ated by converting its components (but not the unit dyads) to spherical coordinates, and integrating each over the two spherical angles (see Section A.7). The off-diagonal terms in Eq. (A.6-13) vanish, again due to the symmetry.

A.7 ORTHOGONAL CURVILINEAR COORDINATES

Enormous simplificatons are achieved in solving a partial differential equation if all boundaries in the problem correspond to *coordinate surfaces,* which are surfaces generated by holding one coordinate constant and varying the other two. Accordingly, many special coordinate systems have been devised to solve problems in particular geometries. The most useful of these systems are *orthogonal;* that is, at any point in space the vectors aligned with the three coordinate directions are mutually perpendicular. In general, the variation of a single coordinate will generate a curve in space, rather than a straight line; hence the term *curvilinear.* In this section a general discussion of orthogonal curvilinear systems is given first, and then the relationships for cylindrical and spherical coordinates are derived as special cases. The presentation here closely follows that in Hildebrand (1976).

Base Vectors

Let (u_1, u_2, u_3) represent the three coordinates in a general, curvilinear system, and let $\mathbf{e}_i$ be the unit vector that points in the direction of increasing u_i. A curve produced by varying u_i, with u_j $(j \neq i)$ held constant, will be referred to as a "u_i curve." Although the base vectors are each of constant (unit) magnitude, the fact that a u_i curve is not generally a straight line means that their direction is variable. In other words, $\mathbf{e}_i$ must be regarded as a function of position, in general. This discussion is restricted to coordinate systems in which $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is an *orthonormal* and *right-handed* set. At any point in space, such a set has the properties of the base vectors used in Section A.3, namely,

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}, \tag{A.7-1}$$

$$\mathbf{e}_i \times \mathbf{e}_j = \sum_k \varepsilon_{ijk} \mathbf{e}_k. \tag{A.7-2}$$

Recalling that the multiplication properties of vectors and tensors are derived from these relationships (and their extensions to unit dyads), we see that *all of the relations in Section A.3 apply to orthogonal curvilinear systems in general,* and not just to rectangular coordinates. It is with spatial derivatives that the variations in $\mathbf{e}_i$ come into play, and the main task in this section is to show how the various differential operators differ from those given in Section A.4 for rectangular coordinates. In the process, we will obtain general expressions for differential elements of arc length, volume, and surface area.

Arc Length

The key to deriving expressions for curvilinear coordinates is to consider the arc length along a curve. In particular, let s_i represent arc length along a u_i curve. From Eq. (A.6-2), a vector that is tangent to a u_i curve and directed toward increasing u_i is given by

$$\mathbf{a}_i = \frac{\partial \mathbf{r}}{\partial u_i} = h_i \mathbf{e}_i, \tag{A.7-3}$$

where $h_i \equiv ds_i/du_i$ is called the *scale factor.* In general, u_i will differ from s_i, so that $\mathbf{a}_i$ is not a *unit* tangent (i.e., $\mathbf{a}_i \neq \mathbf{e}_i$). The relationship between a coordinate and the corresponding arc length is embodied in the scale factor, which generally depends on position. For an arbitrary curve in space with arc length s, we find that

$$t = \frac{d\mathbf{r}}{ds} = \sum_i \frac{\partial \mathbf{r}}{\partial u_i} \frac{du_i}{ds} = \sum_i h_i \mathbf{e}_i \frac{du_i}{ds}. \tag{A.7-4}$$

The properties of the unit tangent imply that

$$\mathbf{t} \cdot \mathbf{t} = 1 = \sum_i h_i^2 \left(\frac{du_i}{ds} \right)^2 \tag{A.7-5}$$

or, after rearranging, that

$$ds = \left[\sum_i h_i^2 (du_i)^2 \right]^{1/2}. \tag{A.7-6}$$

The fact that a space curve has an independent geometric significance indicates that the quantity in brackets must be invariant to the choice of coordinate system.

Volume and Surface Area

For a given coordinate system, the differential volume element dV corresponds to the volume of a parallelepiped with adjacent edges $\mathbf{e}_i ds_i$. From Eq. (A.7-3) and the definition of h_i, the edges can be represented also as $\mathbf{a}_i du_i$. As noted in connection with Eq. (A.3-20), the volume is given by the *scalar triple product,* or

$$dV = (\mathbf{a}_1 du_1) \times (\mathbf{a}_2 du_2) \cdot (\mathbf{a}_3 du_3) = h_1 h_2 h_3 \, du_1 \, du_2 \, du_3. \tag{A.7-7}$$

Expressions for differential surface elements are obtained in a similar manner, by using the geometric interpretation of the *cross product.* Thus, letting dS_i refer to a surface on which the coordinate u_i is held constant, we obtain

$$dS_1 = |\mathbf{a}_2 \times \mathbf{a}_3| \, du_2 \, du_3 = h_2 h_3 \, du_2 \, du_3, \tag{A.7-8a}$$

$$dS_2 = |\mathbf{a}_3 \times \mathbf{a}_1| \, du_1 \, du_3 = h_1 h_3 \, du_1 \, du_3, \tag{A.7-8b}$$

$$dS_3 = |\mathbf{a}_1 \times \mathbf{a}_2| \, du_1 \, du_2 = h_1 h_2 \, du_1 \, du_2. \tag{A.7-8c}$$

