Finite difference analysis of the seepage problem

Physical Problem:

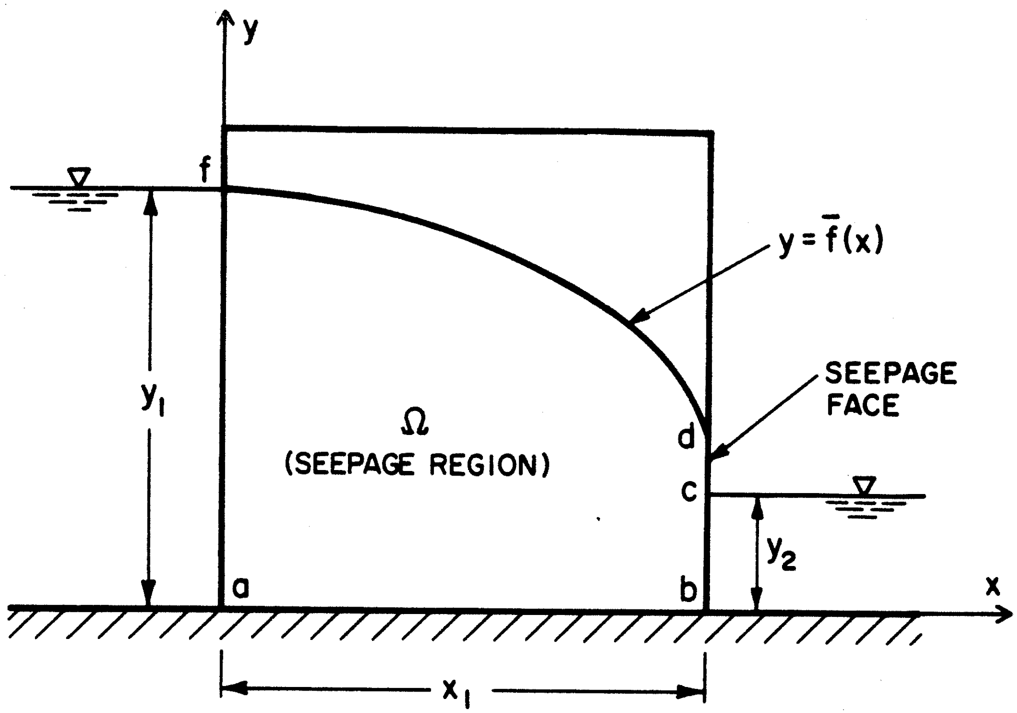


Figure 1: Seepage through a rectangular dam.

Assumptions:

1. Soil in the flowfield is homogeneous and isotropic
2. Capillary and evaporation effects are neglected
3. Flow obeys Darcy's Law
4. Two-dimensional
5. Steady state

Mathematical Formulation:

