

# Structured Stochastic Uncertainty

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**Abstract**—We consider linear time invariant systems in feedback with structured stochastic uncertainties. This setting encompasses linear systems with both additive and multiplicative noise. We provide a purely input-output treatment of these systems without recourse to state space models, and thus our results are applicable to certain classes of distributed systems. We derive necessary and sufficient conditions for mean square stability in terms of the spectral radius of a linear matrix operator whose dimension is that of the number of uncertainties, rather than the dimension of any underlying state space models. Our condition is applicable to the case of correlated uncertainties, and reproduces earlier results for uncorrelated uncertainties.

## I. INTRODUCTION

The setting we consider is that of a discrete-time Linear Time Invariant (LTI) system  $\mathcal{G}$  in feedback with gains  $\delta_1, \dots, \delta_n$  (see Figure 3). These gains are random processes that are temporally independent, but possibly mutually correlated. The setting of LTI systems in feedback with structured uncertainties is common in the robust controls literature where the uncertainties are typically norm-bounded operators, real or complex deterministic gains [1]–[8]. The setting where the uncertainties are stochastic has been relatively less studied [9]–[12], but it is well known that the necessary and sufficient condition for mean square stability in the presence of structured stochastic uncertainties is a bound on the spectral radius of a matrix of  $H^2$  norms of all the subsystems of  $\mathcal{G}$ .

Our aim is to provide a rather elementary and purely input-output treatment and derivation of the necessary and sufficient condition for mean square stability. In the process, we define a new object, a linear matrix operator, which captures how a feedback system amplifies covariances of signals in a loop. A pleasant side effect is that the conditions in the case of correlated uncertainties (which have been unknown) are almost as easy to state as the ones for uncorrelated uncertainties. Those earlier results on uncorrelated uncertainties are easy to reproduce from the conditions we provide.

## II. PRELIMINARIES

All the signals we consider are defined on the half-infinite, discrete-time interval  $\mathbb{Z}^+ = [0, \infty) \subset \mathbb{Z}$ . The dynamical systems we consider are maps between various signal spaces

over the time interval  $\mathbb{Z}^+$ . This is done in contrast with the standard setting over  $\mathbb{Z}$  since stability arguments involve the growth of signals starting from some initial time.

For any random variable (or vector)  $v$ , we use  $\sigma_v := \mathcal{E}\{v^*v\}$  to denote its variance, and  $\Sigma_v := \mathcal{E}\{vv^*\}$  to denote its covariance matrix. A stochastic process  $u$  is a one-sided sequence of random variables  $\{u_k; k \in \mathbb{Z}^+\}$ . We will thus denote by  $\sigma_{u_k} := \mathcal{E}\{u_k^*u_k\}$  the sequence of its variances, and by  $\Sigma_{u_k} = \mathcal{E}\{u_k u_k^*\}$  the sequence of its inter-component correlation matrices. A process  $u$  is termed *second order* if it has finite covariances  $\Sigma_{u_k}$  for each  $k \in \mathbb{Z}^+$ . Although the processes we consider are technically not stationary (stationary processes are defined over the doubly infinite time axis), it can be shown that they are asymptotically stationary in the sense that their statistics become approximately stationary in the limit of large time, or quasi-stationary in the terminology of [13]. This fact is not used in our treatment here and the preceding comment is only included for clarification.

### A. Input-output definition of mean square stability

Let  $\mathcal{G}$  be a linear time invariant (MIMO) system. The system  $\mathcal{G}$  is completely characterized by its impulse response which is a matrix valued sequence  $\{G_k; k \in \mathbb{Z}^+\}$ . The action of  $\mathcal{G}$  on an input signal  $u$  to produce an output signal  $y$  is given by the convolution sum

$$y_k = \sum_{l=0}^k G_{k-l} u_l. \quad (1)$$

If the input  $u$  is a second order stochastic process, then it is clear from (1) that  $y_k$  has finite variance for any  $k$ , even in the cases where this variance may be unbounded in time. This leads to the following input-output definition of Mean-Square Stability.

*Definition 1:* The linear time invariant system  $\mathcal{G}$  is called *Mean-Square Stable* (MSS) if for each second order white input process  $u$  with uniformly bounded variance, the corresponding output process  $y = \mathcal{G}u$  has uniformly bounded variance

$$\sigma_{y_k} := \mathcal{E}\{y_k^*y_k\} \leq M \left( \sup_k \sigma_{u_k} \right), \quad (2)$$

where  $M$  is a constant independent of  $k$  and the process  $u$ .

In this paper we deal exclusively with this kind of stability, and we therefore refer to MSS stable systems as simply *stable*.