Gradient

An expression for the gradient is obtained by examining the differential change in a scalar function associated with a differential change in position. Letting $f = f(u_1, u_2, u_3)$, we have

$$df = \sum_i \frac{\partial f}{\partial u_i} du_i. \tag{A.7-9}$$

From Eqs. (A.6-5) and (A.7-4), df can also be written as

$$df = \mathbf{dr} \cdot \nabla \mathbf{f} = \left(\sum_i h_i \mathbf{e}_i \, du_i \right) \cdot \left(\sum_j \lambda_j \mathbf{e}_j \right) = \sum_i h_i \lambda_i \, du_i, \quad \text{(A.7-10)}$$

where the quantities λ_i are to be determined. Comparing Eqs. (A.7-9) and (A.7-10), we see that $\lambda_i = (1/h_i) \, \partial f / \partial u_i$ and

$$\nabla = \sum_i \frac{\mathbf{e}_i}{h_i} \frac{\partial}{\partial u_i}. \quad \text{(A.7-11)}$$

This is the general expression for the gradient operator, valid for any orthogonal, curvilinear coordinate system.

Several identities involving u_i, h_i, and $\mathbf{e}_i$ are useful in deriving expressions for the other differential operators. From Eq. (A.7-11) we obtain

$$\nabla u_i = \sum_j \frac{\mathbf{e}_j}{h_j} \frac{\partial u_i}{\partial u_j} = \sum_j \frac{\mathbf{e}_j}{h_j} \delta_{ij} = \frac{\mathbf{e}_i}{h_i}. \quad \text{(A.7-12)}$$

From this and identity (7) of Table A-1, it follows that

$$\nabla \times \nabla u_i = \nabla \times \frac{\mathbf{e}_i}{h_i} = \mathbf{0}. \quad \text{(A.7-13)}$$

Using Eqs. (A.7-2) and (A.7-12) we find, for example, that

$$\frac{\mathbf{e}_1}{h_2 h_3} = \frac{\mathbf{e}_2}{h_2} \times \frac{\mathbf{e}_3}{h_3} = \nabla u_2 \times \nabla u_3. \quad \text{(A.7-14)}$$

Two analogous relations are obtained by cyclic permutation of the subscripts. From these and identity (8) of Table A-1 it is found that

$$\nabla \cdot \frac{\mathbf{e}_1}{h_2 h_3} = \nabla \cdot \frac{\mathbf{e}_2}{h_3 h_1} = \nabla \cdot \frac{\mathbf{e}_3}{h_1 h_2} = 0. \quad \text{(A.7-15)}$$

Divergence

To evaluate the divergence of the vector $\mathbf{v}$, we first consider just one component. Bearing in mind that the unit vectors are not necessarily constants, and using identity (2) of Table A-1, the divergence of $v_1 \mathbf{e}_1$ is expanded as

$$\nabla \cdot (v_1 \mathbf{e}_1) = \nabla \cdot \left[(h_2 h_3 v_1) \left(\frac{\mathbf{e}_1}{h_2 h_3} \right) \right] = \nabla (h_2 h_3 v_1) \cdot \frac{\mathbf{e}_1}{h_2 h_3} + h_2 h_3 v_1 \nabla \cdot \left(\frac{\mathbf{e}_1}{h_2 h_3} \right). \quad \text{(A.7-16)}$$

It is seen from Eq. (A.7-15) that the term on the far right of Eq. (A.7-16) is identically zero. Using Eq. (A.7-11) to evaluate the gradient in the remaining term, we find that

$$\nabla \cdot (v_1 \mathbf{e}_1) = \sum_i \frac{\mathbf{e}_i}{h_i} \frac{\partial}{\partial u_i} (h_2 h_3 v_1) \cdot \frac{\mathbf{e}_1}{h_2 h_3} = \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u_1} (h_2 h_3 v_1). \quad \text{(A.7-17)}$$

Treating the other components in a similar manner results in

$$\nabla \cdot \mathbf{v} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(h_2 h_3 v_1) + \frac{\partial}{\partial u_2}(h_3 h_1 v_2) + \frac{\partial}{\partial u_3}(h_1 h_2 v_3)\right], \tag{A.7-18}$$

which is the general expression for the divergence.

Curl

The curl of $\mathbf{v}$ is evaluated in a similar manner. Expanding one component, we obtain

$$\nabla \times (v_1 \mathbf{e}_1) = \nabla \times \left[(h_1 v_1)\left(\frac{\mathbf{e}_1}{h_1}\right)\right] = \nabla(h_1 v_1) \times \frac{\mathbf{e}_1}{h_1} + h_1 v_1 \nabla \times \frac{\mathbf{e}_1}{h_1}. \tag{A.7-19}$$

Again, the term on the far right vanishes [see Eq. (A.7-13)]. The remaining term is expanded further as

$$\nabla \times (v_1 \mathbf{e}_1) = \sum_i \frac{\mathbf{e}_i}{h_i}\frac{\partial}{\partial u_i}(h_1 v_1) \times \frac{\mathbf{e}_1}{h_1} = \frac{1}{h_1 h_2 h_3}\left(h_2 \mathbf{e}_2 \frac{\partial}{\partial u_3} - h_3 \mathbf{e}_3 \frac{\partial}{\partial u_2}\right)(h_1 v_1). \tag{A.7-20}$$

