Three Examples of Godunov Schemes

Suppose ϕ_x^- and ϕ_x^+ are computed at each grid point.

1 Motion in an Externally Generated Velocity

We seek to write the Godunov scheme for $\phi_t + \vec{V}_{ext} \cdot \nabla\phi = 0$, with $\vec{V}_{ext} = (u, v, w)$. This equation can be written as $\phi_t + u\phi_x + v\phi_y + w\phi_z = 0$. The x component is discretized as follows:

1. If $u \geq 0$ discretize $u\phi_x$ by $u\phi_x^-$.
2. If $u \leq 0$ discretize $u\phi_x$ by $u\phi_x^+$.

The y and z components are discretized similarly.

2 Motion in the Normal Direction

We seek to write the Godunov scheme for $\phi_t + \vec{n} \cdot \nabla\phi = 0$, with $\vec{n} = \nabla\phi/|\nabla\phi|$ (thus equivalent to $\phi_t + |\nabla\phi| = 0 \Leftrightarrow \phi_t + \sqrt{\phi_x^2 + \phi_y^2 + \phi_z^2} = 0$). In order to follow the correct characteristic directions, we write this equation as:

$$\phi_t + \frac{\phi_x^2}{\sqrt{\phi_x^2 + \phi_y^2 + \phi_z^2}} + \frac{\phi_y^2}{\sqrt{\phi_x^2 + \phi_y^2 + \phi_z^2}} + \frac{\phi_z^2}{\sqrt{\phi_x^2 + \phi_y^2 + \phi_z^2}} = 0.$$

The term ϕ_x is discretized as follows:

1. If $\phi_x^- \leq 0$ and $\phi_x^+ \leq 0$ discretize ϕ_x by ϕ_x^+.
2. If $\phi_x^- \geq 0$ and $\phi_x^+ \geq 0$ discretize ϕ_x by ϕ_x^-.
3. If $\phi_x^- < 0$ and $\phi_x^+ > 0$ set $\phi_x = 0$.
4. If $\phi_x^- > 0$ and $\phi_x^+ < 0$
   (a) If $|\phi_x^+| > |\phi_x^-|$ discretize ϕ_x by ϕ_x^+.
   (b) If $|\phi_x^+| < |\phi_x^-|$ discretize ϕ_x by ϕ_x^-.

The y and z components are discretized similarly.

3 Reinitialization equation

We seek to write the Godunov scheme for $\phi_t + \text{Sign}(\phi_0)\left(|\nabla\phi| - 1\right) = 0$ (thus equivalent to $\phi_t + \text{Sign}(\phi_0)\sqrt{\phi_x^2 + \phi_y^2 + \phi_z^2} = \text{Sign}(\phi_0)$). In order to follow the correct characteristic directions, we write this equation as:

$$\phi_t + \frac{\text{Sign}(\phi_0)\phi_x^2}{\sqrt{\phi_x^2 + \phi_y^2 + \phi_z^2}} + \frac{\text{Sign}(\phi_0)\phi_y^2}{\sqrt{\phi_x^2 + \phi_y^2 + \phi_z^2}} + \frac{\text{Sign}(\phi_0)\phi_z^2}{\sqrt{\phi_x^2 + \phi_y^2 + \phi_z^2}} = \text{Sign}(\phi_0).$$

The term ϕ_x is discretized as follows:

1. If $\text{Sign}(\phi_0)\phi_x^- \leq 0$ and $\text{Sign}(\phi_0)\phi_x^+ \leq 0$ discretize ϕ_x by ϕ_x^+.
2. If $\text{Sign}(\phi_0)\phi_x^- \geq 0$ and $\text{Sign}(\phi_0)\phi_x^+ \geq 0$ discretize ϕ_x by ϕ_x^-.
3. If $\text{Sign}(\phi_0)\phi_x^- \leq 0$ and $\text{Sign}(\phi_0)\phi_x^+ \geq 0$ set $\phi_x = 0$.
4. If $\text{Sign}(\phi_0)\phi_x^- \geq 0$ and $\text{Sign}(\phi_0)\phi_x^+ \leq 0$
   (a) If $|\text{Sign}(\phi_0)(\phi_x^+)^2| > |\text{Sign}(\phi_0)(\phi_x^-)^2|$ discretize ϕ_x by ϕ_x^+.
   (b) If $|\text{Sign}(\phi_0)(\phi_x^+)^2| < |\text{Sign}(\phi_0)(\phi_x^-)^2|$ discretize ϕ_x by ϕ_x^-.

The terms ϕ_y^2 and ϕ_z^2 are discretized similarly. Note that we can use $\text{Sign}(\phi_0) = \phi_0/\sqrt{\phi_0^2 + \triangle x^2}$.