A standard calculation shows that

$$\sigma_{y_k} = \sum_{l=0}^k |G_l|^2 \sigma_{u_{k-l}} \quad (3)$$

when  $u$  is a white process. A uniform bound can be deduced from the following inequality

$$\sup_k \sigma_{y_k} \leq \left( \sum_{l=0}^{\infty} |G_l|^2 \right) \left( \sup_k \sigma_{u_k} \right). \quad (4)$$

Such quantities will occur often in the sequel, so we adopt the notation

$$\|\sigma_y\|_{\infty} := \sup_k \sigma_{y_k},$$

and note that the bound (4) can be rewritten in terms of the  $H^2$  norm of  $\mathcal{G}$  as

$$\|\sigma_y\|_{\infty} \leq \|\mathcal{G}\|_2^2 \|\sigma_u\|_{\infty}. \quad (5)$$

It is easy to see that equality holds when  $u$  has constant variance. Conversely, if  $\mathcal{G}$  does not have finite  $H^2$  norm, equation (3) shows that any input with constant variance causes  $\sigma_{y_k}$  to grow unboundedly, and thus the bound (2) will not hold for any finite  $M$ .

In summary, we can conclude that a linear time invariant system  $\mathcal{G}$  is MSS if and only if it has finite  $H^2$  norm, and in that case the inequality (5) holds, with equality in the case of the input having equal (in time) variance.

For a feedback interconnection, we define the MSS stability of the overall system in a manner parallel to the conventional scheme of injecting exogenous disturbance signals into all loops. Consider the feedback system 1 with  $d_1$  and  $d_2$  being white second order processes, and  $\mathcal{G}_1$  and  $\mathcal{G}_2$  are linear causal systems. We say that the feedback system is MSS if all signals  $u_1$ ,  $u_2$ ,  $y_1$  and  $y_2$  have finite variance uniformly in time.

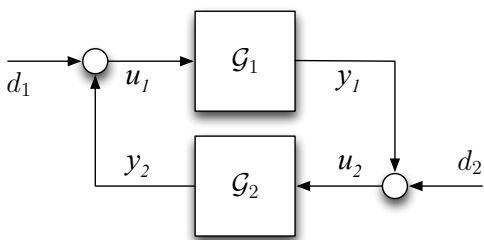


Fig. 1. MSS stability for a feedback interconnection

### B. The MIMO case

When  $\mathcal{G}$  is a MIMO system, the convolution input-output description (1) still holds with  $u$  and  $y$  as vector signals and  $\{G_k\}$  a sequence of matrices. Consider input signals that are temporally white but with correlated components such as

$$\mathcal{E} \{u_{k_1} u_{k_2}^*\} = \delta_{k_1 - k_2} \Sigma_{u_{k_1}},$$

where  $\delta$  is the Kronecker delta function, and  $\Sigma_{u_k}$  is the time varying, inter-component correlation matrix of  $u_k$ . By thinking of the vector component index as a “spatial” variable, we refer to such signals as temporally white and spatially correlated.

A standard calculation then shows that for  $y$  and  $u$  related by (1), their instantaneous correlations matrices are related by

$$\mathcal{E} \{y_k y_k^*\} =: \Sigma_{y_k} = \sum_{l=0}^k G_{k-l} \Sigma_{u_l} G_{k-l}^*. \quad (6)$$

### C. Multiplication of processes

Given any two vector-valued stationary stochastic processes  $u$  and  $h$ , their element-by-element product process

$$\begin{bmatrix} y_1(k) \\ \vdots \\ y_n(k) \end{bmatrix} =: \begin{bmatrix} h_1(k) u_1(k) \\ \vdots \\ h_n(k) u_n(k) \end{bmatrix}$$

is also a stationary process.

## III. SISO UNSTRUCTURED UNCERTAINTY

We now consider the simplest case of uncertainty analysis depicted in Figure 2.  $\mathcal{G}$  is a strictly causal LTI system,  $d$

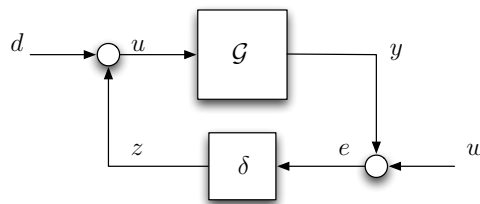


Fig. 2. LTI system in feedback with unstructured stochastic uncertainty

and  $w$  are exogenous white processes with uniform variance, and  $\delta$  is a white process with uniform variance  $\sigma_{\delta}$  and independent of the signals  $d$  and  $w$ . We assume  $\mathcal{G}$  to have finite  $H^2$  norm.

