Level Set Methods and Their Applications

1. Consider a domain $\Omega = [0, 1]$ and the advection equation in one spatial dimension

$$u_t + u_x = 0, \tag{1}$$

with initial data $u(x, 0) = \sin 2\pi x$.

   (a) Implement the upwind scheme assuming periodic boundary conditions.

   (b) Implement the Beam-Warming scheme and the Lax-Wendroff scheme, assuming periodic boundary conditions.

   (c) Plot the result for the numerical solution at $t = 1$ obtained with $\triangle t = .5\triangle x$ using 50 grid nodes and 100 grid nodes. Compare with the exact solution on the same graph (hint: use the method of characteristics to find the analytical solution).

   (d) Change the initial data to be the square wave:

$$u_0(x) = \begin{cases} 1 & \text{if } |x| \leq \frac{1}{3} \\ 0 & \text{if } |x| > \frac{1}{3} \end{cases}$$

   and plot the result for the numerical solution at $t = 1$ obtained with $\triangle t = .5\triangle x$ using 50 grid nodes and 100 grid nodes. Compare with the exact solution on the same graph. Comment on your results.

Notes:

The Lax-Wendroff scheme is:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -\frac{1}{2\Delta x}\left(u_{i+1}^n - u_{i-1}^n\right) + \frac{\Delta t}{2\Delta x^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right).$$

The Beam-Warming scheme is:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -\frac{1}{2\Delta x}\left(3u_i^n - 4u_{i-1}^n + u_{i-2}^n\right) + \frac{\Delta t}{2\Delta x^2}\left(u_i^n - 2u_{i-1}^n + u_{i-2}^n\right).$$