PURPOSE OF LESSON: To show how to solve the interior Dirichlet problem for the circle

\[
\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0 \quad 0 < r < 1
\]

BC \quad u(1, \theta) = g(\theta) \quad 0 \leq \theta < 2\pi

by separation of variables. The solution can be interpreted as expanding the boundary function as

\[
g(\theta) = \sum_{n=0}^{\infty} [a_n \cos(n\theta) + b_n \sin(n\theta)]
\]

and then finding the solution to each of the problems

\[
\begin{align*}
\nabla^2 u &= 0 \\
\nabla^2 u &= 0 \\
u(1, \theta) &= \sin(n\theta) \\
u(1, \theta) &= \cos(n\theta)
\end{align*}
\]

Since these two problems have solutions

\[
\begin{align*}
u(r, \theta) &= r^n \sin(n\theta) \\
u(r, \theta) &= r^n \cos(n\theta)
\end{align*}
\]

the solution to the interior Dirichlet problem is

\[
u(r, \theta) = \sum_{n=0}^{\infty} r^n [a_n \cos(n\theta) + b_n \sin(n\theta)]
\]

After this series is obtained, some algebraic manipulations are performed to arrive at an alternative integral-form of the solution. This new form, the Poisson integral solution, brings out some interesting ideas.

This lesson presents a number of

PDE

\[
\begin{align*}
\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} &= 0 \\
0 < r < 1 \\
u(1, \theta) &= g(\theta) \\
0 \leq \theta < 2\pi
\end{align*}
\]

The method of separation of variables is used to find this series solution, which is known as the Poisson integral formula.

Problem (33.1) is very interesting as it arises in finding the electrostatic potential on the boundary of a circular wire hoop and dip it into a soap solution. The displacement \(g(\theta)\) is small.

The reader should be careful to note that some algebraic manipulations are performed to arrive at an alternative integral-form of the solution. This new form, the Poisson integral solution, brings out some interesting ideas.
This lesson presents a number of new ideas as we solve the Dirichlet problem

\[ \begin{align*} 
\text{PDE} & \quad u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta \theta} = 0 \quad 0 < r < 1 \\
\text{BC} & \quad u(1, \theta) = g(\theta) \quad 0 \leq \theta < 2\pi 
\end{align*} \]  

(33.1)

The method of separation of variables will be the usual procedure, but after we
find this series solution, we then manipulate it to get an alternative formulation
(Poisson integral formula).

Problem (33.1) is very important in physical applications. It can be interpreted
as finding the electrostatic potential inside a circle when the potential is given
on the boundary. Another interpretation is the soap film model. If we start with
a circular wire loop and distort it so that the distortion is measured by \( g(\theta) \) and
dip it into a soap solution, a film of soap is formed within the wire. The height
of the film is represented by the solution of problem (33.1), provided the displace­ment \( g(\theta) \) is small.

The reader should be well aware of the separation-of-variables technique
outlined in Figure 33.1 and should work out the details (problem 1).

A few comments on the outline in Figure 33.1

![Figure 33.1 Outline of the solution for the interior Dirichlet problem.](image)

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1. The reader should verify that the separation constant must be nonnegative (that's why it's called $\lambda^2$). If the separation constant were negative, the function $\Theta(\theta)$ would not be periodic, while, on the other hand, if it were zero, we would throw out the $\ln r$ term in solution $R(r) = a + b \ln r$.

2. The reader should know how the constants $a_n$ and $b_n$ were obtained. To summarize, the solution to the interior Dirichlet problem (33.1) is

$$u(r, \theta) = \sum_{n=0}^{\infty} r^n [a_n \cos (n \theta) + b_n \sin (n \theta)]$$

Before we rewrite this solution in the alternative form, let's make some observations.

