The Scenario Approach Meets Uncertain Game Theory and Variational Inequalities

Dario Paccagnan
In collaboration with M.C. Campi
Theme of this talk: take decision based on data and quantify their risk
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data $\rightarrow$ decision making process $\rightarrow$ solution

- scenario approach
- robust optimization

[Borelli, Calafiore, Campi, Esfahani, Garatti, Goulart, Kuhn, Lygeros, Margellos, Prandini, Ramponi, Sutter, Tempo, ...]

What if decision making is not an optimization problem?

In this talk: decision making process = variational inequality
Theme of this talk: take decision based on data and quantify their risk

If decision making = optimization problem $\implies$
- scenario approach
- robust optimization
- ...

Diagram:
\[
data \rightarrow \text{decision making process} \rightarrow \text{solution}\]
Theme of this talk: take decision based on data and quantify their risk

Data $\rightarrow$ decision making process $\rightarrow$ solution

If decision making $=$ optimization problem $\implies$
- scenario approach
- robust optimization
- ...

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What if decision making is not an optimization problem?

in this talk: decision making process = variational inequality
Why variational inequalities?

“[...] a multitude of interesting connections to numerous disciplines, and a wide range of important applications in engineering and economics”

F. Facchinei, J-S Pang
Why variational inequalities?

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transportation networks

contact problems

demand-response markets

option pricing
OVERVIEW

1. Interconnected dynamics and stability analysis
2. Projected gradient flow on the power flow manifold
3. Numerical experiments

ROADMAP

1. Robust variational inequalities + scenario approach
   ~→ probabilistic bounds on the risk
   ~→ extension to quasi variational inequalities

2. Uncertain and robust games
   ~→ how likely that a Nash equilibrium remains such?
   ~→ application to demand-response

3. Outlook and opportunities
Variational inequalities

**Definition (VI):** given set $\mathcal{X} \subset \mathbb{R}^n$ and operator $F : \mathcal{X} \rightarrow \mathbb{R}^n$, find $\bar{x} \in \mathcal{X}$ s.t. $F(\bar{x})^T(x - \bar{x}) \geq 0, \forall x \in \mathcal{X}$.
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▷ convex optimization as a special case:

$\bar{x}$ solution of $\min_{x \in \mathcal{X}} g(x) \iff \nabla g(\bar{x})^\top (x - \bar{x}) \geq 0$, $\forall x \in \mathcal{X}$
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Stochastic and robust VIs

Q: How to define a VI that incorporates notion of uncertainty?

\[ \text{tentative: set } X, \text{ operator } F : X \times \Delta \rightarrow \mathbb{R}^n \]

find \( \bar{x} \in X \) s.t.

\[ F(\bar{x}, \delta)^\top (x - \bar{x}) \geq 0 \quad \forall x \in X, \forall \delta \in \Delta \]

\[ \Rightarrow \text{has a solution only exceptionally} \]

\[ \text{literature: 1. expected-value formulation} \]

find \( \bar{x} \in X \) s.t.

\[ E_{\delta \sim P}[F(\bar{x}, \delta)]^\top (x - \bar{x}) \geq 0 \quad \forall x \in X \]

\[ \text{[Gürgan, Jiang, Nedić, Robinson, Shanbhag, Yousefian, ...]} \]

\[ 2. \text{expected residual formulation} \]

find \( \bar{x} \in X \) s.t.

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\[ \text{[Chen, Fukushima, Lin, Shanbhag, Wets, Zhang, ...]} \]

this talk: robust VI, i.e.,

\[ F : \mathbb{R}^n \rightarrow \mathbb{R}^n, (\Delta, F, P), \text{sets } \{X_\delta\}_{\delta \in \Delta} \]

find \( x \in \bigcap_{\delta \in \Delta} X_\delta \) s.t.

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**Q:** How to define a VI that incorporates notion of uncertainty?

**tentative:** set $\mathcal{X}$, operator $F : \mathcal{X} \times \Delta \rightarrow \mathbb{R}^n$

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Robust and sampled VI, risk

RVI: find $x_R \in \cap_{\delta \in \Delta} X_\delta$ s.t. $F(x_R)^\top (x - x_R) \geq 0 \quad \forall x \in \cap_{\delta \in \Delta} X_\delta$
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S-RVI: find $x_S \in \cap_{i=1}^N \mathcal{X}_{\delta_i}$ s.t. $F(x_S)^T (x - x_S) \geq 0 \quad \forall x \in \cap_{i=1}^N \mathcal{X}_{\delta_i}$

