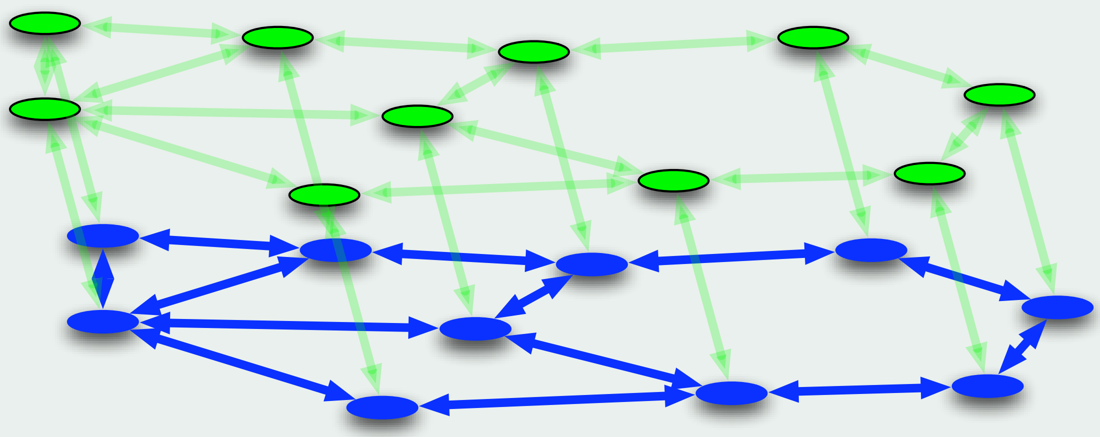
The structure of optimal distributed controllers

What you get for free, and what you can impose

Bassam Bamieh

Mechanical Engineering, UCSB

The Setting



- The Plant has *spatially distributed* dynamics
- The controller also has *spatially distributed* dynamics

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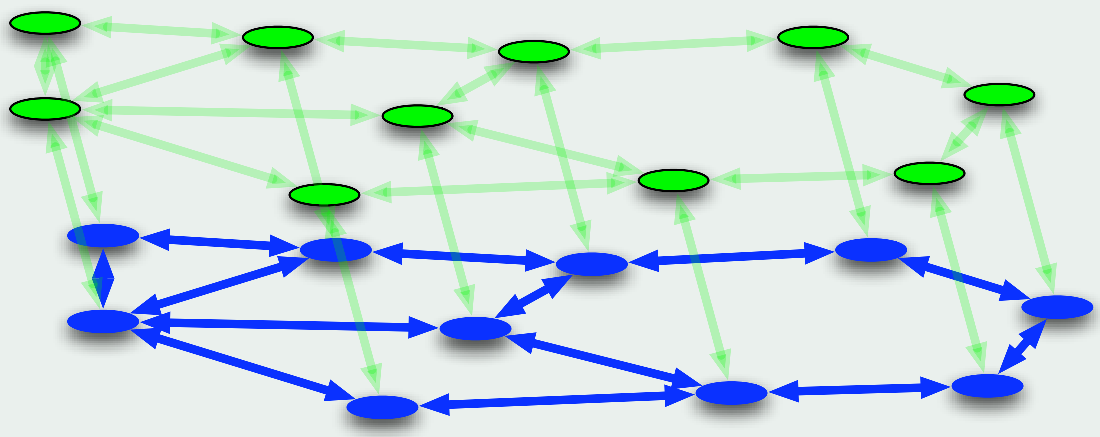
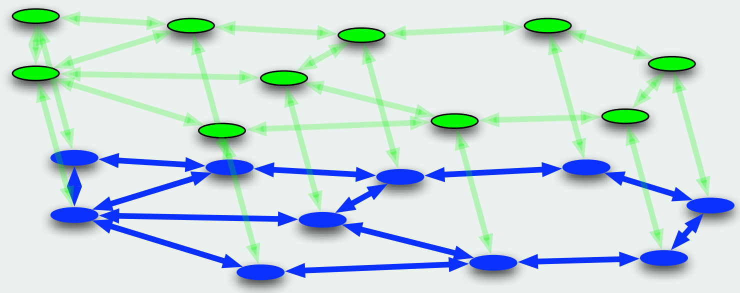


- The Plant has *spatially distributed* dynamics
- The controller also has *spatially distributed* dynamics

- For a given plant structure,
  what's the inherent structure of the *Centralized Controller*?
- If we want to constrain the controller's architecture,
  what type of constraints lead to tractable problems?

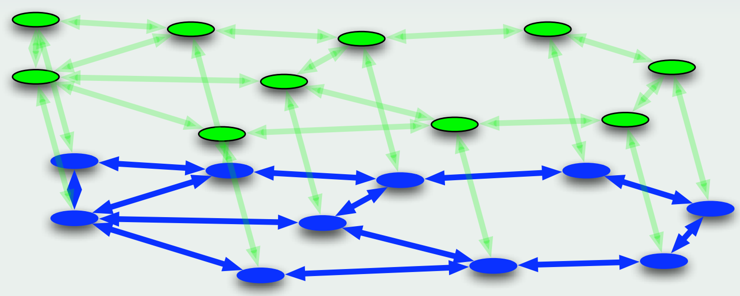
The Approach



WE WILL TAKE A BROAD VIEW OF *spatially distributed dynamics*

- Systems described by Partial Differential Equations (PDEs)
  Continuous Space
- Dynamical systems over lattices and graphs
  Discrete Space

The Approach



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Look for "interesting" special structures

$$\text{Special structure} \longrightarrow \begin{cases} \text{More detailed results} \\ \text{Insight} \end{cases}$$

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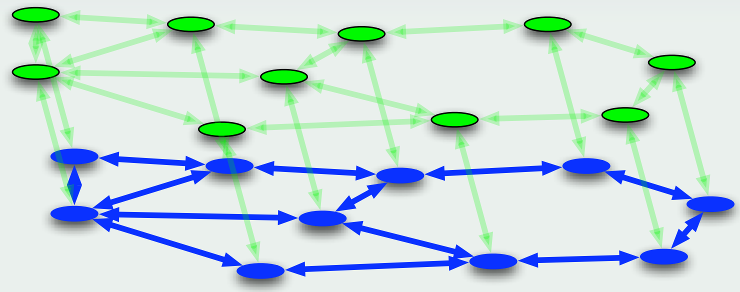

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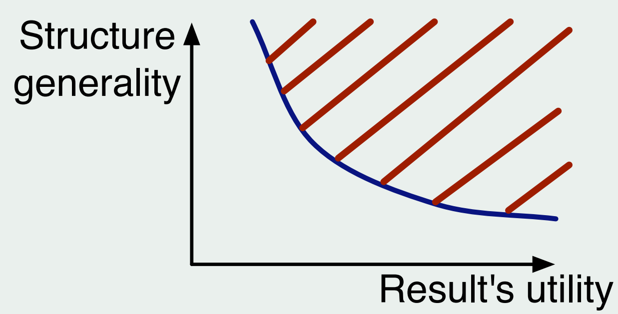

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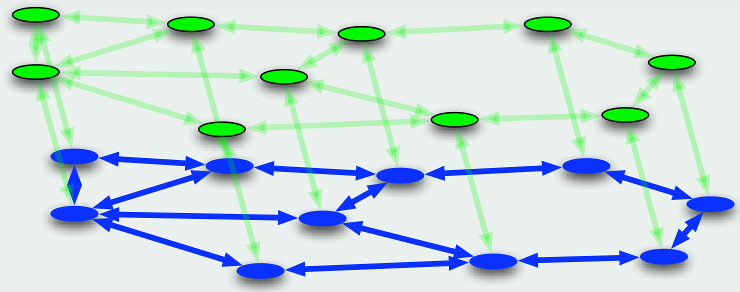

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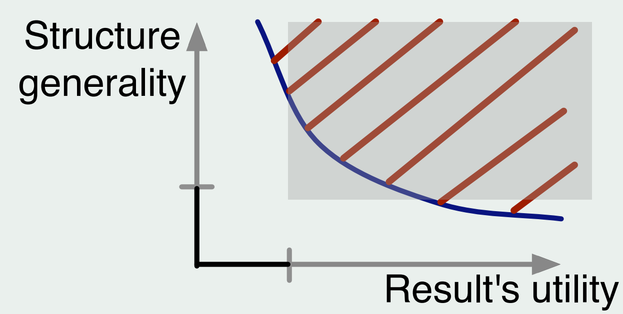


Part I

What you get for free:

The inherent structure of the centralized controller for spatially distributed plants

Outline

Examples

- Vehicular Platoons
- Heat Equation with Distributed Control

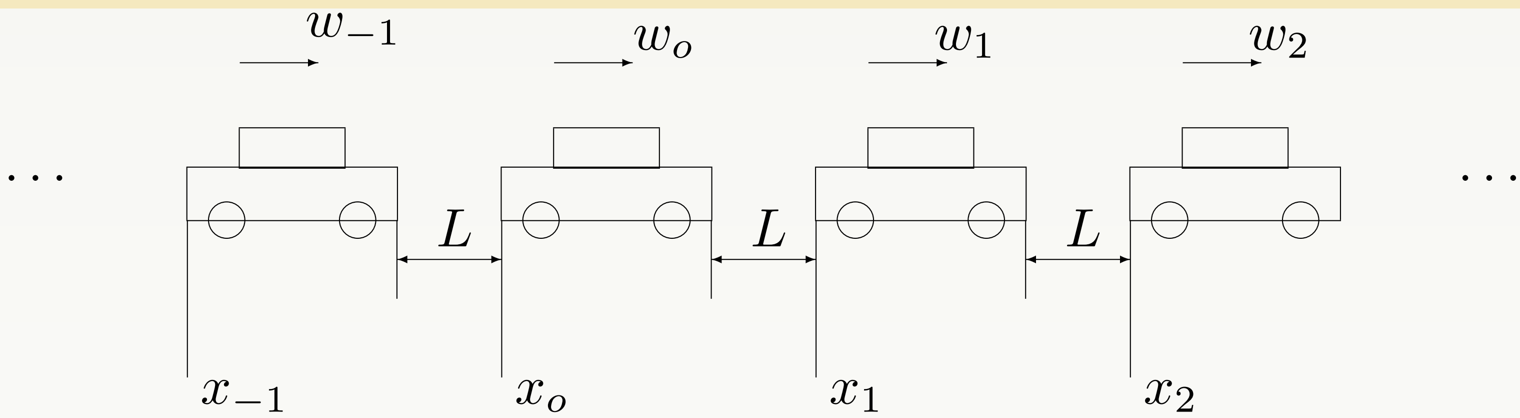
Spatially-Invariant Plants

- Optimal Controllers are Inherently Spatially Invariant
- Optimal Centralized Controllers are Inherently Localized

Spatially-Varying Plants

- Localized Plants over Arbitrary Networks
  - Notions of Distance and Spatial Decay
- Central LQR Controllers are Inherently Localized

Vehicular Platoons



Objective: Design a controller for each vehicle to:

- Maintain constant small slot length L.
- Reject the effect of disturbances $\{w_i\}$ (wind gusts, road conditions, etc...)

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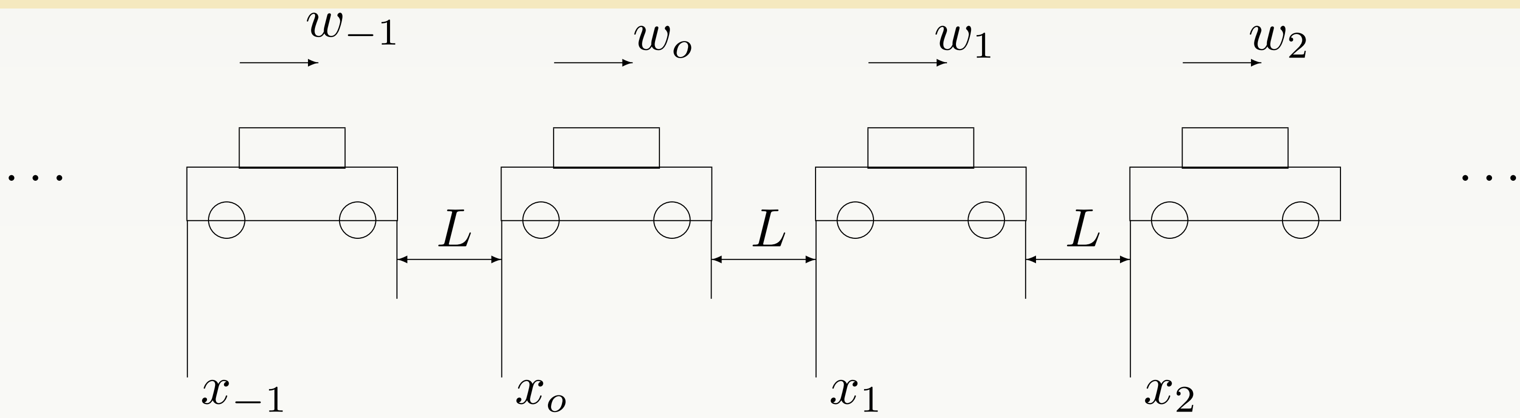

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Warning: Designs based on two vehicle models may lack "string stability", i.e. disturbances get amplified as they propagate through the platoon.

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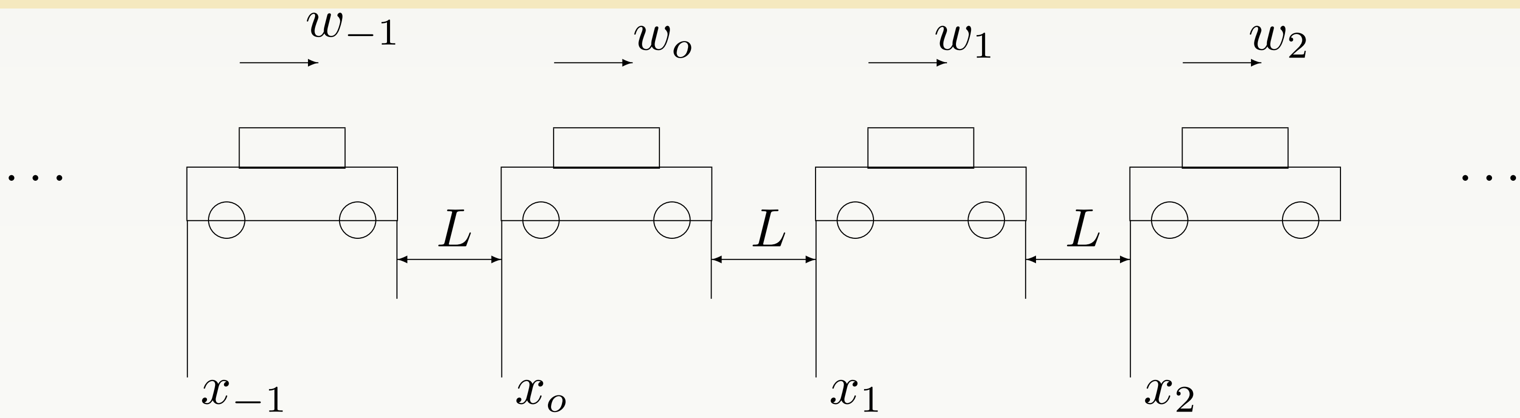

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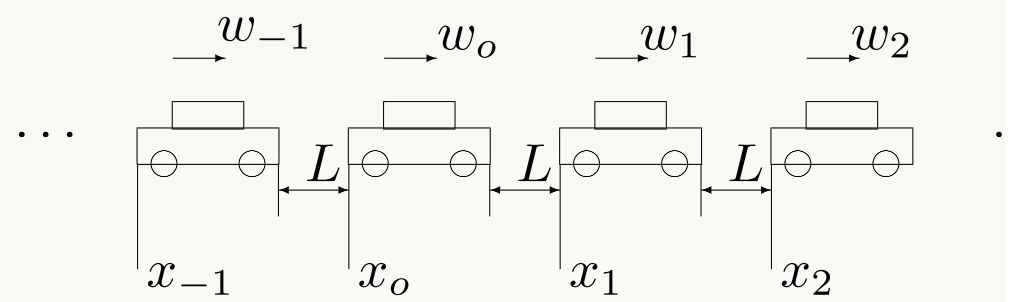
Problem Structure:

- Actuators: each vehicle's throttle input.
- Sensors: position and velocity of each vehicle.

