Adaptive mesh finite element analysis of free surface seepage through a dam

Physical Problem:

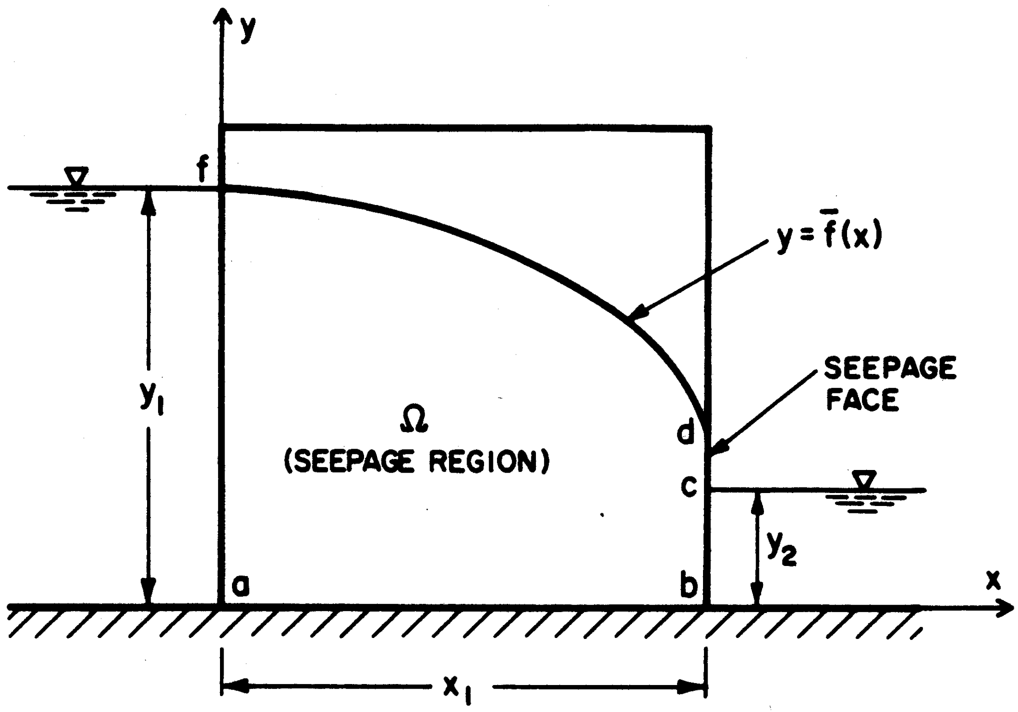


Figure 1: Seepage through a rectangular dam.

Assumptions:

1. Soil in the flowfield is homogeneous and isotropic
2. Capillary and evaporation effects are neglected
3. Flow obeys Darcy's Law
4. Two-dimensional
5. Steady state

Mathematical Formulation:

Darcy's Law: $\vec{q} = -K \text{ grad } h = -K \text{ grad } [(p/\rho g) + y]$

Potential: $\varphi = k[(p/\rho g) + y]$

$\therefore$ Velocity Components: $u = \varphi_x$, $v = \varphi_y$

Continuity Equation: $u_x + v_y = 0$

Irrotationality Condition: $u_y - v_x = 0$

Cauchy Riemann Equation: $\varphi_x = \psi_y$, $-\varphi_y = \psi_x$

Laplace's Equations: $\nabla^2 \varphi = 0$, $\nabla^2 \psi = 0$

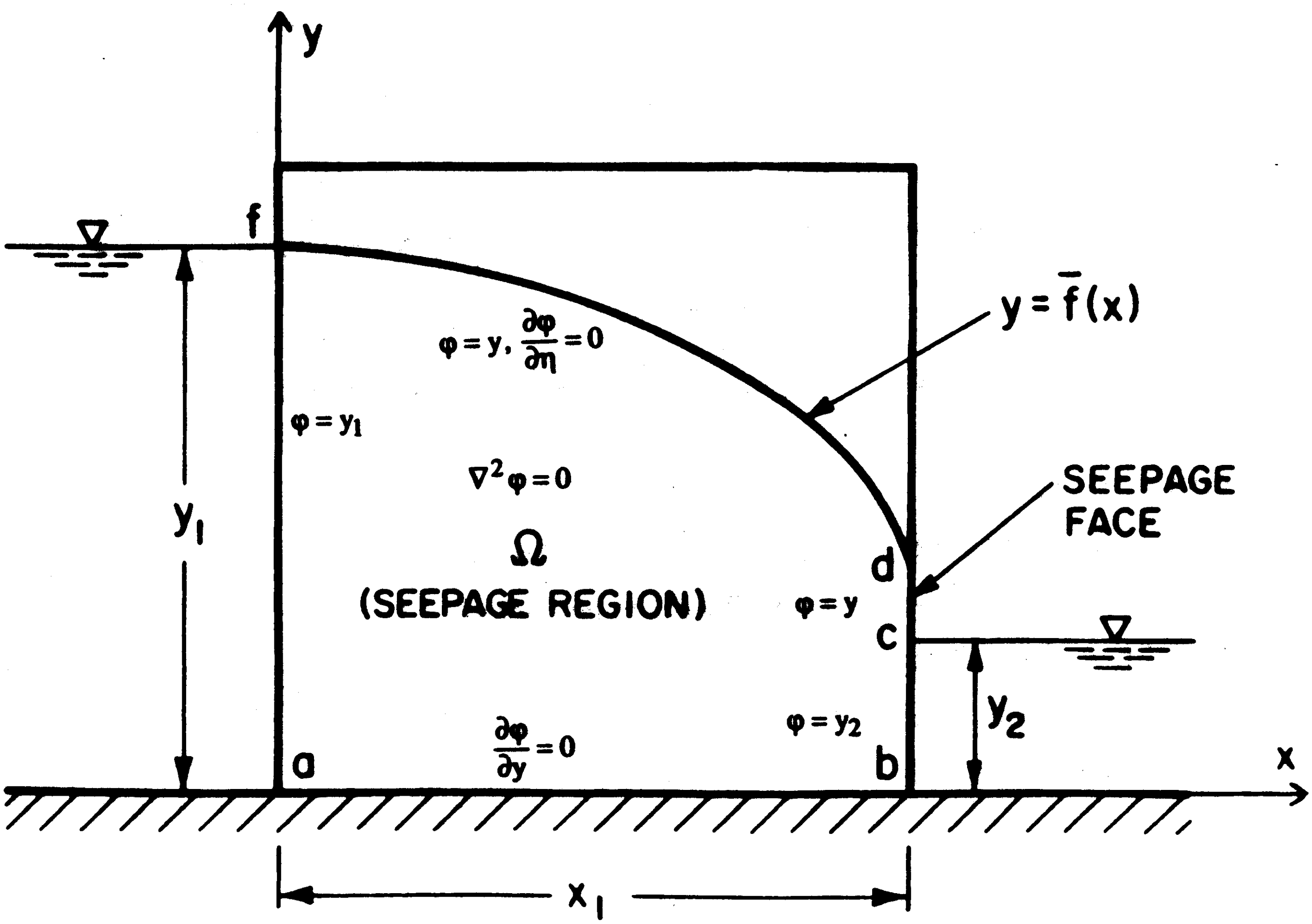


Figure 2: Mathematical formulation of physical problem

Fixed Domain Formulation:

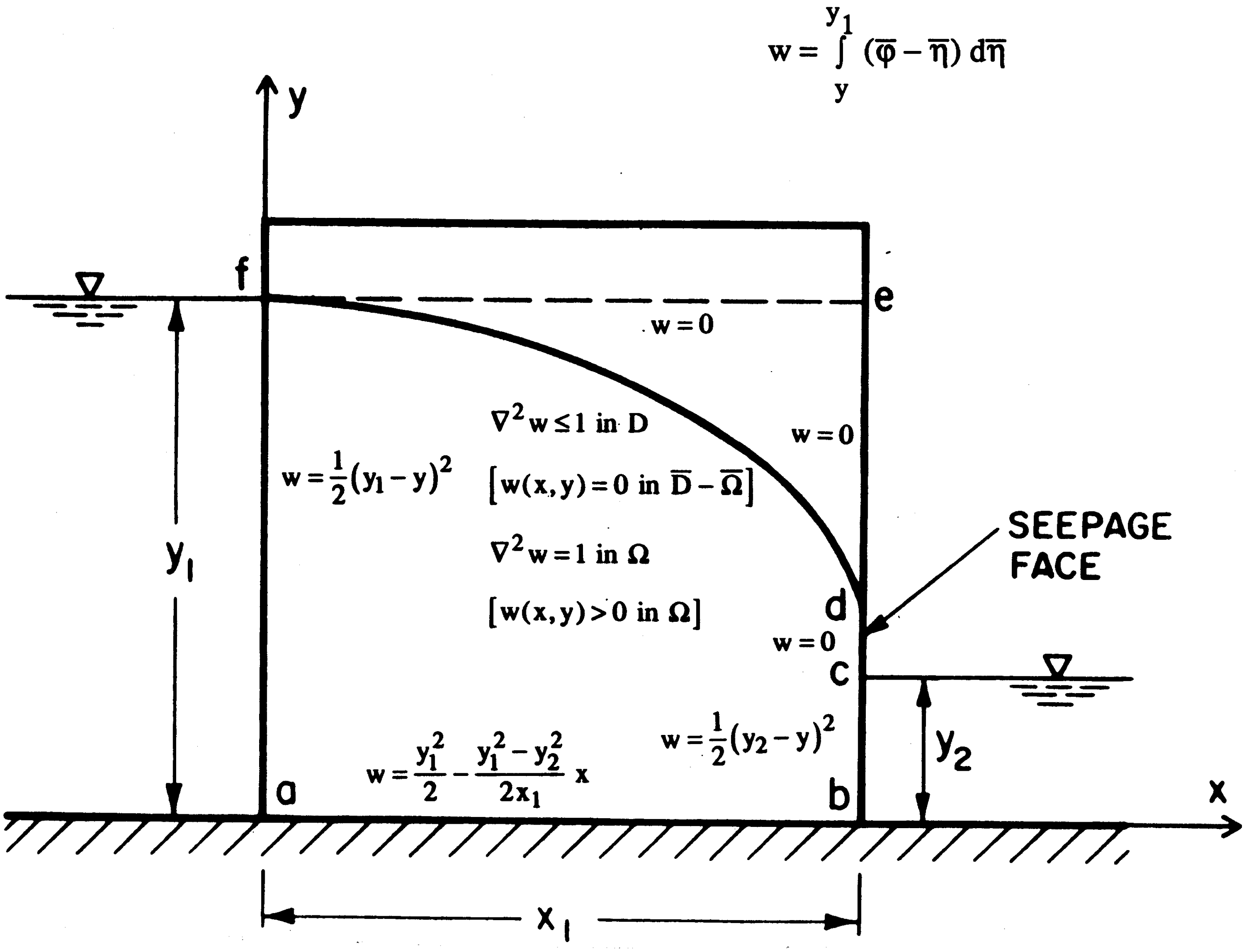


Figure 3: Fixed domain mathematical formulation.

Numerical Algorithm:

A **minimization problem** can be formulated in terms of the functional

$$J(u) = a(u, u) + 2(f, u) \quad , \qquad u \in K,$$

where a is a **bilinear** form, **continuous, symmetric, positive definite** on R and $f \in R$, i.e.,

$$a(u, u) = \iint_D \nabla u \cdot \nabla u \, dx \, dy,$$

$$(f, u) = \iint_D fu \, dx \, dy.$$

The functional J has **one and only one minimum** in a **closed convex set.** The minimum is found using the following algorithm:

$$u_i^{(n+1/2)} = -\frac{1}{a_{ii}}\left(\sum_{j=1}^{i-1} a_{ij}\, u_j^{(n+1)} + \sum_{j=i+1}^{N} a_{ij}\, u_j^{(n)} + f_i \right) ,$$

$$u_i^{(n+1)} = P_i\left(u_i^{(n)} + \alpha\left(u_i^{(n+1/2)} - u_i^{(n)}\right)\right) = \max\left(0, u_i^{(n)} + \alpha\left(u_i^{(n+1/2)} - u_i^{(n)}\right)\right),$$

where $a_{ij} = a(N_i, N_j)$, $f_i = (f, N_i)$, N_i is the canonical basis of R^N, P_i is the projection on the convex set, $i = 1, \ldots, N$, and N is the number of nodal points.