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**Q:** How “likely” is a solution to S-RVI to be a solution of RVI?
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\[ \rightsquigarrow \text{notion of risk:} \] the risk $V(x)$ associated to $x \in \mathbb{R}^n$ is

$V(x) = \mathbb{P}\{\delta \in \Delta \text{ s.t. } x \notin X_\delta\}$
Robust and sampled VI, risk

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$\leadsto$ notion of risk: the risk $V(x)$ associated to $x \in \mathbb{R}^n$ is

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$$(\delta_1, \ldots, \delta_N) \quad \longrightarrow \quad \text{Sampled RVI} \quad \longrightarrow \quad \text{solution } x_S \text{ risk } V(x_S)$$
Robust and sampled VI, risk

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\text{RVI: } \text{find } x_R \in \cap_{\delta \in \Delta} X_\delta \quad \text{s.t.} \quad F(x_R)^T (x - x_R) \geq 0 \quad \forall x \in \cap_{\delta \in \Delta} X_\delta
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\[V(x) = \mathbb{P}\{\delta \in \Delta \text{ s.t. } x \notin X_\delta\}\]

\[\{\delta_1, \ldots, \delta_N\} \quad \xrightarrow{\text{Sampled RVI}} \quad \text{solution } x_S \quad \text{risk } V(x_S)\]

\[\leadsto \textbf{assume:} \text{ existence \& uniqueness of solution } x_S \text{ for all } \{\delta_i\}_{i=1}^N\]
First result

For any $\beta \in (0, 1)$, $k \in \{0, \ldots, N - 1\}$, let $\varepsilon(k)$ be the unique solution of

$$
\frac{\beta}{N + 1} \sum_{l=k}^{N} \binom{l}{k} (1 - \varepsilon)^{l-k} - \binom{N}{k} (1 - \varepsilon)^{N-k} = 0.
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For any $\beta \in (0, 1)$, $k \geq N$, let $\varepsilon(k) = 1$. 

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$$\beta N + 1 = \sum_{l=k}^{N-1} (l_k)(1 - \varepsilon(l - k) - k - \binom{N-k}{1 - \varepsilon(N-k))}-0$$

For any $\beta \in (0, 1)$, let $k \geq N$, let $\varepsilon(k) = 1$.

Theorem: assume existence + uniqueness & non-degeneracy

\[ \Delta \]

$a$-priori bound on risk:

$$P_N \left[ V(x_S) \leq \varepsilon(n) \right] \geq 1 - \beta$$

$a$-posteriori bound on risk:

$$P_N \left[ V(x_S) \leq \varepsilon(s) \right] \geq 1 - \beta$$

where $s$ is the number of support constraints.

"with high probability (larger than $1 - \beta$), the risk is small (below $\varepsilon$)"
First result

For any $\beta \in (0, 1)$, let $k \in \{0, \ldots, N-1\}$, let $\epsilon(k)$ be the unique solution of

$$
\beta N + \sum_{l=k}^{N-1} (l-k)(1-\epsilon)^{l-k} - k - (N-k)(1-\epsilon)^{N-k} = 0.
$$

For any $\beta \in (0, 1)$, let $k \geq N$, let $\epsilon(k) = 1$.

**Theorem:** assume existence + uniqueness & non-degeneracy

\[ n = 10, \ N = 100 \]

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\[ \beta \]

\[ \cdot 10^{-2} \]

\[ \epsilon \]

\[ \beta \]

\[ 0 \]

\[ 0.2 \]

\[ 0.4 \]

\[ 0.6 \]

\[ 0.8 \]

\[ 1 \]

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**Theorem:** assume existence + uniqueness & non-degeneracy
For any $\beta \in (0, 1)$, let $\varepsilon(k)$ be the unique solution of

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**Theorem:** Assume existence + uniqueness & non-degeneracy

▷ $a$-priori bound on risk: $\mathbb{P}^N[V(x_s) \leq \varepsilon(n)] \geq 1 - \beta$

\[ n = 10, N = 100 \]
First result

For any $\beta \in (0, 1)$, let $\epsilon(k)$ be the unique solution of

$$\beta N + 1 = N \sum_{l=k}^{N-1} (1 - \epsilon(l)) l - k - (N-k)(1 - \epsilon) N - k = 0.$$ 

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**Theorem:** assume existence + uniqueness & non-degeneracy

- a-priori bound on risk: $\mathbb{P}^N[V(x_S) \leq \epsilon(n)] \geq 1 - \beta$
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First result

For any $\beta \in (0, 1)$, let $\varepsilon(k)$ be the unique solution of

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where $s$ is the number of support constraints

“with high probability (larger than $1 - \beta$), the risk is small (below $\varepsilon$)”
The result extends to quasi-variational inequalities