Vehicular Platoons Set-up

x_i: i'th vehicle's position.



$$
\begin{aligned}
\tilde{x}_i &:= x_i - x_{i-1} - L - C \\
\tilde{x}_{1,i} &:= \tilde{x}_i \\
\tilde{x}_{2,i} &:= \dot{\tilde{x}}_i
\end{aligned}
$$

Vehicular Platoons (Optimal LQR)

Centralized LQR design *(Melzer & Kuo '70, Athans & Levine '66)*

$$\begin{bmatrix} \dot{\tilde{x}} \\ \dot{\tilde{v}} \end{bmatrix} = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{v} \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} w,$$

$$J = \int_0^\infty \sum_k \left(q_1 \left(\tilde{x}_k - \tilde{x}_{k-1} \right)^2 + q_2 \, \tilde{v}_k^2 + u_k^2 \right)$$

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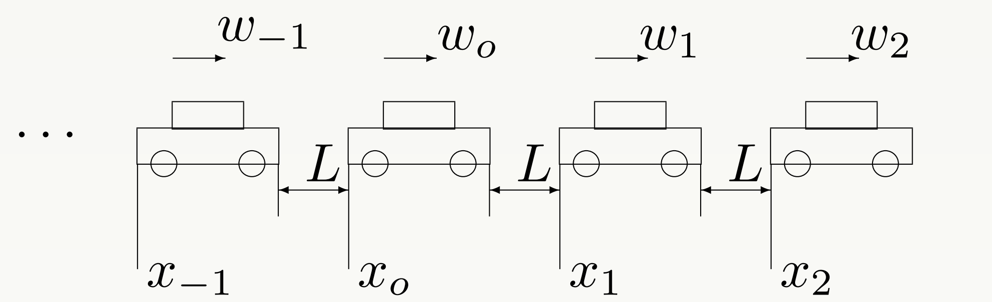

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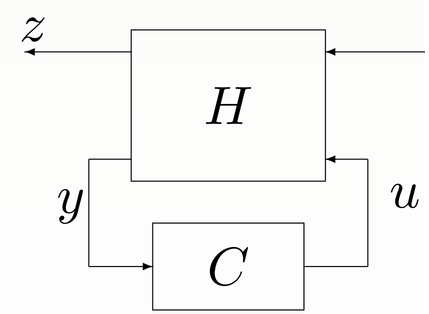
Structure of generalized plant:

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} = \begin{bmatrix} \times & & \times & \\ & \ddots & & 0 \\ & \ddots & h_o & \\ \times & & h_1 & \ddots \\ & 0 & & \ddots \end{bmatrix}$$



The generalized plant has a Toeplitz structure!

$$z = \mathcal{F}(H, C)$$

Optimal Controller for Vehicular Platoon

Example: Centralized $\mathcal{H}^2$ optimal controller gains for a 50 vehicle platoon
(From: Shu and Bamieh '96)

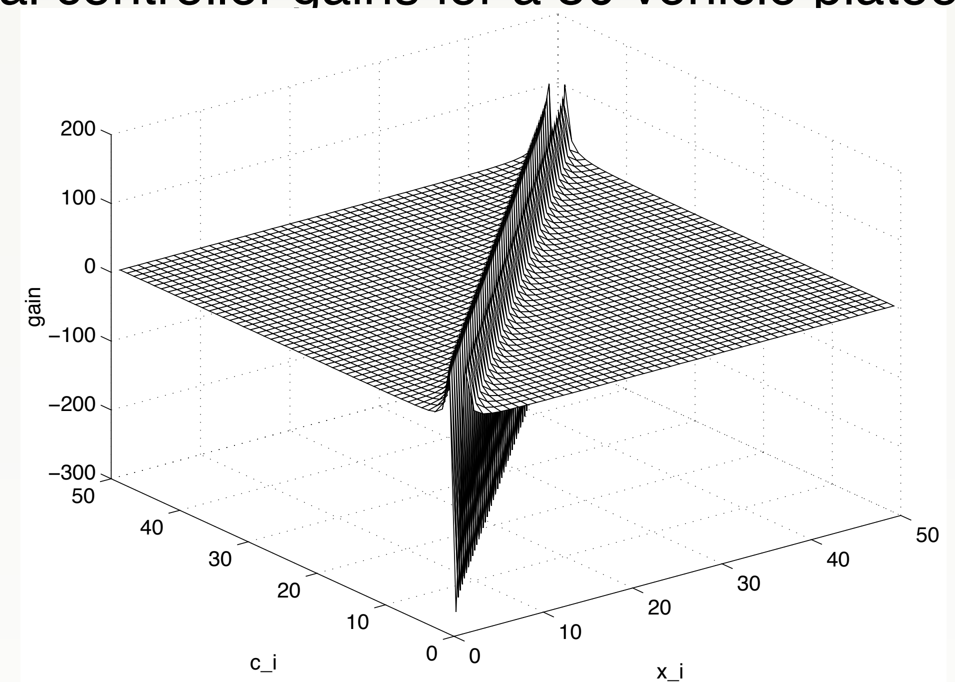


Figure 1: Position error feedback gains for a 50 vehicle platoon

Remarks:

- For large platoons, optimal controller is approximately Toeplitz
- Optimal centralized controller has some inherent decentralization ("localization")
  Controller gains decay away from the diagonal

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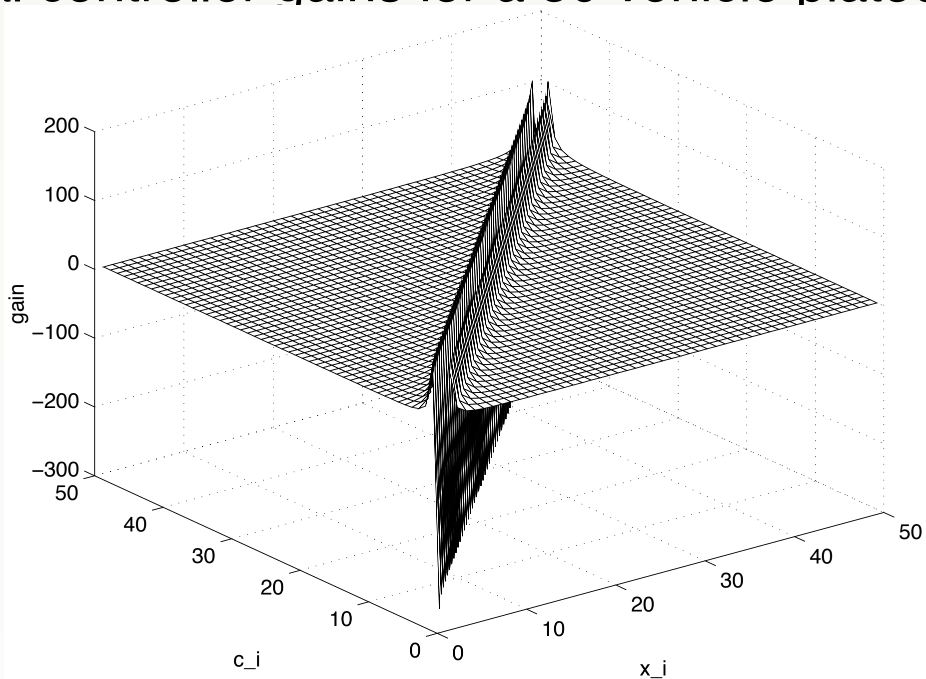

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Q: Do the above 2 results occur in all "such" problems?

Simple Example; Distributed LQR Control of Heat Equation

$$\frac{\partial}{\partial t}\psi(x,t) \;=\; c\frac{\partial^2}{\partial x^2}\psi(x,t) + u(x,t) \qquad \longrightarrow \qquad \frac{d}{dt}\hat{\psi}(\lambda,t) = -c\lambda^2\hat{\psi}(\lambda,t) + \hat{u}(\lambda,t)$$

Solve the LQR problem with $Q = qI$, $R = I$. The corresponding ARE family:

$$-2c\lambda^2\,\hat{p}(\lambda) - \hat{p}(\lambda)^2 + q = 0,$$

and the positive solution is:

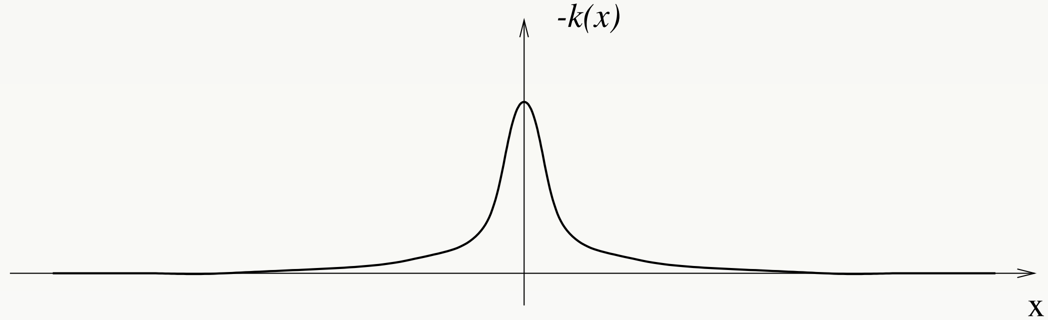
$$\hat{p}(\lambda) = -c\lambda^2 + \sqrt{c^2\lambda^4 + q}.$$

Remark: In general $\hat{P}(\lambda)$ an irrational function of λ, even if $\hat{A}(\lambda), \hat{B}(\lambda)$ are rational.
i.e. PDE systems have optimal feedbacks which are *not* PDE operators.

Let $\{k(x)\}$ be the inverse Fourier transform of the function $\{-\hat{p}(\lambda)\}$.

Then *optimal (temporally static) feedback*

$$u(x,t) \;=\; \int_{\mathbb{R}} k(x-\xi)\;\psi(\xi,t)\;d\xi$$



Remark: The "spread" of $\{k(x)\}$ indicates information required from distant sensors.

Distributed LQR Control of Heat Equation (Cont.)

Important Observation: $\{k(x)\}$ is "localized". It decays exponentially!!

$$\hat{k}(\lambda) = c\lambda^2 - \sqrt{c^2\lambda^4 + q}.$$

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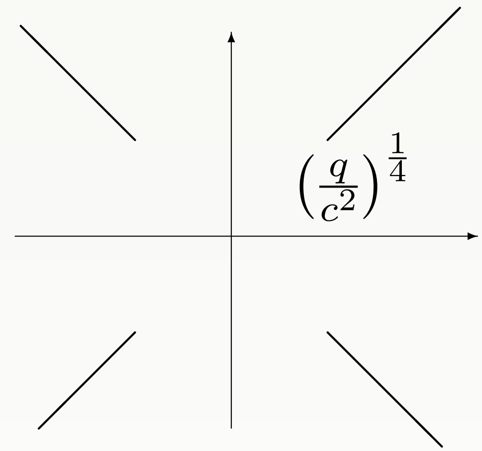
$$\hat{k}(\lambda) = c\lambda^2 - \sqrt{c^2\lambda^4 + q}.$$

This can be analytically extended by:

$$\hat{k}_e(s) = cs^2 - \sqrt{c^2 s^4 + q},$$



which is analytic in the strip

$$\left\{ s \in \mathbb{C}\ ;\ Im\{s\} < \frac{\sqrt{2}}{2}\left(\frac{q}{c^2}\right)^{\frac{1}{4}} \right\}.$$

Therefore: $\exists M$ such that

$$|k(x)| \ \leq \ Me^{-\alpha|x|}, \quad \text{for any } \ \alpha < \frac{\sqrt{2}}{2}\left(\frac{q}{c^2}\right)^{\frac{1}{4}}.$$

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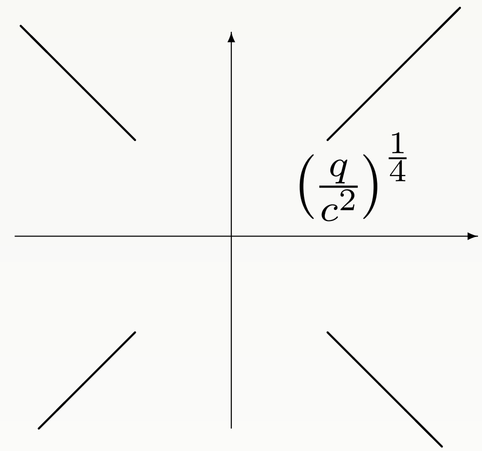

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This results is true in general: under mild conditions
Solutions of AREs always inverse transform to exponentially decaying convolution kernels

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- Special Structure
  - Distributed control and measurement *(now more feasible)*
  - Regular (lattice) arrangement of devices

  Together $\Longrightarrow$ *Spatial Invariance*

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Together $\Longrightarrow$ *Spatial Invariance*

- Control of "Vehicular Strings", (Melzer & Kuo, 71)
- Discretized PDEs, (Brockett, Willems, Krishnaprasd, El-Sayed, '74, '81)
- "Systems over rings", (Kamen, Khargonekar, Sontag, Tannenbaum, ...)
- Systems with "Dynamical Symmetry", (Fagniani & Willems)

More recently:

- Controller architecture and localization, (Bamieh, Paganini, Dahleh)
- LMI techniques, localization, (D'Andrea, Dullerud, Lall)

System Representations

All signals are spatio-temporal, e.g. $u(x,t),\ \psi(x,t),\ y(x,t)$, etc.