FINITE ELEMENT ERROR ANALYSIS

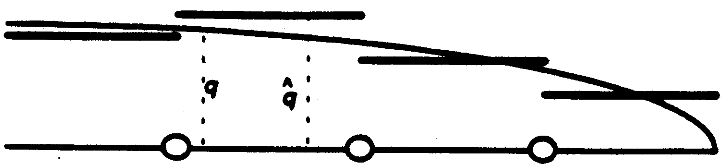
Adaptive Mesh Finite Element Analysis (FEA)

General Equation for FEA:

$$K\bar{u} = f$$

Error Estimate

Error Definition:

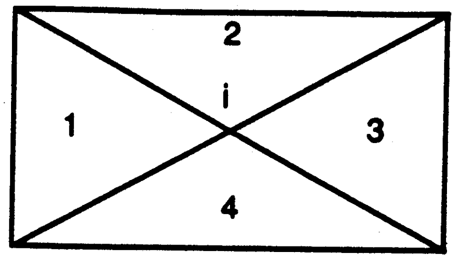


$$e_q = q - \hat{q} = \tilde{q} - SN\bar{u}$$

where $\tilde{q}$ is the approximation of exact solution q;
$\hat{q}$ is the calculated q of an element (constant);
N is the shape function;

and $$S = \begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{bmatrix}$$

Averaging Technique:



$$\bar{q}_i = (\hat{q}_1 + \hat{q}_2 + \hat{q}_3 + \hat{q}_4)/4$$

Error Estimate in An Element

$$e_q = \tilde{q} - \hat{q} = \hat{e}_q \quad \text{where} \quad \tilde{q} = N\bar{q}$$

Error Norm of the Whole Computation Domain

$$\|\hat{e}_q\|_{L2} = \left[\int_R (\hat{e}_q)^T (\hat{e}_q) dR\right]^{\frac{1}{2}}$$

$$\|\hat{e}_q\|_{L2}^2 = \sum_{i=1}^{Ne} \|\hat{e}_q\|_i^2$$

Percentage Error

$$^0\eta = \frac{\|\hat{e}_q\|}{\|\tilde{q}\|} \times 100\% \quad \text{where} \quad \|\tilde{q}\|^2 = \int_R q^2 dR = \sum_{i=1}^{Ne} \|\tilde{q}\|_i$$

Local Mesh Refinement

Desired Criteria

$$^{0}\eta \leq \eta_{max} \quad \text{where } \eta_{max} \text{ is the desired error}$$

Desired Local Error Criteria

$$\|\hat{e}_q\|_i \leq \eta_{max}\left[\frac{\|\tilde{q}\|^2}{Ne}\right]^{\frac{1}{2}} = e_{max}$$

where e_{max} is the maximum allowable element error.

Error Ratio

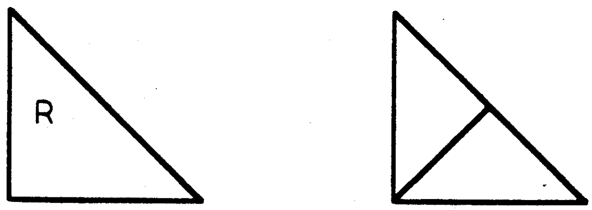
$$\xi_i = \frac{\|\hat{e}_q\|_i}{e_{max}}, \quad \xi_i > 1 \quad \text{refine the element}$$

New Element Size

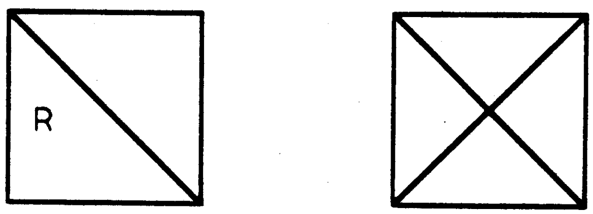
$$[A_i]_{new} \leq \frac{A_i}{\xi_i}$$

Mesh Refinement

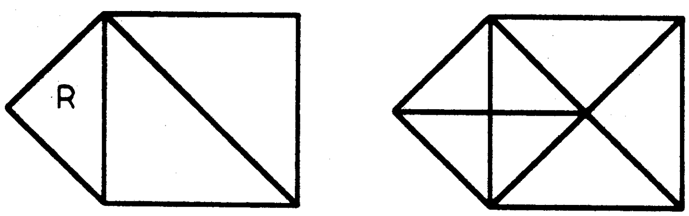
(a) refine one element



(b) refine one element and it's neighbor element



(c) refine elements recursively



Results:

pass	1	2	3	4
no. of nodes	297	549	1427	2968
η_{max}	0.07	0.05	0.03	0.02
${}^{0}\eta$	0.08	0.06	0.036	0.0197
time (sec)	4.86	25.65	175.14	691.58

Table 1. Results for Case 1.

pass	1	2	3
no. of nodes	297	1893	4604
η_{max}	0.03	0.03	0.03
${}^{0}\eta$	0.08	0.031	0.0184
time (sec)	4.87	305.01	

Table 2. Results for Case 2.

pass	1	2	3	4
no. of nodes	149	446	976	2563
η_{max}	0.09	0.06	0.03	0.02
${}^{0}\eta$	0.13	0.0681	0.0385	0.022
time (sec)	2.58	27.61	124.36	794.96

Table 3. Results for Case 3.

pass	1	2	3
no. of nodes	149	1561	4598
η_{max}	0.03	0.03	0.03
${}^{0}\eta$	0.13	0.0359	0.0186
time (sec)	2.58	304.85	

Table 4. Results for Case 4.

pass	1	2	3	4	5
no. of nodes	149	297	553	2129	4257
η_{max}	0.03	0.03	0.03	0.03	0.03
${}^{0}\eta$	0.13	0.08	0.065	0.032	0.019
time (sec)	2.58	4.87	22.59	220.30	322.78

Table 5. Results for Case 5.

pass	1	2	3	4
1 proc.	70	89	99	87
2 proc.	70	88	95	85
4 proc.	69	87	95	86
8 proc.	70	83	94	85
16 proc.	71	80	90	82

Table 6. Iteration numbers for Case 1.

pass	1	2	3	4
1 proc.	80	89	88	88
2 proc.	80	89	86	88
4 proc.	79	87	84	88
8 proc.	75	86	83	87
16 proc.			77	84

Table 7. Iteration numbers for Case 3.

