Networked and Distributed Parameter Systems
(Some) New Directions, Opportunities & Challenges

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## Networked vs. Distributed Parameter Systems

<table>
<thead>
<tr>
<th><strong>Spatially Distributed Systems</strong></th>
</tr>
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<tbody>
<tr>
<td>Networked/Cooperative/Distributed Control</td>
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<tr>
<td><img src="image1.png" alt="Networked System" /></td>
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### Networked vs. Distributed Parameter Systems

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### Correspondences

(Physics/Numerical Analysis perspective)

<table>
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<tr>
<th>Discrete space described by graph structure</th>
<th>Continuum space</th>
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<tr>
<td><img src="image3" alt="Graph Structures" /></td>
<td><img src="image4" alt="Continuum" /></td>
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<tr>
<th>Differential equations over large graphs</th>
<th>Numerical Methods</th>
<th>Partial Differential Equations</th>
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Continuum Models
## Networked vs. Distributed Parameter Systems

### Spatiotemporally Distributed Systems

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### Analogy with Temporal Systems  
(Systems & Controls perspective)

- **Discrete space described by graph structure**
- **Continuum space**

| ![Discrete space](image) | ![Continuum space](image) |

### Unifying Perspective:

- **Spatio-temporal** systems over discrete or continuum space
  - Signals over continuous and/or discrete time and space
  - Investigate systems properties (e.g. system norms & responses)
## Outline

### Spatially Distributed Systems

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### Look at Specific Problems

- Vehicular Strings and Consensus
- Structured Control Design
- Synchrony in AC Power Networks
- Flow Turbulence & Control
- Spatio-temporal

- Impulse Responses
- Frequency Responses
**Spatially Distributed Systems**

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<td><img src="image" alt="Distributed Parameter Systems Diagram" /></td>
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**Some Common Themes Emerge**

- The use of system norms and responses
- Large-scale (even linear) systems exhibit some surprising phenomena
- Large-scale & Regular Networks → Asymptotic statements (in system size)
- Network topology imposes asymptotic “hard performance limits”
Networked/Cooperative/Distributed Control

- Aircraft formation flight
- Formation flight in nature
- Large telescope arrays
- Robotic networks
- Flocks & swarms
- Automated highways

An area rich in deep and interesting problems.

Rapidly evolving: Applications $\cap$ Theory = incomplete, many difficult problems.
An area rich in deep and interesting problems

- Rapidly evolving: \( \text{Applications} \cap \text{Theory} = \text{incomplete} \) many difficult problems
### Vehicular Strings (Platoons)

#### Spatially Distributed Systems

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ACC, June 2014
Vehicular Platoons

Automated control of each vehicle, tight spacing at highway speeds

- Is it enough to look at neighbors? Should information be broadcast to all?
- How does performance scale with size?
- Are there any fundamental limitations?

A fundamentally difficult problem (scales badly with size) due to the network topology
Vehicular Platoons (setting)

\[ \ddot{p}_k = u_k + w_k \]

\[ \uparrow \text{control} \]
\[ \uparrow \text{disturbance} \]

- **Desired trajectory:** \( \bar{p}_k := \bar{v}t + k\Delta \) \( \text{constant velocity} \)
- **Deviations:**
\[ \tilde{p}_k := p_k - \bar{p}_k, \quad \tilde{v}_k := \dot{p}_k - \bar{v} \]
- **Controls:**
\[ u = K\tilde{p} + F\tilde{v} \]
- **Closed loop:**
\[ \frac{d}{dt} \begin{bmatrix} \dot{\tilde{p}} \\ \dot{\tilde{v}} \end{bmatrix} = \begin{bmatrix} 0 & I \\ K & F \end{bmatrix} \begin{bmatrix} \tilde{p} \\ \tilde{v} \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} w \]

*\( K, F \): matrix feedback gains \( \text{(look like “Laplacians” } \approx 2\text{nd order consensus)} \)
Relative vs. Absolute Feedback

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<th>Velocity feedback</th>
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<td>$u = K \hat{p} + F \tilde{v}$</td>
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$$
u_k = K_+ (p_{k+1} - p_k - \Delta) + F_+ (v_{k+1} - v_k)$$

$$K_o (p_k - (vt + \Delta k)) + F_o (v_k - \bar{v})$$

**RELATIVE MEASUREMENTS:**
- Requires ranging devices

**ABSOLUTE MEASUREMENTS:**
- Position: Requires knowing position relative to leader
- Velocity: Requires measurement of own velocity

$\Rightarrow$ row_sums$(K) = 0$
$\Rightarrow$ row_sums$(F) = 0$
Disorder Phenomenon in Platoons (w. only relative meas.)