The stability conditions we derive follow from a type of “small gain” analysis of the feedback interconnection in Figure 2 based on the variances of the signals in the loop. We therefore begin by deriving some basic relations between those variances.

An important consequence of the independence of  $\delta$  and the exogenous signals is that the  $\delta$  block “whitens” its input signal  $e$ , i.e. even though  $e$  will in general be colored,  $z$  is white. This can be seen by letting  $k_2 > k_1$  and calculating

$$\begin{aligned} \mathcal{E} \{z_{k_1} z_{k_2}\} &= \mathcal{E} \{\delta_{k_1} e_{k_1} \delta_{k_2} e_{k_2}\} \\ &= \mathcal{E} \{\delta_{k_1} e_{k_1} e_{k_2}\} \mathcal{E} \{\delta_{k_2}\} = 0, \end{aligned}$$

due to the independence of  $\delta_{k_2}$  of the other signals. In fact, a consequence of the strict causality assumption is that current and future values  $\delta$ ,  $d$  and  $w$  are independent of past values of any of the internal signals. When  $k_1 = k_2$ , we get

$$\sigma_{z_k} = \mathcal{E} \{e_k^2\} = \mathcal{E} \{\delta_k^2 e_k^2\} = \sigma_{\delta} \sigma_{e_k}. \quad (7)$$

Similarly, the signal  $u$  is also white as can be seen from the following calculation with  $k_2 > k_1$

$$\begin{aligned}\mathcal{E}\{u_{k_1}u_{k_2}\} &= \mathcal{E}\{(d_{k_1} + z_{k_1})(d_{k_2} + z_{k_2})\} \\ &= \mathcal{E}\{d_{k_1}z_{k_2}\} + \mathcal{E}\{d_{k_2}z_{k_1}\} \\ &= \mathcal{E}\{d_{k_1}\delta_{k_2}e_{k_2}\} + 0 \\ &= \mathcal{E}\{d_{k_1}e_{k_2}\}\mathcal{E}\{\delta_{k_2}\} = 0.\end{aligned}$$

For the case of  $k_2 = k_1$  we get

$$\sigma_{u_k} = \sigma_{d_k} + \sigma_{z_k} = \sigma_d + \sigma_\delta \sigma_{e_k}. \quad (8)$$

For the other summing junction we observe that even though  $y$  is colored in general, it is uncorrelated with  $w$ , which results in the relation

$$\sigma_{e_k} = \sigma_{y_k} + \sigma_w. \quad (9)$$

Finally we recall the variance inequality (5) between the signals  $u$  and  $y$  which follows from the assumption that  $\mathcal{G}$  is MSS together with the conclusion above that  $u$  is white.

We are now in a position to state the main stability result for unstructured stochastic perturbations.

*Lemma 3.1:* Consider the system in Figure 2 with  $\mathcal{G}$  a stable LTI system and  $\delta$  a white process with variance  $\sigma_\delta$ . The feedback system is Mean-Square Stable if and only if

$$\|\mathcal{G}\|_2^2 \sigma_\delta < 1.$$

*Proof:* We assume  $\sigma_\delta = 1$ . The general case follows by the usual simple scaling.

“*if*”) This is similar to standard sufficiency small gain arguments, but using variances rather than signal norms. First observe that (8) yields

$$\|\sigma_u\|_\infty \leq \|\sigma_e\|_\infty + \sigma_d,$$

while (9) with (5) yields

$$\|\sigma_e\|_\infty \leq \|\mathcal{G}\|_2^2 \|\sigma_u\|_\infty + \sigma_w.$$

These two bounds can be combined as

$$\begin{aligned}\|\sigma_u\|_\infty &\leq \|\mathcal{G}\|_2^2 \|\sigma_u\|_\infty + \sigma_w + \sigma_d \\ \|\sigma_e\|_\infty &\leq \|\mathcal{G}\|_2^2 \|\sigma_e\|_\infty + \|\mathcal{G}\|_2^2 \sigma_d + \sigma_w.\end{aligned}$$

Combining these bounds with the condition  $\|\mathcal{G}\|_2^2 < 1$  gives bounds for the internal signals  $u$  and  $e$  in terms of the exogenous signals  $d$  and  $w$

$$\begin{aligned}\|\sigma_u\|_\infty &\leq \frac{1}{1 - \|\mathcal{G}\|_2^2} (\sigma_w + \sigma_d), \\ \|\sigma_e\|_\infty &\leq \frac{1}{1 - \|\mathcal{G}\|_2^2} (\|\mathcal{G}\|_2^2 \sigma_d + \sigma_w).\end{aligned}$$

In addition, the remaining internal signals  $z$  and  $y$  also have bounded variances as follows from (7) and (5) respectively.

