26. Find the eigenvalues and eigenfunctions of the integral operator

$$Ku(x) = \int_{-1}^{1} (1 - |x - y|)u(y)\, dy.$$

27. (This exercise requires perturbation methods from Chapter 2.) Consider the differential-integral equation

$$\varepsilon u' = u - u^2 - u \int_0^t u(s)ds, \quad u(0) = a < 1,$$

where ε is a small, positive parameter, which models a population $u = u(t)$ undergoing logistics growth and the cumulative effect of a toxin on the population. Find a uniformly valid approximation for $t > 0$.

28. Consider the differential–integral operator

$$Ku = -u'' + 4\pi^2 \int_0^1 u(s)\, ds, \quad u(0) = u(1) = 0.$$

Prove that eigenvalues of K, provided they exist, are positive. Find the eigenfunctions corresponding to the eigenvalue $\lambda = 4\pi^2$.

29. Use (4.40) to numerically solve the integral equation

$$\int_0^1 (1 - 3xy)u(y)\, dy - u(x) = x^3.$$

4.4 Green's Functions

What is a Green's function? Mathematically, it is the kernel of an integral operator that represents the inverse of a differential operator; physically, it is the response of a system when a unit point source is applied to the system. In this section we show how these two apparently different interpretations are actually the same.

4.4.1 Inverses of Differential Operators

To fix the notion we consider a regular Sturm–Liouville problem (SLP), which we write in the form

$$Au \equiv -(pu')' + qu = f, \qquad a < x < b, \tag{4.41}$$

$$B_1 u(a) \equiv \alpha_1 u(a) + \alpha_2 u'(a) = 0, \tag{4.42}$$

$$B_2 u(b) \equiv \beta_1 u(b) + \beta_2 u'(b) = 0, \tag{4.43}$$

where p, p', q, and f are continuous on $[a, b]$, and $p > 0$. So that the boundary conditions do not disappear, we assume that not both α_1 and α_2 are zero, and similarly for β_1 and β_2. For conciseness we represent this problem in operator notation

$$Lu = f, \tag{4.44}$$

where we consider the differential operator L as acting functions in $C^2[a, b]$ that satisfy the boundary conditions (4.42)–(4.43). Thus, L contains the boundary conditions in its definition, as well as the Sturm–Liouville operator A.

Recall the procedure in matrix theory when we have a *matrix* equation $L\mathbf{u} = \mathbf{f}$, where $\mathbf{u}$ and $\mathbf{f}$ are vectors and L is a square, invertible matrix. We immediately have the solution $\mathbf{u} = L^{-1}\mathbf{f}$, where L^{-1} is the inverse matrix. The inverse matrix exists if $\lambda = 0$ is not an eigenvalue of L, or when $\det L \neq 0$. We want to perform a similar calculation with the differential equation (4.44) and write the solution

$$u = L^{-1} f,$$

where L^{-1} is the inverse operator of L. Since L is a differential operator, we expect that the inverse operator to be an integral operator of the form

$$(L^{-1} f)(x) = \int_a^b g(x, \xi) f(\xi)\, d\xi, \tag{4.45}$$

with kernel g. Again drawing inferences from matrix theory, we expect the inverse to exist when $\lambda = 0$ is not an eigenvalue of L, that is, when there are no nontrivial solutions to the differential equation equation $Lu = 0$. If there are nontrivial solutions to $Lu = 0$, then L is not a one-to-one transformation ($u = 0$ is always a solution) and so L^{-1} does not exist.

If the inverse L^{-1} of the differential operator L exists, then the kernel function $g(x, \xi)$ in (4.45) is called the **Green's function** associated with L (recall that L contains in its definition the boundary conditions). This is the mathematical characterization of a Green's function. Physically, as we observe subsequently, the Green's function $g(x, \xi)$ is the solution to (4.44) when f is a unit point source acting at the point ξ; thus, it is the response of the system at x to a unit, point source at ξ. Our approach here will be to present the mathematical result first and then have the methodology unfold with a heuristic physical discussion. The following theorem summarizes our preceding comments and gives a precise representation of the Green's function.

