Three Examples of Godunov Schemes

Suppose $\phi^-$ and $\phi^+$ are computed at each grid point.

1 Motion in an Externally Generated Velocity

We seek to write the Godunov scheme for $\phi_t + \vec{V}_{ext} \cdot \nabla \phi = 0$, with $\vec{V}_{ext} = (u, v, w)$. This equation can be written as $\phi_t + u\phi_x + v\phi_y + w\phi_z = 0$. The $x$ component is discretized as follows:

1. If $u \geq 0$ discretize $u\phi_x$ by $u\phi^-_x$.
2. If $u \leq 0$ discretize $u\phi_x$ by $u\phi^+_x$.

The $y$ and $z$ components are discretized similarly.

2 Motion in the Normal Direction

We seek to write the Godunov scheme for $\phi_t + \vec{n} \cdot \nabla \phi = 0$, with $\vec{n} = \nabla \phi/|\nabla \phi|$ (thus equivalent to $\phi_t + |\nabla \phi| = 0 \iff \phi_t + \sqrt{\phi_x^2 + \phi_y^2 + \phi_z^2} = 0$). In order to follow the correct characteristic directions, we write this equation as:

$$
\phi_t + \frac{\phi_x^2}{\sqrt{\phi_x^2 + \phi_y^2 + \phi_z^2}} + \frac{\phi_y^2}{\sqrt{\phi_x^2 + \phi_y^2 + \phi_z^2}} + \frac{\phi_z^2}{\sqrt{\phi_x^2 + \phi_y^2 + \phi_z^2}} = 0.
$$

The term $\phi_x$ is discretized as follows:

1. If $\phi^{-}_x \leq 0$ and $\phi^{+}_x \leq 0$ discretize $\phi_x$ by $\phi^{+}_x$.
2. If $\phi^{-}_x \geq 0$ and $\phi^{+}_x \geq 0$ discretize $\phi_x$ by $\phi^{-}_x$.
3. If $\phi^{-}_x < 0$ and $\phi^{+}_x > 0$ set $\phi_x = 0$.
4. If $\phi^{-}_x > 0$ and $\phi^{+}_x < 0$
   
   (a) If $|\phi^{+}_x| > |\phi^{-}_x|$ discretize $\phi_x$ by $\phi^{+}_x$.
   
   (b) If $|\phi^{+}_x| < |\phi^{-}_x|$ discretize $\phi_x$ by $\phi^{-}_x$.

The $y$ and $z$ components are discretized similarly.
3 Reinitialization equation

We seek to write the Godunov scheme for $\phi_t + \text{Sign}(\phi_0)(|\nabla \phi| - 1) = 0$ (thus equivalent to $\phi_t + \text{Sign}(\phi_0)\sqrt{\phi_x^2 + \phi_y^2 + \phi_z^2} = \text{Sign}(\phi_0)$). In order to follow the correct characteristic directions, we write this equation as:

$$\phi_t + \frac{\text{Sign}(\phi_0)\phi_x^2}{\sqrt{\phi_x^2 + \phi_y^2 + \phi_z^2}} + \frac{\text{Sign}(\phi_0)\phi_y^2}{\sqrt{\phi_x^2 + \phi_y^2 + \phi_z^2}} + \frac{\text{Sign}(\phi_0)\phi_z^2}{\sqrt{\phi_x^2 + \phi_y^2 + \phi_z^2}} = \text{Sign}(\phi_0).$$

The term $\phi_x$ is discretized as follows:

1. If $\text{Sign}(\phi_0)\phi_x^- \leq 0$ and $\text{Sign}(\phi_0)\phi_x^+ \leq 0$ discretize $\phi_x$ by $\phi_x^+$.  
2. If $\text{Sign}(\phi_0)\phi_x^- \geq 0$ and $\text{Sign}(\phi_0)\phi_x^+ \geq 0$ discretize $\phi_x$ by $\phi_x^-$.  
3. If $\text{Sign}(\phi_0)\phi_x^- \leq 0$ and $\text{Sign}(\phi_0)\phi_x^+ \geq 0$ set $\phi_x = 0$.  
4. If $\text{Sign}(\phi_0)\phi_x^- \geq 0$ and $\text{Sign}(\phi_0)\phi_x^+ \leq 0$
   (a) If $|\text{Sign}(\phi_0)(\phi_x^+)^2| > |\text{Sign}(\phi_0)(\phi_x^-)^2|$ discretize $\phi_x$ by $\phi_x^+$.  
   (b) If $|\text{Sign}(\phi_0)(\phi_x^+)^2| < |\text{Sign}(\phi_0)(\phi_x^-)^2|$ discretize $\phi_x$ by $\phi_x^-$.  

The terms $\phi_y^2$ and $\phi_z^2$ are discretized similarly. Note that we can use $\text{Sign}(\phi_0) = \phi_0/\sqrt{\phi_0^2 + \Delta x^2}$. 