Globally stable formation, but exhibits “accordion-like” large-scale modes

Time trajectories of vehicles’ positions relative to leader (bird’s-eye view)

-100 vehicles

-A large formation in a thunderstorm
Disorder Phenomenon in Platoons (w. only relative meas.)

Zoomed in (small-scale) behavior

Seems well regulated. No collisions.
Disorder Phenomenon in Platoons (w. only relative meas.)

String instability?
Let disturbances enter only at lead vehicle

Unrelated to string instability!
Disorder Phenomenon in Platoons (w. only relative meas.)

String instability? Let disturbances enter only at lead vehicle

- temporally high frequency disturbances well regulated
- temporally low frequency disturbances penetrate further into formation
Is this due to bad design, or is it inherent to this problem?

Note: Also occurs in LQR controllers that yield “localized” feedbacks

- Original formulations:
  - Athans & Levine ’66
  - Melzer & Kuo ’70

- Reexamined as $N \rightarrow \infty$
  - Jovanovic & Bamieh, TAC ’05
Disorder and Feedback “Granularity”

- Disturbances are spatially white (contain all spatial wavelengths)

- Intuition:
  - *Local feedback* can only suppress *short-scale disturbances*
  - Local feedback ineffective against large-scale (& slow) disturbances
  - Looks like *global feedback* is needed for *global regulation*
Disturbances are spatially white (contain all spatial wavelengths)

Intuition:
- *Local feedback* can only suppress *short-scale disturbances*
- Local feedback ineffective against large-scale (& slow) disturbances
- Looks like *global feedback* is needed for *global regulation*

**Surprise:** In higher spatial dimensions:
*Local feedback CAN suppress large-scale disturbances*

*cf. Harmonic Solids*
Harmonic solid: A $d$-dimensional lattice of masses and springs

**Q:** Can short range interaction lead to long range order?

- “short range interaction” $\leftrightarrow$ local feedback
- “long range order” $\leftrightarrow$ tightness of formation
Harmonic solid: A $d$-dimensional lattice of masses and springs

Q: Can short range interaction lead to long range order?

- “short range interaction” $\iff$ local feedback
- “long range order” $\iff$ tightness of formation

Studied using long range correlations

- for $d = 1, 2$ short range interactions $\Rightarrow$ no long range order
- for $d \geq 3$ long range order possible!
- i.e., solids can only exist in $d \geq 3$
Statistical Mechanics of Harmonic Solids

Harmonic solid: A $d$-dimensional lattice of masses and springs

**Q:** Can short range interaction lead to long range order?

- “short range interaction” $\leftrightarrow$ local feedback
- “long range order” $\leftrightarrow$ tightness of formation

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*Similar dimensional-dependencies occur in networked control systems?*
Related Concepts

- Optimal Performance of Distributed Estimation [Barooah, Hespanha]
- Effective Resistance in a Resistor Network
- Global Mean First Passage Time of Simple Random Walk
- Wiener Index for Molecules

Common mathematical problem: calculate sums like (cont. time)

\[ \sum_{n \neq 1} \frac{1}{\lambda_n} \]

\( \lambda_n \): eigenvalues of a graph Laplacian
Performance Limitations of Formations in d Dimensions

Setting:

- $N = M^d$ vehicles arranged in d-dimensional torus $\mathbb{Z}_M^d$
- Desired trajectory: $\bar{p}_k := vt + k\Delta$ constant speed & heading

Structural Constraints

- **Spatial Invariance:**
  State-feedbacks $K$ and $F$ are spatial-convolution operators

- **Locality:**
  $K_{(k_1,\ldots,k_d)} = 0$, if for any $i \in \{1,\ldots,d\}$, $|k_i| > q$

feedback from *local neighbors only*
Performance Measures

- Two measures of “disorder”
  - **Microscopic**: local position deviation
    \[
    \text{var} \left( p_{k+1} - p_k - \Delta \right)
    \]
  - **Macroscopic**: long range deviation
    \[
    \text{var} \left( p_N - p_1 - \Delta N \right)
    \]
    or \[
    \text{var} \left( \bar{p}_k - \frac{1}{N} \sum_l \bar{p}_l \right)
    \]