Theorem 4.19

Consider the SLP (4.41)–(4.43), and assume that $\lambda = 0$ is not an eigenvalue of

L. Then L^{-1} exists and is given by (4.45) with

$$g(x,\xi) = \begin{cases} -\frac{u_1(x)u_2(\xi)}{p(\xi)W(\xi)}, & x < \xi \\ -\frac{u_1(\xi)u_2(x)}{p(\xi)W(\xi)}, & x > \xi \end{cases} \tag{4.46}$$

Here, u_1 and u_2 are independent solutions of the homogeneous differential equation $Au = -(pu')' + qu = 0$ with $B_1u_1(a) = 0$ and $B_2u_2(b) = 0$, and $W = u_1u_2' - u_1'u_2$ is the Wronskian of u_1 and u_2.

Before giving the proof, we make some remarks and give an example. Clearly, knowing the Green's function g allows us to immediately write down the unique solution to the inhomogeneous problem (4.41)–(4.43) as

$$u(x) = \int_a^b g(x,\xi)f(\xi)\,d\xi. \tag{4.47}$$

Also, the Green's function can be expressed as a single equation in terms of the Heaviside step function $H(x)$ (where $H(x) = 0$ if $x < 0$, and $H(x) = 1$ if $x \geq 0$). Then

$$g(x,\xi) = -\frac{1}{p(\xi)W(\xi)}(H(x-\xi)u_1(\xi)u_2(x) + H(\xi - x)u_1(x)u_2(\xi)). \tag{4.48}$$

Let us write down some of the basic properties of the Green's function. First it is clear that: (a) $g(x,\xi)$ satisfies the differential equation $Ag(x,\xi) = 0$ for $x \neq \xi$; (b) $g(x,\xi)$ satisfies the boundary conditions (4.42) and (4.43); (c) g is continuous on $[a,b]$ and, in particular, it is continuous at $x = \xi$; (d) however, g is not differentiable at $x = \xi$. It is easy to verify that there is a jump in the derivative at $x = \xi$ given by $g'(\xi^+,\xi) - g'(\xi^-,\xi) = -\frac{1}{p(\xi)}$. Here, prime denotes differentiation with respect to x. Thus g is a continuous curve with a corner at $x = \xi$.

Example 4.20

Consider the boundary value problem

$$-u'' = f(x), \quad 0 < x < 1; \quad u(0) = u(1) = 0. \tag{4.49}$$

Here $Au = -u''$, and solutions to the homogeneous equation $Au = 0$ are linear functions $u(x) = ax + b$, where a and b are constants. Thus, we take $u_1(x) = ax$ and $u_2(x) = b(1-x)$, so that u_1 satisfies the left boundary condition and u_2 satisfies the right boundary condition. Also, $p = 1$ and the Wronskian is $W = -ab$. Therefore, the Green's function is given, according to (4.48), by

$$g(x,\xi) = \xi(1-x)H(x-\xi) + x(1-\xi)H(\xi - x).$$

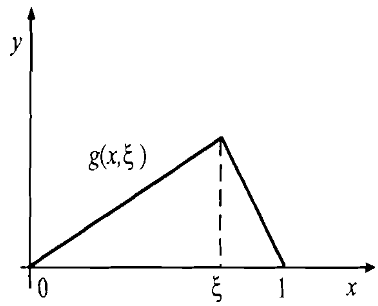

Figure 4.1 Green's function.

Consequently, the solution to (4.49) is given by

$$u(x) = \int_0^1 g(x,\xi) f(\xi)\, d\xi$$

The reader should verify that this is indeed the solution to (4.49). The Green's function is shown in Fig. 4.1.

Finally, the proof of the theorem is a straightforward calculation. First we write

$$\begin{aligned} v(x) &= \int_a^b g(x,\xi) f(\xi)\, d\xi \\ &= -\int_a^x \frac{u_1(\xi)u_2(x)}{p(\xi)W(\xi)} f(\xi)\, d\xi - \int_x^b \frac{u_1(x)u_2(\xi)}{p(\xi)W(\xi)} f(\xi)\, d\xi. \end{aligned}$$

To prove the theorem, we must show that $Av = f$ and that the boundary conditions hold. The calculation of the derivatives of v can be accomplished by a direct application of the Leibniz rule, and we leave the details of this calculation to the reader.