Spatially distributed inputs, states, and outputs

- State space description

$$\begin{aligned} \tfrac{d}{dt}\psi(x,t) &= \mathcal{A}\,\psi(x,t) + \mathcal{B}\,u(x,t) \\ y(x,t) &= \mathcal{C}\,\psi(x,t) + \mathcal{D}\,u(x,t) \end{aligned}$$

$\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ translation invariant operators

$\longrightarrow$ spatially invariant system

- Spatio-temporal impulse response $h(x,t)$

$$y(x,t) = \int\int h(x-\xi, t-\tau)\, u(\xi,\tau)\, d\tau\, d\xi,$$

- Transfer function description

$$Y(\kappa,\omega) = H(\kappa,\omega)\, U(\kappa,\omega)$$

Spatio-temporal Impulse Response

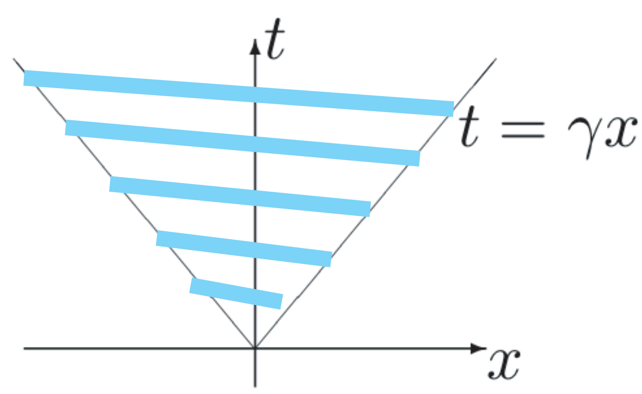
Spatio-temporal impulse response $h(x,t)$

$$y(x,t) \;=\; \int\int h(x-\xi, t-\tau)\, u(\xi,\tau)\, d\tau\, d\xi,$$

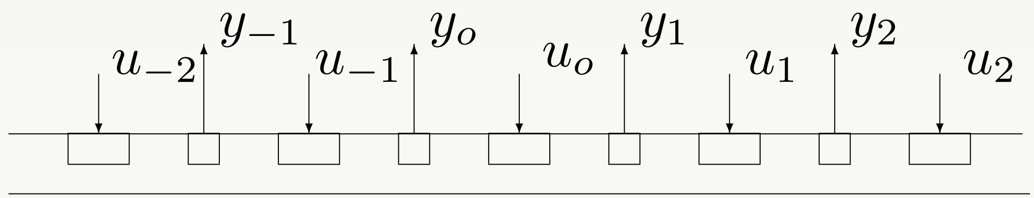
Interpretation

$h(x,t)$: effect of input on output a distance x away and time t later

Example: Constant maximum speed of effects



Example: Distributed Control of the Heat Equation



u_i: input to heating elements. y_i: signal from temperature sensor.
Dynamics are given by:

$$\begin{bmatrix} \vdots \\ y_{-1} \\ y_o \\ y_1 \\ \vdots \end{bmatrix} = \begin{bmatrix} & \vdots & & \vdots & \\ \cdots & & H_{-1,0} & & \cdots \\ & H_{0,-1} & H_{0,0} & H_{0,1} & \\ \cdots & & H_{1,0} & & \cdots \\ & \vdots & & \vdots & \end{bmatrix} \begin{bmatrix} \vdots \\ u_{-1} \\ u_o \\ u_1 \\ \vdots \end{bmatrix}$$

Each $H_{i,j}$ is an infinite-dimensional SISO system.

Fact: Dynamics are spatially invariant $\Rightarrow$ H is Toeplitz

The input-output relation can be written as a *convolution over the actuator/sensor index*:

$$y_i = \sum_{j=-\infty}^{\infty} \bar{H}_{(i-j)} \, u_j,$$

The limit of large actuator sensor array:

$$\frac{\partial \psi}{\partial t}(x,t) \;=\; c\frac{\partial^2 \psi}{\partial x^2}(x,t) \;+\; u(x,t) \qquad\qquad \psi_x \;=\; \int_{-\infty}^{\infty} H_{x-\zeta}\, u_\zeta d\zeta,$$

Spatial Invariance of Dynamics

Indexing of actuator and sensor signals:

$$u_i(t) \;:=\; u_{(i_1,\ldots,i_n)}(t), \qquad\qquad y_i(t) \;:=\; y_{(i_1,\ldots,i_n)}(t).$$

$i \;:=\; (i_1, \ldots, i_n)$ a spatial multi-index, $\quad i \in \mathbb{G} \;:=\; \mathbb{G}_1 \times \ldots \times \mathbb{G}_n$.

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Linear input-output relations:

A general linear system from u to y:

$$y_i \;=\; \sum_{j \in \mathbb{G}} H_{i,j}\, u_j, \qquad \Leftrightarrow \qquad y_{(i_1,\ldots,i_n)} \;=\; \sum_{j_1 \in \mathbb{G}_1} \ldots \sum_{j_n \in \mathbb{G}_n} H_{(i_1,\ldots,i_n),(j_1,\ldots,j_n)}\, u_{(j_1,\ldots,j_n)},$$

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Spatial Invariance:

Assumption 1: Set of spatial indices $=$ commutative group

$\mathbb{G} \;:=\; \mathbb{G}_1 \times \ldots \times \mathbb{G}_n$, $\qquad$ each $\mathbb{G}_i$ a commutative group.

Remark: “spatial shifting” of signals

$(S_\sigma u)_i \;:=\; u_{i-\sigma}$ $\qquad$ Compare with: *Time shift by* τ $\;(S_\tau u)(t) := u(t-\tau)$

Assumption 2: Spatial invariance $\longleftrightarrow$ Commute with spatial shifts

$$\forall \sigma \in \mathbb{G}, \qquad H\, S_\sigma = S_\sigma\, H \qquad \Leftrightarrow \qquad S_\sigma^{-1} H S_\sigma = H$$

Examples of Spatial Invariance

Generally: Spatial invariance easily ascertained from basic physical symmetry!

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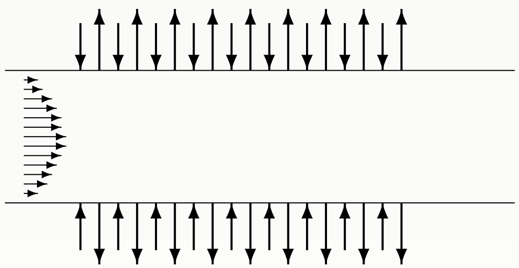
- Channel flow: Signals indexed over $\{0,1\} \times \mathbb{Z}$:

$$y_{(l,i)} = \sum_{j=-\infty}^{\infty} H_{(l-0,i-j)}\, u_{(0,j)} + \sum_{j=-\infty}^{\infty} H_{(l-1,i-j)}\, u_{(1,j)}, \qquad l = 0, 1.$$



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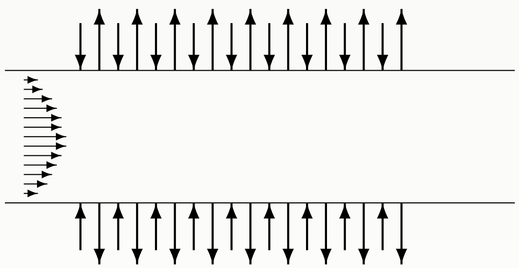

Remark: The input-output mapping of a spatially invariant system can be rewritten:

$$y_i \;=\; \sum_{j\in\mathbb{G}} \bar{G}_{i-j}\, u_j, \qquad \Leftrightarrow \qquad y_{(i_1,\ldots,i_n)} \;=\; \sum_{j_1\in\mathbb{G}_1} \cdots \sum_{j_n\in\mathbb{G}_n} \bar{G}_{(i_1-j_1,\ldots,i_n-j_n)}\, u_{(j_1,\ldots,j_n)}.$$

A spatial convolution

Symmetry in Dynamical Systems and Control Design

- Many-body systems always have some inherent dynamical symmetries: e.g. equations of motion are invariant to certain coordinate transformations
- **Question:** Given an unstable dynamical system with a certain symmetry, is it possible to stabilize it with a controller that has the same symmetry? (i.e. without "breaking the symmetry")
- **Answer:** Yes! (Fagnani & Willems '93)

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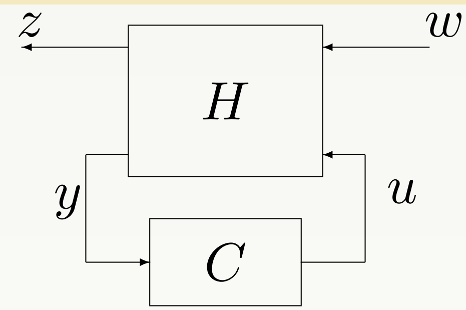
Remark: Spatial invariance is a dynamical symmetry
This answer applies to optimal design as well

i.e.
For best achievable performance, need only consider spatially-invariant controllers

The Standard Problem of Optimal and Robust Control

The standard problem:



$$z \ = \ \mathcal{F}(H, C)\ w$$

Signal norms:

$$\|w\|_p^p \ := \ \sum_{i \in \mathbb{G}} \int_{\mathbb{R}} |w_i(t)|^p dt \ = \ \sum_{i \in \mathbb{G}} \|w\|_p^p$$

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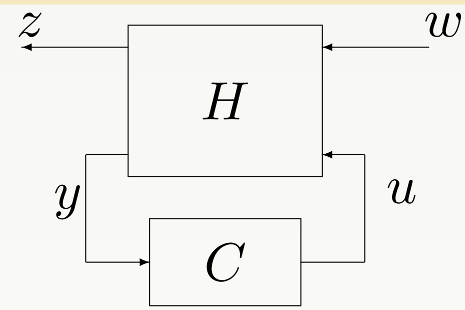

$$z \;=\; \mathcal{F}(H,C)\; w$$

The standard problem:

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$$\|w\|_p^p \;:=\; \sum_{i\in\mathbb{G}} \int_{\mathbb{R}} |w_i(t)|^p dt \;=\; \sum_{i\in\mathbb{G}} \|w\|_p^p$$

Induced system norms:

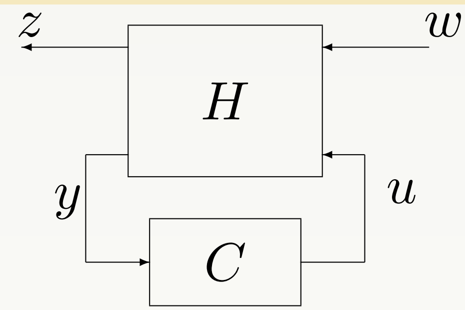
$$\|\mathcal{F}(G,C)\|_{p-i} \;:=\; \sup_{w\in L^P} \frac{\|z\|_p}{\|w\|_p}.$$

The $\mathcal{H}^2$ norm:

$$\|\mathcal{F}(G,C)\|_{\mathcal{H}^2}^2 \;=\; \|z\|_2^2 \;=\; \sum_{i\in\mathbb{G}} \|z_i\|_{L^2}^2,$$

with impulsive disturbance input $w_i(t) \;=\; \delta(i)\delta(t)$.

The Standard Problem of Optimal and Robust Control



$$z = \mathcal{F}(H, C)\, w$$

The standard problem:

Signal norms:

$$\|w\|_p^p := \sum_{i \in \mathbb{G}} \int_{\mathbb{R}} |w_i(t)|^p dt = \sum_{i \in \mathbb{G}} \|w\|_p^p$$

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Note: In the platoon problem: finite system norm $\Rightarrow$ string stability.

Spatially-Invariant vs. Spatially-Varying Controllers

Question: Are spatially-varying controllers better than spatially-invariant ones?

Answer: If plant is spatially invariant, no!

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$LSI :=$ The class of Linear Spatially-Invariant systems.

$LSV :=$ The class of Linear Spatially-Varying systems.

Compare the two problems:

$$\gamma_{si} \quad := \quad \inf_{\substack{\text{stabilizing } C \\ C \in LSI}} \quad \|\mathcal{F}(G,C)\|_{p-i}$$

The best achievable performance with spatially-invariant controllers

$$\gamma_{sv} \quad := \quad \inf_{\substack{\text{stabilizing } C \\ C \in LSV}} \quad \|\mathcal{F}(G,C)\|_{p-i}$$

The best achievable performance with spatially-varying controllers

Theorem 1. *If the plant and performance objectives are spatially invariant, i.e. if the generalized plant G is spatially invariant, then the best achievable performance can be approached with a spatially invariant controller. More precisely*

$$\gamma_{si} \;=\; \gamma_{sv}.$$

Spatially-Invariant vs. Spatially-Varying Controllers (Cont.)

Related Problem: *Time-Varying vs. Time-Invariant Controllers*

Fact: For time-invariant plants, time-varying controllers offer no advantage over time-invariant ones! *for norm minimization problems*

Proofs based on use of YJBK parameterization. Convert to

$$\gamma_{ti} := \inf_{\substack{\text{stable } Q \\ Q \in LTI}} \|T_1 - T_2 Q T_3\| \qquad \gamma_{tv} := \inf_{\substack{\text{stable } Q \\ Q \in LTV}} \|T_1 - T_2 Q T_3\| ,$$

T_1, T_2, T_3 determined by plant, therefore time invariant.

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- The $\mathcal{H}^\infty$ case: (Feintuch & Francis, '85), (Khargonekar, Poolla, & Tannenbaum, '85). *A consequence of Nehari's theorem*
- The ℓ^1 case: (Shamma & Dahleh, '91). *Using an averaging technique*
- Any induced ℓ^p norm: (Chapellat & Dahleh, '92). *Generalization of the averaging technique*

Spatially-Invariant vs. Spatially-Varying Controllers (Cont.)

Idea of proof: After YJBK parameterization:

$$\gamma_{si} \; := \inf_{\substack{\text{stable } Q \\ Q \in LSI}} \|T_1 - T_2 Q T_3\| \quad \geq \quad \gamma_{sv} \; := \inf_{\substack{\text{stable } Q \\ Q \in LSV}} \|T_1 - T_2 Q T_3\|$$

Let $\bar{Q}$ achieve a performance level $\bar{\gamma} = \|T_1 - T_2 \bar{Q} T_3\|$.