- All above obtained *asymptotically* (as \( N \to \infty \)) from \( H^2 \) norm calculations
Asymptotic Performance Lower Bounds

Tori networks, \( N \), network size = \( N \), spatial dimension = \( d \), control effort = \( \mathcal{E}\{u_k^2\} \leq U \)

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<th>Macroscopic Disorder</th>
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<td>( \frac{1}{U} )</td>
<td>( \frac{1}{U} )</td>
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<tr>
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<td>( \frac{1}{U} )</td>
<td>( \frac{1}{U} )</td>
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"Coherence in Large-Scale Networks: Dimension-Dependent Limitations of Local Feedback"

BB, Jovanovic, Mitra, Patterson  TAC, 2012
### Implications for Vehicular Platoons

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Using only local feedback:

**cannot have 1 dimensional, large and yet coherent formations!**
Role of Node Dynamics

- Each node a chain of $n$ integrators
- Controllers use local static state feedback

Critical dimension needed for global coherence $= 2n + 1$
- Tradeoff between network connectivity and node memory
### Spatial Dimension and Network Connectivity

<table>
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<th>Dimension</th>
<th>d = 1</th>
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<th>d = 3</th>
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<tr>
<td>Torus (Lattice)</td>
<td>d-dimensional</td>
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#### Macroscopic Disorder

\[
\frac{1}{N} \sum_{n \neq 1} \frac{1}{\lambda_n}
\]

#### 1st-Order Consensus

- Node degree does not quantify this phenomenon
- e.g. compare with

![Diagram of network connectivity](image)
### Spatial Dimension and Network Connectivity

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#### Macroscopic Disorder

- \( \frac{1}{N} \sum_{n \neq 1} 1/\lambda_n \)

#### 1st-order Consensus

- \( N \)
- \( \log(N) \)
- Bounded
- Bounded

- Node degree does not quantify this phenomenon

- e.g. compare

- Note: \( \frac{1}{N} \sum_{n \neq 1} 1/\lambda_n \) scales differently from \( 1/\lambda_2 \)
For general graphs, what is the corresponding notion of “spatial dimension”?

- The Hausdorff dimension of a fractal graph does not fully characterize coherence. 
  
  \[ d=1.4? \] 

  \[ d=3.2? \]

- Open question: a purely topological measure of coherence for general graphs.

\( \text{Patterson, BB, ’11 CDC} \)
Swarms and Flocks in Nature

Network dimensionality determines coherence of motion?

Starling Flocks:  Young, Scardovi, Cavagna, Giardina, Leonard, ’13, PLOS CB
Further Questions

- Can more general control laws break these limitations?
  - Spatial varying control gains?
  - Nonlinear feedback?
  - Dynamic feedback?

- Must have global feedback to address coherence problem
  - *Vulnerability to errors in global feedback (as N → ∞)?*
### Structured, Distributed Control Design

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- Vehicular Strings and Consensus
- Structured Control Design
- Synchrony in AC Power Networks
- Flow Turbulence & Control
- Spatio-temporal

- Impulse Responses
- Frequency Responses
Controller Architecture: Constraints on controller information flow

\[ \dot{x}_k = f(x_k, u_p, u_c) \]

Optimal Constrained Controller Design

- In general: difficult, non-convex, non-scalable
- Some Exceptions:
  - Partially Nested Info. Structure, Funnel Causality, Quadratic Invariance
  - Sparsity Promoting ($\ell^1$-regularized) designs
- Often possible to propose (non-optimal), scalable algorithms that “work”
  - e.g. Consensus-like algorithms (cf. multi-agent systems)
Distributed Control Systems Design

- **Controller Architecture**: Constraints on controller information flow

\[ \dot{x}_k = f(x_k, u_p, u_c) \]

- **Optimal Constrained Controller Design**
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- **Q: Why care about optimality?**
  Quantify fundamental limitations-of-performance due to *network topology*? akin to those due to RHP poles/zeros
Why care about difficult \textit{optimal/robust} control problems?