4.4.2 Physical Interpretation

Next we investigate the physical interpretation of the Green's function (George Green, 1793–1841). Our discussion is heuristic and intuitive, but it is made precise in the next section. To fix the context we consider a steady-state heat flow problem. Consider a cylindrical bar of length L and cross-sectional area A, and let $u = u(x)$ denote the temperature at cross section x. Further, let $q = q(x)$ denote the energy flux across the face at x, measured in energy/(area

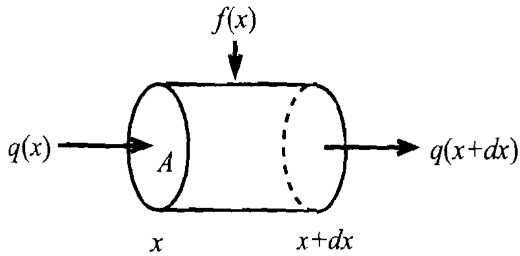

Figure 4.2 Small region.

· time), and let $f(x)$ be a given, distributed heat source over the length of the bar, measured in energy/(volume · time). By convention, the flux q is positive when the heat flow is to the right. Then, if x and $x + dx$ denote two arbitrary locations in the bar, in a steady-state mode the heat energy entering the cross section at x, minus the heat energy exiting the cross section at $x + dx$, plus the heat generated by the source is zero. See Fig. 4.2. That is,

$$Aq(x) - Aq(x + dx) + f(x)A\,dx = 0,$$

or, rearranging,

$$\frac{q(x + dx) - q(x)}{dx} = f(x).$$

These equations are merely a statement of conservation of energy. Now, taking the limit as $dx \to 0$, we obtain

$$q'(x) = f(x).$$

To obtain and equation for temperature, we assume Fourier's heat conduction law, which states that the flux is proportional to the negative temperature gradient, or

$$q(x) = -Ku'(x),$$

where K is the thermal conductivity of the bar, a physical constant. Then

$$-Ku''(x) = f(x),$$

which is the steady-state heat equation. For simplicity, we assume that the ends of a bar of unit length are held at zero degrees, and $K = 1$. Then the steady temperature distribution $u = u(x)$ along the length of the bar satisfies the boundary value problem

$$-u'' = f(x), \quad 0 < x < 1; \quad u(0) = u(1) = 0,$$

which is the same as (4.49). Now imagine that f is an idealized heat source of unit strength that acts only at a single point $x = \xi$ in $(0,1)$, and we denote this source by $f \equiv \delta(x,\xi)$. Thus, it is assumed to have the properties

$$\delta(x,\xi) = 0, \quad x \neq \xi$$

and

$$\int_0^1 \delta(x,\xi)\,dx = 1.$$

A little reflection indicates that there is no ordinary function δ with these two properties; a function that is zero everywhere but one point must have zero integral. Nevertheless, this δ symbol has been used since the inception of quantum mechanics in the late 1920s (it was introduced by the mathematician-physicist P. Dirac); in spite of not having a rigorous definition of the symbol until the early 1950s, when the mathematician L. Schwartz gave a precise characterization of point sources, both physicists and engineers used the "δ function" with great success. In the next section we give a careful definition of the delta function, but for the present we continue with an intuitive discussion.

The differential equation for steady heat flow becomes, symbolically,

$$-u'' = \delta(x,\xi). \tag{4.50}$$

Thus, for $x \neq \xi$ we have $-u'' = 0$, which has solutions of the form $u = Ax + B$. To satisfy the boundary conditions we take

$$u = Ax, \quad x < \xi; \qquad u = B(1-x), \quad x > \xi.$$

The reader should be reminded of the calculation we made in the last example. To determine the two constants A and B we use the fact that the temperature should be continuous at $x = \xi$, giving the relation $A\xi = B(1-\xi)$. To obtain another condition on A and B we integrate the symbolic differential equation (4.50) over an interval $[\xi - \varepsilon, \xi + \varepsilon]$ containing ξ to get

$$-\int_{\xi-\varepsilon}^{\xi+\varepsilon} u''(x)\,dx = \int_{\xi-\varepsilon}^{\xi+\varepsilon} \delta(x,\xi)\,dx.$$

