligned} p^2(x) &= \frac{k_t}{\beta} \left(-x^2 - \frac{2(1-\beta)}{\beta} x + \frac{2-\beta}{\beta} \frac{e^{\beta(x-1)} - e^{-\beta}}{1 - e^{-\beta}} \right) (H_0 - H_1) \\ &\quad + 2 \frac{k_t}{\beta} \left(x - \frac{e^{\beta(x-1)} - e^{-\beta}}{1 - e^{-\beta}} \right) H_1 \\ &\quad + p_0^2 - \left(\frac{e^{\beta(x-1)} - e^{-\beta}}{1 - e^{-\beta}} \right) (p_0^2 - p_1^2), \end{aligned} \quad (56)$$

where

$$\beta := \frac{k_t}{\alpha} := \frac{\epsilon_f \bar{\gamma} L k_g}{\gamma R \bar{\rho} \bar{v}} \frac{1}{\alpha}.$$

The details of this derivation are in Appendix A.

Equation (56) gives $p(x, t)$ as a function of the time varying boundary conditions $p_0(t), p_1(t)$, but it also requires the parameter $\alpha(t)$. One can solve for the latter from its definition and enforcing the density boundary conditions ρ_0 or ρ_1

$$\alpha(t) \rho_0 = p_x(0, t) \quad \text{or} \quad \alpha(t) \rho_1 = p_x(1, t). \quad (57)$$

The choice of which boundary to use is dictated by the conditions (53) and (54). Equations (56)-(57) form a coupled system that can be solved for α using a root finding routine. This is done at each time step, with the value of α at the previous time step used as the starting point for root finding. In more detail, note for example the case $\alpha \rho_0 = p_x(0)$. Differentiating (56) with respect to x gives the condition

$$\alpha \rho_0 = p_x(0) = f(p_0, p_1, \alpha),$$

for some function f which is a combination of polynomials and exponentials in α . Therefore, given the values of ρ_0, p_0, p_1 , a root finding routine can be used at each time step to solve for the corresponding α .

We finally note that once p and α are solved for at each t , velocity and density can then be determined from

$$v(x, t) = -\frac{1}{\epsilon_f} p_x(x, t), \quad \text{equation (38)}$$

$$\rho(x, t) = \frac{\alpha(t)}{p_x(x, t)}, \quad \text{definition of } \alpha,$$

while the temperature follows from the ideal gas law.

Numerical Results for QSS Model: A comparison was carried out of the above QSS model and the full model of Section II using the ENO scheme. For each of the models, the gas sections had 2 states each and the flywheel dynamics had 2 states giving a total of 6 states for the lumped portion of the engine. The ENO scheme for the full model used 100 grid points, while the QSS model of the regenerator is memoryless. Therefore the total state count

is $4 \cdot 100 + 6 = 406$ for the full model and 6 for the QSS model, giving a significant reduction in state dimension.

For comparison, both beta engine models were given the same initial condition and simulated until a steady state limit cycle was reached. Figure 6 shows a comparison of the limit cycles of the sections' states (the profiles of the fields in the regenerator are not shown). The trajectories

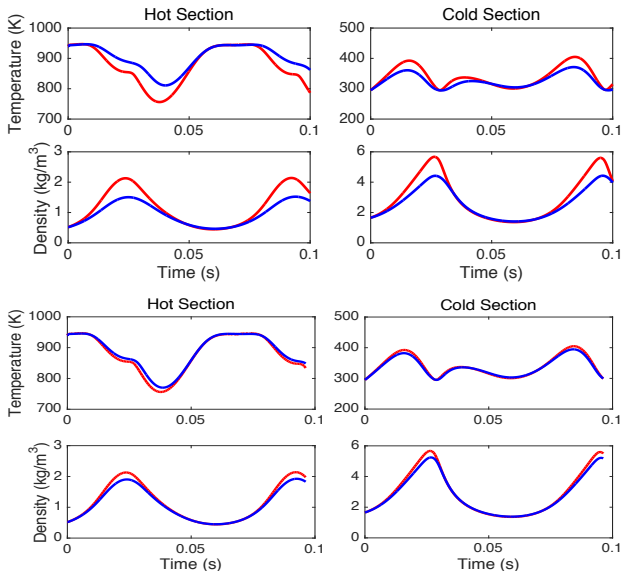


Fig. 6. A qualitative comparison of time histories of the sections' states for both the full (blue) and Quasi-Steady State (QSS) models (red). The top diagram is for the case of a moderate value of $\epsilon_1\epsilon_f$ for which the QSS model is a crude approximation, while the bottom diagram is for a smaller value of $\epsilon_1\epsilon_f$, for which the QSS model is a relatively more accurate approximation.

show that the QSS model is very close to the full model for small values $\epsilon_1\epsilon_f$ while still being qualitatively close for moderate values of $\epsilon_1\epsilon_f$.