- \textbf{Optimality} gives \textit{Best Achievable Limits of performance}
  - e.g. a plant $G$ with a RHP pole $p$ and zero $z$
    \[
    \inf_{C \text{ stabilizing}} \left\| (1 + PC)^{-1} \right\|_{\infty} = \frac{|z + p|}{|z - p|} \checkmark
    \]
  - If $z \neq p$, system is both controllable/observable, the rank tests
    \[
    \text{rank} \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \quad \text{rank} \begin{bmatrix} C; CA; \cdots; CA^{n-1} \end{bmatrix}
    \]
    give a deceptive answer! (especially for large-scale systems!)
Why care about difficult \textit{optimal/robust} control problems?

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\]

give a deceptive answer! (especially for large-scale systems!)

- Use $\inf_{C \text{ stabilizing}} \| \mathcal{F}(G; C) \|$ to measure
  approximate network controllability/observability

- Optimal/Robust Control is useful to
  \textit{design/characterize a good plant, not just controller!}

A point recognized in 80’s-90’s, but has not made it into networks literature
Case Study: Vehicular Formations

Vehicular string control with only local (no leader) information

- Corresponds to banded controller structure
- This exact problem is non-convex (currently unsolved)
- *Can find lower bounds (hard performance limits) as function of topology!*
- *The platoons problem is fundamentally difficult because of the 1d topology*
Structured Optimal Control in the Limit of Large System Size

- The problem \( \inf_{C \text{ structured}} \| \mathcal{F}(G; C) \| \)
  - very difficult for finite \( N \)
  - may admit simple answers as \( N \to \infty \)
  - cf. *Statistical Mechanics*

- Use structured Robust/Optimal control problems
  - not to design network controllers, but to quantify *limits of performance*

- Implications:
  - In engineered systems: allows for selection of network structures
  - In natural systems (e.g. biological):
    - may explain naturally evolved network structures
  - *Quantify network controllability/observability*
SYNCHRONY IN AC POWER NETWORKS

SPATIALLY DISTRIBUTED SYSTEMS

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- Flow Turbulence & Control
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  - Impulse Responses
  - Frequency Responses
Machines “tug” on each other to achieve phase synchrony

Linearized dynamics (swing equations) similar to vehicle formations

\[
\frac{d}{dt} \begin{bmatrix} \theta \\ \omega \end{bmatrix} = \begin{bmatrix} 0 & I \\ -L_B & -\beta I \end{bmatrix} \begin{bmatrix} \theta \\ \omega \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} w
\]
Phase Synchronization in AC Networks

- Machines “tug” on each other to achieve phase synchrony
  Linearized dynamics (swing equations) similar to vehicle formations

\[
\frac{d}{dt} \begin{bmatrix} \theta \\ \omega \end{bmatrix} = \begin{bmatrix} 0 & I \\ -L_B & -\beta I \end{bmatrix} \begin{bmatrix} \theta \\ \omega \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} w
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- Electrical power flows back and forth as a signaling mechanism
A Thought Experiment: Network with Identical Generators

Assume identical generators but general topology

\[
\frac{d}{dt} \begin{bmatrix} \theta \\ \omega \end{bmatrix} = \begin{bmatrix} 0 & I \\ -L_B & -\beta I \end{bmatrix} \begin{bmatrix} \theta \\ \omega \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} w
\]

\[y = \begin{bmatrix} C_1 & 0 \end{bmatrix}\]

Resistive power loss over \((i, k)\) link

Total resistive losses \(\tilde{P}_{\text{loss}} = y^* y\)

\[
\tilde{P}_{\text{loss}_{ik}} = g_{ik} |\theta_i - \theta_k|^2
\]

Notes

- Network Admittance Matrix: \(Y = \text{Re}\{Y\} + j\text{Im}\{Y\} =: L_G + jL_B\)
- Linearized dynamics
- Keep only quadratic part of loss term
- Model too simple? Note: Modeling best case scenario, no instabilities
Calculating the $H^2$ Norm

Assumption: $L_G$ is a multiple of $L_B$

$$\alpha := \frac{g_{ik}}{b_{ik}} = \frac{r_{ik}}{x_{ik}} = \text{ratio of line resistance to reactance}$$

Then total resistive power loss

$$E\{y^*y\} = \frac{\alpha}{\beta} (N - 1)$$

$N$: Network Size

Total resistive losses are independent of the network topology!!
Implications

Compare:
less coherent  <  more coherent
larger phase fluctuations  >  small phase fluctuations
less links  <  more links
Resistive losses = Resistive losses

A fundamental limitation, independent of network topology
A consequence of using electrical power flows as the signaling mechanism!