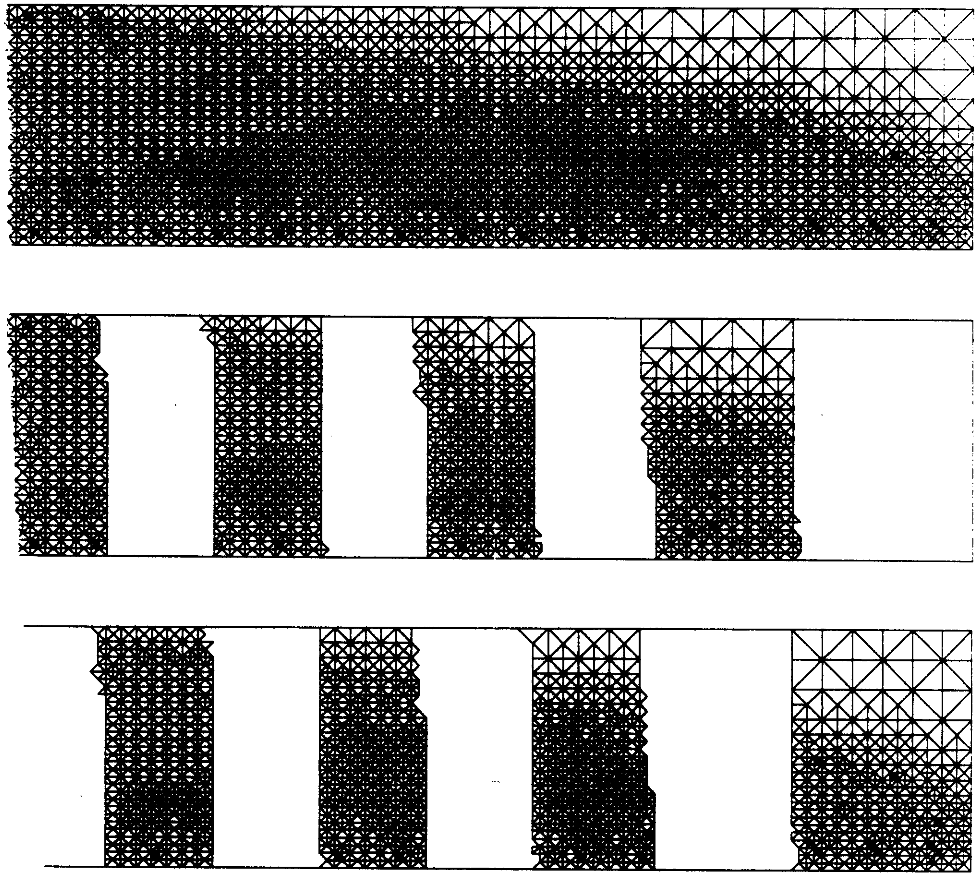


Figure 4: Domain Subdividing for Pass 4 of Case 1.

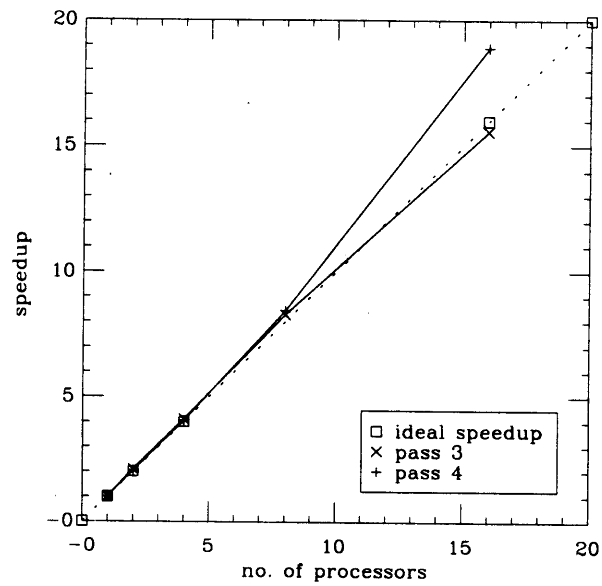
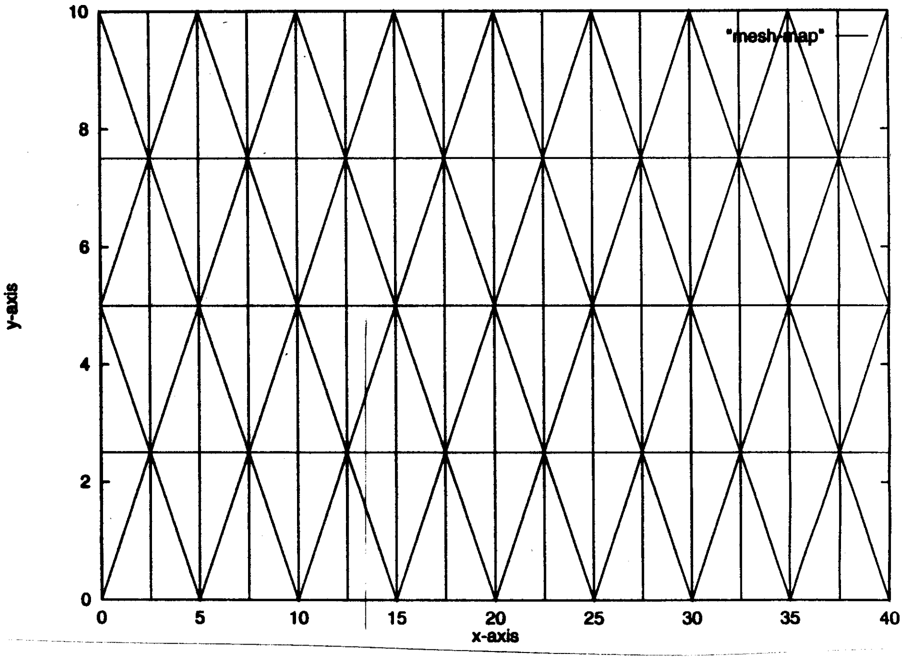


