MODULE

for

Plane Stress

and

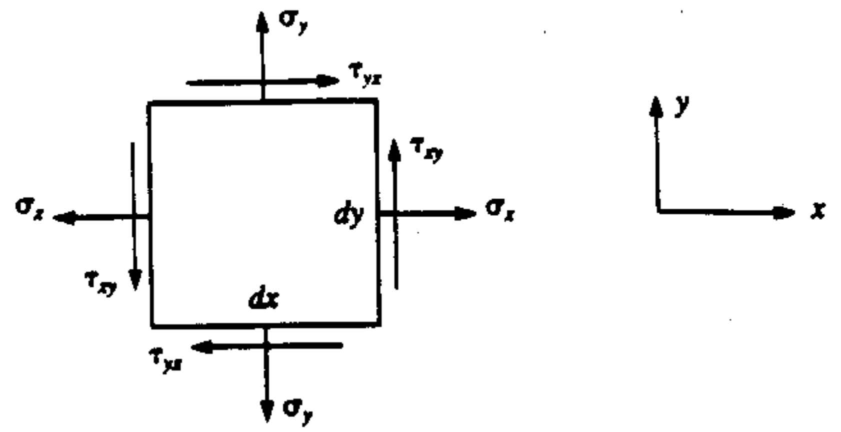
Plane Strain

Analysis

TWO-DIMENSIONAL ELASTICITY

Many problems in elasticity may be treated satisfactorily by a **two-dimensional**, or *plane theory of elasticity*. There are two general types of problems involved in this plane analysis, **plane stress** and **plane strain**. These two types will be defined by setting down certain restrictions and assumptions on the stress and displacement fields. They will also be introduced descriptively in terms of their physical prototypes.

The two-dimensional state of stress is illustrated below where σ_x and σ_y are normal stresses and τ_{xy} and τ_{yx} are the shear stresses.



As seen three independent stresses, σ_x, σ_y and τ_{xy}, exist which can be written as the following stress vector:

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

From these stresses the maximum and minimum normal stresses, the principal stresses, in the two-dimensional plane are

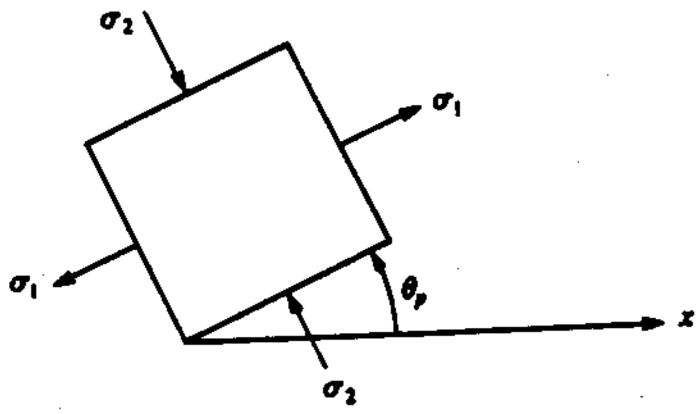
$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sigma_{max}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sigma_{min}$$

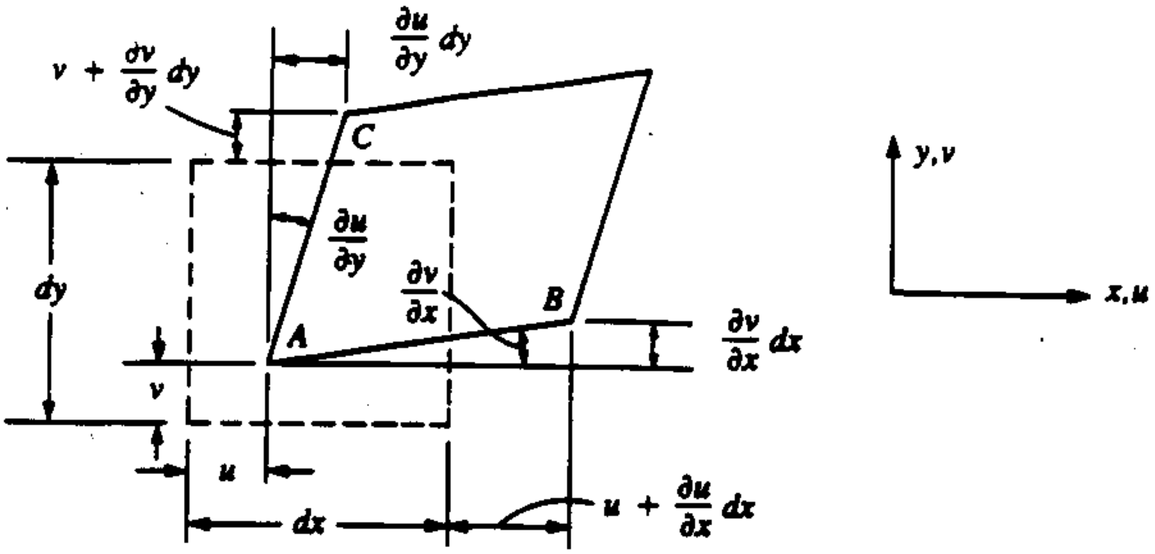
and the principle angle is

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

The principal stresses and their directions are shown below



The general two-dimensional state of strain at some point in a structure is represented by the shown infinitesimal element, dxdy, where u and v are the x and y-displacements at point A, respectively, and lines AC and AB have been extended and displaced.