"The Price of Synchrony", BB, Gayme, '13, ACC
Losses proportional to network size $N$
What if $N \approx$ millions in a future highly-distributed-generation smart grid??
Another argument for a communications layer in the smart grid
Implications

Compare:

- less coherent < more coherent
- larger phase fluctuations > small phase fluctuations
- less links < more links
- Resistive losses = Resistive losses

A fundamental limitation, independent of network topology

A consequence of using *electrical power flows* as the signaling mechanism!

“The Price of Synchrony”, BB, Gayme, ’13, ACC

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*What if $N \approx$ millions in a future highly-distributed-generation smart grid??*

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# Flow Turbulence & Control

## Spatially Distributed Systems

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<thead>
<tr>
<th>Networked/Cooperative/Distributed Control</th>
<th>Distributed Parameter Systems</th>
</tr>
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<tbody>
<tr>
<td><img src="image1" alt="Networked System" /></td>
<td><img src="image2" alt="Distributed Parameter System" /></td>
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</table>

## Look at Specific Problems

- Vehicular Strings and Consensus
  ![Vehicular System](image3)
- Structured Control Design
  ![Structured Control](image4)
- Synchrony in AC Power Networks
  ![AC Power Network](image5)
- Flow Turbulence & Control
  ![Flow Turbulence](image6)
- Spatio-temporal
  ![Spatio-temporal Response](image7)

Impulse Responses  
Frequency Responses
Turbulence in Streamlined Flows (Boundary Layers)

Streamlining a vehicle reduces form drag. Still stuck with: Skin-Friction Drag (higher in Turbulent BL than in Laminar BL). Same in pipe flows (increases required pumping power).

boundary layer turbulence

side view

top view
Turbulence in Streamlined Flows (Boundary Layers)

Streamlining a vehicle reduces *form drag*

Still stuck with: *Skin-Friction Drag* (higher in *Turbulent BL* than in *Laminar BL*)

Same in pipe flows (*increases required pumping power*)
Control of Boundary Layer Turbulence

in nature: “passive” control

active control with sensor/actuator arrays

- Intuition: must have ability to actuate at spatial scale comparable to flow structures

- spatial-bandwidth of controller $\geq$ plant’s bandwidth
Control of Boundary Layer Turbulence

in nature: “passive” control

active control with sensor/actuator arrays

- corrugated skin
- compliant skin

Intuition: must have ability to actuate at spatial scale comparable to flow structures

Spatial-bandwidth of controller $\geq$ plant’s bandwidth

Caveat: Plant’s dynamics are not well understood

Obstacles

- not only device technology
- also: dynamical modeling and control design
The Navier-Stokes (NS) equations:

\[ \partial_t \mathbf{u} = -\nabla \mathbf{u} - \text{grad } p + \frac{1}{R} \Delta \mathbf{u} \]
\[ 0 = \text{div } \mathbf{u} \]

- **Hydrodynamic Stability:** view NS as a dynamical system
- **laminar flow** \( \mathbf{u}_R := \) a stationary solution of the NS equations (an *equilibrium*)
Mathematical Modeling of Transition: Hydrodynamic Stability

The Navier-Stokes (NS) equations:

\[
\begin{align*}
\partial_t \mathbf{u} &= -\nabla \mathbf{u} - \nabla \mathbf{p} + \frac{1}{R} \Delta \mathbf{u} \\
0 &= \text{div} \mathbf{u}
\end{align*}
\]

- **Hydrodynamic Stability:** view NS as a dynamical system
- **laminar flow** $\mathbf{u}_R :=$ a stationary solution of the NS equations (an *equilibrium*)

**laminar flow $\mathbf{u}_R$ stable** $\iff$ i.c. $\mathbf{u}(0) \neq \mathbf{u}_R$, $\mathbf{u}(t) \xrightarrow{t \to \infty} \mathbf{u}_R$

- typically done with dynamics linearized about $\mathbf{u}_R$
- various methods to track further “non-linear behavior”
The Navier-Stokes (NS) equations:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= -\nabla_u u - \text{grad } p + \frac{1}{R} \Delta u \\
0 &= \text{div } u
\end{align*}
\]