Figure 5: Speedup for Case 1.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8
10
5
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25
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35
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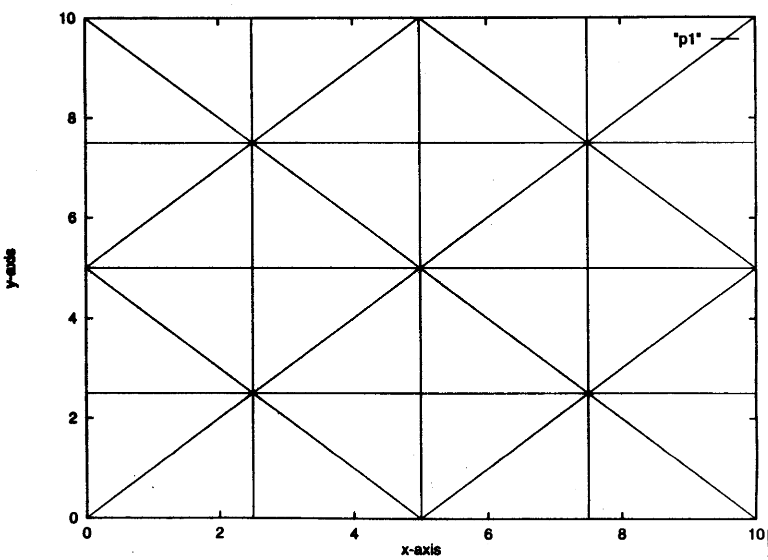
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x-axis
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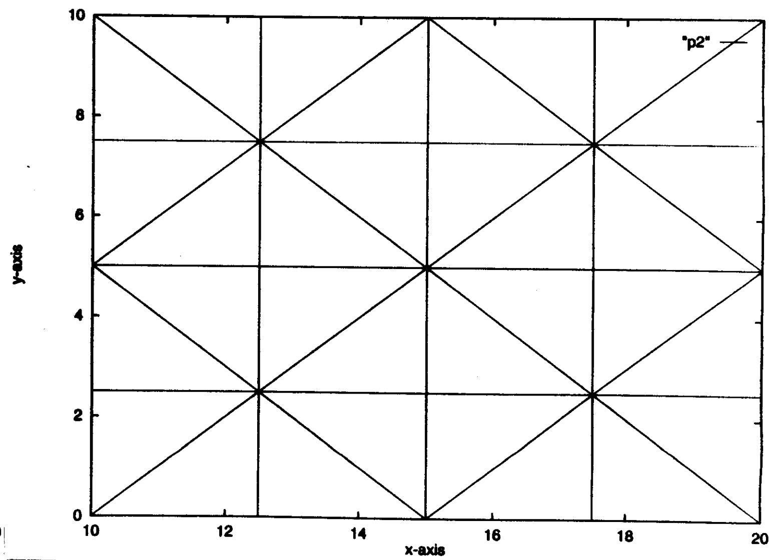
10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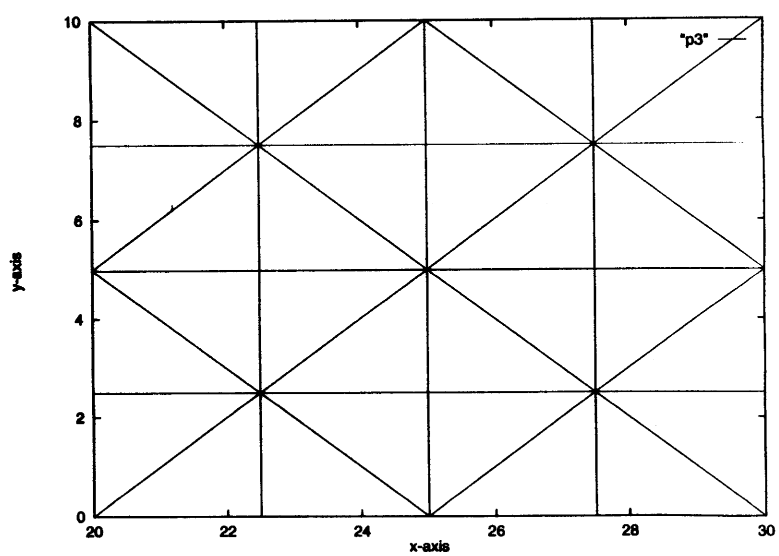
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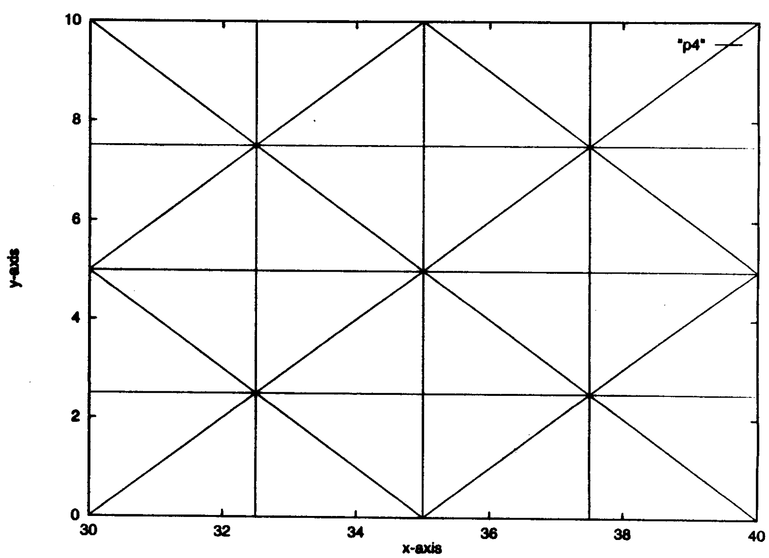