**Observations on the Dirichlet Solution**

1. The interpretation of our solution (33.2) is that we should expand the boundary function $g(\theta)$ as a Fourier series

$$g(\theta) = \sum_{n=0}^{\infty} [a_n \cos (n \theta) + b_n \sin (n \theta)]$$

and then solve the problem for each sine and cosine in the series. Since each of these terms will give rise to solutions $r^n \sin (n \theta)$ and $r^n \cos (n \theta)$, we can then say (by superposition) that

$$u(r, \theta) = \sum_{n=0}^{\infty} r^n [a_n \cos (n \theta) + b_n \sin (n \theta)]$$

2. The solution of

\[
\begin{align*}
\text{PDE} & \quad \nabla^2 u = 0 \quad 0 < r < 1 \\
\text{BC} & \quad u(1, \theta) = 1 + \sin \theta + \frac{1}{2} \sin (3 \theta) + \cos (4 \theta)
\end{align*}
\]

would be

$$u(r, \theta) = 1 + r \sin \theta + \frac{r^3}{2} \sin (3 \theta) + r^4 \cos (4 \theta)$$

Here, the $g(\theta)$ is already in the form of a Fourier series, with

$$\begin{align*}
a_0 & = 1 & b_1 & = 1 \\
a_1 & = 1 & b_2 & = 0.5 \\
\text{All other } a_n \text{'s} & = 0 & \text{All other } b_n \text{'s} & = 0
\end{align*}$$

3. If the radius of the circle would be $R$, the solution of $g(\theta)$ would be

$$u(r, \theta) = \sum_{n=0}^{\infty} \frac{r^n}{R^n} \left(1 + 2 \int_0^{2\pi} g(\theta) d\theta \right)$$

This completes our discussion and gets to the interesting Poisson integral formula.

**Poisson Integral Formula**

We start with the separation of variables $\nabla^2 u = 0$ and $u(1, \theta) = 1 + \sin \theta + \frac{1}{2} \sin (3 \theta) + \cos (4 \theta)$ (we now take an arbitrary radius $R$). After a few manipulations, we have

\[
\begin{align*}
\text{PDE} & \quad \nabla^2 u = 0 \\
\text{BC} & \quad u(1, \theta) = 1 + \sin \theta + \frac{1}{2} \sin (3 \theta) + \cos (4 \theta)
\end{align*}
\]

and so we don't have to use $\lambda^2$. Then the solution for $u(r, \theta)$ would be

$$u(r, \theta) = \sum_{n=0}^{\infty} \frac{r^n}{R^n} \left(1 + 2 \int_0^{2\pi} g(\theta) d\theta \right)$$

(we now take an arbitrary radius $R$)
and so we don’t have to use the formulas for $a_n$ and $b_n$.

3. If the radius of the circle were arbitrary (say $R$), then the solution would be

$$u(r, \theta) = \sum_{n=0}^{\infty} \left( \frac{r}{R} \right)^n \left[ a_n \cos (n\theta) + b_n \sin (n\theta) \right]$$

4. Note that the constant term $a_0$ in solution (33.2) represents the average of $g(\theta)$

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} g(\theta) \, d\theta$$

This completes our discussion of the separation-of-variables solution. We now get to the interesting Poisson integral formula.

**Poisson Integral Formula**

We start with the separation of variables solution

$$u(r, \theta) = \sum_{n=0}^{\infty} \left( \frac{r}{R} \right)^n \left[ a_n \cos (n\theta) + b_n \sin (n\theta) \right]$$

(we now take an arbitrary radius for the circle) and substitute the coefficients $a_n$ and $b_n$. After a few manipulations involving algebra, calculus, and trigonometry, we have

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} g(\theta) \, d\theta + \frac{1}{\pi} \sum_{n=1}^{\infty} \left( \frac{r}{R} \right)^n \int_0^{2\pi} g(\alpha) \left[ \cos (n\alpha) \cos (n\theta) + \sin (n\alpha) \sin (n\theta) \right] d\alpha$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left\{ 1 + 2 \sum_{n=1}^{\infty} \left( \frac{r}{R} \right)^n \cos [n(\theta - \alpha)] \right\} g(\alpha) \, d\alpha$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left\{ 1 + \sum_{n=1}^{\infty} \left( \frac{r}{R} \right)^n \left[ e^{in(\theta - \alpha)} + e^{-in(\theta - \alpha)} \right] \right\} g(\alpha) \, d\alpha$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left\{ 1 + \frac{re^{i(\theta - \alpha)}}{R - re^{-i(\theta - \alpha)}} + \frac{re^{-i(\theta - \alpha)}}{R - re^{i(\theta - \alpha)}} \right\} g(\alpha) \, d\alpha$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left[ \frac{R^2 - r^2}{R^2 - 2rR \cos (\theta - \alpha) + r^2} \right] g(\alpha) \, d\alpha$$