- **Hydrodynamic Stability:** view NS as a dynamical system
- A very successful (*phenomenologically predictive*) approach for many decades
- **However:** *it fails badly in the special (but important) case of streamlined flows*
Mathematical Modeling of Transition: Adding Signal Uncertainty

- Decompose the fields as

\[ u = \bar{u}_R + \tilde{u} \]

\[ \uparrow \text{laminar} \quad \uparrow \text{fluctuations} \]

- Fluctuation dynamics:

In linear hydrodynamic stability, \(-\nabla \tilde{u} \tilde{u}\) is ignored.

\[
\partial_t \tilde{u} = -\nabla \bar{u}_R \tilde{u} - \nabla \tilde{u} \bar{u}_R - \text{grad} \tilde{p} + \frac{1}{R} \Delta \tilde{u} - \nabla \tilde{u} \tilde{u} + d
\]

\[
\text{div} \tilde{u} = 0
\]

- a time-varying \textit{exogenous disturbance} field \(d\)

Input-Output view of the Linearized NS Equations

\(Jovanovic, BB, '05 JFM\)
\[ \partial_t \begin{bmatrix} \Delta \tilde{v} \\ \tilde{\omega} \end{bmatrix} = \begin{bmatrix} U'' \partial_x - U \Delta \partial_x + \frac{1}{R} \Delta^2 & 0 \\ -U' \partial_z & -U \partial_x + \frac{1}{R} \Delta \end{bmatrix} \begin{bmatrix} \tilde{v} \\ \tilde{\omega} \end{bmatrix} + \begin{bmatrix} -\partial_{xy} & \partial_x^2 + \partial_z^2 & -\partial_{zy} \\ \partial_z & 0 & -\partial_x \end{bmatrix} \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix} \]

\[ \begin{bmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{bmatrix} = \left( \partial_x^2 + \partial_z^2 \right)^{-1} \begin{bmatrix} \partial_{xy} & -\partial_z \\ \partial_z & 0 \\ \partial_{xy} & \partial_z \end{bmatrix} \begin{bmatrix} \tilde{v} \\ \tilde{\omega} \end{bmatrix} \]

\[ \partial_t \Psi = A \Psi + B \mathbf{d} \]

\[ \tilde{u} = C \Psi \]
Input-Output Analysis of the Linearized NS Equations

\[
\begin{align*}
\frac{\partial}{\partial t} \begin{bmatrix} \Delta \tilde{v} \\ \tilde{w} \end{bmatrix} &= \begin{bmatrix} U'' \partial_x - U \Delta \partial_x + \frac{1}{R} \Delta^2 & 0 \\ -U' \partial_z & -U \partial_x + \frac{1}{R} \Delta \end{bmatrix} \begin{bmatrix} \tilde{v} \\ \tilde{\omega} \end{bmatrix} + \begin{bmatrix} -\partial_{xy} & \partial_x^2 + \partial_z^2 & -\partial_{zy} \\ \partial_y & 0 & -\partial_x \end{bmatrix} \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix} \\
\begin{bmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{bmatrix} &= \left( \partial_x^2 + \partial_z^2 \right)^{-1} \begin{bmatrix} \partial_{xy} & -\partial_z \\ \partial_y & 0 \\ \partial_{xz} & \partial_x \end{bmatrix} \begin{bmatrix} \tilde{v} \\ \tilde{\omega} \end{bmatrix}
\end{align*}
\]

- **eigs** (\(A\)): determine stability
  
  (standard technique in *Linear Hydrodynamic Stability*)

- **Transfer Function** \(d \rightarrow \tilde{u}\): determines response to disturbances
  \( \text{uncommon in Fluid Mechanics} \)
Input-Output Analysis of the Linearized NS Equations

\[
\partial_t \begin{bmatrix} \Delta \tilde{v} \\ \tilde{\omega} \end{bmatrix} = \begin{bmatrix} U'' \partial_x - U \Delta \partial_x + \frac{1}{R} \Delta^2 & 0 \\ -U' \partial_z & -U \partial_x + \frac{1}{R} \Delta \end{bmatrix} \begin{bmatrix} \tilde{v} \\ \tilde{\omega} \end{bmatrix} + \begin{bmatrix} -\partial_{xy} & \partial_x^2 + \partial_z^2 & -\partial_{zy} \\ \partial_y & 0 & -\partial_x \end{bmatrix} \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix}
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\[
\begin{bmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{bmatrix} = \left( \partial_x^2 + \partial_z^2 \right)^{-1} \begin{bmatrix} \partial_{xy} & \partial_x \\ \partial_{zy} & 0 \end{bmatrix} \begin{bmatrix} \tilde{v} \end{bmatrix}
\]

\[
\partial_t \Psi = A \Psi + B d \\
\tilde{u} = C \Psi
\]