Further Remarks on the QSS Model: The results showed that the simplified QSS model performed similarly to that of the finite difference model as long as the product $\epsilon_1\epsilon_f$ was small. A natural question would be if a higher order perturbation approximation would yield a better match. This is not the case for this application. It turns out that higher order perturbation approximations require that the time derivatives of the inputs to the perturbed model (boundary conditions) be known. The calculation of the time derivatives of the inputs (the section states) requires the use of the lower order approximations of the regenerator profiles. As such these time derivatives will differ from that of the unreduced system. Via experimentation, it was discovered that the higher order profile approximations are very sensitive to errors in these time derivatives. As a result, the higher order approximation performed worse than the lower order approximation. In the case that the inputs to this regenerator model are not states but instead predetermined time histories whose time derivatives are known, then a higher order approximation of these profiles would likely yield better results. However, this was not the case for this application.

V. MODEL REDUCTION USING CHIRP-POD

In this section we investigate a Principal Orthogonal Decomposition (POD) numerical method for model reduction of the no-Acoustics model of section IV-A. The main idea behind our particular POD method is in the choice of simulation conditions used to obtain the time traces from which POD modes are extracted. Since the regenerator will operate in a time-periodic manner once connected to other components of the engine, our approach is to generate snapshots from a simulation where all signals follow a chirp profile (a sinusoid with a linearly time-varying frequency) with a frequency range representative of the engine's potential operating frequency range. This approach has some commonality with that used in [31], [32], and we term it a "chirp-POD" technique.

The POD technique we use is a standard one with appropriate weightings. The no-Acoustics model (42) is simulated and the resulting density and pressure time series are collected in two matrices Y_ρ and Y_p whose columns represent time instants, and rows represent the values of density and pressure at the grid points respectively, more precisely

$$\begin{aligned} (Y_\rho)_{kl} &= \tilde{\rho}(x_k, t_l) \\ (Y_p)_{kl} &= \tilde{p}(x_k, t_l) \end{aligned}$$

where $\{t_l\}$ and $\{x_k\}$ are the time instances and the grid point locations respectively. The fields $\tilde{\rho}$ and $\tilde{p}$ are the fluctuations of density and pressure from the nominal solution corresponding to zero displacer motion.

A weighted POD method finds the singular values and left singular vectors of the matrix

$$W^{\frac{1}{2}}Y,$$

where Y is the data matrix and W is a diagonal weighting matrix. Equivalently, one finds the eigenvalues and eigenvectors of

$$[\Phi_W, \Lambda] = \text{eig}(YWY^T).$$

The POD modes are then obtained from the columns of the matrix

$$\Phi = W^{-\frac{1}{2}}\Phi_W. \quad (58)$$

The above procedure is applied to the density and pressure data separately.

The reduced model is then obtained by choosing a model order N , the first N POD modes (in terms of decreasing size of singular values of $W^{\frac{1}{2}}Y$) as a basis set, and expressing density and pressure fluctuations in the subspace spanned by that basis

$$\begin{aligned} \tilde{\rho}(x, t) &\approx \sum_{k=1}^N \phi_k^\rho(x) a_k(t), \\ \tilde{p}(x, t) &\approx \sum_{k=1}^N \phi_k^p(x) b_k(t), \end{aligned}$$

where $\{\phi_k^\rho\}$ and $\{\psi_k^p\}$ are the first N columns of the corresponding matrix Φ in (58) expressed as functions of a continuous variable (by e.g. interpolation). The reduced regenerator model then has $2N$ states. We will refer to this reduced model as the chirp-POD model.

A. chirp-POD Reduction of a beta-type Engine

We now present results of the chirp-POD model reduction technique performed using snapshots from a simulation of the no-Acoustics regenerator model (42) connected to the beta-type engine model. In order to generate trajectories with a time-varying instantaneous frequency (i.e. a chirp), an input is needed. This was done in the beta engine model (14)-(19) by imposing a time-varying trajectory $\{\theta(t)\}$ on the flywheel. To do this, the dynamics of the flywheel/power piston assembly (equations (14)-(15)) were removed from the model, and the imposed θ trajectory was used to generate piston motions from (16)-(17).