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solutions for different \( n \) we then obtain
\[
u(x, y) = \sum_{n=1}^{\infty} B_n \sinh[\pi(a-x)/b] \sin(n\pi y/b),
\] (21.20)
for some constants \( B_n \). We have omitted negative values of \( n \) in the sum (21.20) since the relevant terms are already included in those obtained for positive \( n \). Again the \( n = 0 \) term is identically zero. Using the final boundary condition \( u(0,y) = f(y) \) as above we find that the constants \( B_n \) must satisfy
\[
f(y) = \sum_{n=1}^{\infty} B_n \sinh(n\pi a/b) \sin(n\pi y/b),
\]
and, remembering the caveats discussed in the previous example, the \( B_n \) are therefore given by
\[
B_n = \frac{2}{b \sinh(n\pi a/b)} \int_0^b f(y) \sin(n\pi y/b) dy.
\] (21.21)

For the case where \( f(y) = u_0 \), following the working of the previous example gives (21.21) as
\[
B_n = \frac{4u_0}{\pi n \sinh(n\pi a/b)} \text{ for } n \text{ odd}, \quad B_n = 0 \text{ for } n \text{ even.}
\] (21.22)
The required solution is thus
\[
u(x, y) = \sum_{n \text{ odd}} \frac{4u_0}{\pi n \sinh(n\pi a/b)} \sinh[\pi(a-x)/b] \sin(n\pi y/b).
\]
We note that, as required, in the limit \( a \to \infty \) this solution tends to the solution of the previous example.

Often the principle of superposition can be used to write the solution to problems with more complicated boundary conditions as the sum of solutions to problems that each satisfy only some part of the boundary condition but when added together satisfy all the conditions.

\begin{itemize}
  \item Find the steady-state temperature in the (finite) rectangular plate of the previous example, subject to the boundary conditions \( u(x, b) = 0 \), \( u(a, y) = 0 \) and \( u(0, y) = f(y) \) as before, but now, in addition, \( u(x, 0) = g(x) \).
\end{itemize}

Figure 21.3(c) shows the imposed boundary conditions for the metal plate. Although we could find a solution to this problem using the methods presented above, we can arrive at the answer almost immediately by using the principle of superposition and the result of the previous example.

Let us suppose the required solution \( u(x, y) \) is made up of two parts:
\[
u(x, y) = v(x, y) + w(x, y),
\]
where \( v(x, y) \) is the solution satisfying the boundary conditions shown in figure 21.3(a).
whilst \( v(x, y) \) is the solution satisfying the boundary conditions in figure 21.3(b). It is clear that \( v(x, y) \) is simply given by the solution to the previous example,

\[
\v(x, y) = \sum_{n \text{ odd}} B_n \sinh \left( \frac{n\pi(a-x)}{b} \right) \sin \left( \frac{n\pi y}{b} \right),
\]

where \( B_n \) is given by (21.21). Moreover, by symmetry, \( w(x, y) \) must be of the same form as \( v(x, y) \) but with \( x \) and \( a \) interchanged with \( y \) and \( b \), respectively, and with \( f(y) \) in (21.21) replaced by \( g(x) \). Therefore the required solution can be written down immediately without further calculation as

\[
u(x, y) = \sum_{n \text{ odd}} B_n \sinh \left( \frac{n\pi(a-x)}{b} \right) \sin \left( \frac{n\pi y}{b} \right) + \sum_{n \text{ odd}} C_n \sinh \left( \frac{n\pi(b-y)}{a} \right) \sin \left( \frac{n\pi x}{a} \right),
\]

the \( B_n \) being given by (21.21) and the \( C_n \) by

\[
C_n = -\frac{2}{a \sinh(ab/a)} \int_0^b g(x) \sin(n\pi x/a) \, dx.
\]

Clearly, this method may be extended to cases in which three or four sides of the plate have non-zero boundary conditions.\footnote{A bar of length \( L \) at 0 \( ^\circ \text{C} \) and the temperature distribution}

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with \( \kappa = k/\rho c \) the last of which cause difficulty equivalent takes the form

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which are homogeneous

From (21.12) the corresponding solutions are to add to \( v(x, y) \) to we require \( A \) and

As a final example of the usefulness of the principle of superposition we now consider a problem that illustrates how to deal with inhomogeneous boundary conditions by a suitable change of variables.

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