**Definition (QVI):** given set-valued map $\mathcal{X} : \mathbb{R}^n \rightrightarrows 2^{\mathbb{R}^n}$ and $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$, find $\bar{x} \in \mathcal{X}(\bar{x})$ s.t. $F(\bar{x})^\top(x - \bar{x}) \geq 0$, $\forall x \in \mathcal{X}(\bar{x})$
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▷ informal: *a VI where the feasible set depends on the point $x$*
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**Robust/Sampled QVI:** $F : \mathbb{R}^n \to \mathbb{R}^n$, $(\Delta, \mathcal{F}, \mathbb{P})$, set val. maps $\{\mathcal{X}_\delta\}_{\delta \in \Delta}$
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- RQVI: find \( x_R \in \bigcap_{\delta \in \Delta} \mathcal{X}_\delta(x_R) \) s.t. \( F(x_R)^\top(x - x_R) \geq 0 \) \( \forall x \in \bigcap_{\delta \in \Delta} \mathcal{X}_\delta(x_R) \)
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**Robust/Sampled QVI:** $F : \mathbb{R}^n \to \mathbb{R}^n$, $(\Delta, \mathcal{F}, \mathbb{P})$, set val. maps $\{\mathcal{X}_\delta\}_{\delta \in \Delta}$

- **RQVI:** find $x_R \in \cap_{\delta \in \Delta} \mathcal{X}_\delta(x_R)$ s.t. $F(x_R)^\top (x - x_R) \geq 0 \ \forall x \in \cap_{\delta \in \Delta} \mathcal{X}_\delta(x_R)$
- **S-RQVI:** find $x_S \in \cap_{i=1}^N \mathcal{X}_{\delta_i}(x_S)$ s.t. $F(x_S)^\top (x - x_S) \geq 0 \ \forall x \in \cap_{i=1}^N \mathcal{X}_{\delta_i}(x_S)$ where $\delta_i$ iid $\sim \mathbb{P}$

Risk: the risk associated to $x \in \mathbb{R}^n$ is $V(x) = \mathbb{P}\{\delta \in \Delta \text{ s.t. } x \notin \mathcal{X}_\delta(x)\}$

**Theorem (informal):** the same bounds on the risk hold for QVI.
The result extends to quasi-variational inequalities

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- S-RQVI: find $x_S \in \cap_{i=1}^N \mathcal{X}_{\delta_i}(x_S)$ s.t. $F(x_S)^\top (x - x_S) \geq 0$, $\forall x \in \cap_{i=1}^N \mathcal{X}_{\delta_i}(x_S)$, $\delta_i$ iid $\sim P$
The result extends to quasi-variational inequalities

**Definition (QVI):** given set-valued map $\mathcal{X} : \mathbb{R}^n \rightrightarrows 2^{\mathbb{R}^n}$ and $F : \mathbb{R}^n \to \mathbb{R}^n$, find $\bar{x} \in \mathcal{X}(\bar{x})$ s.t. $F(\bar{x})^\top (x - \bar{x}) \geq 0, \ \forall x \in \mathcal{X}(\bar{x})$

- informal: a VI where the feasible set depends on the point $x$
- we will use QVI to describe games with uncertain costs

**Robust/Sampled QVI:** $F : \mathbb{R}^n \to \mathbb{R}^n$, $(\Delta, \mathcal{F}, \mathbb{P})$, set val. maps $\{\mathcal{X}_\delta\}_{\delta \in \Delta}$

- **RQVI:** find $x_R \in \cap_{\delta \in \Delta} \mathcal{X}_\delta(x_R)$ s.t. $F(x_R)^\top (x - x_R) \geq 0, \ \forall x \in \cap_{\delta \in \Delta} \mathcal{X}_\delta(x_R)$
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- $\delta_i$ iid $\sim \mathbb{P}$

**Risk:** the risk associated to $x \in \mathbb{R}^n$ is $V(x) = \mathbb{P}\{\delta \in \Delta \text{ s.t. } x \notin \mathcal{X}_\delta(x)\}$
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**Theorem (informal):** the same bounds on the risk hold for QVI.
Uncertain games
Uncertain games

- Each agent's decision $x_j \in X_j \subseteq \mathbb{R}^n$, let $X = X_1 \times \cdots \times X_M$.
- Each agent's cost function $C_j(x_j, x_{-j}; \delta)$:

  Robust NE ([Aghassi and Berstimas]):
  $x_R \in X$ is a robust NE if
  $$\max_{\delta \in \Delta} C_j(x_R; \delta) \leq \max_{\delta \in \Delta} C_j(x_j, x_{-j_R}; \delta)$$
  $\forall x_j \in X_j, \forall j$.