Darcy's Law: $\vec{q} = -K \text{ grad } h = -K \text{ grad } [(p/\rho g) + y]$

Potential: $\varphi = k[(p/\rho g) + y]$

$\therefore$ Velocity Components: $u = \varphi_x$, $v = \varphi_y$

Continuity Equation: $u_x + v_y = 0$

Irrotationality Condition: $u_y - v_x = 0$

Cauchy Riemann Equation: $\varphi_x = \psi_y$, $-\varphi_y = \psi_x$

Laplace's Equations: $\nabla^2 \varphi = 0$, $\nabla^2 \psi = 0$

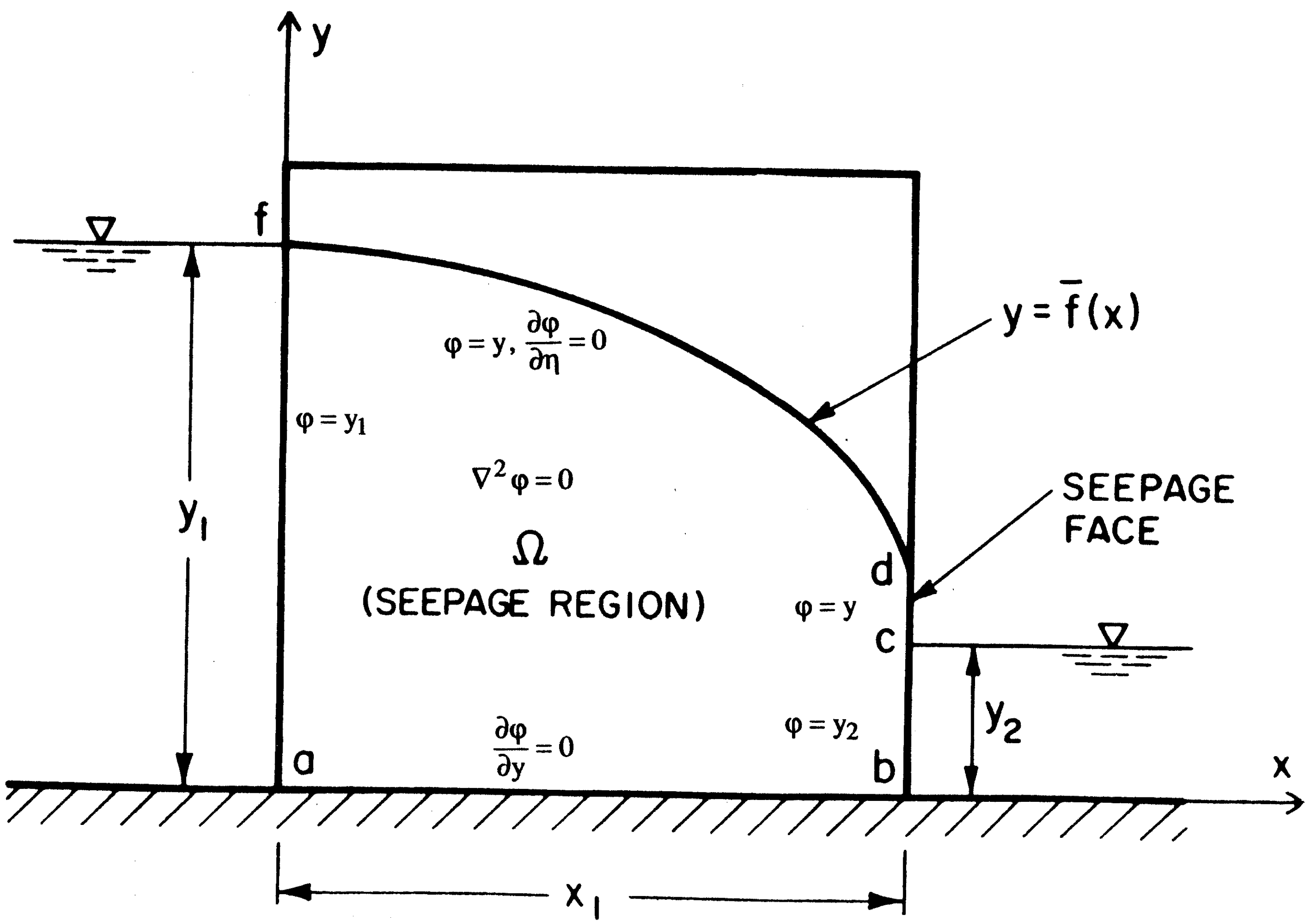


Figure 2: Mathematical formulation of physical problem.

Fixed Domain Formulation:

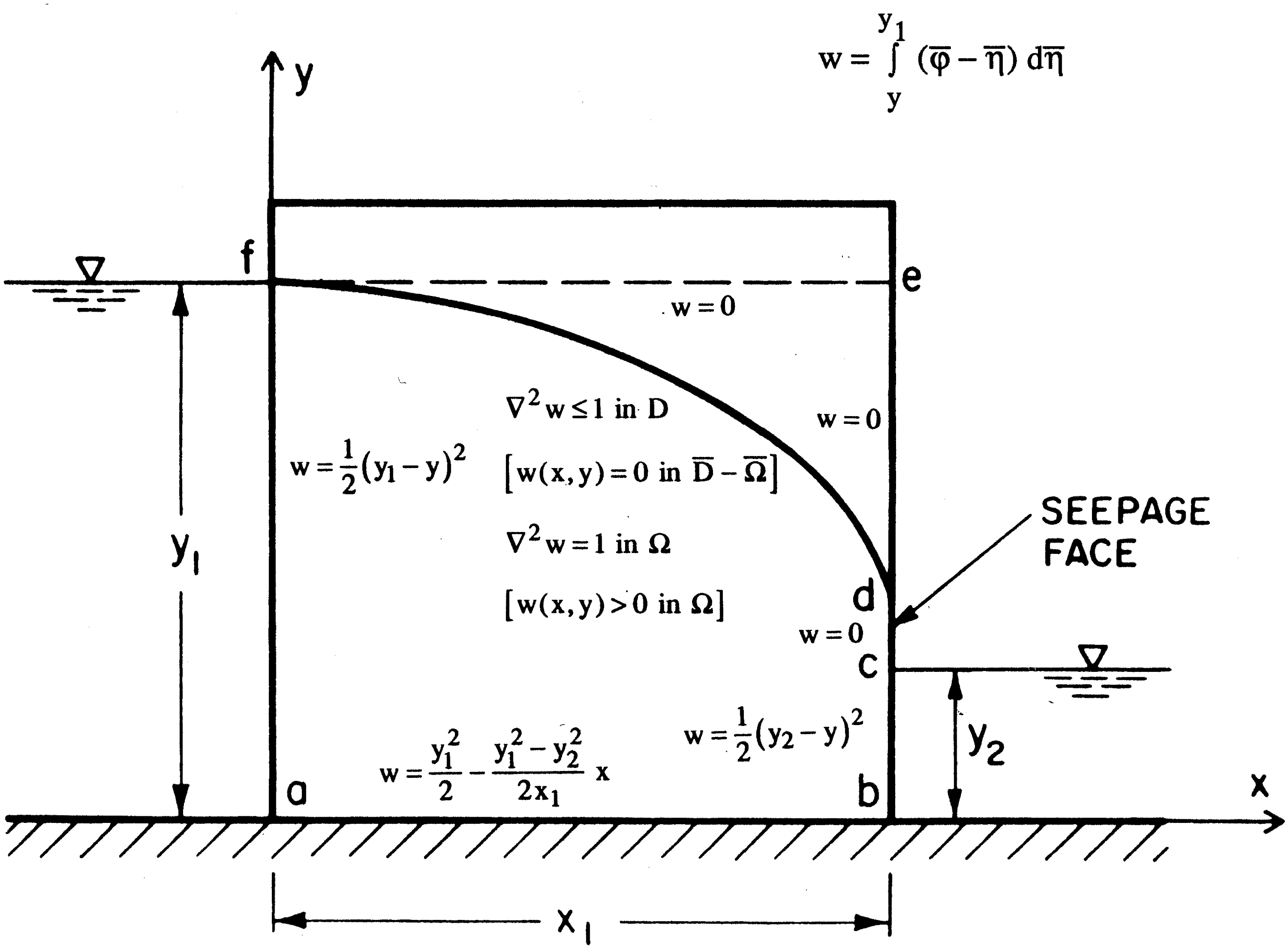


Figure 3: Fixed domain mathematical formulation.

Numerical Algorithm:

SOR iterative method to compute at all points within the computation domain (i = 2, . . ., *nc* – 1 and j = 2, . . ., nr – 1):

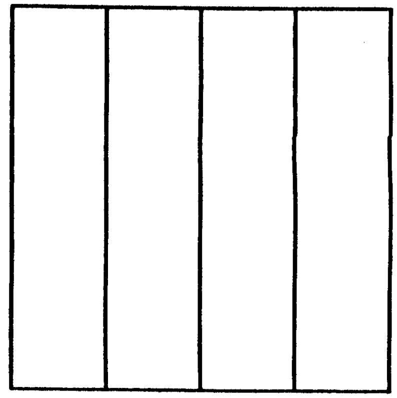
$$w_{i,j}^{(n+1/2)} = \frac{\bar{c}_1}{c_3}\left(w_{i+1,j}^{(n)} + w_{i-1,j}^{(n+1)}\right) + \frac{c_2}{c_3}\left(w_{i,j+1}^{(n)} + w_{i,j-1}^{(n+1)}\right) - \frac{1}{c_3},$$

and

$$w_{i,j}^{(n+1)} = (1-\alpha)\, w_{i,j}^{(n)} + \alpha w_{i,j}^{(n+1/2)}.$$

$$w(i,j)^{n+1} = \max\left(0,\, w(i,j)^{n+1}\right)$$

where $\bar{c}_1 = 1/\Delta x^2$, $c_2 = 1/\Delta y^2$ and $c_3 = (2/\Delta x^2) + (2/\Delta y^2)$. The subscript i stands for the horizontal location, and the subscript j denotes the vertical location. Thus, i = 1, . . ., *nc* (number of columns) and j = 1, . . ., *nr* (number of rows). Moreover, values for points located along i = 1, i = *nc*, j = 1, and j = *nr* are known since they are boundary points.