The chirp input consisted of increasing the flywheel's speed linearly in time from 40 to 600 rad/s. This frequency range was chosen to include 80 rad/s, which is the nominal operating frequency of this engine model. It was found that a choice of truncation order of $N = 3$ produced very good results, while $N < 3$ resulted in unrealistic trajectories. This assessment was carried out as follows. The chirp-POD modes were obtained from simulations with a forced flywheel trajectory, while the comparison of trajectories was done with dynamics of the beta engine (14)-(19) connected to the regenerator models. These models were simulated until stable limit cycles emerged, and the comparison is displayed in Figure 7 (Left) showing a very close match.

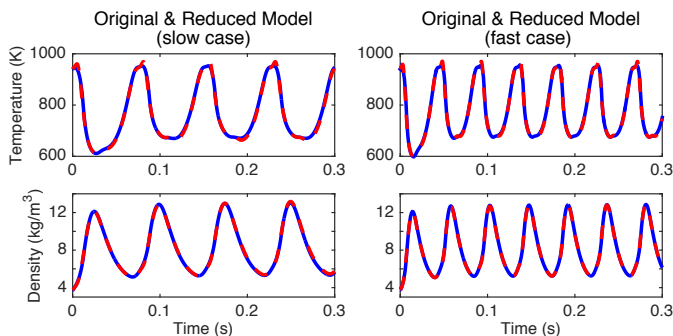


Fig. 7. (Left) Time histories of the hot section states for full (with-Acoustics, in blue) and chirp-POD reduced model (in dashed red) with 3 regenerator states. The two sets of trajectories are indistinguishable. (Right) The trajectories produced when the same chirp-POD reduced model is coupled to gas sections with wall conduction/convection coefficients of twice the magnitude of the sections used to generate model reduction POD trajectories. Note that despite the increase in engine frequency, the reduced models still match closely, indicating the efficacy of the chirp-POD technique over a wide range of engine parameters. The cold sections' states are not shown, but their behavior is similar to the above.

The reduced model was further tested by connecting it with a sections' model that has double the wall heat transfer coefficient compared to that used in obtaining the chirp-POD modes. The results are shown in Figure 7 (Right), and indicate that the technique of using chirped inputs appears effective in capturing POD modes that work for a large frequency range rather than a single operating condition.

B. chirp-POD Reduction and Frequency Response of a Driven Engine

In the previous subsection we compared the reduced and full model by comparing trajectories in two operating regimes. The Driven Stirling Engine model however offers the possibility of performing a more comprehensive comparison since it has a natural input, and therefore one can compare input-output behavior. Since the anticipated operation is periodic, it is natural to attempt to quantify this input-output behavior using a notion of frequency response. For nonlinear systems, a full frequency response would require analyzing the full harmonic content input-output periodic pairs at all possible amplitudes. We instead use a simpler notion of a frequency response assuming inputs to be pure sinusoids. The details are described below.

First we describe the POD step. In the previous subsection, the kinematically linked engine was forced to have a flywheel trajectory which followed a chirp signal. For the driven engine, the displacer motion is a direct input, and we therefore used the displacer position as a chirped input with the same range as previously stated, 40 rad/s to 600 rad/s. We used the driven model (13) where the displacer velocity was chosen so that the displacer motion is the required chirp signal. A POD analysis was done on the resulting data and a reduced model using the first four modes was chosen. These POD modes are shown in Figure 8.

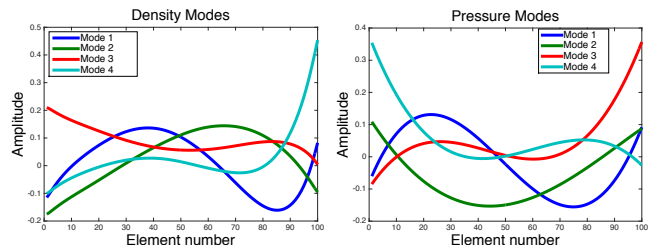


Fig. 8. The first four POD modes for the density (left) and pressure (right) profiles of the driven engine model. These modes were obtained with a chirp input driving signal. The horizontal axis is the element number in the spatial discretization of the no-Acoustics model of section IV-A. Note the approximate odd reflection symmetry (between pressure and density) of modes 1 and 2, and the approximate even reflection symmetry of modes 3 and 4.