The right side is unity because of the unit heat source assumption; the fundamental theorem of calculus is then applied to the left side to obtain

$$-u'(\xi+\varepsilon) + u'(\xi-\varepsilon) = 1.$$

Taking the limit as $\varepsilon \to 0$ and multiplying by -1 yields

$$u'(\xi^+) - u'(\xi^-) = -1.$$

Observe that this condition is precisely the jump condition on the derivative at the point $x = \xi$ required of a Green's function. This last condition forces $-B - A = -1$. Therefore, solving the two equations for A and B simultaneously gives

$$A = (1 - \xi), \quad B = \xi.$$

Therefore the steady-state temperature in the bar, caused by a point source at $x = \xi$ is

$$u = (1 - \xi)x, \quad x < \xi; \quad u = \xi(1 - x), \quad x > \xi$$

or

$$u(x, \xi) = \xi(1 - x)H(x - \xi) + x(1 - \xi)H(\xi - x).$$

This is precisely the Green's function for the operator $L = -d^2/dx^2$ with boundary conditions $u(0) = u(1) = 0$ that we calculated in the last example. Consequently, this gives a physical interpretation for the Green's function; it is the response of a system (in this case the steady temperature response) to a point source (in this case a unit heat source). See Fig. 4.1.

This idea is fundamental in applied mathematics. The Green's function $g(x, \xi)$ is the solution to the *symbolic* boundary value problem

$$Ag(x, \xi) \equiv (-pg')' + qg = \delta(x, \xi), \tag{4.51}$$

$$B_1 g(a, \xi) = 0, \quad B_2 g(b, \xi) = 0, \tag{4.52}$$

where δ represents a point source at $x = \xi$ of unit strength. Furthermore, the solution (4.47) to the boundary value problem $Lu = f$ can therefore be regarded as a superposition of point sources of magnitude $f(\xi)$ over the entire interval $a < \xi < b$.

Example 4.21

Consider the differential operator $A = -d^2/dx^2$ on $0 < x < 1$ with the boundary conditions $u'(0) = u'(1) = 0$. In this case the Green's function does not exist since the equation $Lu = 0$ has nontrivial solutions (any constant function will satisfy the differential equation and the boundary conditions); stated differently, $\lambda = 0$ is an eigenvalue. Physically we can also see why the Green's function does not exist. This problem can be interpreted in the context of steady-state heat flow. The zero-flux boundary conditions imply that both ends of the bar are insulated, and so heat cannot escape from the bar. Thus it is impossible to impose a point source, which would inject heat energy at a constant, unit rate, and have the system respond with a time-independent temperature distribution; energy would build up in the bar, precluding a steady state.

In this heuristic discussion we noted that δ is not an ordinary function. Furthermore, the Green's function is not differentiable at $x = \xi$, so it remains to determine the meaning of applying a differential operator to g as in formulas (4.51)–(4.52). Our goal in the next section is to put these notions on firm ground.

Example 4.22

(Causal Green's function) The **causal Green's function** is the Green's function for an initial value problem. Consider the problem

$$Au \equiv -(pu')' + qu = f(t), \quad t > 0; \quad u(0) = u'(0) = 0 \tag{4.53}$$

We assume p, p', and q are continuous for $t \geq 0$ and $p > 0$. The causal Green's function, also called the *impulse response function*, is the solution to (4.53) when f is a unit impulse applied at time τ, or, in terms of the delta function notation, when $f = \delta(t, \tau)$. Thus, symbolically, $Ag(t, \tau) = \delta(t, \tau)$, where g denotes the causal Green's function. We shall give a physical argument to determine g. Since the initial data is zero the response of the system is zero up until time τ; therefore, $g(t, \tau) = 0$ for $t < \tau$. For $t > \tau$ we require $Ag(t, \tau) = 0$, and we demand that g be continuous at $t = \tau$, or

$$g(\tau^+, \tau) = 0.$$

At $t = \tau$, the time when the impulse is given, we demand that g have a jump in its derivative of magnitude

$$g'(\tau^+, \tau) = -1/p(\tau).$$

This jump condition is derived in the same way that the jump condition is obtained in earlier. The continuity condition and the jump condition, along with the fact that g satisfies the homogeneous differential equation, are enough to determine $g(x, \tau)$ for $t > \tau$.