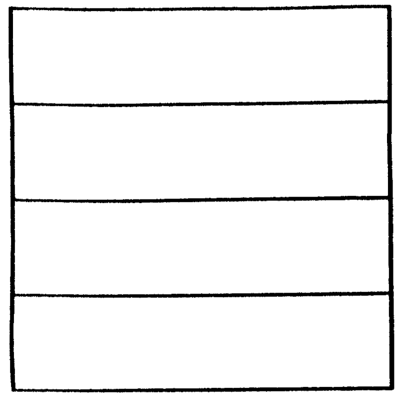
Columns of subdomains (vertical splitting)

Values at the interface points are updated concurrently by:

$$w_{Int,j}^{(n+1/2)} = \frac{\bar{c}_1}{c_3}\left(w_{Int+1,j}^{(n+1)} + w_{Int-1,j}^{(n+1)}\right) + \frac{c_2}{c_3}\left(w_{Int,j+1}^{(n)} + w_{Int,j-1}^{(n+1)}\right) - \frac{1}{c_3}$$

and

$$w_{Int,j}^{(n+1)} = (1-\alpha)\, w_{Int,j}^{(n)} + \alpha w_{Int,j}^{(n+1/2)}$$



$$w_{Int,j}^{(n+1)} = \max\left(0,\, w_{Int,j}^{(n+1)}\right)$$

Rows of subdomains (horizontal splitting)

Interface values are updated by:

$$w_{i,Int}^{(n+1/2)} = \frac{\bar{c}_1}{c_3}\left(w_{i+1,Int}^{(n)} + w_{i-1,Int}^{(n+1)}\right) + \frac{c_2}{c_3}\left(w_{i,Int-1}^{(n+1)} + w_{i,Int+1}^{(n+1)}\right) - \frac{1}{c_3}$$

and

$$w_{i,Int}^{(n+1)} = (1-\alpha)\, w_{i,Int}^{(n)} + \alpha w_{i,Int}^{(n+1/2)}$$

$$w_{i,Int}^{(n+1)} = \max\left(0,\, w_{i,Int}^{(n+1)}\right)$$

Blocks of subdomains (vertical and horizontal splitting)

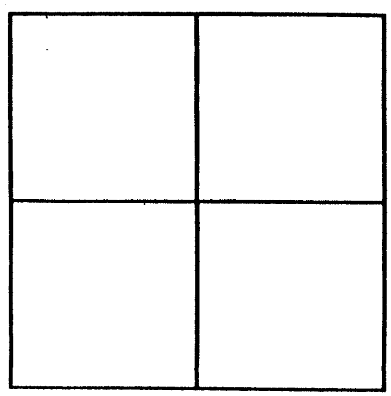
Intersection point:

$$w_{Inti,Intj}^{(n+1/2)} = \frac{\bar{c}_1}{c_3}\left(w_{Int,Intj+1}^{(n+1)} + w_{Inti,Intj-1}^{(n+1)}\right) + \frac{c_2}{c_3}\left(w_{Inti-1,Intj}^{(n+1)} + w_{Inti+1,Intj}^{(n+1)}\right) - \frac{1}{c_3}$$

and

$$w_{Inti,Intj}^{(n+1)} = (1-\alpha)\, w_{Inti,Intj}^{(n)} + \alpha w_{Inti,Intj}^{(n+1/2)}$$

$$w_{Inti,Intj}^{(n+1)} = \max\left(0,\, w_{Inti,Intj}^{(n+1)}\right)$$



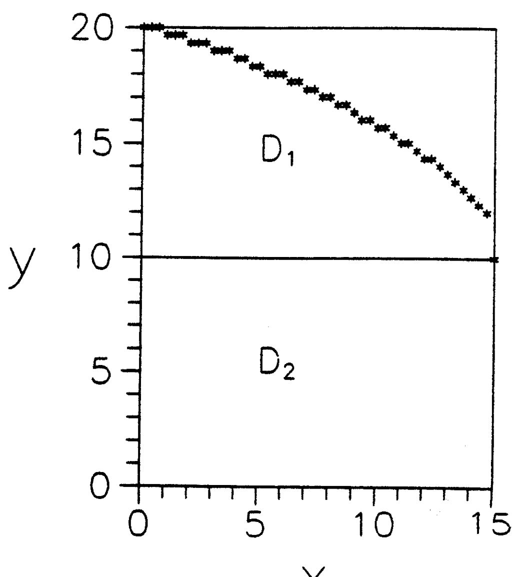
Results:

The numerical example problem investigated used the following data:

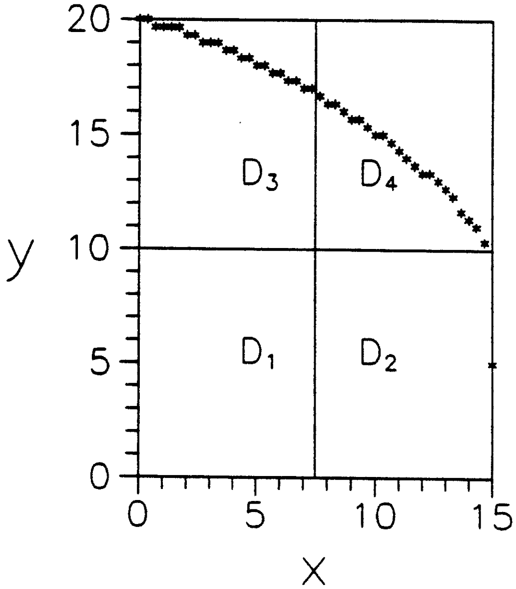
$$y_1 = 20.0,\ y_2 = 10.0 \text{ or } 5.0,\ x_1 = 15.0,\ \overline{\alpha} = 1.85,\ \varepsilon = 0.001,\ \Delta x = \Delta y = 1/3.$$

The first case analyzed is that shown where $y_2 = 10.0$, and the extended solution domain

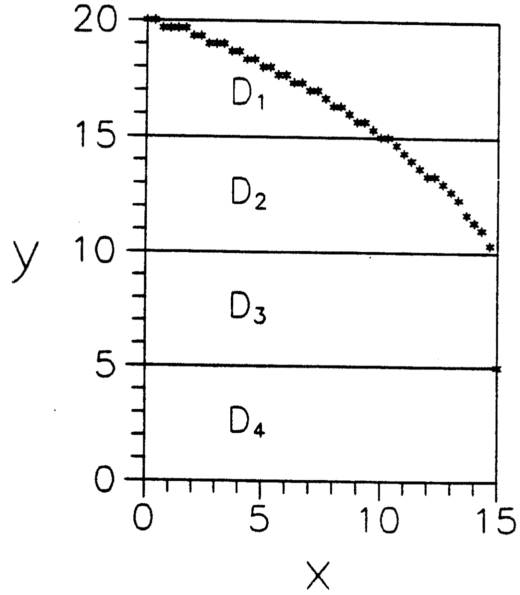
$$D = \{(x,y) \mid 0 < x < 15.0,\ 0 < y < 20.0\}$$
$$D_1 = \{(x,y) \mid 0 < x < 15.0,\ 10 < y < 20.0\},$$
$$D_2 = \{(x,y) \mid 0 < x < 15.0,\ 0 < y < 10.0\}$$
$$\Gamma = \{(x,y) \mid 0 < x < 15.0,\ y = 10.0\}.$$



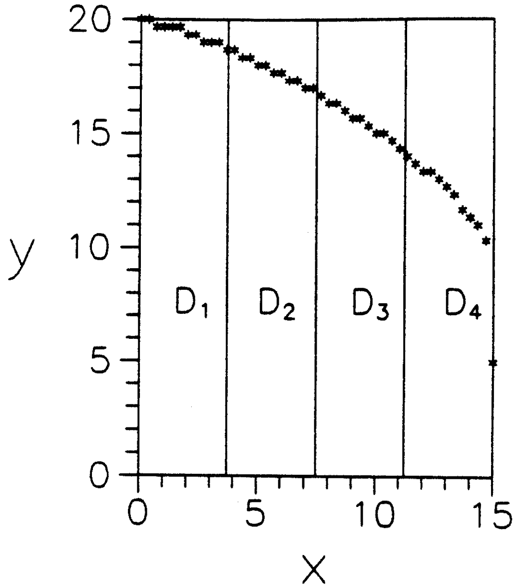
Horizontal decomposition.



Block decomposition.



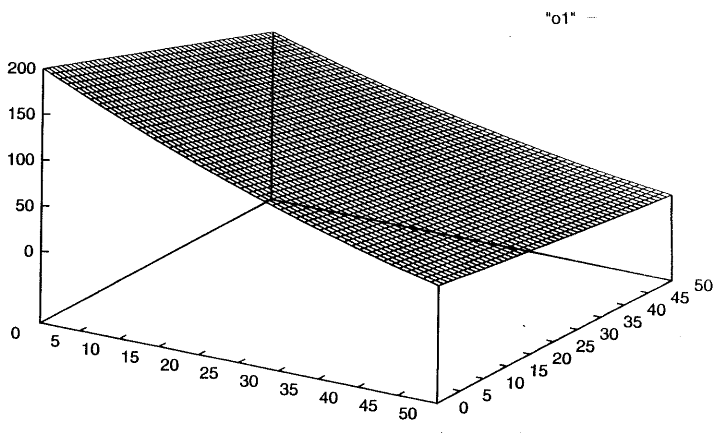
Horizontal decomposition.