When all components are included, the curl is written most compactly as a determinant:

$$\nabla \times \mathbf{v} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1 \mathbf{e}_1 & h_2 \mathbf{e}_2 & h_3 \mathbf{e}_3 \\ \partial/\partial u_1 & \partial/\partial u_2 & \partial/\partial u_3 \\ h_1 v_1 & h_2 v_2 & h_3 v_3 \end{vmatrix}. \tag{A.7-21}$$

Laplacian

The Laplacian for curvilinear coordinates is derived from Eq. (A.7-18) by setting $\mathbf{v} = \nabla$, or $v_i = (1/h_i)\ \partial/\partial u_i$. The result is

$$\nabla \cdot \nabla = \nabla^2 = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial}{\partial u_3}\right)\right]. \tag{A.7-22}$$

Material Derivative

The material derivative is given by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla = \frac{\partial}{\partial t} + \sum_i \frac{v_i}{h_i}\frac{\partial}{\partial u_i}. \tag{A.7-23}$$

This completes the general results for orthogonal curvilinear coordinates. The remainder of this section is devoted to the three most useful special cases.

Rectangular Coordinates

In rectangular coordinates, which we have written as either (x, y, z) or (x_1, x_2, x_3), the position vector is given by Eq. (A.6-1). In this case it is readily shown from Eq. (A.7-3) that $h_x = h_y = h_z = 1$, and it is found that the general expressions for the differential operators reduce to the forms given in Section A.4. For convenient reference, the principal results are summarized in Table A-2.

TABLE A-2
Differential Operations in Rectangular Coordinates[a]

(1)	$\nabla f=\frac{\partial f}{\partial x}\mathbf{e}_x+\frac{\partial f}{\partial y}\mathbf{e}_y+\frac{\partial f}{\partial z}\mathbf{e}_z$
(2)	$\nabla\cdot\mathbf{v}=\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}$
(3)	$\nabla\times\mathbf{v}=\left[\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right]\mathbf{e}_x+\left[\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right]\mathbf{e}_y+\left[\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right]\mathbf{e}_z$
(4)	$\nabla^2 f=\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}$
(5)	$(\nabla\mathbf{v})_{xx}=\frac{\partial v_x}{\partial x}$
(6)	$(\nabla\mathbf{v})_{xy}=\frac{\partial v_y}{\partial x}$
(7)	$(\nabla\mathbf{v})_{xz}=\frac{\partial v_z}{\partial x}$
(8)	$(\nabla\mathbf{v})_{yx}=\frac{\partial v_x}{\partial y}$
(9)	$(\nabla\mathbf{v})_{yy}=\frac{\partial v_y}{\partial y}$
(10)	$(\nabla\mathbf{v})_{yz}=\frac{\partial v_z}{\partial y}$
(11)	$(\nabla\mathbf{v})_{zx}=\frac{\partial v_x}{\partial z}$
(12)	$(\nabla\mathbf{v})_{zy}=\frac{\partial v_y}{\partial z}$
(13)	$(\nabla\mathbf{v})_{zz}=\frac{\partial v_z}{\partial z}$

[a] In these relationships, f is any differentiable scalar function and $\mathbf{v}$ is any differentiable vector function.

Cylindrical Coordinates

Circular cylindrical coordinates, denoted as (r, θ, z), are shown in relation to rectangular coordinates in Fig. A-3(a). Using

$$x=r\cos\theta,\quad y=r\sin\theta,\quad z=z, \tag{A.7-24}$$

the position vector is expressed as

$$\mathbf{r}=r\cos\theta\mathbf{e}_x+r\sin\theta\mathbf{e}_y+z\mathbf{e}_z. \tag{A.7-25}$$

Alternatively, the position vector is given by

$$\mathbf{r}=r\mathbf{e}_r+z\mathbf{e}_z, \tag{A.7-26}$$

where $\mathbf{e}_r$, the unit vector in the radial direction, is given below. Equation (A.7-25) is more convenient for derivations involving differentiation or integration because it involves only constant base vectors. Whichever expression is used, note that in cylindrical coordinates there is an irregularity in our notation, such that $|\mathbf{r}|=(r^2+z^2)^{1/2}\neq r$.

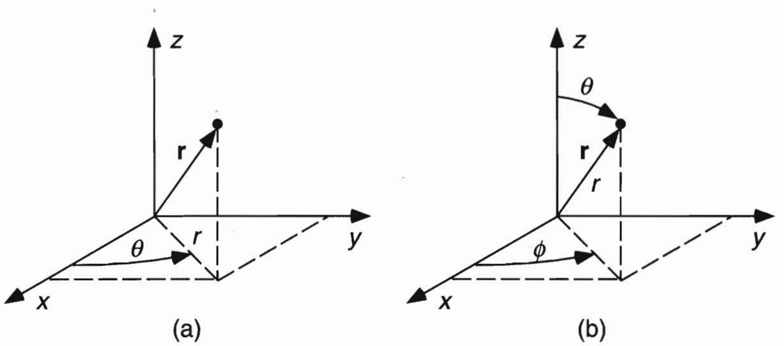

Figure A-3. Cylindrical coordinates (a) and spherical coordinates (b). The ranges of the angles are: cylindrical, $0 \leq \theta \leq 2\pi$; spherical, $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$.