- Often agents have access to past realizations $\delta_i$ from $(\Delta, F, P)$.

  Sampled robust NE: \{ $\delta_i$ \}_{N_i=1} \text{iid} \sim P, x_S \in X$ is a sampled robust NE if
  $$\max_i C_j(x_S; \delta_i) \leq \max_i C_j(x_j, x_{-j_S}; \delta_i)$$
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Uncertain games

- $M$ agents
Uncertain games

- $M$ agents
- each agent’s decision $x^j \in \mathcal{X}^j \subseteq \mathbb{R}^n$, let $\mathcal{X} = \mathcal{X}^1 \times \cdots \times \mathcal{X}^M$
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▷ often agents have access to past realizations $\delta_i$ from $(\Delta, \mathcal{F}, \mathbb{P})$

Sampled robust NE: $\{\delta_i\}_{i=1}^N$ iid $\sim \mathbb{P}$, $x_S \in \mathcal{X}$ is a sampled robust NE if

$$\max_i C^j(x_S; \delta_i) \leq \max_i C^j(x^j, x_S^{-j}; \delta_i) \quad \forall x^j \in \mathcal{X}^j, \forall j$$
Risk associated to sampled robust NE

**Setup:** samples $\{\delta_i\}_{i \in N}$ are known to the agents, which decide to play $x_S$. 

**Answer:** an application of the previous theory.

Let agent's risk be $V_j(x_S) = P\{\delta \in \Delta \text{ s.t. } C_j(x_S; \delta) \geq \max_i C_j(x_S; \delta_i)\}$

**Theorem:** existence, uniqueness, non-degeneracy $\Rightarrow \Delta$-a-priori bound on risk: $P_N[V_j(x_S) \leq \epsilon (nM + M)] \geq 1 - \beta$

$\Delta$-a-posteriori bound on risk: $P_N[V_j(x_S) \leq \epsilon (s)] \geq 1 - \beta$
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Setup: samples \( \{\delta_i\}_{i \in N} \) are known to the agents, which decide to play \( x_S \).

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(\delta_1, \ldots, \delta_N) \rightarrow \text{Sampled robust Nash} \rightarrow \text{solution } x_S
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**Q:** how likely is each agent to incur a higher cost than what predicted? i.e., higher than $\max_i C^i(x_S; \delta_i)$?
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\[ V^j(x) = \mathbb{P}\{\delta \in \Delta \text{ s.t. } C^j(x; \delta) \geq \max_i C^j(x; \delta_i)\} \]
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**Theorem:** existence, uniqueness, non-degeneracy \( \implies \)

- a-priori bound on risk: \( \mathbb{P}^N[V^j(x_S) \leq \varepsilon(nM + M)] \geq 1 - \beta \)
- a-posteriori bound on risk: \( \mathbb{P}^N[V^j(x_S) \leq \varepsilon(s)] \geq 1 - \beta \)
Robust Charging games
Robust Charging games

- A fleet of EVs to recharge
Robust Charging games

- A fleet of EVs to recharge

\[ \begin{align*}
\text{players:} & \quad j \in \{1, \ldots, M\} \\
\end{align*} \]
Robust Charging games

- A fleet of EVs to recharge
- Each vehicle min bill in $[1, n]$ players: $j \in \{1, \ldots, M\}$

\[
x_1^j \quad x_2^j \quad x_{n-1}^j \quad x_n^j
\]
Robust Charging games

- A fleet of EVs to recharge
- Each vehicle min bill in $[1, n]$

\[
x^j_1 x^j_2 x^j_{n-1} x^j_n
\]

players: $j \in \{1, \ldots, M\}$

cost of $j$: $p(\sum_j x^j + d)^T x^j$
Robust Charging games

- A fleet of EVs to recharge
- Each vehicle min bill in $[1, n]$

$$x_j^1 x_j^2 \ldots x_{n-1}^j x_n^j$$

players: $j \in \{1, \ldots, M\}$
cost of $j$: $p(\sum_j x_j^j + d)^T x_j^j$
Robust Charging games