The “frequency response” comparison between the full and reduced model was performed as follows. Both models were simulated with the same pure sinusoid as input. The outputs were then periodic signals with several harmonic components. The amplitude of each of those harmonics was found, and this analysis was repeated for a range of input signal frequencies. Figure 9 (Top) shows the results where only the first four harmonics of the outputs are shown (higher harmonics' amplitudes were too small to be relevant). As the results show, the first two or three harmonics match up quite well and deviation only begins once the spectral content has dropped by at least an order of magnitude. Because both models are nonlinear, changing the amplitude of the input signal may result in

a different frequency response. To test this, the displacer amplitude was increased by 50 percent and the experiment was repeated with the same POD modes used previously. Those results are shown in Figure 9 (Bottom) and indicate similarly good reduced model fidelity.

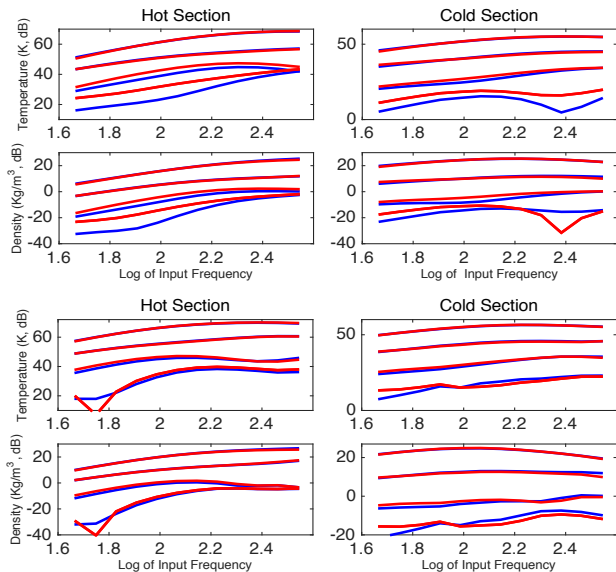


Fig. 9. Frequency response comparison of the full (blue) and a 4 state chirp-POD model (red). The horizontal axes are the frequency of the displacer input, which is a pure sinusoid. The vertical axes are the amplitudes of the first 4 harmonics of the respective outputs. The bottom figure is the response of the systems to an input with amplitude 50% larger than the top figure. The two figures would not necessarily be similar since this is the frequency response of a nonlinear system.

It is worth mentioning that this reduced chirp-POD model of the driven engine has a total state dimension of 15 (8 states for the regenerator model, 4 for the two gas sections and 3 for the piston dynamics). This model gives trajectories that are essentially indistinguishable from the full order 406 states model. In addition to a smaller state dimension, the reduced model also runs much faster. For a particular displacer motion, one cycle of the actuated engine was simulated with the original regenerator model, the acoustics removed version, and the reduced chirp-POD model. The simulation time for the original model was 1203 seconds, the acoustics removed version took 72 seconds, and the reduced chirp-POD model took 12 seconds (100 times faster). Most of this time savings comes from the fact that the time stepping routine can now take larger steps, as there are no longer fast acoustic phenomena that require a very small time step to accurately capture.

VI. CONCLUSIONS

We have shown that effective and significant model order reduction is possible for Stirling engine regenerators. Depending on the operating regime of the engine, reductions from model orders in the hundreds to single digits is possible. More generally, our main contribution is the development of a methodology for model reduction

of systems with compressible gas dynamics. By identifying several non-dimensional parameters that characterize different types of engines, we used singular perturbation methods to obtain a hierarchy of simplified models.

The regenerator has distributed dynamics, while the remaining sections of the engine have lumped models. We therefore had to address the issue of how one does POD, or balanced truncation type model reduction, with the objective that the reduced model works well when connected with the remainder of the engine. This might be thought of as a feedback-aware model reduction objective. For the Stirling engine, it appears that the chirp-POD technique we used is particularly suited since the overall system operates eventually in a limit cycling mode. We anticipate that this technique might be useful for other cyclically operating devices common in energy harvesting and thermoacoustics. In particular, when the limit cycle itself is to be designed such as in Optimal Periodic Control, reduced models such as the ones we presented can significantly reduce the computational complexity of optimal control calculations, while retaining the requisite fidelity in dynamics. This is the intended application for this work.