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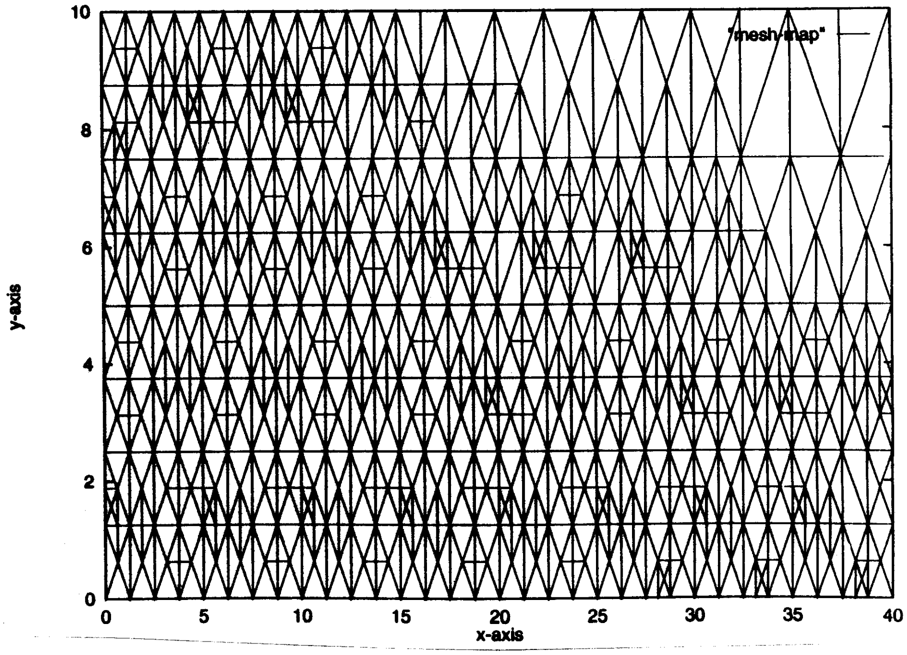
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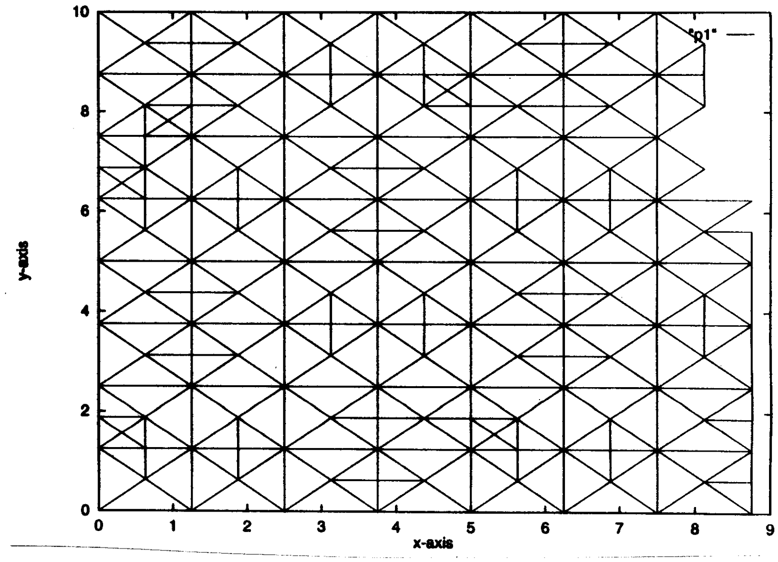
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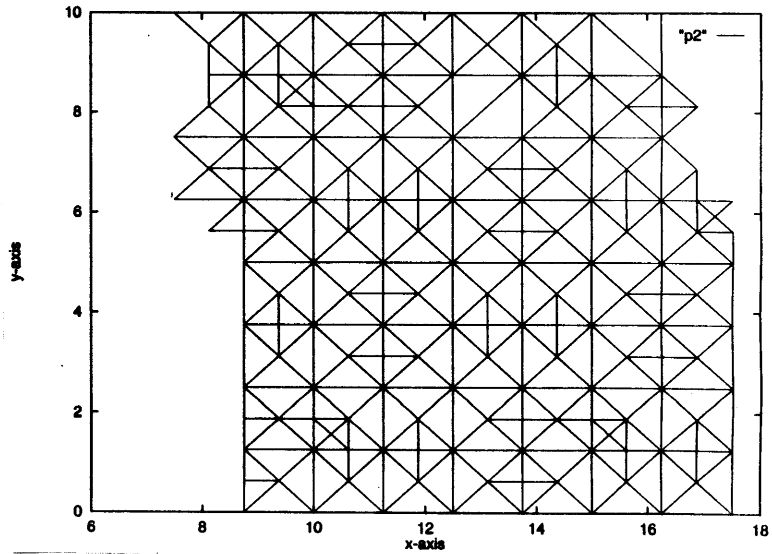
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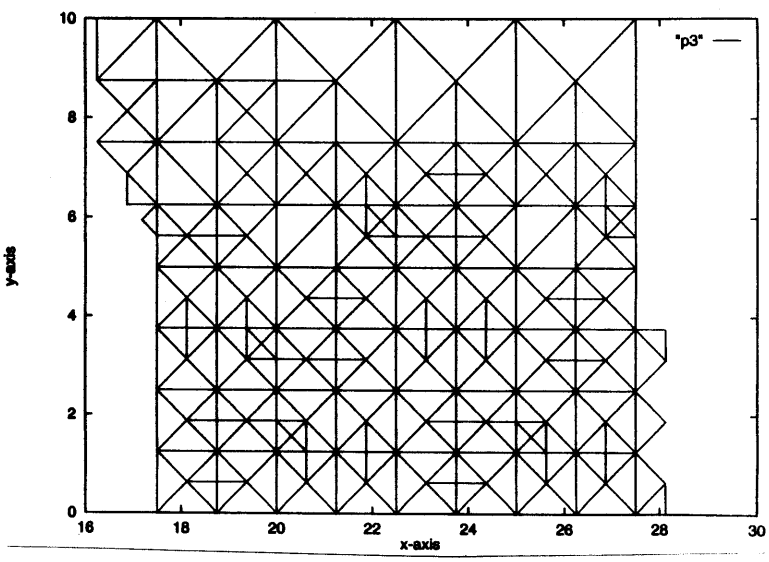
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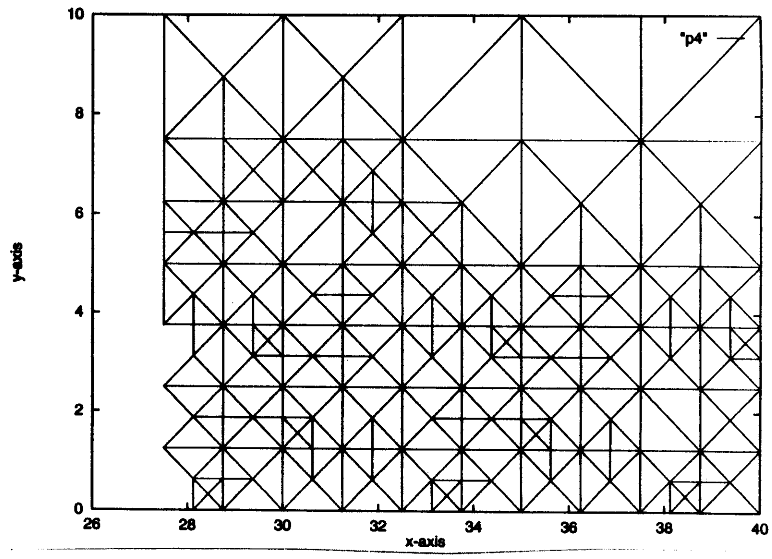
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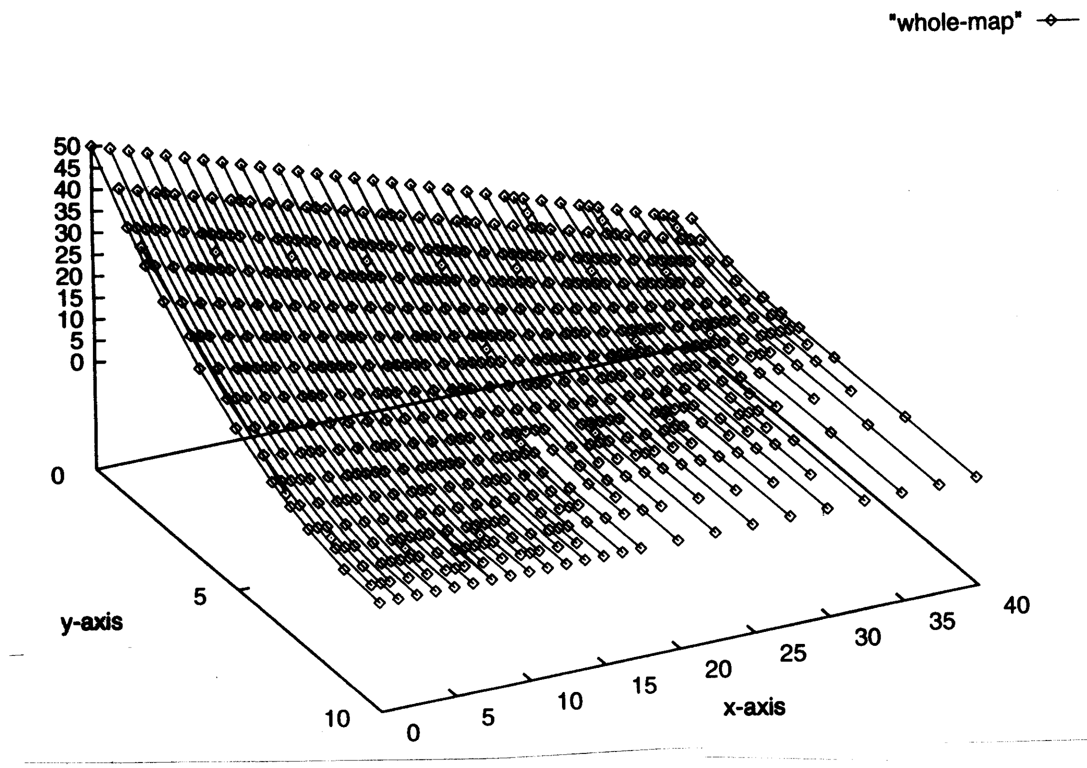
y-axis
x-axis