Surprises:

- Even when \(A\) is stable
  
  the gain \(d \rightarrow \tilde{u}\) can be very large
  
  \((H^2 \text{ norm})^2\) scales with \(R^3\)
- Input-output resonances
  
  very different from least-damped modes of \(A\)
Modal vs. Input-Output Response

Typically: underdamped poles $\leftrightarrow$ frequency response peaks

cf. The “rubber sheet analogy”: 
However:

Pole Locations $\iff$ Frequency Response Peaks

**Theorem:** Given any desired pole locations

$$z_1, \ldots, z_n \in \mathbb{C}_- \ (LHP),$$

and any stable frequency response $H(j\omega)$, arbitrarily close approximation is achievable with

$$\left\| H(s) - \left( \sum_{i=1}^{N_1} \frac{\alpha_{1,i}}{(s - z_1)^i} + \cdots + \sum_{i=1}^{N_n} \frac{\alpha_{n,i}}{(s - z_n)^i} \right) \right\|_{\mathcal{H}^2} \leq \epsilon$$

by choosing any of the $N_k$'s large enough.
However:

**Theorem:** Given any desired **pole locations**

\[ z_1, \ldots, z_n \in \mathbb{C}_{-} \text{ (LHP)}, \]

and any stable frequency response \( H(j\omega) \), arbitrarily close approximation is achievable with

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\left\| H(s) - \left( \sum_{i=1}^{N_1} \frac{\alpha_{1,i}}{(s - z_1)^i} + \cdots + \sum_{i=1}^{N_n} \frac{\alpha_{n,i}}{(s - z_n)^i} \right) \right\|_{\mathcal{H}^2} \leq \epsilon
\]

by choosing any of the \( N_k \)'s large enough

**Remarks:**

- No necessary relation between **pole locations** and **system resonances**
- \( (\epsilon \to 0 \Rightarrow N_k \to \infty) \), i.e. this is a **large-scale systems** phenomenon
- Large-scale systems: IO behavior not always predictable from modal behavior
Modal vs. Input-Output Response

However:

<table>
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<th>Pole Locations</th>
<th>↔</th>
<th>Frequency Response Peaks</th>
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MIMO case: \( H(s) = (sI - A)^{-1} \)

- If \( A \) is normal (has orthogonal eigenvectors), then
  \[
  \sigma_{\text{max}} \left( (j\omega I - A)^{-1} \right) = \frac{1}{\text{distance} (j\omega, \text{nearest pole})}
  \]

- If \( A \) is non-normal: no clear relation between singular value plot \( \leftrightarrow \) \( \text{eigs}(A) \)
Translation invariance in $x$ & $z$ implies

- **Impulse Response** (Green’s Function)

\[
\tilde{u}(t, x, y, z) = \int G(t - \tau, x - \xi, y, y', z - \zeta) \, d(\tau, \xi, y', \zeta) \, d\tau \, d\xi \, dy' \, d\zeta
\]

\[
\tilde{u}(t, x, ., z) = \int G(t - \tau, x - \xi, ., z - \zeta) \, d(\tau, \xi, ., \zeta) \, d\tau \, d\xi \, d\zeta
\]

\[G(t, x, z) : \text{Operator-valued impulse response}\]

- **Frequency Response**

\[
\tilde{u}(\omega, k_x, k_z) = G(\omega, k_x, k_z) \, d(\omega, k_x, k_z)
\]

\[G(\omega, k_x, k_z) : \text{Operator-valued frequency response}\] (Packs lots of information!)