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Averaging $\bar{Q}$:

- If $\mathbb{G}$ is finite: define

$$Q_{av} := \frac{1}{|\mathbb{G}|} \sum_{\sigma \in \mathbb{G}} \sigma^{-1} \bar{Q} \sigma. \rightarrow$$ Q_{av} *is spatially invariant, i.e.* $\forall \sigma \in \mathbb{G}, \quad \sigma^{-1} Q_{av}\, \sigma = Q_{av}$

  Then

$$\begin{aligned} \|T_1 - T_2 Q_{av} T_3\| &= \left\|T_1 - T_2 \left(\frac{1}{|\mathbb{G}|} \sum_{\sigma \in \mathbb{G}} \sigma^{-1} \bar{Q} \sigma \right) T_3 \right\| = \left\| \frac{1}{|\mathbb{G}|} \sum_{\sigma \in \mathbb{G}} \sigma^{-1} \left(T_1 - T_2 \bar{Q} T_3 \right) \sigma \right\| \\ &\leq \frac{1}{|\mathbb{G}|} \sum_{\sigma \in \mathbb{G}} \left\| \sigma^{-1} \left(T_1 - T_2 \bar{Q} T_3 \right) \sigma \right\| = \|T_1 - T_2 \bar{Q} T_3\| \end{aligned}$$

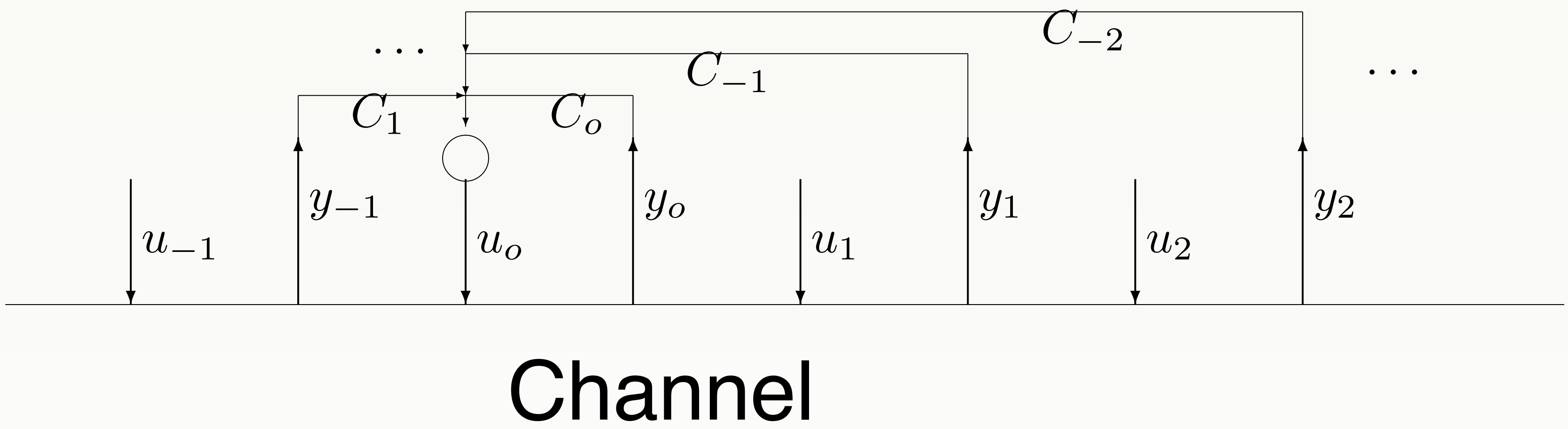
- If $\mathbb{G}$ is infinite, take a sequence of finite subsets $M_1 \subset M_2 \subset \cdots$, with $\bigcup_n M_n = \mathbb{G}$,

  Then define: $$Q_n := \frac{1}{|M_n|} \sum_{\sigma \in M_n} \sigma^{-1} \bar{Q} \sigma.$$

  Q_n converges weak $*$ to a spatially-invariant Q_{av} with the required norm bound.

Implications of the Structure of Spatial Invariance

Poiseuille flow stabilization:



$$u_i \;=\; \sum_j C_{i-j}\, y_j$$

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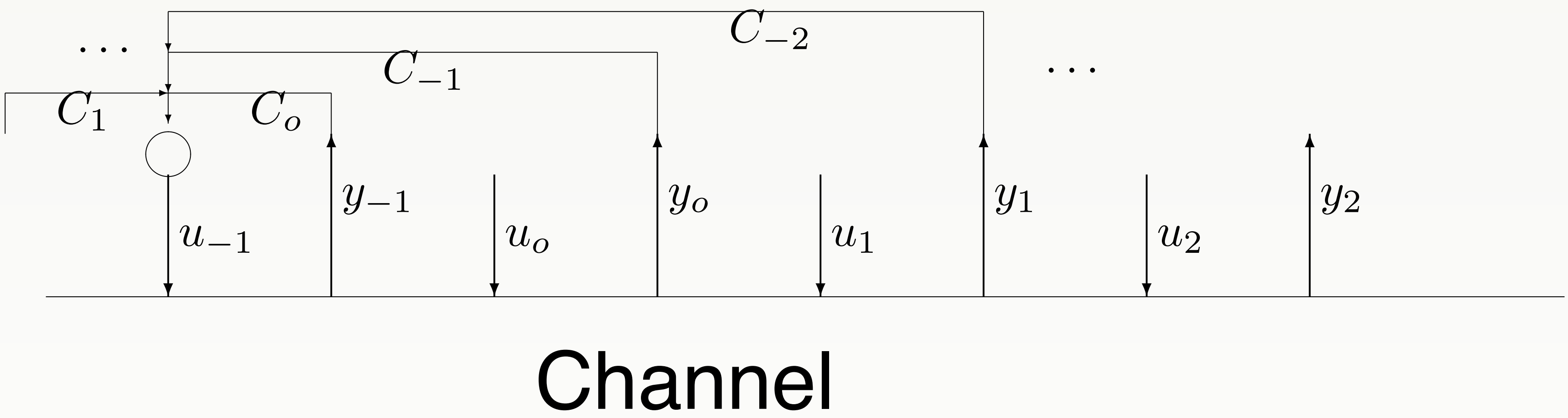

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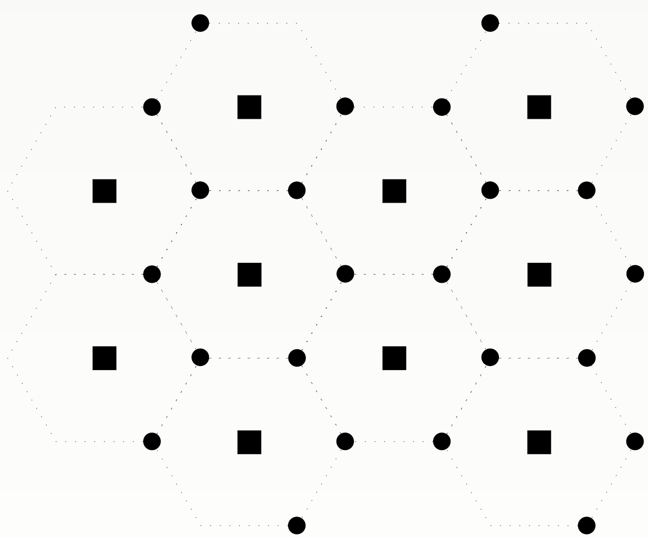
Implications of the Structure of Spatial Invariance (Cont.)

Uneven distribution of sensors and actuators

Consider the following geometry of sensors and actuators:

- Sensor

▪ Actuator



What kind of spatial invariance do optimal controllers have?

Implications of the Structure of Spatial Invariance (Cont.)

Uneven distribution of sensors and actuators (Cont.)

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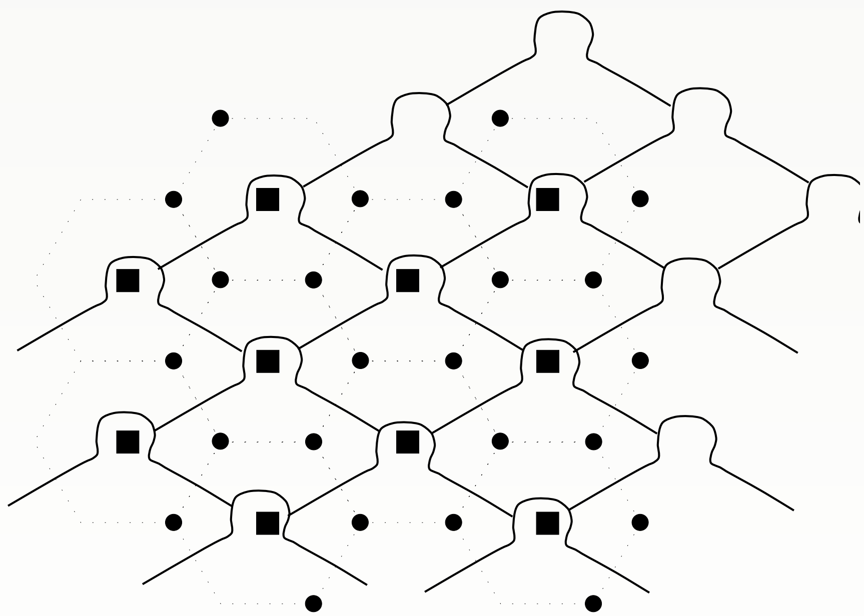

Each “cell” is a 1-input, 2-output system. **underlying group is** $\mathbb{Z} \times \mathbb{Z}$

Transform Methods

Consider the following PDE with distributed control:

$$\begin{aligned}\frac{\partial\psi}{\partial t}(x_1,\ldots,x_n,t) &= \mathcal{A}\left(\tfrac{\partial}{\partial x_1},\ldots,\tfrac{\partial}{\partial x_n}\right)\psi(x_1,\ldots,x_n,t) \;+\; \mathcal{B}\left(\tfrac{\partial}{\partial x_1},\ldots,\tfrac{\partial}{\partial x_n}\right)u(x_1,\ldots,x_n,t)\\ y(x_1,\ldots,x_n,t) &= \mathcal{C}\left(\tfrac{\partial}{\partial x_1},\ldots,\tfrac{\partial}{\partial x_n}\right)\psi(x_1,\ldots,x_n,t),\end{aligned}$$

where $\mathcal{A},\mathcal{B},\mathcal{C}$ are matrices of polynomials in $\frac{\partial}{\partial x_i}$.

Consider also combined PDE difference equations such as:

$$\begin{aligned}\frac{\partial\psi}{\partial t}(x_1,\ldots,x_m,k_1,\ldots,k_n,t) &= \mathcal{A}\left(\tfrac{\partial}{\partial x_1},\ldots,\tfrac{\partial}{\partial x_n},z_1^{-1},\ldots,z_n^{-1}\right)\psi(x_1,\ldots,x_n,k_1,\ldots,k_n,t)\\ &+ \mathcal{B}\left(\tfrac{\partial}{\partial x_1},\ldots,\tfrac{\partial}{\partial x_n},z_1^{-1},\ldots,z_n^{-1}\right)u(x_1,\ldots,x_n,k_1,\ldots,k_n,t)\end{aligned}$$

We only require that the spatial variables x, k, belong to a commutative group

Taking the Fourier transform:

$$\hat{\psi}(\lambda,t) \;:=\; \int_{\mathbb{G}} e^{-j<\lambda,x>}\psi(x,t)\,dx,$$

The above system equations become:

$$\begin{aligned}
\frac{d\hat{\psi}}{dt}(\lambda,t) &= \mathcal{A}(\lambda)\,\hat{\psi}(\lambda,t) \;+\; \mathcal{B}(\lambda)\,\hat{u}(\lambda,t) \\
\hat{y}(\lambda,t) &= \mathcal{C}(\lambda)\,\hat{\psi}(\lambda,t),
\end{aligned}$$

where $\lambda \in \hat{\mathbb{G}}$, the dual group to $\mathbb{G}$.

Remark: This can be thought of as a parameterized family of finite-dimensional systems.

Block Diagonalization by Fourier Transforms

The Fourier transform converts:

spatially-invariant operators on $\mathcal{L}_2(\mathbb{G})$ $\longrightarrow$ multiplication operators on $\mathcal{L}_2(\hat{\mathbb{G}})$

In general:

group: $\mathbb{G}$	dual group: $\hat{\mathbb{G}}$	Transform
$\mathbb{R}$	$\mathbb{R}$	Fourier Transform
$\mathbb{Z}$	$\partial\mathbb{D}$	Z-Transform
$\partial\mathbb{D}$	$\mathbb{Z}$	Fourier Series
$\mathbb{Z}_n$	$\mathbb{Z}_n$	Discrete Fourier Transform

and the transforms preserve $\mathcal{L}_2$ norms:

$$\|f\|_2^2 = \int_{\mathbb{G}} |f(x)|^2 dx = \int_{\hat{\mathbb{G}}} |\hat{f}(\lambda)|^2 d\lambda = \|\hat{f}\|_2^2$$

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The system operation is then spatially decoupled or "block diagonalized":

$$\begin{aligned} \tfrac{\partial}{\partial t}\psi(x,t) &= A\,\psi(x,t) + B\,u(x,t) \\ y(x,t) &= C\,\psi(x,t) + D\,u(x,t) \end{aligned} \quad \longrightarrow \quad \begin{aligned} \tfrac{d}{dt}\hat{\psi}(\lambda,t) &= \hat{A}(\lambda)\hat{\psi}(\lambda,t) + \hat{B}(\lambda)\hat{u}(\lambda,t) \\ \hat{y}(\lambda,t) &= \hat{C}(\lambda)\hat{\psi}(\lambda,t) + \hat{D}(\lambda)\hat{u}(\lambda,t) \end{aligned}$$

A distributed,
spatially-invariant system

A parameterized family
of finite-dimensional systems

Transform Methods

In physical space

$$\frac{d}{dt}\psi_n = A_n \star \psi_n + B_n \star u_n$$
$$y_n = C_n \star \psi_n$$

After spatial Fourier trans. (FT)

$$\frac{d}{dt}\hat{\psi}(\theta) = \hat{A}(\theta)\,\hat{\psi}(\theta) + \hat{B}(\theta)\,\hat{u}(\theta)$$
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Transfer functions

operator-valued transfer function

$$\mathcal{H}(s) = \mathcal{C}\,(sI - \mathcal{A})^{-1}\,\mathcal{B}$$

spatio-temporal transfer function

$$H(s,\theta) = \hat{C}(\theta)\left(sI - \hat{A}(\theta)\right)^{-1}\hat{B}(\theta)$$

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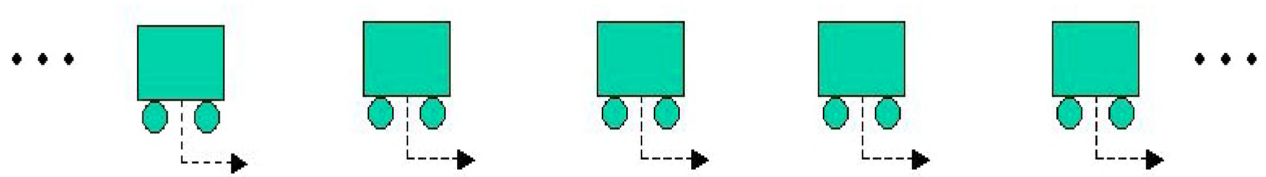
A multi-dimensional system with temporal, but not spatial causality

Optimal Control of Infinite Platoons

☞ GOOD APPROXIMATION OF LARGE BUT FINITE PLATOONS



MAIN IDEA: EXPLOIT SPATIAL INVARIANCE

LA:

$$\begin{bmatrix} \dot{\zeta}_n \\ \dot{\phi}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 - T_{-1} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \zeta_n \\ \phi_n \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_n, \quad n \in \mathbb{Z}$$

$\downarrow$ SPATIAL $\mathcal{Z}_\theta$-TRANSFORM

$$\begin{bmatrix} \dot{\zeta}_\theta \\ \dot{\phi}_\theta \end{bmatrix} = \begin{bmatrix} 0 & 1 - \mathrm{e}^{-j\theta} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \zeta_\theta \\ \phi_\theta \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_\theta, \quad 0 \leq \theta < 2\pi$$

☞ NOT STABILIZABLE AT $\theta = 0$

Parameterized ARE solutions yield "localized" operators!