To illustrate the derivation of scale factors and base vectors, consider the θ quantities in cylindrical coordinates. From Eqs. (A.7-3) and (A.7-25) it is found that

$$\frac{\partial \mathbf{r}}{\partial \theta} = h_\theta \mathbf{e}_\theta = -r \sin\theta \mathbf{e}_x + r \cos\theta \mathbf{e}_y, \qquad \text{(A.7-27)}$$

$$h_\theta = \left| \frac{\partial \mathbf{r}}{\partial \theta} \right| = \left(r^2 \sin^2\theta + r^2 \cos^2\theta\right)^{1/2} = r, \qquad \text{(A.7-28)}$$

and $\mathbf{e}_\theta = (1/h_\theta)(\partial \mathbf{r}/\partial\theta)$. Repeating the calculations for the r and z quantities, the scale factors and base vectors for cylindrical coordinates are found to be

$$h_r = 1, \qquad h_\theta = r, \qquad h_z = 1, \qquad \text{(A.7-29)}$$

$$\mathbf{e}_r = \cos\theta \mathbf{e}_x + \sin\theta \mathbf{e}_y, \qquad \text{(A.7-30a)}$$

$$\mathbf{e}_\theta = -\sin\theta \mathbf{e}_x + \cos\theta \mathbf{e}_y, \qquad \text{(A.7-30b)}$$

$$\mathbf{e}_z = \mathbf{e}_z. \qquad \text{(A.7-30c)}$$

The dependence of $\mathbf{e}_r$ and $\mathbf{e}_\theta$ on θ is shown in Eq. (A.6-30); the expression for $\mathbf{e}_r$ confirms the equivalence of Eqs. (A.7-25) and (A.7-26). Inverting the relationships in Eq. (A.7-30) to find expressions for the rectangular base vectors, we obtain

$$\mathbf{e}_x = \cos\theta \mathbf{e}_r - \sin\theta \mathbf{e}_\theta, \qquad \text{(A.7-31a)}$$

$$\mathbf{e}_y = \sin\theta \mathbf{e}_r + \cos\theta \mathbf{e}_\theta, \qquad \text{(A.7-31b)}$$

$$\mathbf{e}_z = \mathbf{e}_z. \qquad \text{(A.7-31c)}$$

The differential volume and surface elements are evaluated using Eqs. (A.7-7) and (A.7-8) as

$$dV = r\, dr\, d\theta\, dz, \qquad \text{(A.7-32)}$$

$$dS_r = r\, d\theta\, dz, \qquad dS_\theta = dr\, dz, \qquad dS_z = r\, dr\, d\theta. \qquad \text{(A.7-33)}$$

A summary of differential operations in cylindrical coordinates is presented in Table A-3. Several quantities not shown, including $\nabla^2 \mathbf{v}$, $\nabla \cdot \boldsymbol{\tau}$, and $\mathbf{v} \cdot \nabla \mathbf{v}$, may be obtained from the tables in Chapter 5.

TABLE A-3
Differential Operations in Cylindrical Coordinates[a]

(1)	$\nabla f=\frac{\partial f}{\partial r}\mathbf{e}_r+\frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta+\frac{\partial f}{\partial z}\mathbf{e}_z$
(2)	$\nabla\cdot\mathbf{v}=\frac{1}{r}\frac{\partial}{\partial r}(rv_r)+\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial v_z}{\partial z}$
(3)	$\nabla\times\mathbf{v}=\left[\frac{1}{r}\frac{\partial v_z}{\partial \theta}-\frac{\partial v_\theta}{\partial z}\right]\mathbf{e}_r+\left[\frac{\partial v_r}{\partial z}-\frac{\partial v_z}{\partial r}\right]\mathbf{e}_\theta+\left[\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)-\frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]\mathbf{e}_z$
(4)	$\nabla^2 f=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}+\frac{\partial^2 f}{\partial z^2}$
(5)	$(\nabla\mathbf{v})_{rr}=\frac{\partial v_r}{\partial r}$
(6)	$(\nabla\mathbf{v})_{r\theta}=\frac{\partial v_\theta}{\partial r}$
(7)	$(\nabla\mathbf{v})_{rz}=\frac{\partial v_z}{\partial r}$
(8)	$(\nabla\mathbf{v})_{\theta r}=\frac{1}{r}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta}{r}$
(9)	$(\nabla\mathbf{v})_{\theta\theta}=\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r}{r}$
(10)	$(\nabla\mathbf{v})_{\theta z}=\frac{1}{r}\frac{\partial v_z}{\partial \theta}$
(11)	$(\nabla\mathbf{v})_{zr}=\frac{\partial v_r}{\partial z}$
(12)	$(\nabla\mathbf{v})_{z\theta}=\frac{\partial v_\theta}{\partial z}$
(13)	$(\nabla\mathbf{v})_{zz}=\frac{\partial v_z}{\partial z}$

[a] In these relationships, f is any differentiable scalar function and $\mathbf{v}$ is any differentiable vector function.

Spherical Coordinates

Spherical coordinates, denoted as (r, θ, ϕ), are shown in relation to rectangular coordinates in Fig. A-3(b). Note that $0\leq\theta\leq\pi$ and $0\leq\phi\leq 2\pi$. Using

$$x=r\sin\theta\cos\phi, \qquad y=r\sin\theta\sin\phi, \qquad z=r\cos\theta \tag{A.7-34}$$

the position vector is expressed as

$$\mathbf{r}=r\sin\theta\cos\phi\,\mathbf{e}_x+r\sin\theta\sin\phi\,\mathbf{e}_y+r\cos\theta\,\mathbf{e}_z. \tag{A.7-35}$$

The position vector is also given by

$$\mathbf{r}=r\mathbf{e}_\mathbf{r}. \tag{A.7-36}$$

Either expression indicates that $|\mathbf{r}|=r$, consistent with our usual notation.