“*only if*”) We assume that  $\|\mathcal{G}\|_2^2 \geq 1$  and show that if  $d$  is a white, constant variance process and  $w = 0$ , then  $\sigma_{u_k}$  is an unbounded sequence.

From (8), (9), and (5) we have

$$\sigma_{u_k} = \sum_{l=0}^k G_{k-l}^2 \sigma_{u_l} + \sigma_d.$$

Consider any time horizon  $\bar{k}$ , and note that the quantity  $\alpha := \sum_{l=0}^{\bar{k}} G_l^2$  can be made arbitrarily close to  $\|\mathcal{G}\|_2 \geq 1$ . The monotonicity of the sequence  $\sigma_u$  gives the following lower bounds

$$\begin{aligned}\sigma_{u_{n\bar{k}}} &\geq \sum_{l=0}^{n\bar{k}} G_{n\bar{k}-l}^2 \sigma_{u_l} + \sigma_d \\ &\geq \sum_{l=(n-1)\bar{k}}^{n\bar{k}} G_{n\bar{k}-l}^2 \sigma_{u_l} + \sigma_d \\ &\geq \left( \sum_{l=0}^{\bar{k}} G_l^2 \right) \min_{(n-1)\bar{k} \leq l \leq n\bar{k}} \sigma_{u_l} + \sigma_d \\ &= \alpha \sigma_{u_{(n-1)\bar{k}}} + \sigma_d.\end{aligned}$$

This is a difference inequality (in  $n$ ) which has the initial condition  $\sigma_{u_0} = \sigma_d$  (this follows from the strict causality of  $\mathcal{G}$ ). A simple induction argument gives

$$\sigma_{n\bar{k}} \geq (\alpha^n + \dots + \alpha + 1) \sigma_d. \quad (10)$$

Now if  $\|\mathcal{G}\|_2 > 1$ , then we can choose a time horizon  $\bar{k}$  such that  $\alpha > 1$ , and (10) shows that  $\sigma_{u_{n\bar{k}}}$  (and thus  $\sigma_{u_k}$ ) is a geometrically increasing sequence. The case  $\|\mathcal{G}\|_2 = 1$  is slightly more delicate. We can choose  $\bar{k}$  such that  $\alpha$  is as close to 1 as desired. For  $\alpha < 1$  we also have that

$$\lim_{n \rightarrow \infty} (\alpha^n + \dots + \alpha + 1) = \frac{1}{1 - \alpha}.$$

Thus  $n$  can be chosen such that

$$\sigma_{n\bar{k}} \geq \frac{1}{1 - \alpha} - \epsilon$$

for any  $\epsilon > 0$ . Now given any lower bound  $B$ , choose  $\bar{k}$  and  $n$  such that  $\alpha$  is sufficiently close to 1 and  $\epsilon$  is sufficiently small so that

$$\sigma_{n\bar{k}} \geq \frac{1}{1 - \alpha} - \epsilon > B.$$

This proves that  $\sigma_u$  is an unbounded sequence even though it may not have geometric growth.

Two remarks are in order regarding the necessity part of the previous proof. First is that we did not need to construct a so-called “destabilizing” perturbation as is typical in worst case perturbation analysis. Perturbations here are described statistically rather than members of sets, and variances will always grow when the stability condition is violated. Second, the necessity argument can be interpreted as showing that  $\|\mathcal{G}\|_2 \geq 1$  implies that the transfer function  $(1 - \mathcal{G}(z))$  has a zero in the interval  $[0, \infty)$ , and thus  $(1 - \mathcal{G}(z))^{-1}$  has an unstable pole. The argument presented above however is more easily generalizable to the MIMO case we consider in the sequel.

#### IV. STRUCTURED UNCERTAINTY

We now consider the situation where the uncertainty  $\Delta$  is diagonal as in Figure 3, i.e.

$$\Delta = \text{diag}(\delta_1, \dots, \delta_n).$$

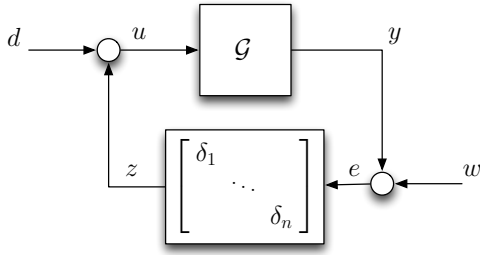


Fig. 3. A feedback system with a structured, diagonal perturbation

Assume the  $\delta_i$ 's to be temporally white, but possibly mutually correlated. Let  $\delta(k)$  denote the vector  $\delta(k) := [\delta_1(k) \ \cdots \ \delta_n(k)]^T$ . The instantaneous correlations of the  $\delta$ 's can be expressed with the matrix

$$\Sigma_\delta := \mathcal{E} \{ \delta(k) \delta^*(k) \},$$

which we will assume to be independent of  $k$ .