Consider unbounded domains, i.e. $\mathbb{G} = \mathbb{R}$ (or $\mathbb{Z}$).

Theorem 2. *Consider the parameterized family of Riccati equations:*

$$A^*(\lambda)P(\lambda) + P(\lambda)A(\lambda) - P(\lambda)B(\lambda)R(\lambda)B^*(\lambda)P(\lambda) + Q(\lambda) = 0, \qquad \lambda \in \hat{\mathbb{G}}.$$

Under mild conditions:
there exists an analytic continuation $P(s)$ of $P(\lambda)$ in a region

$$\{|Im(s)| < \alpha\}, \quad \alpha > 0.$$

Convolution kernel resulting from Parameterized ARE has exponential decay.
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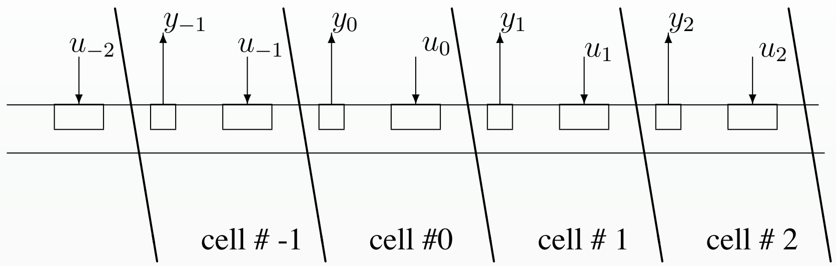
Comparison:

- **Modal truncation:** In the transform domain, ARE solutions decay algebraically.
- **Spatial truncation:** In the spatial domain, convolution kernel of ARE solution decays exponentially.

Therefore: Use transform domain to design $\forall\lambda$. Approximate in the spatial domain!

EXAMPLE: one dimensional array of systems indexed in $\mathbb{Z}$.



ψ_n: the state of the system in the n'th cell

total state: $\{\ldots, \psi_{-1}, \psi_o, \psi_1, \ldots\}$

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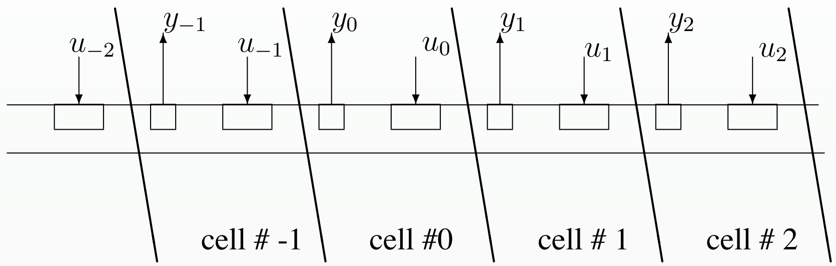

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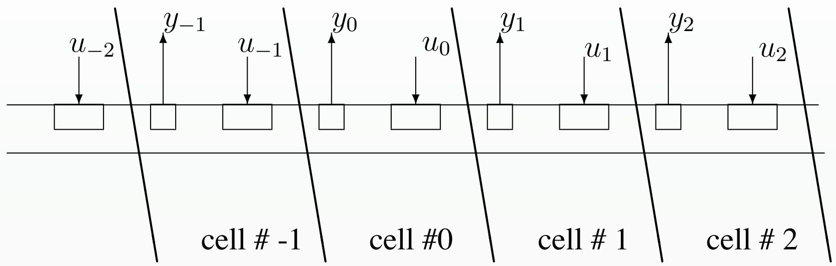

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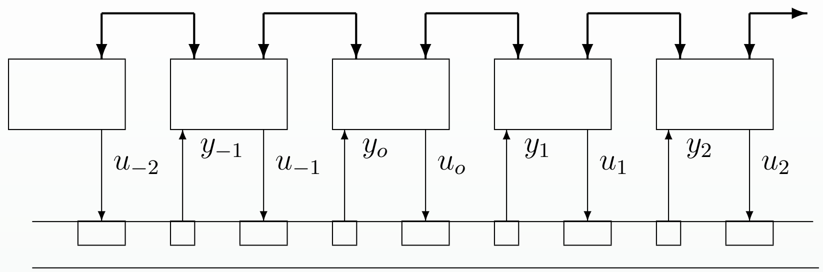

Observer based controller has the following structure:

Plant

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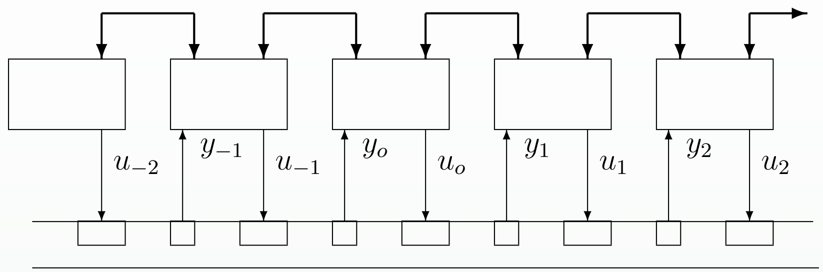

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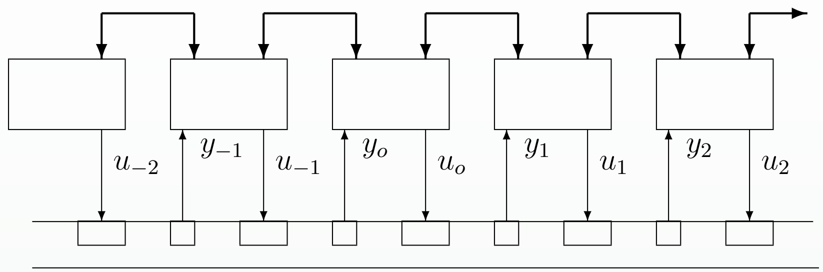
REMARKS:

- Optimal Controller is “locally” finite dimensional.

Distributed Architecture of Quadratically Optimal Controllers



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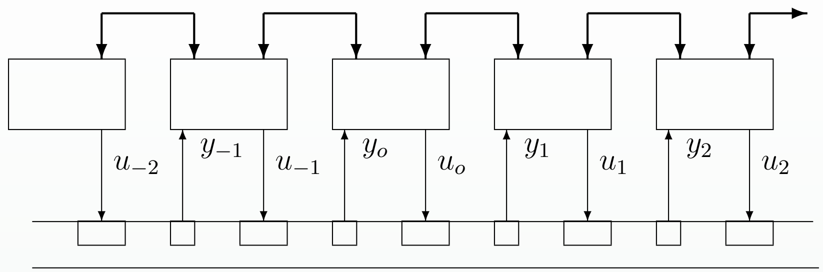

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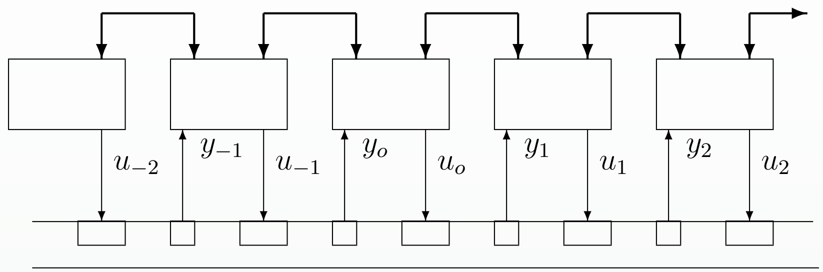
REMARKS:

- Optimal Controller is "locally" finite dimensional.
- The gains $\{K_i\}$, $\{L_i\}$ are localized (exponentially decaying) $\rightarrow$ "spatial truncation"
- After truncation, local controller need only receive information from neighboring subsystems.



Observer based controller has the following structure:

Plant

$$\frac{d}{dt}\psi_n = A_n \star \psi_n + B_n \star u_n$$
$$y_n = C_n \star \psi_n$$

Controller

$$u_i = K_i \star \hat{\psi}_i$$
$$\frac{d}{dt}\hat{\psi}_n = A_n \star \hat{\psi}_n + B_n \star u_n + L_n \star (y_n - \hat{y}_n)$$

REMARKS:

- Optimal Controller is “locally” finite dimensional.
- The gains $\{K_i\}$, $\{L_i\}$ are localized (exponentially decaying) $\rightarrow$ “spatial truncation”
- After truncation, local controller need only receive information from neighboring subsystems.
- Quadratically optimal controllers are inherently distributed and semi-decentralized (*localized*)

Outline

Examples

- Vehicular Platoons
- Heat Equation with Distributed Control

Spatially-Invariant Plants

- Optimal Controllers are Inherently Spatially Invariant
- Optimal Centralized Controllers are Inherently Localized

Spatially-Varying Plants

- Localized Plants over Arbitrary Networks
  - Notions of Distance and Spatial Decay
- Central LQR Controllers are Inherently Localized
  - PhD Thesis of Nader Motee
  - Motee & Jadbabaie, *Optimal control of spatially distributed systems*

Spatially Distributed Dynamical Systems

- Engineered systems involve finite number of subsystems.
- Infinite-dimensional abstractions allows for a precise mathematical Analysis.
- Our focus will be on spatially distributed linear systems:

$$
\begin{aligned}
\frac{d}{dt}\psi(i,t) &= (A\psi)(i,t) + (Bu)(i,t) \\
y(i,t) &= (C\psi)(i,t) + (Du)(i,t)
\end{aligned}
$$

ψ, u, y : state, input, and output variables

i : spatial variable

t : temporal variable

A, B, C, D : infinite-dimensional matrices

Spatially Distributed Dynamical Systems

- Spatially decaying (SD) matrices

$$\begin{aligned} \frac{d}{dt}\psi &= A\psi + Bu \\ y &= C\psi + Du \end{aligned}$$

- Infinite-dimensional matrices: $A, B, C, D : \ell_2 \to \ell_2$

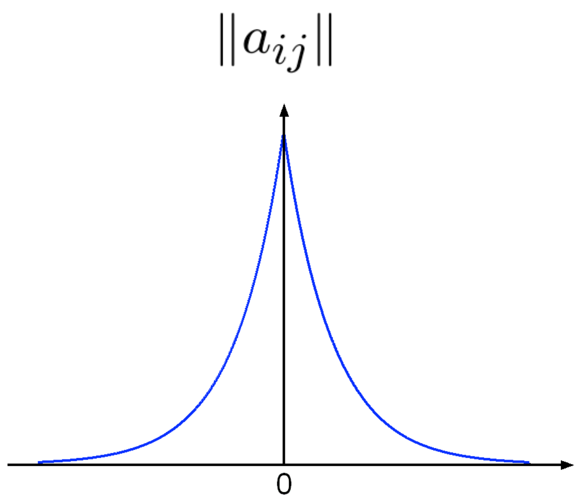
$$A = (a_{i,j}) = \begin{pmatrix} & & \vdots & & \\ & & a_{i-1,j} & & \\ \cdots & a_{i,j-1} & a_{i,j} & a_{i,j+1} & \cdots \\ & & a_{i+1,j} & & \\ & & \vdots & & \end{pmatrix}$$

- Banach spaces: $\ell_p := \{x \ : \ \|x\|_p < \infty\}$ where $\|x\|_p = \left(\sum_i |x_i|^p\right)^{\frac{1}{p}}$

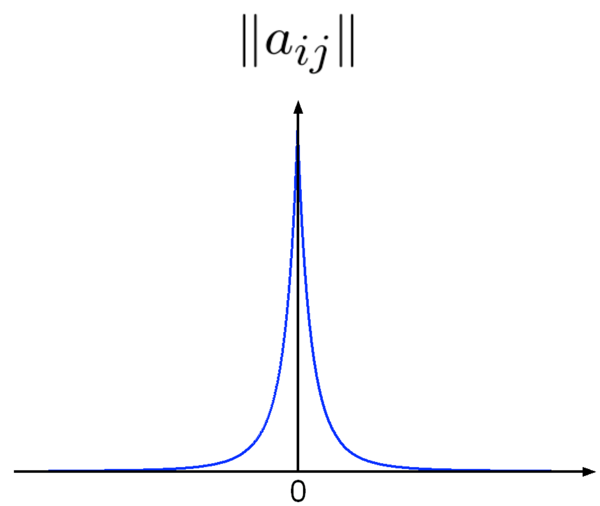
Spatially Decaying (SD) Operators

- In many applications the corresponding matrices are spatially decaying:

$$A = (a_{ij})$$



distance(i,j)



$$\chi_\alpha(x) = (1 + |x|)^\alpha$$

$$\chi_\alpha(x) = e^{\alpha|x|^\beta} \quad , \quad 0 < \beta < 1$$

Spatially Decaying (SD) matrices

Optimal Control of Spatially Decaying Systems

Structural Properties of Spatially Decaying Systems:

$$\underset{K}{\text{minimize}} \quad \int_0^\infty \langle \psi, Q\psi \rangle + \langle u, Ru \rangle dt$$

$$\text{subject to:} \quad \begin{aligned} \frac{d}{dt}\psi &= A\psi + Bu \\ u &= K\psi \end{aligned}$$

Our goal:

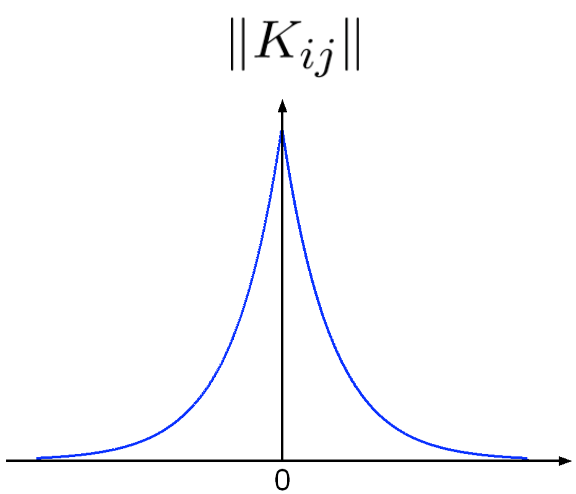
Assume that the corresponding LQR problem is optimizable and exponentially detectable. If A, B, Q, R are spatially decaying (SD), then K is also SD.