Employing Eq. (A.7-35) in the general relationships for curvilinear coordinates, the scale factors and base vectors for spherical coordinates are evaluated as

$$h_r = 1, \qquad h_\theta = r, \qquad h_\phi = r \sin\theta, \tag{A.7-37}$$

$$\mathbf{e}_r = \sin\theta\cos\phi\,\mathbf{e}_x + \sin\theta\sin\phi\,\mathbf{e}_y + \cos\theta\,\mathbf{e}_z, \tag{A.7-38a}$$

$$\mathbf{e}_\theta = \cos\theta\cos\phi\,\mathbf{e}_x + \cos\theta\sin\phi\,\mathbf{e}_y - \sin\theta\,\mathbf{e}_z, \tag{A.7-38b}$$

$$\mathbf{e}_\phi = -\sin\phi\,\mathbf{e}_x + \cos\phi\,\mathbf{e}_y. \tag{A.7-38c}$$

The dependence of the three base vectors on θ and ϕ is shown in Eq. (A.7-38); the expression for $\mathbf{e}_r$ confirms the equivalence of Eqs. (A.7-35) and (A.7-36). The complementary expressions for the rectangular base vectors are

$$\mathbf{e}_x = \sin\theta\cos\phi\,\mathbf{e}_r + \cos\theta\cos\phi\,\mathbf{e}_\theta - \sin\phi\,\mathbf{e}_\phi, \tag{A.7-39a}$$

$$\mathbf{e}_y = \sin\theta\sin\phi\,\mathbf{e}_r + \cos\theta\sin\phi\,\mathbf{e}_\theta + \cos\phi\,\mathbf{e}_\phi, \tag{A.7-39b}$$

$$\mathbf{e}_z = \cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta. \tag{A.7-39c}$$

Finally, the differential volume and surface elements are evaluated as

$$dV = r^2 \sin\theta\, dr\, d\theta\, d\phi \tag{A.7-40}$$

$$dS_r = r^2 \sin\theta\, d\theta\, d\phi, \qquad dS_\theta = r\sin\theta\, dr\, d\phi, \qquad dS_\phi = r\, dr\, d\theta. \tag{A.7-41}$$

A summary of differential operations in spherical coordinates is presented in Table A-4. As already mentioned, several other quantities, including $\nabla^2\mathbf{v}$, $\boldsymbol{\nabla}\cdot\boldsymbol{\tau}$, and $\mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v}$, may be obtained from the tables in Chapter 5.

Many other orthogonal coordinate systems have been developed. A compilation of scale factors, differential operators, and solutions of Laplace's equation in 40 such systems is provided in Moon and Spencer (1961).

A.8 SURFACE GEOMETRY

In Section A.5 a number of integral transformations were presented involving vectors which are normal or tangent to a surface. The objective of this section is to show how those vectors are computed and how they are used to define such quantities as surface gradients. This completes the information needed to understand the integral forms of the various conservation equations. Much of the material in this section is adapted from Brand (1947).

Normal and Tangent Vectors

We begin by assuming that positions on an arbitrary surface are described by two coordinates, u and v, which are not necessarily orthogonal. The position vector at the surface is denoted as $\mathbf{r}_s(u, v)$. From Eq. (A.7-3), two vectors that are tangent to the surface are

$$\mathbf{A} = \frac{\partial \mathbf{r}_s}{\partial u}, \qquad \mathbf{B} = \frac{\partial \mathbf{r}_s}{\partial v}. \tag{A.8-1}$$

Specifically, $\mathbf{A}$ is tangent to a "u curve" on the surface (i.e., a curve where v is held constant) and $\mathbf{B}$ is tangent to a "v curve." In general, these vectors are not orthogonal to one another and they are not of unit length. However, their cross product is orthogonal

TABLE A-4
Differential Operations in Spherical Coordinates[a]

(1)	$\nabla f=\frac{\partial f}{\partial r}\mathbf{e}_r+\frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta+\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\mathbf{e}_\phi$
(2)	$\nabla\cdot\mathbf{v}=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2v_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(v_\theta\sin\theta)+\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\theta}$
(3)	$\nabla\times\mathbf{v}=\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(v_\phi\sin\theta)-\frac{\partial v_\theta}{\partial\phi}\right]\mathbf{e}_r+\left[\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial\phi}-\frac{1}{r}\frac{\partial}{\partial r}(rv_\phi)\right]\mathbf{e}_\theta+\left[\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)-\frac{1}{r}\frac{\partial v_r}{\partial\theta}\right]\mathbf{e}_\phi$
(4)	$\nabla^2 f=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$
(5)	$(\nabla\mathbf{v})_{rr}=\frac{\partial v_r}{\partial r}$
(6)	$(\nabla\mathbf{v})_{r\theta}=\frac{\partial v_\theta}{\partial r}$
(7)	$(\nabla\mathbf{v})_{r\phi}=\frac{\partial v_\phi}{\partial r}$
(8)	$(\nabla\mathbf{v})_{\theta r}=\frac{1}{r}\frac{\partial v_r}{\partial\theta}-\frac{v_\theta}{r}$
(9)	$(\nabla\mathbf{v})_{\theta\theta}=\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_r}{r}$
(10)	$(\nabla\mathbf{v})_{\theta\phi}=\frac{1}{r}\frac{\partial v_\phi}{\partial\theta}$
(11)	$(\nabla\mathbf{v})_{\phi r}=\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial\phi}-\frac{v_\phi}{r}$
(12)	$(\nabla\mathbf{v})_{\phi\theta}=\frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi}-\frac{v_\phi\cot\theta}{r}$
(13)	$(\nabla\mathbf{v})_{\phi\phi}=\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}+\frac{v_r}{r}+\frac{v_\theta\cot\theta}{r}$