whole map - four processors



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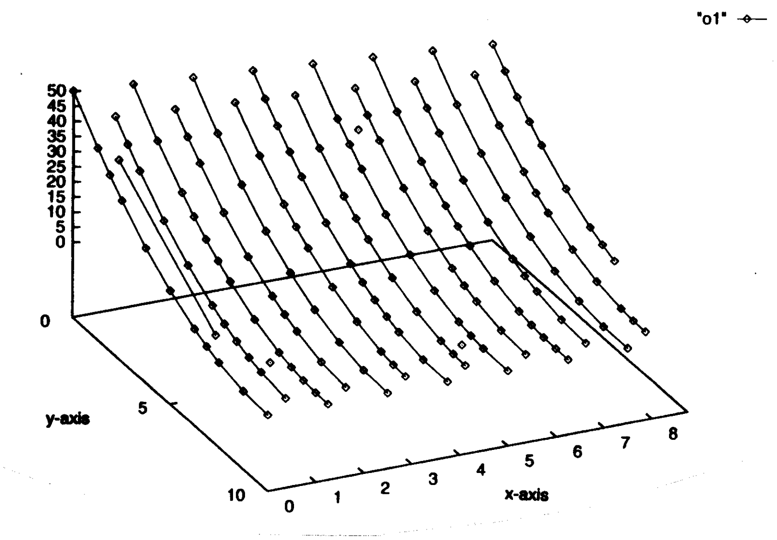
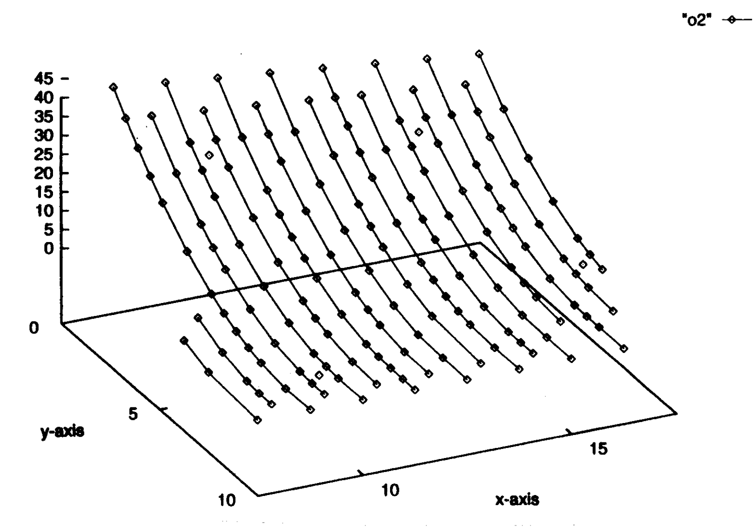
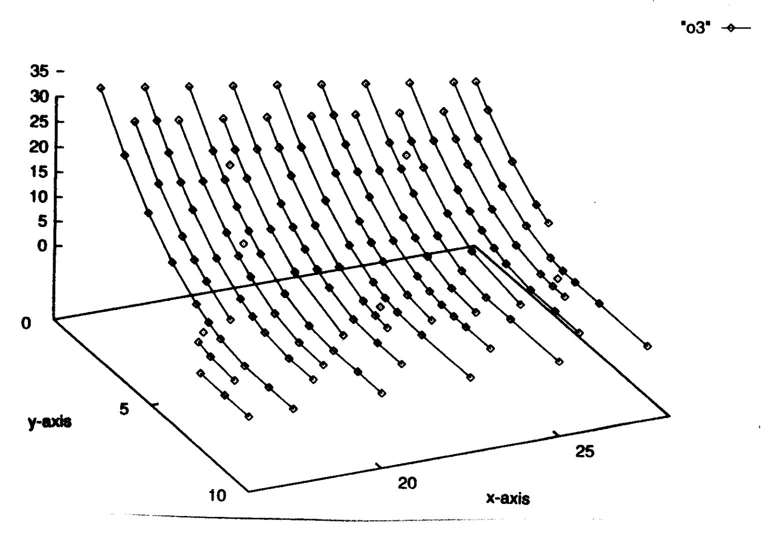