Finally we note that although this paper is primarily concerned with Stirling engine regenerators, much of our model reduction work is likely applicable to the regenerators (stacks) of thermoacoustic engines and heat pumps [33]–[36]. The underlying physical mechanisms are quite similar, though the operating regimes in parameter space might be different.

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APPENDIX

APPENDIX

Equation (2) is simplified by using equation (1) as follows

$$\begin{aligned} (v\rho)_t &= -(v^2\rho)_x - p_x - \beta v, \\ v_t\rho + v\rho_t &= -2vv_x\rho - v^2\rho_x - p_x - \beta v, \\ v_t\rho + v(-v\rho_x - v_x\rho) &= -2vv_x\rho - v^2\rho_x - p_x - \beta v, \\ v_t\rho &= -vv_x\rho - p_x - \beta v, \\ v_t &= -vv_x - \frac{1}{\rho}(p_x + \beta v), \end{aligned}$$

where we have used equation (1) to go from the 2nd line to the 3rd. The rest follows from the chain rule.

For notational simplicity we will replace the matrix/gas heat exchange term with the variable q . We start from equation (3) and use equation (1) and (2) (4th line below) as follows

$$\left(\rho c_v T + \frac{\rho v^2}{2}\right)_t = -\left(v\left(\rho c_v T + \frac{\rho v^2}{2}\right) + pv\right)_x + q,$$

$$(\rho c_v T)_t + \frac{1}{2} (\rho v^2)_t = - (v \rho c_v T)_x - \frac{1}{2} (\rho v^3)_x - (pv)_x + q,$$

$$(\rho c_v T)_t + \frac{1}{2} (\rho_t v^2 + 2 \rho v v_t) = - (v \rho c_v T)_x - \frac{1}{2} (\rho_x v^3 + 3 \rho v^2 v_x) - (p_x v + p v_x) + q,$$

$$\frac{1}{2} \left((-v_x \rho - v \rho_x) v^2 + 2 \rho v \left(-v v_x - \frac{1}{\rho} (p_x + \beta v) \right) \right) + (\rho c_v T)_t = - (v \rho c_v T)_x - \frac{1}{2} (\rho_x v^3 + 3 \rho v^2 v_x) - (p_x v + p v_x) + q,$$

$$(\rho c_v T)_t - \frac{1}{2} (v^2 v_x \rho + v^3 \rho_x) - v^2 v_x \rho - p_x v - \beta v^2 = - (v \rho c_v T)_x - \frac{1}{2} (\rho_x v^3 + 3 \rho v^2 v_x) - (p_x v + p v_x) + q,$$

$$(\rho c_v T)_t - \frac{3}{2} v^2 v_x \rho - \frac{1}{2} v^3 \rho_x - p_x v - \beta v^2 = - (v \rho c_v T)_x - \frac{1}{2} \rho_x v^3 - \frac{3}{2} \rho v^2 v_x - (p_x v + p v_x) + q,$$

$$(\rho c_v T)_t = - (v \rho c_v T)_x - p v_x + q + \beta v^2.$$

We now use the relation $c_v \rho T = \frac{c_v}{R} p$, which is derived from the ideal gas law, and we arrive at

$$p_t = -\gamma p v_x - v p_x + \bar{\gamma} q + \beta v^2,$$

which is the desired result.

Recall equation (55)

$$\gamma \alpha(t) (p^2)_{xx} + \epsilon_f \frac{\bar{\gamma} L k_g}{R \bar{\rho} \bar{v}} (2\alpha(t) \Phi(x) - (p^2)_x) = 0,$$

and rewrite in the form

$$\left(f'' - \frac{k_t}{\alpha} f' \right) = -2k_t \Phi \quad (59)$$

$$k_t := \frac{\epsilon_f \bar{\gamma} L k_g}{\gamma R \bar{\rho} \bar{v}}, \quad (60)$$

where $f := p^2$, and the boundary conditions are $f(0) = p_0^2$ and $f(1) = p_1^2$. In addition, observe that the matrix temperature

$$\Phi(x) = H_1 + (H_1 - H_0)x,$$

satisfies the following ODE

$$\Phi''(x) = 0,$$

with the boundary conditions $\Phi(0) = H_0$ and $\Phi(1) = H_1$.

$$\begin{aligned} \frac{d}{dx} \begin{bmatrix} \Phi \\ \Phi' \\ f \\ f' \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -2k_t & 0 & 0 & \frac{k_t}{\alpha} \end{bmatrix} \begin{bmatrix} \Phi \\ \Phi' \\ f \\ f' \end{bmatrix} \\ \begin{bmatrix} H_0 \\ H_1 \\ p_0^2 \\ p_1^2 \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Phi \\ \Phi' \\ f \\ f' \end{bmatrix}_0 + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \Phi \\ \Phi' \\ f \\ f' \end{bmatrix}_1 \end{aligned}$$

where the second equation represents the 4 boundary conditions as algebraic constraints.