[a] In these relationships f is any differentiable scalar function and $\mathbf{v}$ is any differentiable vector function.

to both, and hence is normal to the surface. It follows that a *unit normal* vector is given by

$$\mathbf{n}=\frac{\mathbf{A}\times\mathbf{B}}{|\mathbf{A}\times\mathbf{B}|}. \tag{A.8-2}$$

Depending on how u and v are selected, it may be necessary to reverse the order of $\mathbf{A}$ and $\mathbf{B}$ (thereby changing the sign of $\mathbf{n}$) to make the unit normal point *outward* from a closed surface.

Suppose now that $u=x$, $v=y$, and the surface is represented as the function $z=F(x, y)$. In this case the position vector at the surface is given by

$$\mathbf{r}_s=x\mathbf{e}_x+y\mathbf{e}_y+F(x, y)\mathbf{e}_z. \tag{A.8-3}$$

The two tangent vectors are found to be

$$\mathbf{A}=\frac{\partial\mathbf{r}_s}{\partial x}=(1)\mathbf{e}_x+(0)\mathbf{e}_y+\frac{\partial F}{\partial x}\mathbf{e}_z, \tag{A.8-4}$$

$$\mathbf{B} = \frac{\partial \mathbf{r}_s}{\partial y} = (0)\mathbf{e}_x + (1)\mathbf{e}_y + \frac{\partial F}{\partial y}, \tag{A.8-5}$$

and the unit normal is computed as

$$\mathbf{n} = \frac{\mathbf{A} \times \mathbf{B}}{|\mathbf{A} \times \mathbf{B}|} = \frac{-(\partial F/\partial x)\mathbf{e}_x - (\partial F/\partial y)\mathbf{e}_y + \mathbf{e}_z}{[(\partial F/\partial x)^2 + (\partial F/\partial y)^2 + 1]^{1/2}}. \tag{A.8-6}$$

An alternate way to compute a unit normal is to represent the surface as $G(x, y, z) = 0$ and to use (Hildebrand, 1976, p. 294)

$$\mathbf{n} = \frac{\nabla G}{|\nabla G|}. \tag{A.8-7}$$

It is readily confirmed that if $G(x, y, z) \equiv z - F(x, y)$, this formula yields the same result as in Eq. (A.8-6). As already mentioned, $\mathbf{n}$ may have to be replaced by $-\mathbf{n}$ to give the outward normal.

Reciprocal Bases

Until now we have employed base vectors, denoted as $\mathbf{e}_i$, which form orthonormal, right-handed sets. The special properties of such sets, as given by Eqs. (A.7-1) and (A.7-2), make them very convenient for the representation of other vectors. However, base vectors need not be orthonormal, or even orthogonal. Indeed, an arbitrary vector can be expressed in terms of any three vectors which are not coplanar or, equivalently, not linearly dependent (Brand, 1947). The volume of a parallelepiped which has the three vectors as adjacent edges will be nonzero only if the vectors are not coplanar; as discussed in Section A.3, that volume is equal to the scalar triple product. Thus, any three vectors with a nonvanishing scalar triple product constitute a possible *basis*. In describing the local curvature of a surface and other surface-related quantities, it proves convenient to employ two complementary sets of base vectors, neither of which is orthogonal. The special properties of these base vectors leads them to be termed *reciprocal bases*. Accordingly, a brief discussion of the properties of reciprocal bases is needed.

One basis of interest is the set of vectors defined above, $(\mathbf{A}, \mathbf{B}, \mathbf{n})$. Their scalar triple product is

$$H = \mathbf{A} \times \mathbf{B} \cdot \mathbf{n}. \tag{A.8-8}$$

Because $\mathbf{A}$ and $\mathbf{B}$ are tangent to the surface and $\mathbf{n}$ is normal to it, they are obviously not coplanar; thus, $H \neq 0$. Suppose now that $(\mathbf{A}, \mathbf{B}, \mathbf{n})$ and a second set $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ are reciprocal bases. Then, by definiton, they satisfy

$$\begin{aligned} &\mathbf{a} \cdot \mathbf{A} = 1, \quad &\mathbf{a} \cdot \mathbf{B} = 0, \quad &\mathbf{a} \cdot \mathbf{n} = 0, \\ &\mathbf{b} \cdot \mathbf{A} = 0, \quad &\mathbf{b} \cdot \mathbf{B} = 1, \quad &\mathbf{b} \cdot \mathbf{n} = 0, \\ &\mathbf{c} \cdot \mathbf{A} = 0, \quad &\mathbf{c} \cdot \mathbf{B} = 0, \quad &\mathbf{c} \cdot \mathbf{n} = 1. \end{aligned} \tag{A.8-9}$$

Notice that each base vector is orthogonal to two members of the *reciprocal* set. It is straightforward to verify that these relationships will hold if

$$\mathbf{a} = \frac{\mathbf{B} \times \mathbf{n}}{H}, \qquad \mathbf{b} = \frac{\mathbf{n} \times \mathbf{A}}{H}, \qquad \mathbf{c} = \frac{\mathbf{A} \times \mathbf{B}}{H} = \mathbf{n}. \tag{A.8-10}$$

An orthonormal basis is its own reciprocal; compare Eqs. (A.8-9) and (A.7-1). For an orthonormal basis we have $H=1$.