For later reference, we will need to calculate quantities like  $\mathcal{E} \{ \Delta M \Delta \}$  for some matrix  $M$

$$\begin{aligned} \mathcal{E} \{ \Delta M \Delta^* \} &= \mathcal{E} \left\{ \begin{bmatrix} \delta_1 & & \\ & \ddots & \\ & & \delta_n \end{bmatrix} M \begin{bmatrix} \delta_1 & & \\ & \ddots & \\ & & \delta_n \end{bmatrix} \right\} \\ &= \Sigma_\delta \circ M, \end{aligned}$$

the Hadamard (element-by-element) product of  $\Sigma_\delta$  and  $M$ . Thus, if  $e$  and  $z$  are the input and output signals (respectively) to the  $\Delta$  block then

$$\Sigma_{z_k} = \mathcal{E} \{ \Delta_k e_k e_k^* \Delta_k^* \} = \Sigma_\delta \circ \Sigma_{e_k}.$$

In the special case where the perturbations are uncorrelated and all have unit variance, then  $\Sigma_\delta = I$ , and we get the simple expression

$$\Sigma_{z_k} = \text{diag}(\Sigma_{e_k}),$$

where  $\text{diag}(M)$  is a diagonal matrix made up of the diagonal entries of the matrix  $M$ . Observe that if the  $\delta$ 's are white and mutually uncorrelated, then the vector signal  $z$  is temporally and spatially uncorrelated even though  $e$  may have both types of correlations. In other words, a structured perturbation with uncorrelated components will “spatially whiten” its input.

The key to the mean square stability analysis of the system in Figure 3 is to consider the deterministic system of evolution of covariance matrices in Figure 4. The signals in this feedback system are matrix-valued and they take values in the cone of positive semi-definite matrices. We now give the main result of this paper which is the mean square stability condition for the system in Figure 3. The proof of this result is contained in the paragraphs of the remainder of this section. The specialization of this result to uncorrelated uncertainties which recovers the standard result is shown in Appendix D.

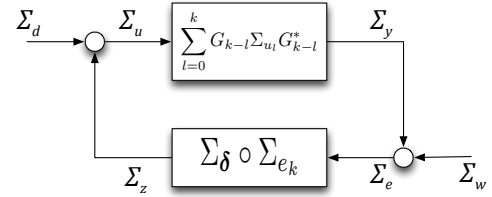


Fig. 4. The feedback dynamics of the covariance matrices. The forward path is a matrix convolution operation while the feedback path is a memoryless matrix-valued gain represented by the Hadamard product with  $\Sigma_\delta$ . When signals are considered to take values in the cone of positive semi-definite matrices, then this feedback system can be shown to be *monotone*.

*Theorem 4.1:* Consider the system in Figure 3 and the linear matrix operator

$$\mathcal{L}(X) := \Sigma_\delta \circ \left( \sum_{l=0}^{\infty} G_l X G_l^* \right), \quad (11)$$

where  $\Sigma_\delta$  is correlation matrix of the uncertainties and  $\{G_k\}$  is the matrix-valued impulse response sequence of the LTI system  $\mathcal{G}$ . The system is Mean Square Stable (MSS) if and only if

$$\rho(\mathcal{L}) < 1.$$

We note that  $\mathcal{L}$  is a finite-dimensional object, it is a linear operator mapping  $n \times n$  matrices to  $n \times n$  matrices. It has a finite number of eigenvalues. In the absence of any additional structure, this calculation involves at worst the calculation of the eigenvalues of an  $n^2 \times n^2$  matrix as follows. Let  $\text{vec}(X)$  denote the “vectorization” operation of converting a matrix  $X$  into a vector by stacking up its columns. It is then not difficult to show that (11) can be equivalently written as

$$\text{vec}(\mathcal{L}(X)) = \underbrace{\left( \text{diag}(\text{vec}(\Sigma_\delta)) \sum_{l=0}^{\infty} G_l \otimes G_l \right)}_{\text{matrix representation of } \mathcal{L}} \text{vec}(X).$$

Therefore, the eigenvalues (and corresponding eigenmatrices) of  $\mathcal{L}$  can be found by calculating the eigenvalues/vectors of its  $n^2 \times n^2$  representation above using standard methods. However, with special structure, this calculation can be significantly simplified (as in the case of uncorrelated uncertainties. See Appendix D).