Locality Features of the Optimal Controller

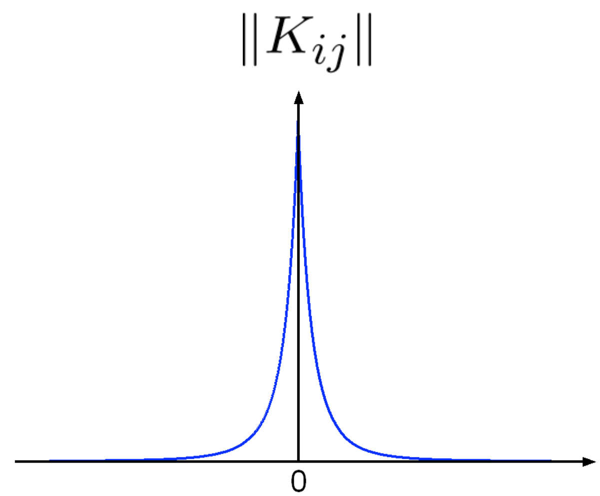
- The state feedback $K = (K_{ij})$ is SD:

$$\lim_{\text{dis}(k,i)\to\infty} \|K_{ij}\| \ \chi_\alpha(\text{dis}(i,j)) = 0$$

Coupling function



$\chi_\alpha(x) = (1 + |x|)^\alpha$



$\chi_\alpha(x) = e^{\alpha|x|^\beta} \quad , \quad 0 \leq \beta < 1$

Coupling function

- Properties of a coupling characteristic function:
  - $\chi_\alpha(0) = 1$ for all $\alpha \geq 0$ and $\chi_0(x) = 1$ for all $x \geq 0$.
  - Continuous and nondecreasing in x.
  - $\chi_\alpha(x+y) \leq \chi_\alpha(x)\, \chi_\alpha(y)$ (submultiplicative)

Examples:

- Sub-exponential: $\chi_\alpha(x) = e^{\alpha|x|^\beta}$, $0 \leq \beta < 1$
- Polynomial: $\chi_\alpha(x) = (1+|x|)^\alpha$
- Logarithm: $\chi_\alpha(x) = \Big(\log(e+|x|)\Big)^\alpha$
- Product of coupling functions, e.g. $\chi_\alpha(x) = e^{\alpha|x|^\beta}(1+|x|)^\alpha$

Subspace of Spatially Decaying Operators

- Consider the following subspace of infinite-dimensional matrices:

$$\mathcal{S}_{\mathcal{T}}^{\infty}(\mathscr{C}) = \{A \ : \ |||A|||_{\mathcal{T}} \ < \ \infty\}$$

- An operator norm can be defined:

$$|||A|||_{\alpha} = \max\left(\sup_{k} \sum_{i} \|a_{ki}\| \chi_{\alpha}(\mathrm{dis}(k,i)), \ \sup_{i} \sum_{k} \|a_{ki}\| \chi_{\alpha}(\mathrm{dis}(k,i)) \right)$$

- Structure of this subspace:

$(\mathcal{S}_{\mathcal{T}}^{\infty}(\mathscr{C}), |||.|||)$ forms a Banach Algebra,

$$|||AB||| \ \leq \ |||A||| \ |||B|||$$

for all $A, B \in \mathcal{S}_{\mathcal{T}}^{\infty}(\mathscr{C})$.

Banach Algebra of Spatially Decaying Operators

$(\mathcal{S}_{\mathcal{T}}^{\infty}(\mathscr{C}), |||.|||)$ forms a Banach Algebra

- Properties: For all $A, B \in \mathcal{S}_{\mathcal{T}}^{\infty}(\mathscr{C})$, it follows
  - Closed under addition: $A + B \in \mathcal{S}_{\mathcal{T}}^{\infty}(\mathscr{C})$
  - Closed under multiplication: $AB \in \mathcal{S}_{\mathcal{T}}^{\infty}(\mathscr{C})$
  - Closed under inversion: $A^{-1} \in \mathcal{S}_{\mathcal{T}}^{\infty}(\mathscr{C})$
  - Convergence of Cauchy sequences

Spectral Properties of SD operators

Theorem (Groechenig 2006):

Assume that χ_α satisfies

$$\lim_{n\to\infty} \chi_\alpha(nx)^{\frac{1}{n}} = 1$$

and the weak growth condition

$$\chi_\alpha(x) \geq C(1+|x|)^\delta \quad \text{for some} \quad 0 < \delta \leq 1.$$

Then

The spectral radius w.r.t. the Banach Algebra $= \rho_{\mathcal{S}_\tau}(A) = \rho_{\ell_2}(A) = \|A\|_{2,2}$

for all $A = A^* \in \mathcal{S}_\tau^\infty$. Consequently,

$$\sigma_{\mathcal{S}_\tau^\infty}(A) = \sigma_{\ell_2}(A).$$

for all $A \in \mathcal{S}_\tau^\infty$.

Applications of the Spectral Properties

Lemma:

Assume that $A = A^* \in \mathcal{S}_\tau^\infty$ is the infinitesimal generator of e^{At} and e^{At} is exponentially stable. Then

$$|||e^{At}|||_\alpha \;\leq\; C\; e^{-\mu t}$$

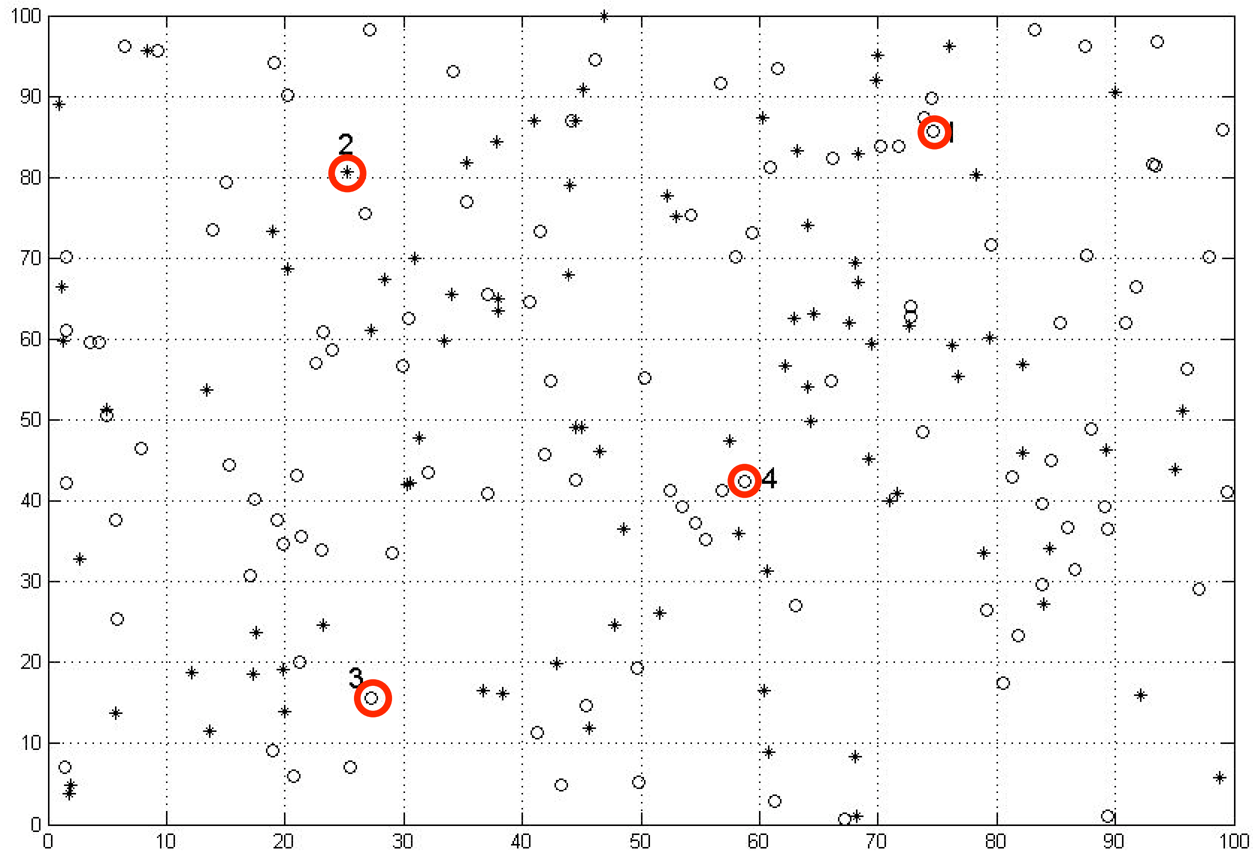
for some $C, \mu > 0$.

- The result holds for any exponentially stable semigroup.
- The unique solution of the Lyapunov equation is SD:

$$P(t)\phi = \int_0^t e^{A^* s} Q e^{As} \phi ds$$

Form a Cauchy sequence

Simulations



Systems marked by '*':

$$A_{kk} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad B_{kk} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Systems marked by 'o':

$$A_{kk} = \begin{bmatrix} -2 & 1 \\ 1 & -3 \end{bmatrix}, \quad B_{kk} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

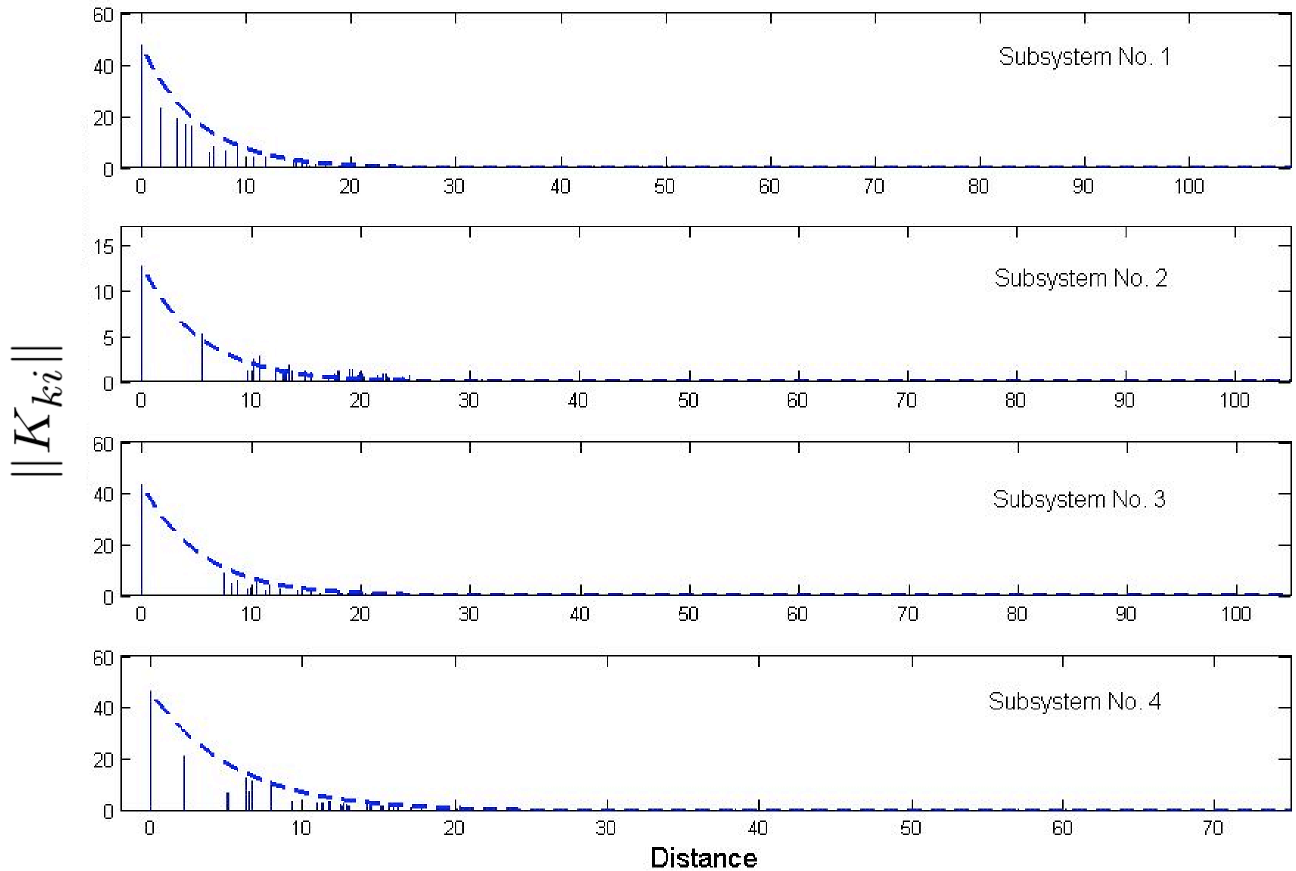
Coupling matrix:

$$A_{ki} = \frac{1}{\chi_\alpha(\mathrm{dis}(k,i))} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Coupled systems: $\dot{x}_k = A_{kk}\, x_k + B_{kk}\, u_k + \sum_{i=1}^{N} A_{ki}\, x_i \quad , \quad N = 200$
- In quadratic cost functional, the weighting matrices are defined as:

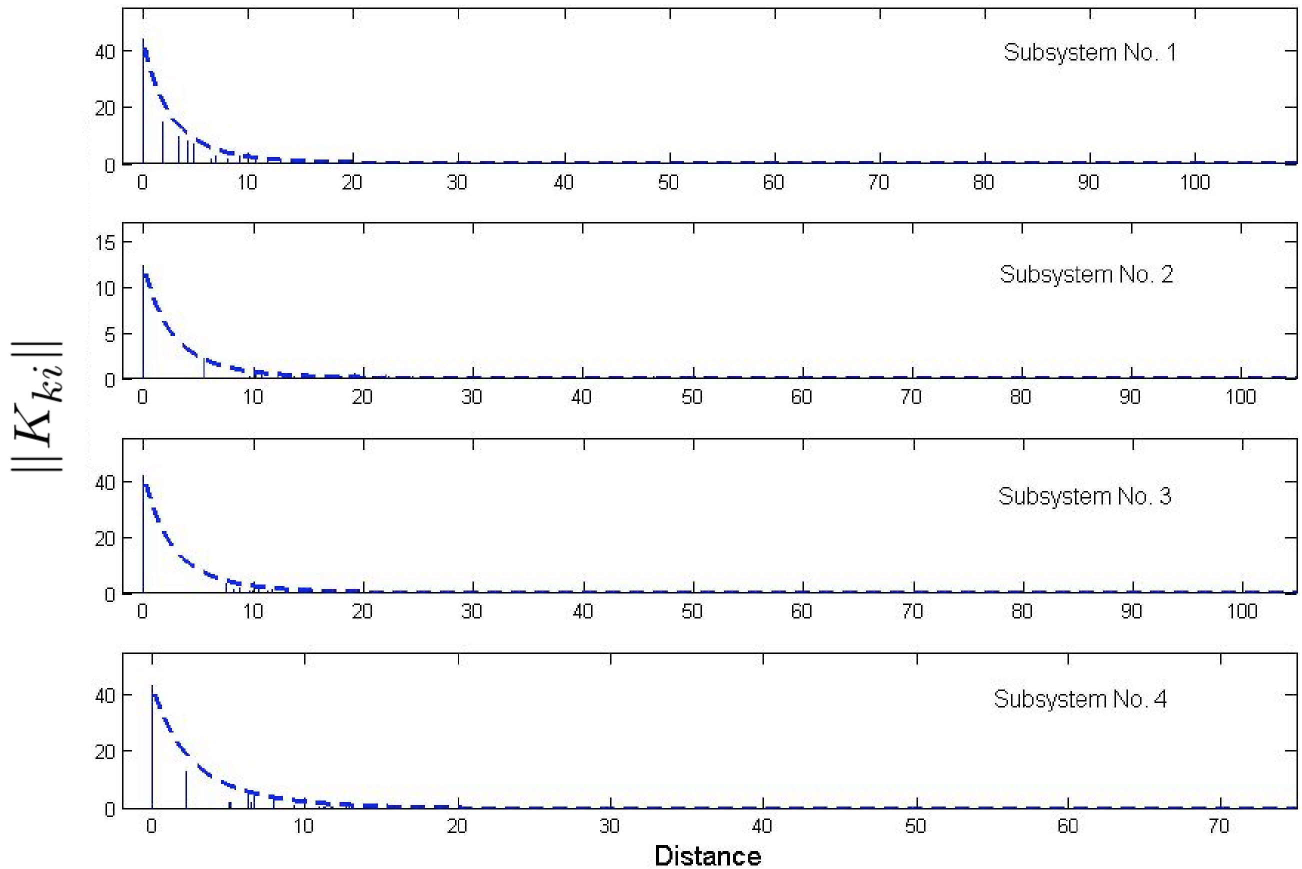
$$\text{(graph Laplacian)}\ Q_{ij} = \begin{cases} -1 & \text{if} \quad i \sim j \\ d_{ii} & \text{if} \quad i = j \end{cases} \quad , \quad R = I$$

Exponentially Decaying Couplings



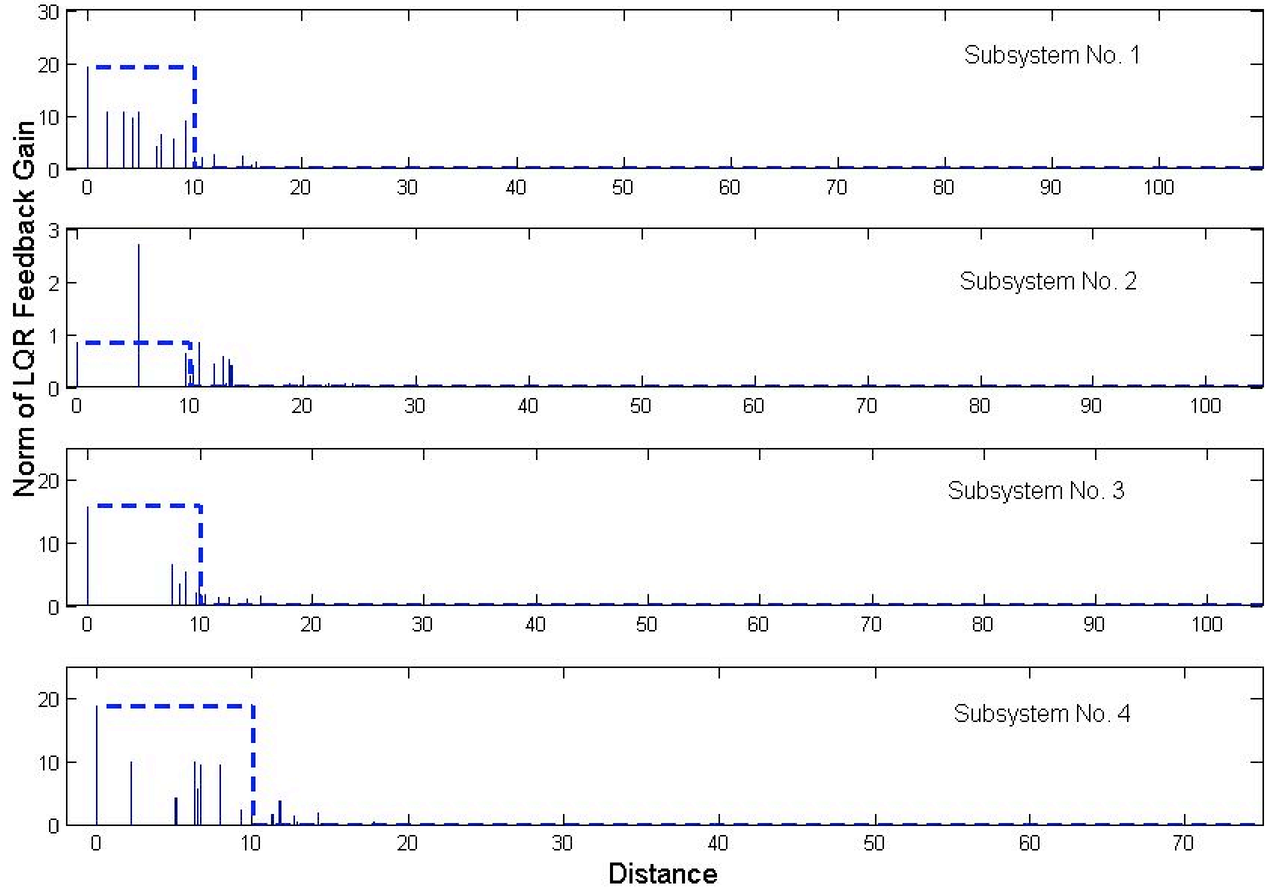
- The coupling function is $\chi_\alpha(x) = e^{\alpha x}$ where $\alpha = 0.1823$
- Optimal state-feedback: $u_k = K_{kk}\, x_k + \sum_{i \neq k} K_{ki}\, x_i$

Algebraically Decaying Couplings



- The coupling function is $\chi_\alpha(x) = (1 + 0.1x)^\alpha$ where $\alpha = 4$
- Optimal state-feedback: $u_k = K_{kk}\, x_k + \sum_{i \neq k} K_{ki}\, x_i$

Nearest Neighbor Couplings

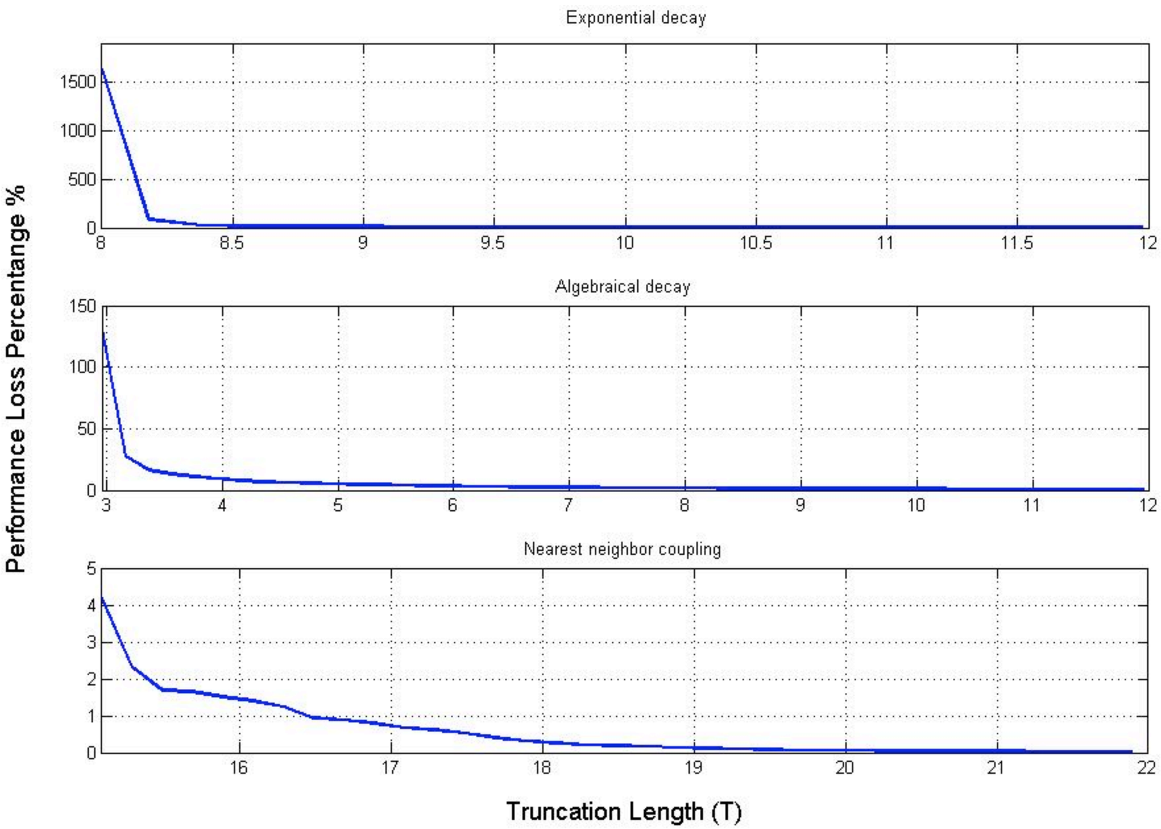


- The coupling is defined as $[\mathcal{A}]_{ki} = \begin{cases} \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} & \text{if} \quad \text{dis}(k,i) \leq 10 \\ \mathbf{0} & \text{otherwise} \end{cases}$

- Optimal state-feedback: $u_k = K_{kk}\, x_k + \sum_{i \neq k} K_{ki}\, x_i$

Spatial Truncation vs. Performance Loss

- Spatial truncation of the optimal controller: $[K_T]_{ki} = \begin{cases} [K]_{ki} & \text{if} \quad \text{dis}(k,i) \le T \\ \mathbf{0} & \text{if} \quad \text{dis}(k,i) > T. \end{cases}$



- Stabilizing truncation length:
  - Exp. decaying: $T_s = 7.9785$
  - Algeb. decaying: $T_s = 2.9603$
  - Nearest Neighbor: $T_s = 15.0934$

- Performance criteria: $$\left| \frac{\text{Trace}(P_T) - \text{Trace}(P)}{\text{Trace}(P)} \right| \times 100$$

where $$(A+BK_T)^* P_T + P_T(A+BK_T) + Q + K_T^* R K_T = \mathbf{0}$$

Part II

What you can impose

Architectural constraints that lead to convex optimal control problems

Outline

Controller Constraints that Lead to Convex Problems

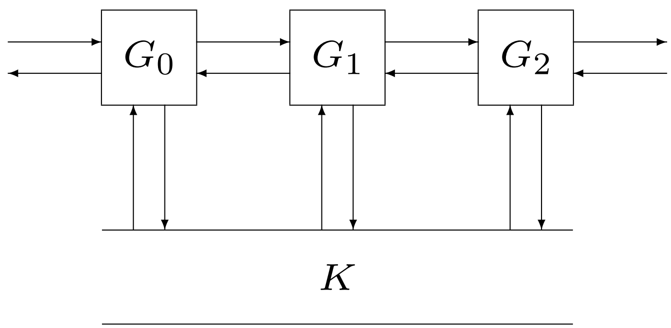
The YJBK Parameterization

Funnel Causality

Controller Architecture

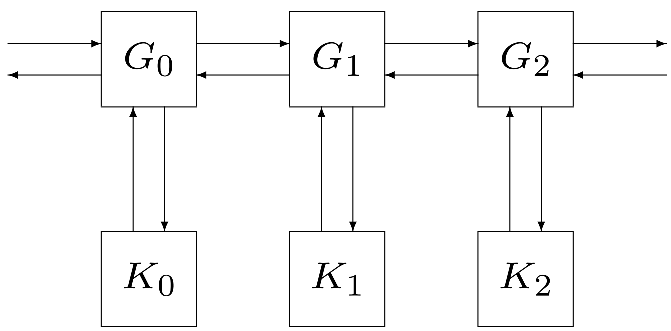
Centralized vs. Decentralized control: An old and difficult problem

CENTRALIZED:



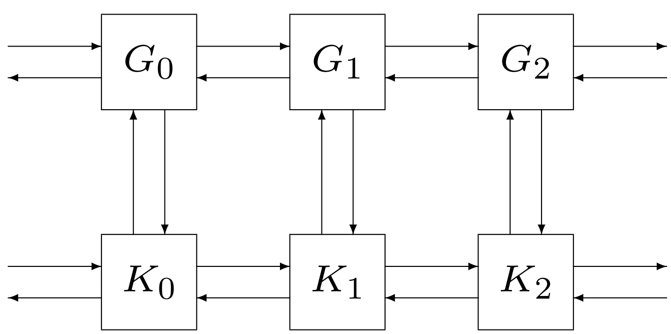
BEST PERFORMANCE
EXCESSIVE COMMUNICATION

FULLY DECENTRALIZED:



WORST PERFORMANCE
NO COMMUNICATION

LOCALIZED:



MANY POSSIBLE ARCHITECTURES

Reasoning with the YJBK Parameterization

Let G be a *stable* MIMO plant

- All stabilizing controllers (Internal Model Control)

$$K = Q(I + GQ)^{-1} \qquad Q \text{ stable}$$

- If G and Q belong to a CLASS closed under
  additions, multiplications, inversions

  Then $\quad Q \in \text{CLASS} \Leftrightarrow K \in \text{CLASS}$

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  Then $Q \in \text{CLASS} \Leftrightarrow K \in \text{CLASS}$

- Optimal design becomes

$$\inf_{Q \text{ stable},\ Q \in \text{CLASS}} \|H - UQV\|$$

Convex CLASS $\Rightarrow$ Convex problem

Reasoning with the YJBK Parameterization

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- All stabilizing controllers (Internal Model Control)

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$$\inf_{Q \text{ stable},\, Q \in \text{CLASS}} \|H - UQV\|$$