For a surface represented as $z=F(x, y)$, it is found that

$$H=\left[\left(\frac{\partial F}{\partial x}\right)^2+\left(\frac{\partial F}{\partial y}\right)^2+1\right]^{1/2}, \tag{A.8-11}$$

$$\mathbf{a}=\frac{1}{H^2}\left\{\left[1+\left(\frac{\partial F}{\partial y}\right)^2\right]\mathbf{e}_x-\frac{\partial F}{\partial x}\frac{\partial F}{\partial y}\mathbf{e}_y+\frac{\partial F}{\partial x}\mathbf{e}_z\right\}, \tag{A.8-12}$$

$$\mathbf{b}=\frac{1}{H^2}\left\{-\frac{\partial F}{\partial x}\frac{\partial F}{\partial y}\mathbf{e}_x+\left[1+\left(\frac{\partial F}{\partial x}\right)^2\right]\mathbf{e}_y+\frac{\partial F}{\partial y}\mathbf{e}_z\right\}. \tag{A.8-13}$$

Surface Gradient

The key to describing variations of geometric quantities or field variables over a surface is the *surface gradient* operator, denoted as $\boldsymbol{\nabla}_\mathbf{s}$; this operator is for surfaces what $\boldsymbol{\nabla}$ is for three-dimensional space. To derive an expression for $\boldsymbol{\nabla}_\mathbf{s}$ we consider a surface curve with arc length s. Evaluating the unit tangent using Eq. (A.6-2) and employing the coordinates u and v, we obtain

$$\mathbf{t}=\frac{d\mathbf{r_s}}{ds}=\frac{\partial \mathbf{r_s}}{\partial u}\frac{du}{ds}+\frac{\partial \mathbf{r_s}}{\partial v}\frac{dv}{ds}=\mathbf{A}\frac{du}{ds}+\mathbf{B}\frac{dv}{ds}. \tag{A.8-14}$$

From Eqs. (A.8-14) and (A.8-9) it is found that

$$\mathbf{a}\cdot\mathbf{t}=\frac{du}{ds}, \qquad \mathbf{b}\cdot\mathbf{t}=\frac{dv}{ds}. \tag{A.8-15}$$

Accordingly, the rate of change of a scalar function f along the curve is given by

$$\frac{\partial f}{\partial s}=\frac{\partial f}{\partial u}\frac{du}{ds}+\frac{\partial f}{\partial v}\frac{dv}{ds}=\mathbf{t}\cdot\left(\mathbf{a}\frac{\partial f}{\partial u}+\mathbf{b}\frac{\partial f}{\partial v}\right). \tag{A.8-16}$$

By analogy with Eq. (A.6-3), we define the quantity in parentheses as $\boldsymbol{\nabla}_\mathbf{s}f$. Accordingly, the surface gradient operator is

$$\boldsymbol{\nabla}_\mathbf{s}=\mathbf{a}\frac{\partial}{\partial u}+\mathbf{b}\frac{\partial}{\partial v}. \tag{A.8-17}$$

For a surface described by $z=F(x, y)$, the result is

$$\begin{aligned}\boldsymbol{\nabla}_\mathbf{s}=&\frac{1}{H^2}\left\{\left[1+\left(\frac{\partial F}{\partial y}\right)^2\right]\mathbf{e}_x-\frac{\partial F}{\partial x}\frac{\partial F}{\partial y}\mathbf{e}_y+\frac{\partial F}{\partial x}\mathbf{e}_z\right\}\frac{\partial}{\partial x}\\&+\frac{1}{H^2}\left\{-\frac{\partial F}{\partial x}\frac{\partial F}{\partial y}\mathbf{e}_x+\left[1+\left(\frac{\partial F}{\partial x}\right)^2\right]\mathbf{e}_y+\frac{\partial F}{\partial y}\mathbf{e}_z\right\}\frac{\partial}{\partial y}\end{aligned} \tag{A.8-18}$$

where H is given by Eq. (A.8-11). As a simple example, consider a planar surface corresponding to a constant value of z. In this case $H=1$ and the surface gradient operator reduces to

$$\boldsymbol{\nabla}_\mathbf{s}=\mathbf{e}_x\frac{\partial}{\partial x}+\mathbf{e}_y\frac{\partial}{\partial y} \qquad (z=F=\text{constant}), \tag{A.8-19}$$

which is simply a two-dimensional form of ∇ involving the surface coordinates x and y.

As shown in Brand (1947, p. 209), the gradient and surface gradient operators are related by

$$\nabla = \nabla_s + \mathbf{nn}\cdot\nabla, \tag{A.8-20}$$

so that an alternative expression for ∇_s is

$$\nabla_s = (\boldsymbol{\delta} - \mathbf{nn})\cdot\nabla. \tag{A.8-21}$$

Subtracting the term $\mathbf{nn}\cdot\nabla$ from ∇ has the effect of removing from the operator any contributions which are not in the tangent plane.