We now present the proof of this theorem. It amounts to small gain calculations for the arrangement in Figure 3. Expressing  $\Sigma_{u_k}$  by following signals in the loop

$$\begin{aligned} \Sigma_{u_k} &= \Sigma_d + \Sigma_{z_k} = \Sigma_d + \Sigma_\delta \circ \Sigma_{e_k} \\ &= \Sigma_d + \Sigma_\delta \circ (\Sigma_w + \Sigma_{y_k}) \\ &= \Sigma_d + \Sigma_\delta \circ \left( \Sigma_w + \sum_{l=0}^k G_{k-l} \Sigma_{u_l} G_{k-l}^* \right), \end{aligned} \quad (12)$$

where the last equation follows from (6) and the fact that  $u_k$  is temporally white. Since  $\Sigma_{u_l}$  is a non-decreasing sequence we can bound the summation by

$$\sum_{l=0}^k G_{k-l} \Sigma_{u_l} G_{k-l}^* \leq \sum_{l=0}^k G_l \Sigma_{u_k} G_l^*.$$

A Theorem of Schur [14, Thm 2.1] implies that for any matrices  $M_1 \leq M_2$  and  $H \geq 0$ , we have  $H \circ M_1 \leq H \circ M_2$ . The last two facts allow us to replace (12) with the bounds

$$\begin{aligned} \Sigma_{u_k} &\leq \Sigma_\delta \circ \left( \sum_{l=0}^k G_l \Sigma_{u_k} G_l^* \right) + \Sigma_d + \Sigma_\delta \circ \Sigma_w \\ &\leq \Sigma_\delta \circ \left( \sum_{l=0}^{\infty} G_l \Sigma_{u_k} G_l^* \right) + \Sigma_d + \Sigma_\delta \circ \Sigma_w. \end{aligned}$$

Note that now all quantities other than  $\Sigma_{u_k}$  are independent of  $k$  and the next step is see under what conditions this last bound gives a uniform bound on the sequence  $\Sigma_{u_k}$ . The key is to rewrite the above bounds in the following form

$$(I - \mathcal{L})(\Sigma_{u_k}) \leq \Sigma_d + \Sigma_\delta \circ \Sigma_w,$$

where  $\mathcal{L}$  is the linear matrix operator

$$\mathcal{L}(X) := \Sigma_\delta \circ \left( \sum_{l=0}^{\infty} G_l X G_l^* \right).$$

It is easy to show that this operator maps positive semi-definite matrices to positive semi-definite matrices. It is thus ‘‘cone-invariant’’ in the terminology of [15] for the cone of positive semi-definite matrices. It then follows [15, Thm. 4] that we can bound

$$(1 - \rho(\mathcal{L})) \Sigma_{u_k} \leq (I - \mathcal{L})(\Sigma_{u_k}) \leq \Sigma_d + \Sigma_\delta \circ \Sigma_w.$$

We thus arrive at the sufficient condition

$$\rho(\mathcal{L}) < 1$$

for the MSS of the feedback system. This gives the uniform (in  $k$ ) bound

$$\Sigma_{u_k} \leq \frac{1}{(1 - \rho(\mathcal{L}))} (\Sigma_d + \Sigma_\delta \circ \Sigma_w).$$

The stability of  $G$  then implies in addition that all other signals in Figure 3 have bounded covariances.

For the converse, we assume  $w$  to be zero and recall equation (12)

$$\begin{aligned} \Sigma_{u_k} &= \Sigma_\delta \circ \left( \sum_{l=0}^k G_{k-l} \Sigma_{u_l} G_{k-l}^* \right) + \Sigma_d \\ &\geq \Sigma_\delta \circ \left( \sum_{l=0}^k G_{k-l} \Sigma_d G_{k-l}^* \right) + \Sigma_d. \end{aligned} \quad (13)$$

The lower bound follows from the fact that  $\Sigma_{u_l}$  is monotonic,  $\Sigma_{u_0} = \Sigma_d$ , and the previously invoked theorem of Schur stating that  $\Sigma_\delta \circ M_1 \geq \Sigma_\delta \circ M_2$  whenever  $M_1 \geq M_2$ . To study (13) we define the sequence of linear matrix operators

$$\mathcal{L}_k(X) := \Sigma_\delta \circ \left( \sum_{l=0}^k G_{k-l} X G_{k-l}^* \right). \quad (14)$$