Vertical decomposition.

no. proc.	c	r	no. ite.	time(sec)	S_p	e_p
1	1	1	180	494.49	1	1
2	2	1	179	255.97	1.93	0.97
2	1	2	181	254.54	1.94	0.97
4	4	1	177	132.42	3.73	0.93
4	2	2	180	133.46	3.71	0.93
4	1	4	183	130.48	3.79	0.95
8	8	1	166	64.18	7.70	0.96
8	4	2	178	70.07	7.06	0.88
8	2	4	182	71.35	6.93	0.87
8	1	8	189	75.25	6.57	0.82
16	16	1	163	33.10	14.94	0.93
16	8	2	167	34.50	14.33	0.90
16	4	4	180	39.02	12.67	0.79
16	2	8	188	41.62	11.88	0.74
16	1	16	194	41.15	12.02	0.75
32	32	1	166	19.07	25.93	0.81
32	16	2	163	18.43	26.83	0.84
32	8	4	169	19.31	25.61	0.80
32	4	8	186	21.68	22.81	0.71
32	2	16	193	22.54	21.94	0.69
32	1	32	207	25.19	19.63	0.61

Results for (201,201) grids, α=1.96.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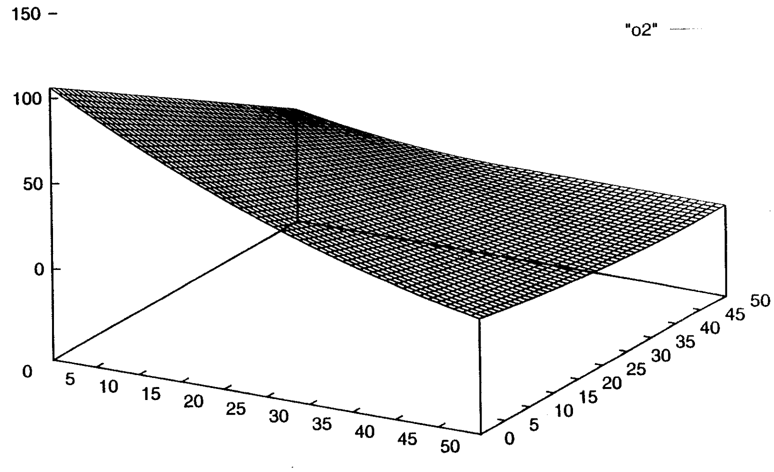
200
150
100
50
0
0
5 10 15 20 25 30 35 40 45 50
0 5 10 15 20 25 30 35 40 45 50