The above equations are of the form of the Two Point Boundary Value State Space Realization (TPBVS)

$$\begin{aligned} \frac{d}{dx} \psi(x) &= A \psi(x) \\ u &= N_i \psi(x_i) + N_f \psi(x_f) \end{aligned} \quad (61)$$

where the vector u (the boundary conditions) can be thought of as an input. The following formula for the solution is easily derived by (a) combining the relation $\psi(x_f) = e^{A(x_f-x_i)} \psi(x_i)$ with (61) to solve for $\psi(x_i)$ and $\psi(x_f)$ in terms of u , and then (b) expressing $\psi(x) = e^{A(x-x_i)} \psi(x_i) = e^{A(x-x_f)} \psi(x_f)$ as an average of the two equal quantities

$$\psi(x) = \frac{1}{2} \begin{bmatrix} e^{A(x-x_i)} & e^{A(x-x_f)} \end{bmatrix} \begin{bmatrix} -e^{A(x_f-x_i)} & I \\ N_i & N_f \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ u \end{bmatrix}.$$

This implies that $p(x)$ in particular must have the form

$$p^2(x) = \begin{bmatrix} c_{H_0}(x) & c_{H_1}(x) & c_{p_0}(x) & c_{p_1}(x) \end{bmatrix} \begin{bmatrix} H_0 \\ H_1 \\ p_0^2 \\ p_1^2 \end{bmatrix},$$

where the functions c are a combination of exponentials, x and x^2 in the variable x , while α appears as a parameter. In more detail

$$\begin{aligned} c_{H_0}(x) &= \alpha \left(-x^2 - \frac{2(\alpha-k_t)}{k_t} x + \frac{2\alpha-k_t}{k_t} \frac{e^{\beta(x-1)} - e^{-\beta}}{1-e^{-\beta}} \right) \\ c_{H_1}(x) &= \alpha \left(x^2 + \frac{2\alpha}{k_t} x - \frac{2\alpha+k_t}{k_t} \frac{e^{\beta(x-1)} - e^{-\beta}}{1-e^{-\beta}} \right) \\ c_{p_0}(x) &= 1 - \frac{e^{\beta(x-1)} - e^{-\beta}}{1-e^{-\beta}} \\ c_{p_1}(x) &= \frac{e^{\beta(x-1)} - e^{-\beta}}{1-e^{-\beta}} \end{aligned}$$

where

$$\beta := k_t/\alpha.$$

Note that these equations can be written in a compact form in terms of the differences $H_0 - H_1$ and $p_0^2 - p_1^2$ as follows

$$\begin{aligned} p^2(x) &= \alpha \left(-x^2 - \frac{2(\alpha-k_t)}{k_t} x + \frac{2\alpha-k_t}{k_t} \frac{e^{\beta(x-1)} - e^{-\beta}}{1-e^{-\beta}} \right) (H_0 - H_1) \\ &\quad + 2\alpha \left(x - \frac{e^{\beta(x-1)} - e^{-\beta}}{1-e^{-\beta}} \right) H_1 \\ &\quad + p_0^2 - \left(\frac{e^{\beta(x-1)} - e^{-\beta}}{1-e^{-\beta}} \right) (p_0^2 - p_1^2) \end{aligned} \quad (62)$$

An important consideration is the dependence of these functions on the parameter α in the limit as $\alpha \rightarrow 0$, which is equivalent to $\beta \rightarrow \infty$. Each term in the above functions goes to a finite limit as $\beta \rightarrow \infty$. However, the dependence on $p_0^2 - p_1^2$ limits to a function that may have a discontinuity at $x = 1$. Note however that $\alpha = 0$ implies that $p_x(x) = 0$, which in turn implies that $p_0^2 - p_1^2 = 0$, and therefore in that case the solution is continuous, constant function $p(x) = p_0 = p_1$.

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