This sequence is itself non-decreasing, in the sense that  $\mathcal{L}_k(X) \leq \mathcal{L}_{k+1}(X)$  for any symmetric  $X$ . It has been shown [16] that this implies that  $\rho(\mathcal{L}_k) \leq \rho(\mathcal{L}_{k+1})$ . The operator  $\mathcal{L} := \mathcal{L}_\infty$  (replacing the upper limit in the sum

in (14) with infinity) is clearly an upper bound, i.e. for any  $k$

$$\mathcal{L}_k \leq \mathcal{L} \quad \Rightarrow \quad \rho(\mathcal{L}_k) \leq \rho(\mathcal{L}).$$

Since  $\rho(\mathcal{L}_k)$  is thus a non-decreasing sequence of real numbers with an upper bound, it must converge to some real number which can be shown to be  $\rho(\mathcal{L})$ .

We conclude that  $\rho(\mathcal{L}_k)$  is a monotonic sequence with

$$\rho(\mathcal{L}) = \lim_{k \rightarrow \infty} \rho(\mathcal{L}_k).$$

Now we complete the necessity proof in a similar manner to that in Lemma 3.1. If  $\rho(\mathcal{L}) > 1$ , then  $\exists \bar{k}$  such that  $\rho(\mathcal{L}_{\bar{k}}) = \alpha > 1$ . Furthermore, there exists a non-zero semidefinite eigenmatrix  $X$  such that [17]

$$\mathcal{L}_{\bar{k}}(X) = \rho(\mathcal{L}_{\bar{k}}) X = \alpha X. \quad (15)$$

Referring back to (12) we obtain the following bounds

$$\begin{aligned} \Sigma_{u_{n\bar{k}}} &= \Sigma_\delta \circ \left( \sum_{l=0}^{n\bar{k}} G_{n\bar{k}-l} \Sigma_{u_l} G_{n\bar{k}-l}^* \right) + \Sigma_d \\ &\geq \Sigma_\delta \circ \left( \sum_{l=(n-1)\bar{k}}^{n\bar{k}} G_{n\bar{k}-l} \Sigma_{u_l} G_{n\bar{k}-l}^* \right) + \Sigma_d \\ &\geq \mathcal{L}_{\bar{k}} \left( \Sigma_{u_{(n-1)\bar{k}}} \right) + \Sigma_d. \end{aligned}$$

A simple induction argument shows that if we use the eigenmatrix  $X$  from (15) for  $\Sigma_d$ , we obtain

$$\Sigma_{u_{n\bar{k}}} \geq (\alpha^n + \dots + \alpha + 1) X.$$

Since  $\alpha > 1$  and  $X$  is a non-zero semidefinite matrix, then  $\Sigma_{u_l}$  is a geometrically growing sequence.

We note that the above argument produces a sort of worst case covariance  $\Sigma_d$  as the eigenmatrix of the operator  $\mathcal{L}$ . The significance of this is yet to be investigated.

## REFERENCES

- [1] M. Dahleh and Y. Ohta, ‘‘A necessary and sufficient condition for robust bibo stability,’’ *Systems & control letters*, vol. 11, no. 4, pp. 271–275, 1988.
- [2] A. Megretski, ‘‘Necessary and sufficient conditions of stability: A multiloop generalization of the circle criterion,’’ *Automatic Control, IEEE Transactions on*, vol. 38, no. 5, pp. 753–756, 1993.
- [3] B. Bamieh and M. Dahleh, ‘‘On robust stability with structured time-invariant perturbations,’’ *Systems & control letters*, vol. 21, no. 2, pp. 103–108, 1993.
- [4] J. Doyle, J. Wall, and G. Stein, ‘‘Performance and robustness analysis for structured uncertainty,’’ in *Decision and Control, 1982 21st IEEE Conference on*, vol. 21. IEEE, 1982, pp. 629–636.
- [5] M. Khammash and J. Pearson Jr, ‘‘Performance robustness of discrete-time systems with structured uncertainty,’’ *Automatic Control, IEEE Transactions on*, vol. 36, no. 4, pp. 398–412, 1991.
- [6] J. Shamma, ‘‘Robust stability with time-varying structured uncertainty,’’ *Automatic Control, IEEE Transactions on*, vol. 39, no. 4, pp. 714–724, 1994.
- [7] M. Fan, A. Tits, and J. Doyle, ‘‘Robustness in the presence of mixed parametric uncertainty and unmodeled dynamics,’’ *Automatic Control, IEEE Transactions on*, vol. 36, no. 1, pp. 25–38, 1991.
- [8] J. Doyle, ‘‘Structured uncertainty in control system design,’’ in *Decision and Control, 1985 24th IEEE Conference on*, vol. 24. IEEE, 1985, pp. 260–265.