"o2"
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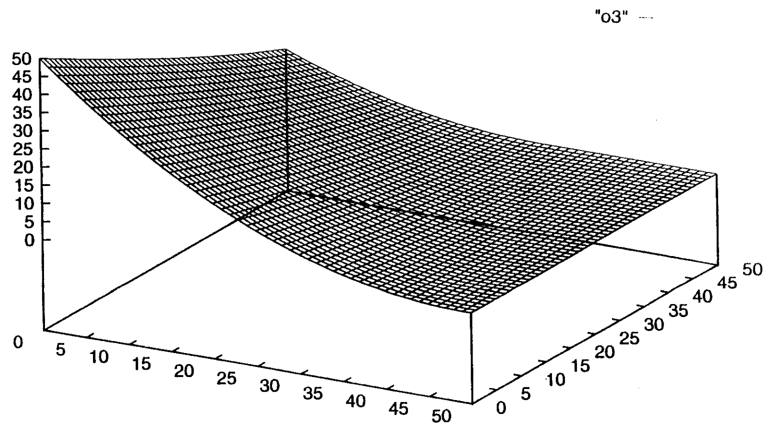
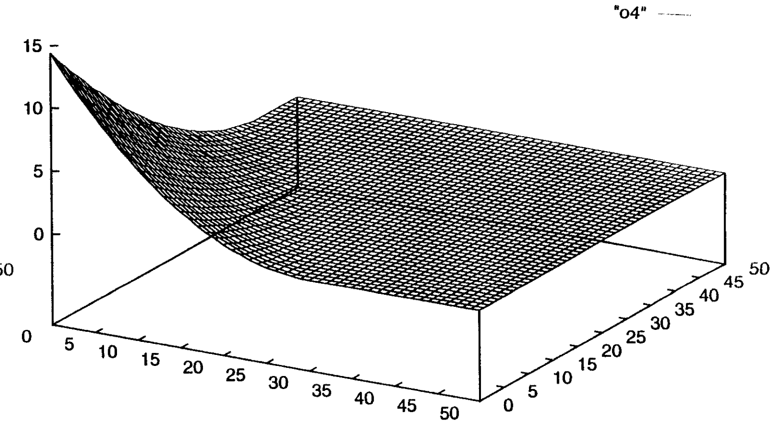
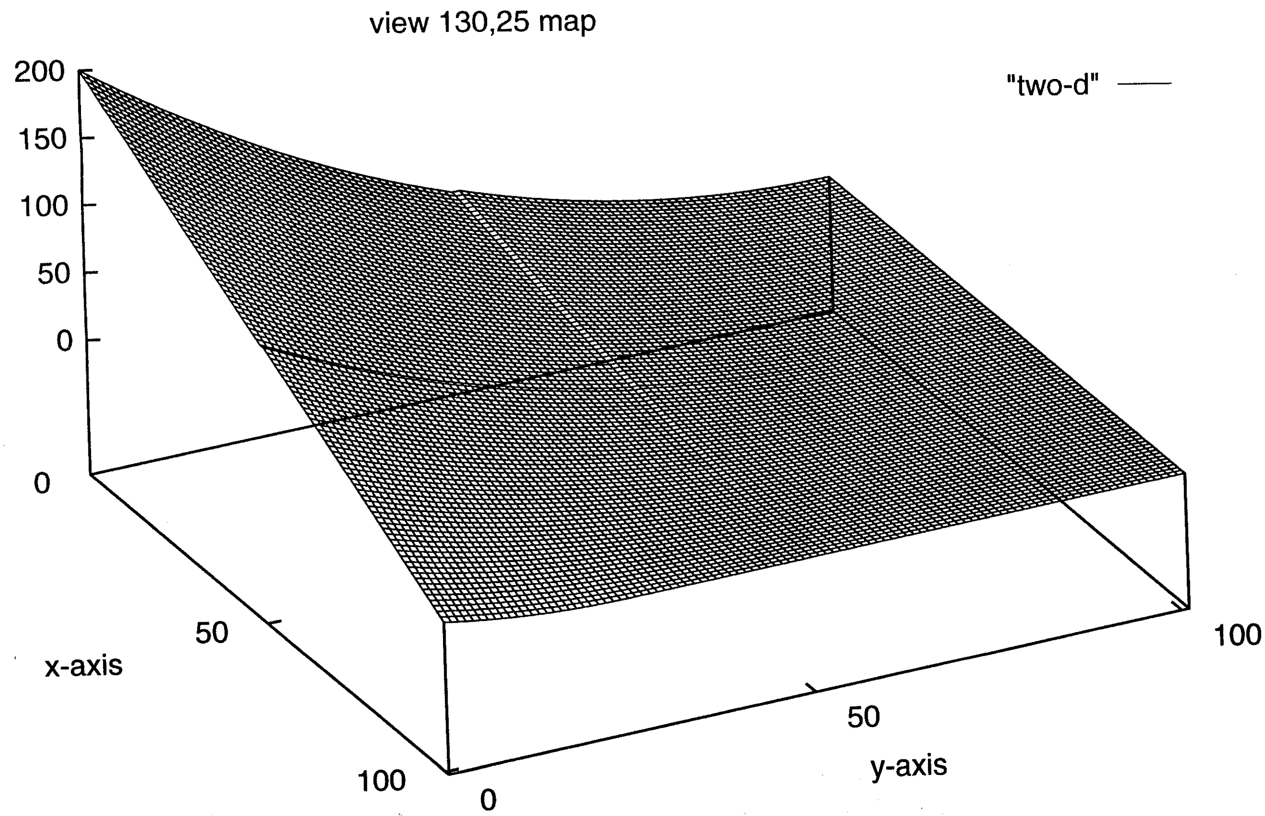
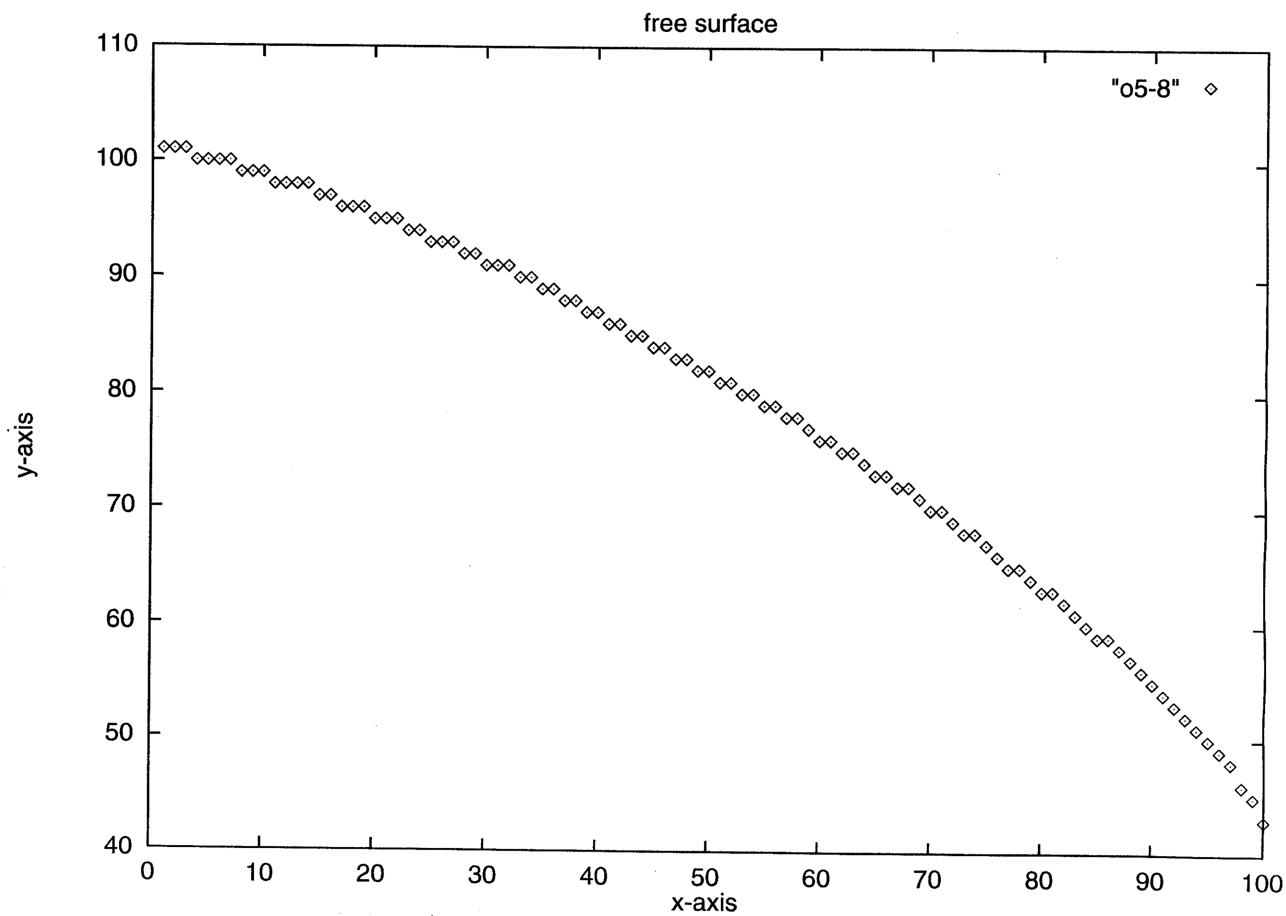
100
50
0
0
5 10 15 20 25 30 35 40 45 50
0 5 10 15 20 25 30 35 40 45 50


"o3"
50 45 40 35 30 25 20 15 10 5 0
0
5 10 15 20 25 30 35 40 45 50
0 5 10 15 20 25 30 35 40 45 50


"o4"
15
10
5
0
0
5 10 15 20 25 30 35 40 45 50
0 5 10 15 20 25 30 35 40 45 50



view 130,25 map
"two-d"
200
150
100
50
0
0
50
x-axis
100
0
50
y-axis
100



free surface
"o5-8"
110
100
90
80
70
60
50
40
y-axis
0
10
20
30
40
50
60
70
80
90
100
x-axis