- **Spectrum of $A$:**

\[
\sigma(A) = \bigcup_{k_x, k_z} \sigma\left(\hat{A}(k_x, k_z)\right)
\]
Modal vs. Input-Output Analysis

\[ \partial_t \Psi = A \Psi + B \mathbf{d} \]
\[ \mathbf{\tilde{u}} = C \Psi \]

- IR: \( G(t, x, z) \)
- FR: \( G(\omega, k_x, k_z) \)
Modal vs. Input-Output Analysis

\[ \partial_t \Psi = A \Psi + B \, d \]
\[ \tilde{\Psi} = C \, \Phi \]

IR: \( G(t, x, z) \)
FR: \( G(\omega, k_x, k_z) \)

**Modal Analysis:** Look for unstable eigs of \( A \)

<table>
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<tr>
<th>Flow type</th>
<th>Classical linear theory ( R_c )</th>
<th>Experimental ( R_c )</th>
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<tbody>
<tr>
<td>Channel Flow</td>
<td>5772</td>
<td>( \approx 1,000-2,000 )</td>
</tr>
<tr>
<td>Plane Couette</td>
<td>( \infty )</td>
<td>( \approx 350 )</td>
</tr>
<tr>
<td>Pipe Flow</td>
<td>( \infty )</td>
<td>( \approx 2,200-100,000 )</td>
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Modal Analysis: Look for unstable eigens of $\mathcal{A}$

- Channel Flow @ $R = 6000$, $k_x = 1$, $k_z = 0$:

  Flow structure of corresponding eigenfunction: Tollmein-Schlichting (TS) waves
Modal vs. Input-Output Analysis

\[ \begin{align*}
\partial_t \Psi &= A \Psi + B \mathbf{d} \\
\tilde{\mathbf{u}} &= C \Psi \\
\text{IR: } G(t, x, y, -1, z) \\
\text{FR: } G(\omega, k_x, k_z)
\end{align*} \]

Impulse Response Analysis: Channel Flow $@ R = 2000$

similar to “turbulent spots”

Jovanovic, BB, ’01 ACC
Spatio-temporal Frequency Response

\( G(\omega, k_x, k_z) \) is a \textit{large} object!

one aggregation method: \( \sup_{\omega} \sigma_{\text{max}} \left( G(\omega, k_x, k_z) \right) \)

\[ \begin{array}{ccc}
    d_1 \rightarrow u & d_2 \rightarrow u & d_3 \rightarrow u \\
    d_1 \rightarrow v & d_2 \rightarrow v & d_3 \rightarrow v \\
    d_1 \rightarrow w & d_2 \rightarrow w & d_3 \rightarrow w
\end{array} \]

Jovanovic, BB, '05 JFM
Spatio-temporal Frequency Response

\( G(\omega, k_x, k_z) \) is a large object!

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What do the corresponding flow structures look like?
$G(\omega, k_x, k_z)$ is a large object!

one aggregation method: $\sup_{\omega} \sigma_{\text{max}} \left( G(\omega, k_x, k_z) \right)$

What do the corresponding flow structures look like?
Spatio-temporal Frequency Response

\( G(\omega, k_x, k_z) \) is a large object!

one aggregation method: \( \sup_{\omega} \sigma_{\text{max}} \left( G(\omega, k_x, k_z) \right) \)

What do the corresponding flow structures look like?

closer (than TS waves) to structures seen in turbulent boundary layers
Flow Control

Some recent related progress in Fluid Dynamics and Controls communities

- Henningson & Co. @ KTH
- Rowley & Co. @ Princeton
- Gayme, Doyle & Mckeon @ Caltech
- Jovanovic & Co. @ Minnesotta

Viscoelastic turbulence
Vibrational Control with Wall Oscillations

FLOW CONTROL remains

- an *under-explored* field
- with may *high-payoff* possibilities
  (both intellectually and technologically)
Recap

**Spatially Distributed Systems**

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<td>![Networked System Diagram]</td>
<td>![Distributed Parameter System Diagram]</td>
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**Some Common Themes emerge**

- The use of system norms and responses
- Large-scale & Regular Networks $\rightarrow$ Asymptotic statements *(in system size)*
- Network topology imposes asymptotic “hard performance limits”
- Large-scale (even linear) systems exhibit some surprising phenomena

- This is a very rich area with many remaining
  - fascinating questions, unsolved problems
  - research problems yet to be properly formulated
Collaborators

- M. Jovanovic
- D. Gayme
- S. Patterson
- J.C. Doyle
- B. McKeon

- M. Dahleh
- P. Mitra
- P. Voulgaris
- F. Paganini
- M.A. Dahleh

Support:

Energy, Power & Adaptive Systems Program (ECCS)
Control Systems (CMMI)
Physics of Living Systems (PHY)

Dynamics & Control Program