Now we return to the SLP problem (4.41)–(4.43) and ask what can be said if $\lambda = 0$ is an eigenvalue, that is, if the homogeneous problem has a nontrivial solution. In this case there may not be a solution, and if there is a solution, it is not unique. The following theorem summarizes the result.

Theorem 4.23

Consider the Sturm–Liouville problem (4.41)–(4.43), and assume there exists a nontrivial solution ϕ of the homogeneous problem $L\phi = 0$. Then (4.41)–

(4.43) has a solution if, and only if,

$$(\phi, f) \equiv \int_a^b \phi f \, dx = 0.$$

To prove necessity we assume a solution u exists. Then

$$(\phi, f) = (\phi, Au) = -\int_a^b \phi(pu')' \, dx + \int_a^b \phi qu \, dx.$$

The first integral may be integrated by parts twice to remove the derivatives from u and put them on ϕ. Performing this calculation gives, after collecting terms,

$$\begin{aligned}(\phi, f) &= [p(u\phi' - \phi u')]_a^b + \int_a^b u(-(p\phi')' + q\phi) \, dx \\ &= [p(u\phi' - \phi u')]_a^b + \int_a^b uA\phi \, dx \\ &= [p(u\phi' - \phi u')]_a^b,\end{aligned}$$

because $A\phi = 0$. Both ϕ and u satisfy the boundary conditions (4.42)–(4.43), and one can easily show that $[p(u\phi' - \phi u')]_a^b = 0$.

The proof of the "if" part of the theorem is less straightforward. We indicate the solution and leave the verification for the reader. Assume that $(\phi, f) = 0$ and let v be independent of ϕ and satisfy only the differential equation $Av = 0$, and not the boundary conditions. Now define

$$G(x, \xi) \equiv -\frac{1}{p(\xi)W(\xi)}(\phi(x)v(\xi)H(\xi - x) + \phi(\xi)v(x)H(x - \xi))$$

where H is the Heaviside function and $W = \phi v' - v\phi'$. Then

$$u(x) = c\phi(x) + \int_a^b G(x, \xi)f(\xi) \, d\xi$$

is a solution to $Lu = f$ for any constant c. (Note here that G is not the Green's function; as we have noted, the Green's function does not exist.)

4.4.3 Green's Function via Eigenfunctions

Suppose Green's function exists for the SLP (4.41)–(4.43). If we can solve the eigenvalue problem associated with the operator L, then we can find the Green's function. We know from Section 4.2 that the SLP problem

$$Lu = \lambda u, \quad a < x < b, \tag{4.54}$$

where L includes the boundary conditions (4.42)–(4.43), has infinitely many eigenvalues and corresponding orthonormal eigenfunctions λ_n and $\phi_n(x), n = 1, 2, \ldots$, respectively. Moreover, the eigenfunctions form a basis for the square-integrable functions on (a, b). Therefore we assume that the solution u of (4.41)–(4.43) is given in terms of the eigenfunctions as

$$u(x) = \sum_{n=1}^{\infty} c_n \phi_n(x), \tag{4.55}$$

where the coefficients c_n are to be determined. Further, we write the given function f in terms of the eigenfunctions as

$$f(x) = \sum_{n=1}^{\infty} f_n \phi_n(x), \quad f_n = \int_a^b f(\xi)\phi_n(\xi)\, d\xi.$$

Substituting these expansions into (4.54) gives

$$L\left(\sum_{n=1}^{\infty} c_n \phi_n(x)\right) = \sum_{n=1}^{\infty} f_n \phi_n(x).$$

But

$$\begin{aligned} L\left(\sum_{n=1}^{\infty} c_n \phi_n(x)\right) &= \sum_{n=1}^{\infty} c_n L(\phi_n(x)) \\ &= \sum_{n=1}^{\infty} c_n \lambda_n \phi_n(x) \\ &= \sum_{n=1}^{\infty} f_n \phi_n(x). \end{aligned}$$