Mean Curvature

The unit normal vector will be independent of position only for a planar surface. The rate of change of $\mathbf{n}$ along a surface is clearly related to the extent of surface curvature: The greater the curvature, the more rapid the variation in $\mathbf{n}$. A measure of the local curvature of a surface that is useful in fluid mechanics is the *mean curvature,* $\mathcal{H}$, which is proportional to the *surface divergence* of $\mathbf{n}$. Specifically,

$$\mathcal{H} \equiv -\frac{1}{2}\nabla_s\cdot\mathbf{n}. \tag{A.8-22}$$

For a surface expressed as $z = F(x, y)$, the mean curvature is given by

$$2\mathcal{H} = \frac{1}{H^3}\left\{\left[1+\left(\frac{\partial F}{\partial y}\right)^2\right]\frac{\partial^2 F}{\partial x^2} - 2\frac{\partial F}{\partial x}\frac{\partial F}{\partial y}\frac{\partial^2 F}{\partial x\,\partial y} + \left[1+\left(\frac{\partial F}{\partial x}\right)^2\right]\frac{\partial^2 F}{\partial y^2}\right\}. \tag{A.8-23}$$

For a surface with $z = F(x)$ only, this simplifies to

$$2\mathcal{H} = \left[\left(\frac{\partial F}{\partial x}\right)^2 + 1\right]^{-3/2}\frac{\partial^2 F}{\partial x^2}. \tag{A.8-24}$$

Two important special cases are cylinders and spheres. For a cylinder of radius R, with the surface defined as $z = F(x) = (R^2 - x^2)^{1/2}$, Eq. (A.8-24) gives

$$\mathcal{H} = -\frac{1}{2R} \quad \text{(cylinder)}. \tag{A.8-25}$$

For a sphere with $z = F(x, y) = (R^2 - x^2 - y^2)^{1/2}$, it is found from Eq. (A.8-23) that

$$\mathcal{H} = -\frac{1}{R} \quad \text{(sphere)}. \tag{A.8-26}$$

Unlike most surfaces, the curvature of cylinders and spheres is uniform (i.e., independent of position). For any piece of a surface described by $z = F(x, y)$, it is found that $\mathcal{H} < 0$ when $\mathbf{e}_z$ points away from the local center of curvature (as in these examples) and $\mathcal{H} > 0$ when $\mathbf{e}_z$ points toward the local center of curvature. For a general closed surface, $\mathcal{H} < 0$ when the outward normal $\mathbf{n}$ points away from the local center of curvature, and $\mathcal{H} > 0$ when $\mathbf{n}$ points toward the local center of curvature. In other words, $\mathcal{H}$ is negative or positive according to whether the surface is locally convex or concave, respectively.

Integral Transformation

One additional integral transformation is needed, which involves the surface gradient and surface curvature. Setting $\mathbf{v}=\mathbf{n}f$ in Eq. (A.5-7) gives

$$\int_S (\mathbf{n}\times\nabla)\times(\mathbf{n}f)\, dS=\int_C \mathbf{t}\times\mathbf{n}f\, dC. \tag{A.8-27}$$

As discussed in connection with Fig. A-2, the unit vector $\mathbf{m}=\mathbf{t}\times\mathbf{n}$ is tangent to the surface S and outwardly normal to the contour C. The left-hand side is rearranged by first noting that $\mathbf{n}\times\nabla=\mathbf{n}\times\nabla_s$; this is shown using Eq. (A.8-20). After further manipulation of the left-hand side, it is found that

$$\int_S (\nabla_s f-\nabla_s\cdot\mathbf{nn}f)\, dS=\int_C \mathbf{m}f\, dC \tag{A.8-28}$$

or, using the symbol for mean curvature,

$$\int_S (\nabla_s f+2\mathcal{H}\mathbf{n}f)\, dS=\int_C \mathbf{m}f\, dC. \tag{A.8-29}$$

This result is used in Chapter 5 in deriving the stress balance at a fluid–fluid interface, including the effects of surface tension.

References

Aris, R. *Vectors, Tensors, and the Basic Equations of Fluid Mechanics.* Prentice-Hall, Englewood Cliffs, NJ, 1962 (reprinted by Dover, New York, 1989).

Bird, R. B., R. C. Armstrong, and O. Hassager. *Dynamics of Polymeric Liquids,* Vol. 1, second edition. Wiley, New York, 1987.

Bird, R. B., W. E. Stewart, and E. N. Lightfoot. *Transport Phenomena.* Wiley, New York, 1960.

Brand, L. *Vector and Tensor Analysis.* Wiley, New York, 1947.

Hay, G. B. *Vector and Tensor Analysis.* Dover, New York, 1953.

Hildebrand, F. B. *Advanced Calculus for Applications,* second edition. Prentice-Hall, Englewood Cliffs, NJ, 1976.

Jeffreys, H. *Cartesian Tensors.* Cambridge University Press, Cambridge, 1963.

Moon, P. and D. E. Spencer. *Field Theory Handbook.* Springer-Verlag, Berlin, 1961.

Morse, P. M. and H. Feshbach. *Methods of Theoretical Physics.* McGraw-Hill, New York, 1953.

Prager, W. *Introduction to Mechanics of Continua.* Ginn, New York, 1961 (reprinted by Dover, New York, 1973).

Wilson, E. B. *Vector Analysis.* Yale University Press, New Haven, 1901.