  Convex CLASS $\Rightarrow$ Convex problem

If G is unstable, use a factorization $G = NM^{-1}$, $XM - YN = I$

- All stabilizing controllers

$$K = (Y + MQ)(X + NQ)^{-1} \qquad Q \text{ stable}$$

Spatio-temporal Impulse Response

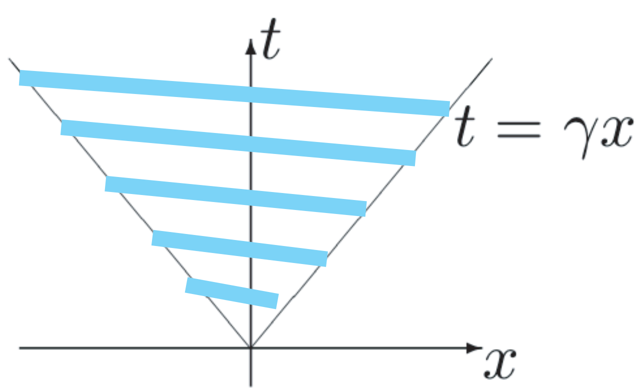
Spatio-temporal impulse response $h(x,t)$

$$y(x,t) = \int\int h(x-\xi, t-\tau)\, u(\xi,\tau)\, d\tau\, d\xi,$$

Interpretation

$h(x,t)$: effect of input on output a distance x away and time t later

Example: Constant maximum speed of effects



Funnel Causality

Def: A system is *funnel-causal* if impulse response $h(.,.)$ satisfies

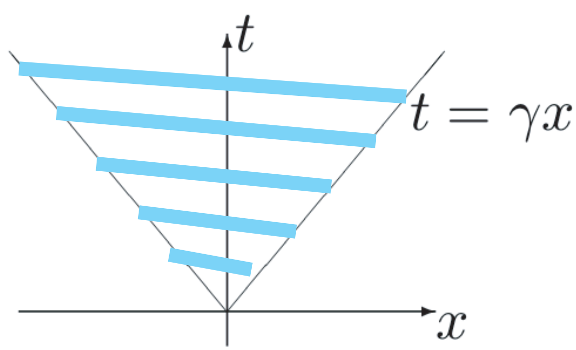
$h(x,t) = 0$ for $t < f(x)$,

where

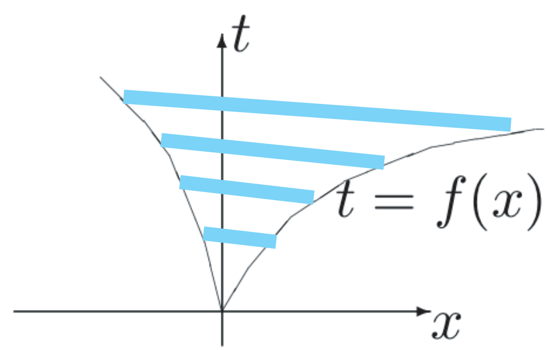
$f(.)$ is (1) non-negative

(2) $f(0) = 0$

(3) $\{f(x),\ x \geq 0\}$ and $\{f(x),\ x \leq 0\}$ are concave



(a) Cone causality



(b) Funnel causality

i.e. supp (h) is a “funnel shaped” region

Properties of funnel causal systems

Let S_f be a funnel shaped set

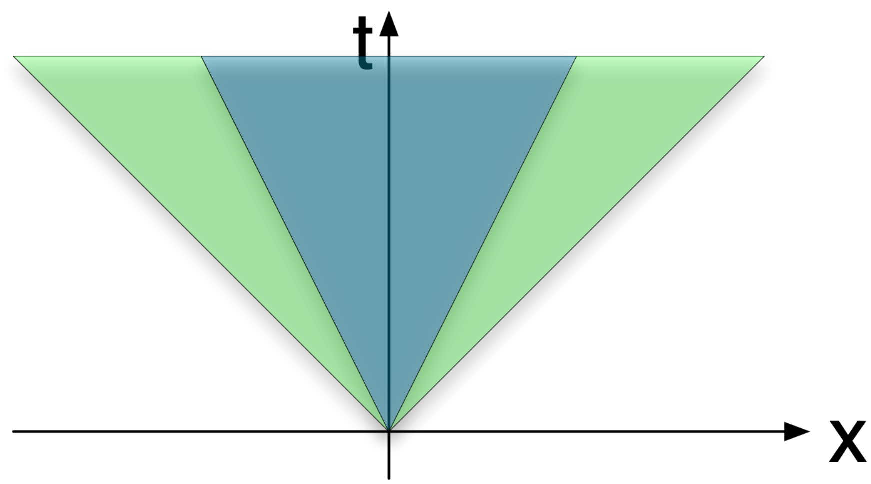
- supp $(h_1) \subset S_f$ & supp $(h_2) \subset S_f \quad \Rightarrow \quad$ supp $(h_1 + h_2) \subset S_f$
- supp $(h_1) \subset S_f$ & supp $(h_2) \subset S_f \quad \Rightarrow \quad$ supp $(h_1 * h_2) \subset S_f$
- $(I+h_1)^{-1}$ exists & supp $(h_1) \subset S_f \quad \Rightarrow \quad$ supp $\left((I+h_1)^{-1}\right) \subset S_f$

i.e.
The class of funnel-causal systems is closed under
Parallel, Serial, & Feedback interconnections

A Class of Convex Problems

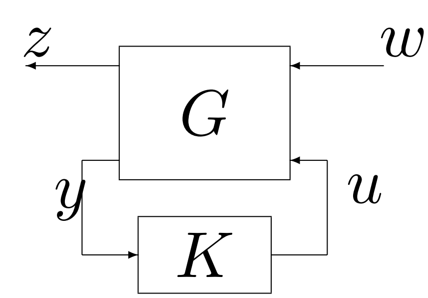
- Given a plant G with supp $(G_{22}) \subset S_{f_g}$
- Let S_{f_k} be a set such that $S_{f_g} \subset S_{f_k}$

  i.e. controller signals travel at least as fast as the plant's



Solve

$$\inf_{\substack{K \text{ stabilizing} \\ \text{supp}(K) \subset S_{f_k}}} \|\mathcal{F}(G;K)\|,$$



YJBK Parameterization and the Model Matching Problem

$L_f :=$ class of linear systems w/ impulse response supported in S_f

- Let $G_{22} \in L_{f_g}$
  $G_{22} = NM^{-1}$ and $XM - YN = I$ with $N, M, X, Y \in L_{f_g}$ and stable
- Let $S_{f_g} \subset S_{f_k}$
- Then all stabilizing controllers K such that $K \in L_{f_k}$ are given by

$$K \;=\; (Y + MQ)(X + NQ)^{-1},$$

  where Q is a stable system in L_{f_k}. ∎
- The problem becomes

$$\inf_{\substack{Q \text{ stable} \\ Q \in L_{f_k}}} \|H - UQV\|,$$

A convex problem!

Coprime Factorizations

Bezout identity: Find K and L such that $A + LC$ and $A + BK$ stable

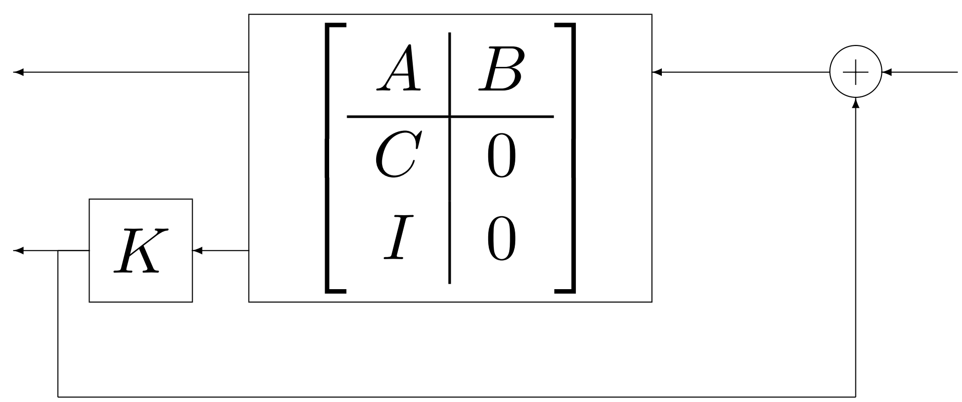
$$\begin{bmatrix} X & -Y \end{bmatrix} := \left[\begin{array}{c|cc} A+LC & -B & L \\ \hline K & I & 0 \end{array}\right], \quad \begin{bmatrix} M \\ N \end{bmatrix} := \left[\begin{array}{c|c} A+BK & B \\ \hline K & I \\ C & 0 \end{array}\right],$$

then $G = NM^{-1}$ and $XM - YN = I$,

If
- $e^{tA}B$, Ce^{tA} and $Ce^{tA}B$ are funnel causal
- K and L are funnel causal (Easy!)

then all elements of Bezout identity are funnel-causal

$$\left[\begin{array}{c|c} A+BK & B \\ \hline C & 0 \\ K & 0 \end{array}\right]$$



Example: Wave Equations with Input

1-d wave equation, $x \in \mathbb{R}$: $\partial_t^2 \psi(x,t) = c^2 \partial_x^2 \psi(x,t) + u(x,t)$

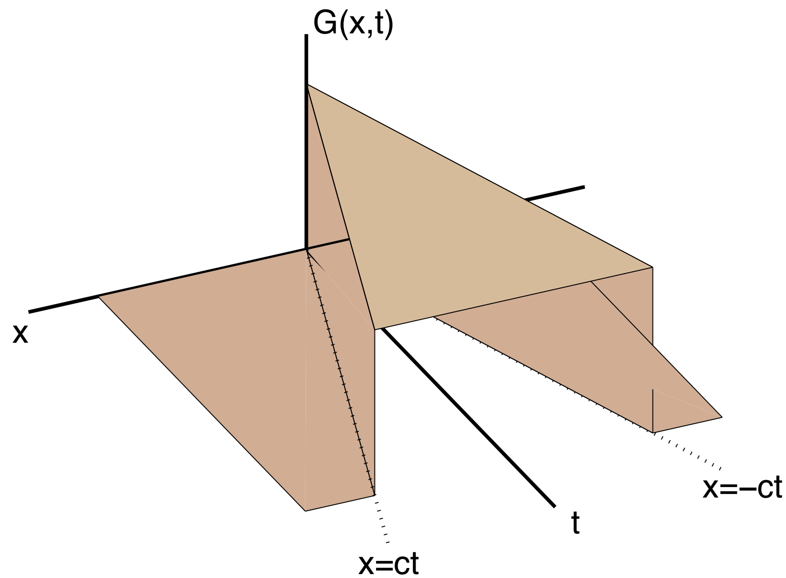
State space representation :

$$\partial_t \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} 0 & I \\ c^2 \partial_x^2 & 0 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} u$$

$$\psi = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}.$$

The semigroup

$$e^{tA} = \frac{1}{2} \begin{bmatrix} T_{ct} + T_{-ct} & \frac{1}{c} R_{ct} \\ c\partial_x^2 R_{ct} & T_{ct} + T_{-ct} \end{bmatrix}.$$



R_{ct} := spatial convolution with rec$(\frac{1}{ct}x)$

T_{ct} := translation by ct

all components are funnel causal

e.g. the impulse response $h(x,t) = \frac{1}{2c}$ rec $\left(\frac{1}{ct}x\right)$.

Example: Wave Equations with Input (cont.)

κ := spatial Fourier transform variable ("wave number")

$$A + BK = \begin{bmatrix} 0 & 1 \\ -c^2\kappa^2 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 1 \\ -c^2\kappa^2 + k_1 & k_2 \end{bmatrix}.$$

Set $k_1 = 0$, then

$$\sigma(A+BK) = \bigcup_{\kappa\in\mathbb{R}} \left(k_2 \pm \frac{1}{2}\sqrt{k_2^2 - 4c^2\kappa^2} \right) = \left[\frac{3}{2}k_2, \frac{1}{2}k_2\right] \bigcup \, (k_2 + j\mathbb{R})$$

Similarly for $A + LC$. Therefore, choose e.g.

$$K = \begin{bmatrix} 0 & -1 \end{bmatrix}, \quad L = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

Elements of the Bezout Identity are thus:

$$\left[\begin{array}{cc} X & -Y \end{array}\right] = \left[\begin{array}{cc|cc} -1 & 1 & 0 & -1 \\ -c^2\kappa^2 & 0 & -1 & 0 \\ \hline 0 & -1 & 1 & 0 \end{array}\right],$$

$$\left[\begin{array}{c} M \\ N \end{array}\right] = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -c^2\kappa^2 & -1 & 1 \\ \hline 0 & -1 & 1 \\ 1 & 0 & 0 \end{array}\right].$$

Equivalently

$$M = \frac{s^2 + c^2\kappa^2}{s^2 + s + c^2\kappa^2}, \qquad X = \frac{s^2 + 2s + c^2\kappa^2 + 1}{s^2 + s + c^2\kappa^2},$$

$$N = \frac{1}{s^2 + s + c^2\kappa^2}, \qquad -Y = \frac{-c^2\kappa^2}{s^2 + s + c^2\kappa^2}.$$

How easily solvable are the resulting convex problems?

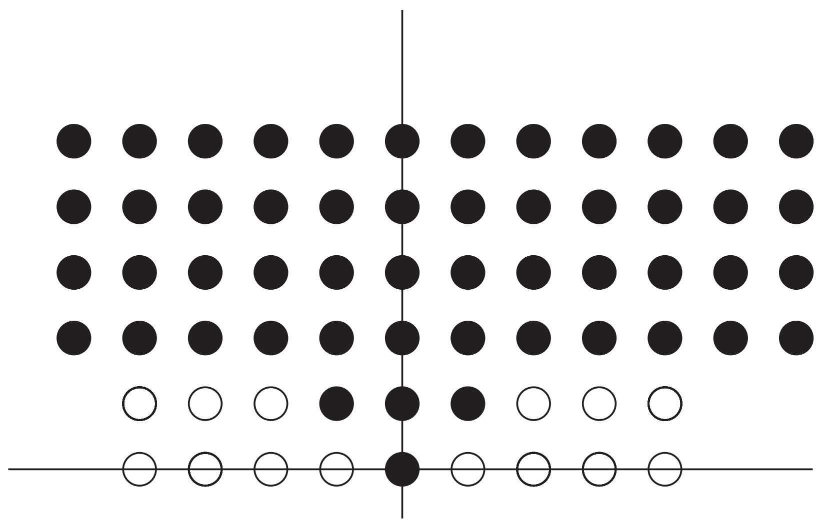
- In general, these convex problems are infinite dimensional

  i.e. worse than standard half-plane causality

- In certain cases, problem similar in complexity to half-plane causality

  e.g. H^2 with the causality structure below

  (Voulgaris, Bianchini, Bamieh, SCL '03)



Generalizations

- Quick generalizations:
  - Several spatial dimensions
  - Spatially-varying systems

    funnel causality $\leftrightarrow$ *non-decreasing speed with distance*
  - Use relative degree in place of time delay

- Quadratic Invariance (*Rotkowitz, Lall*)

- Arbitrary graphs (*Rotkowitz, Cogill, Lall*)

- How to solve the resulting convex problems

Related recent work:

- *Anders Rantzer*