- A fleet of EVs to recharge
- Each vehicle min bill in \([1, n]\)
  \[x_1^j \quad x_2^j \quad x_{n-1}^j \quad x_n^j\]
- Charging requirements

players: \(j \in \{1, \ldots, M\}\)
cost of \(j\): \(p(\sum_j x_j^j + d)^\top x_j^j\)

\[\text{constr: } x_j^j \in \mathcal{X}_j\]
Robust Charging games

- A fleet of EVs to recharge
- Each vehicle minimizes bill in $[1, n]$
  
  $x_1^j \ x_2^j \ x_{n-1}^j \ x_n^j$

- Charging requirements
- Past non-EV demand

players: $j \in \{1, \ldots, M\}$

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constr: $x_j^j \in \mathcal{X}_j$

samples: $\{d_i\}_{i=1}^N$
Robust Charging games

- A fleet of EVs to recharge
- Each vehicle min bill in \([1, n]\)

\[
\begin{align*}
\sum_{j=1}^{n} x_j & \\
\sum_{j=1}^{n-1} x_j & 
\end{align*}
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- Charging requirements
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**literature**: \(d\) known in advance \[\text{[Callaway, Chen, Grammatico, Hiskens, Ma, ...]}\]

- Past non-EV demand

**players**: \(j \in \{1, \ldots, M\}\)

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Robust Charging games

- A fleet of EVs to recharge
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literature: \(d\) known in advance \([\text{Callaway, Chen, Grammatico, Hiskens, Ma, \ldots }\]

Q: What guarantees can we provide the users without this assumption?
Numerical experiments

Charging profile coordinated to a sampled-robust NE $\rightsquigarrow$ prob. guarantees
Numerical experiments

Charging profile coordinated to a sampled-robust NE $\leadsto$ prob. guarantees

- How “likely” are users to pay more than “expected”? Little
Numerical experiments

Charging profile coordinated to a sampled-robust NE $\leadsto$ prob. guarantees

- How “likely” are users to pay more than “expected”? Little
- How “likely” are users to deviate from agreed charging? Little
Numerical experiments

Charging profile coordinated to a sampled-robust NE ⇜ prob. guarantees

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![Graph showing consumption profile over time]
Numerical experiments

Charging profile coordinated to a sampled-robust NE \( \rightsquigarrow \) prob. guarantees

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Simulations with \( M = 100 \) agents, \( N = 500 \) days of history
Numerical experiments

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Simulations with $M = 100$ agents, $N = 500$ days of history
$\leadsto$ a-priori bound is not useful as $nM + M = 2500$
Numerical experiments

Charging profile coordinated to a sampled-robust NE \(\rightsquigarrow\) prob. guarantees

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Simulations with \(M = 100\) agents, \(N = 500\) days of history

\(\rightsquigarrow\) a-priori bound is not useful as \(nM + M = 2500\)

\(\rightsquigarrow\) a-posteriori bound is useful as typically \(3 \leq s \leq 7\)
Numerical experiments

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$\leadsto$ a-posteriori bound is useful as typically $3 \leq s \leq 7$

with $s = 7$, $V^j(x_S) \leq 6.5\%$ with probability larger than $1 - 10^{-6}$
Numerical experiments

Charging profile coordinated to a sampled-robust NE \( \Rightarrow \) prob. guarantees

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V^j(x_S) \leq 6.5\% \quad \text{with probability larger than } 1 - 10^{-6}
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\( \Rightarrow \) exact calculations reveal that \( 0.11\% \leq V^j(x_S) \leq 0.16\% \)
Numerical experiments

Charging profile coordinated to a sampled-robust NE \(\rightsquigarrow\) prob. guarantees

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\(\rightsquigarrow\) exact calculations reveal that \(0.11\% \leq V^j(x_S) \leq 0.16\%\)
Conclusions and Outlook

Theme of this talk: take decision based on data and quantify risk
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data $\rightarrow$ variational inequality $\rightarrow$ solution risk

Opportunities:
- novel and unexplored framework
- explore different applications
- tightly bound agents' private risk?
Conclusions and Outlook

**Theme of this talk:** take decision based on data and quantify risk

```
data → variational inequality → solution risk
```

**Technical results:** - a-priori/a-posteriori bounds for VI and QVI

Opportunities: novel and unexplored framework ⇝ explore different applications ⇝ tightly bound agents' private risk

sites.engineering.ucsb.edu/∼dariop dariop@ucsb.edu
Conclusions and Outlook

**Theme of this talk:** take decision based on data and quantify risk

![Diagram showing data → variational inequality → solution risk]

**Technical results:**
- a-priori/a-posteriori bounds for VI and QVI
- scenario approach for uncertain game theory

Opportunities:
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sites.engineering.ucsb.edu/~dariop dariop@ucsb.edu
Conclusions and Outlook

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![Diagram](data → variational inequality → solution risk)

**Technical results:**
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Conclusions and Outlook

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```
data  →  variational inequality  →  solution
                   | risk
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- data → variational inequality → solution
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⇝ explore different applications
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Conclusions and Outlook

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sites.engineering.ucsb.edu/~dariop    dariop@ucsb.edu