- [9] J. Lu and R. Skelton, "Mean-square small gain theorem for stochastic control: discrete-time case," *Automatic Control, IEEE Transactions on*, vol. 47, no. 3, pp. 490–494, 2002.
- [10] N. Elia, "Remote stabilization over fading channels," *Systems & Control Letters*, vol. 54, no. 3, pp. 237–249, 2005.
- [11] D. Hinrichsen and A. Pritchard, "Stability radii for infinite dimensional systems with stochastic uncertainty," in *Proc. 3rd. European Control Conf.* Citeseer, 1995, pp. 3291–3293.
- [12] A. El Bouhtouri and A. Pritchard, "Stability radii of linear systems with respect to stochastic perturbations," *Systems & control letters*, vol. 19, no. 1, pp. 29–33, 1992.
- [13] L. Ljung, *System identification*. Wiley Online Library, 1999.
- [14] R. Horn and R. Mathias, "Block-matrix generalizations of schur's basic theorems on hadamard products," *Linear Algebra and its Applications*, vol. 172, pp. 337–346, 1992.
- [15] P. Parrilo and S. Khatri, "On cone-invariant linear matrix inequalities," *Automatic Control, IEEE Transactions on*, vol. 45, no. 8, pp. 1558–1563, 2000.
- [16] H. Tigelaar, "On monotone linear operators and the spectral radius of their representing matrices," *SIAM Journal on Matrix Analysis and Applications*, vol. 12, p. 726, 1991.
- [17] A. Berman and R. Plemmons, *Nonnegative matrices in the mathematical sciences*. Society for Industrial Mathematics, 1994, vol. 9.

## APPENDIX

### Misc facts

A.

Given  $C \geq 0$  and  $A \geq B$ , then

$$\text{tr}(AC) \geq \text{tr}(BC).$$

This can be seen by using the dyadic decomposition of  $C$ .

B.

If  $u$  and  $y$  are inputs and outputs respectively, then

$$\Sigma_{u_{k+1}} - \Sigma_{u_k} \geq 0 \Rightarrow \Sigma_{y_{k+1}} - \Sigma_{y_k} \geq 0$$

C.

For any two symmetric matrices  $A$  and  $B$

$$\lambda_{\min}(A) \text{tr}(B) \leq \text{tr}(AB) \leq \lambda_{\max}(A) \text{tr}(B).$$

The roles of  $A$  and  $B$  in the above formula can of course be reversed.

D. The operator  $X \mapsto \text{diag}(\sum_k G_k X G_k^*)$

First, consider the following operator on square matrices

$$\mathcal{D}(X) := \text{diag}(G X G^*).$$

The eigenmatrices of this operator must clearly be diagonal matrices, so we can restrict attention to understanding its action on diagonal matrices. Let  $V := \text{diag}(v_1, \dots, v_n)$  be a diagonal matrix, then  $W = \text{diag}(G X G^*)$  is also a diagonal matrix and can thus be written as  $W := \text{diag}(w_1, \dots, w_n)$ . Evaluating element-by-element leads to the following equation between the entries of  $V$  and  $W$

$$\begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} g_{11}^2 & \cdots & g_{1n}^2 \\ \vdots & \ddots & \vdots \\ g_{n1}^2 & \cdots & g_{nn}^2 \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}.$$

Thus, when restricted to diagonal matrices, the operator  $\mathcal{D}$  has as its matrix representation the matrix  $G \circ G$ , the element-by-element square of the matrix  $G$ .

$\mathcal{D}$  is clearly a linear operator, and thus the matrix representation (on diagonal matrices) of the operator  $\text{diag}(G_1 X G_1^*) + \text{diag}(G_2 X G_2^*)$  is simply  $G_1 \circ G_1 + G_2 \circ G_2$ . Clearly this fact also holds for any finite or infinite sum, e.g. the matrix representation of the operator

$$X \mapsto \text{diag}\left(\sum_k G_k X G_k^*\right) \quad (16)$$

on diagonal matrices  $X$  is

$$\sum_k G_k \circ G_k. \quad (17)$$

In particular, the (possibly non-zero) eigenvalues of the operator (16) are simply the eigenvalues of the matrix (17). Furthermore, since the matrix (17) is non-negative, its spectral radius is achieved by a real positive eigenvalue. This statement is of course also valid for the operator (16).