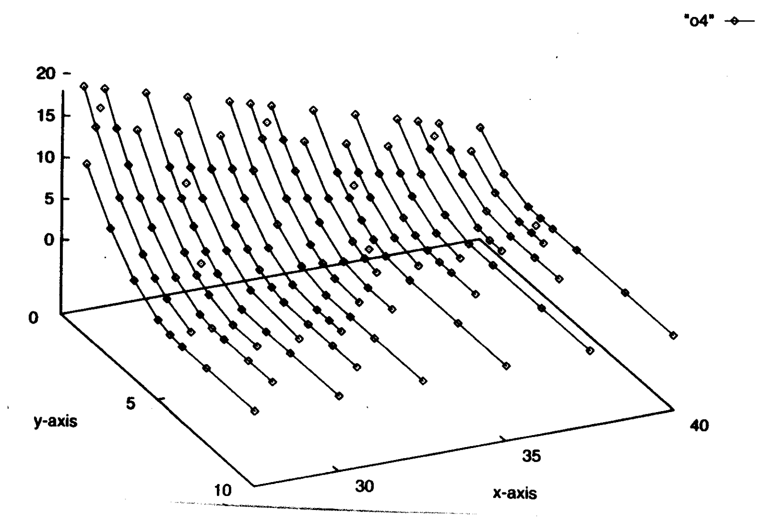
o2 - processor nr. 2

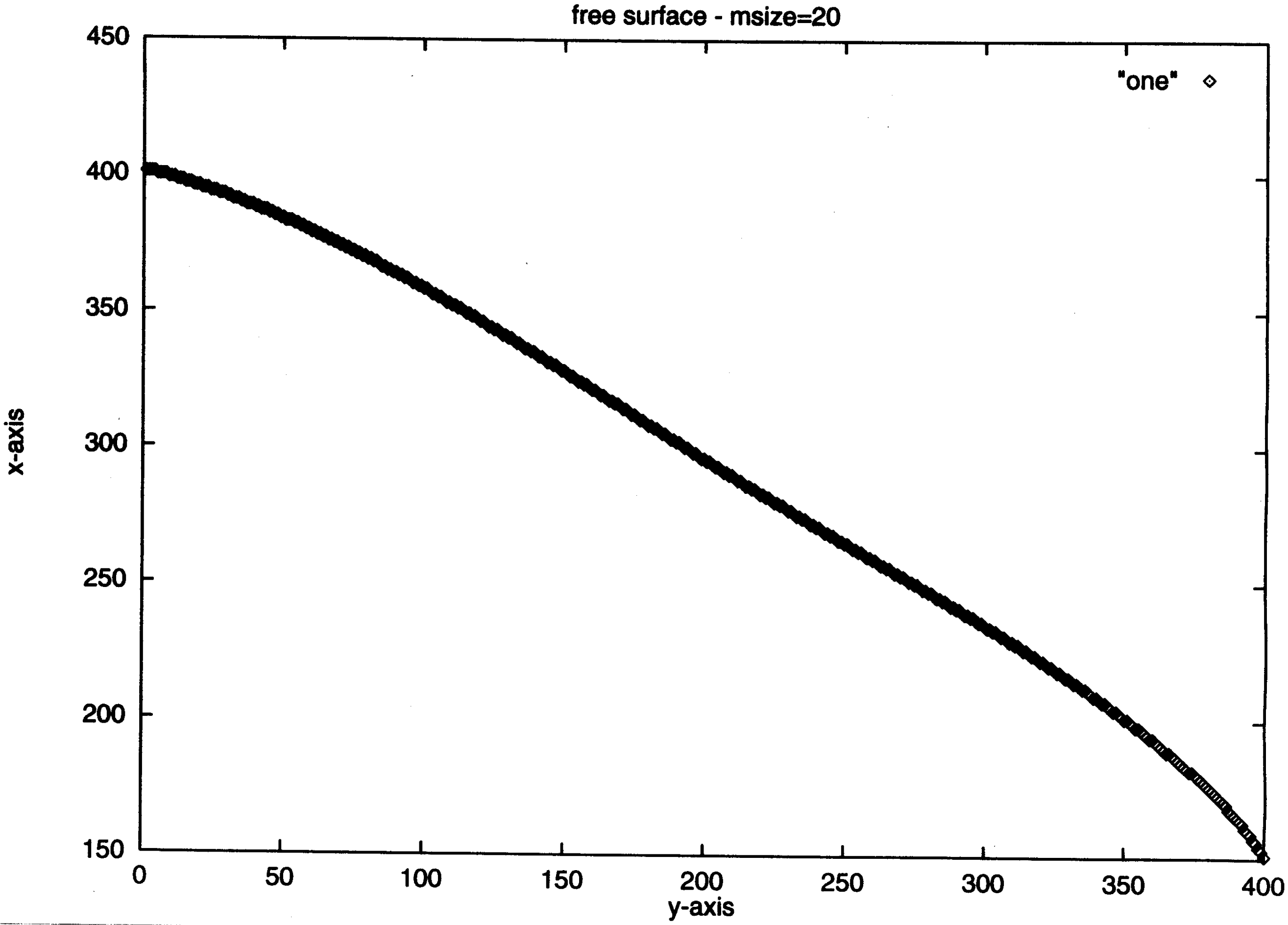


o3 - processor nr. 3



o4 - processor nr. 4

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y-axis
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