Equating coefficients of the independent eigenfunctions gives

$$c_n = \frac{f_n}{\lambda_n} = \frac{1}{\lambda_n} \int_a^b f(\xi)\phi_n(\xi)\, d\xi.$$

Thus, from (4.55),

$$u(x) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \left(\int_a^b f(\xi)\phi_n(\xi)\, d\xi \right) \phi_n(x).$$

Now we formally interchange the summation and integral to obtain

$$u(x) = \int_a^b \left(\sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(\xi)}{\lambda_n} \right) f(\xi)\, d\xi.$$

Consequently, we have inverted the operator L and so we must have

$$g(x,\xi) = \sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(\xi)}{\lambda_n},$$

which gives the Green's function in terms of the eigenvalues and eigenfunctions of the operator L. This expansion for g is called the **bilinear expansion.**

EXERCISES

1. Discuss the solvability of the boundary value problem

$$u'' + \pi^2 u = f(x), \quad 0 < x < 1; \quad u(0) = u(1) = 0.$$

2. Determine if there is a Green's function associated with the operator $Lu = u'' + 4u$, $0 < x < \pi$, with $u(0) = u(\pi) = 0$. Find the solution to the boundary value problem

$$u'' + 4u = f(x), \quad 0 < x < \pi; \quad u(0) = u(\pi) = 0.$$

3. Consider the boundary value problem

$$u'' - 2xu' = f(x), \quad 0 < x < 1; \quad u(0) = u'(1) = 0.$$

Find Green's function or explain why there isn't one.

4. Consider the boundary value problem

$$u'' + u' - 2u = f(x), \quad 0 < x < 1; \quad u(0) = u'(1) = 0.$$

Find Green's function or explain why there isn't one.

5. Use the method of Green's function to solve the problem

$$-(K(x)u')' = f(x), \quad 0 < x < 1; \quad u(0) = u(1) = 0; \quad K(x) > 0.$$

(Note: This differential equation is a steady-state heat equation in a bar with variable thermal conductivity $K(x)$.)

6. Consider a spring–mass system governed by the initial value problem

$$mu'' + ku = f(t), \quad t > 0; \quad u(0) = u'(0) = 0,$$

where $u = u(t)$ is the displacement from equilibrium, f is an applied force, and m and k are the mass and spring constants, respectively.

a) Show that the causal Green's function is

$$g(t,\tau) = \frac{1}{\sqrt{km}} \sin\sqrt{\frac{k}{m}}(t-\tau), \quad t > \tau.$$

b) Find the solution to the initial value problem and write it in the form

$$u(t) = \frac{1}{\sqrt{km}} \int_0^t \sin\sqrt{\frac{k}{m}}(t-\tau) f(\tau)\, d\tau.$$

7. By finding Green's function in two different ways, evaluate the sum

$$\sum_{n=1}^{\infty} \frac{\sin nx \sin n\xi}{n^2}, \quad 0 < x, \xi < \pi.$$

8. Find the inverse of the differential operator $Lu = -(x^2u')'$ on $1 < x < e$ subject to $u(1) = u(e) = 0$.

4.5 Distributions

In the last section we showed, in an intuitive manner, that the Green's function satisfies a differential equation with a unit point source. Because the Green's function is not a smooth function, it leads us to the question of what it means to differentiate such a function. Also, we have not pinned down the properties of a point source (a delta function) in a precise way. If a point source is not a function, then what is it? The goal of this section is to come to answers to these questions. New notation, terminology, and concepts are required.

4.5.1 Test Functions

Let K be a set of real numbers. A real number c is said to be a **limit point** of set K if every open interval containing c, no matter how small, contains at least one point of K. If K is a set, then the **closure** of K, denoted by $\overline{K}$, is the set K along with all its limit points. A set is called **closed** if it contains all of its limit points. For example, the closure of the half-open interval $(a, b]$ is the closed interval $[a, b]$. A useful set in characterizing properties of a function ϕ is the set of points where it takes on nonzero values. The closure of this set is called the **support** of the function; precisely, we define supp $\phi \equiv \overline{\{x|\phi(x) \neq 0\}}$.