The normal or (ex-tensional or longitudinal) strains are defined as

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial v}{\partial y}$$

and the shear strain is

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

These can be written as the strain vector

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

The basic partial differential equations for plane elasticity including body and inertia forces are

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + X = \rho \frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + Y = \rho \frac{\partial^2 v}{\partial t^2}$$

where X and Y denote the body forces per unit volume in the x and y-directions, respectively, and ρ is the density of the material.

Plane Stress

Plane stress *is defined to be a state of stress in which the normal stress,* σ_z *, and the shear stresses,* σ_{xz} *and* σ_{yz}*, directed perpendicular to the x-y plane are assumed to be zero.*

The geometry of the body is essentially that of a plate with one dimension much smaller than the others. The loads are applied uniformly over the thickness of the plate and act in the plane of the plate as shown. The plane stress condition is the simplest form of behavior for continuum structures and represents situations frequently encountered in practice.

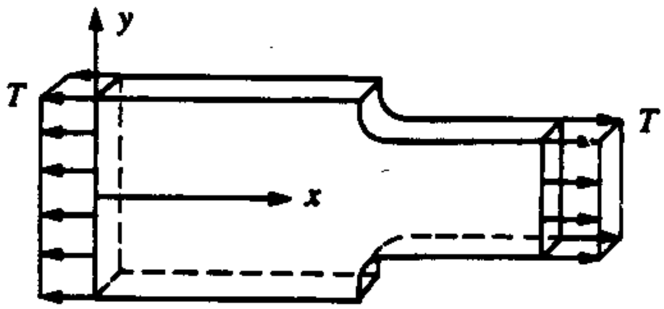


plate with fillet

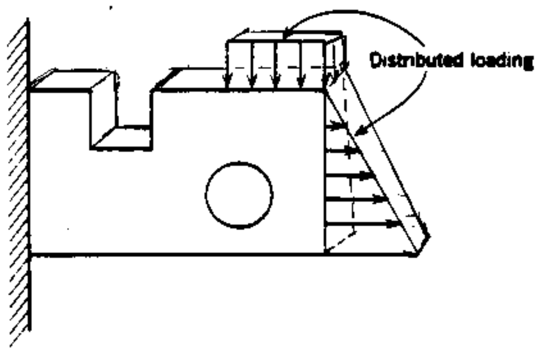
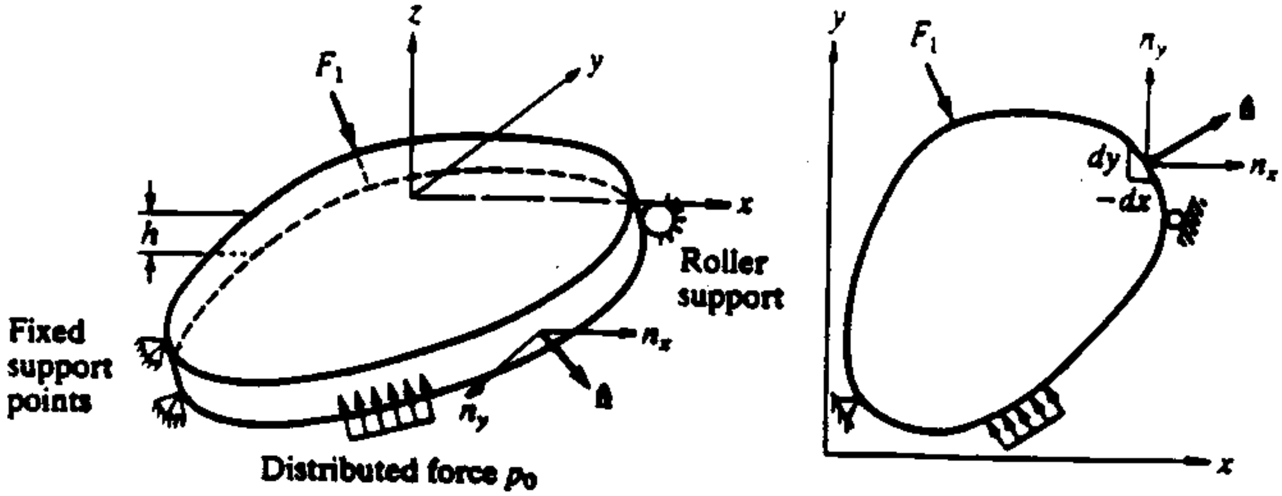


plate with hole

Typical **loading** and **boundary conditions** for plane stress problems in two-dimensional elasticity.

a) Loadings may be point forces or distributed forces applied over the thickness of the plate.

b) Supports may be fixed points or fixed edges or roller supports.



For isotropic materials and assuming

$$\sigma_z = \tau_{xz} = \tau_{yz} = 0$$

and

$$\gamma_{xz} = \gamma_{yz} = 0,$$

yields

$$\{\sigma\} = [D]\ \{\varepsilon\}$$

where

$$[D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

in which [D] is the stress/strain matrix (or constitutive matrix), E is the modulus of elasticity and ν is Poisson's ratio.

The strains in plane stress then are

$$\{\varepsilon\} = [C]\ \{\sigma\}$$

or
$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix}\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix}$$

where $[C]^{-1} = [D]$. Also

$$\varepsilon_y = \frac{1}{E}(-\nu)\left(\sigma_x + \sigma_y\right)$$

The basic partial differential equations for plane stress including body and inertia forces are

$$G\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+G\frac{1+\nu}{1-\nu}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+X=\rho\frac{\partial^2 u}{\partial t^2},$$

$$G\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)+G\frac{1+\nu}{1-\nu}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+Y=\rho\frac{\partial^2 v}{\partial t^2}.$$

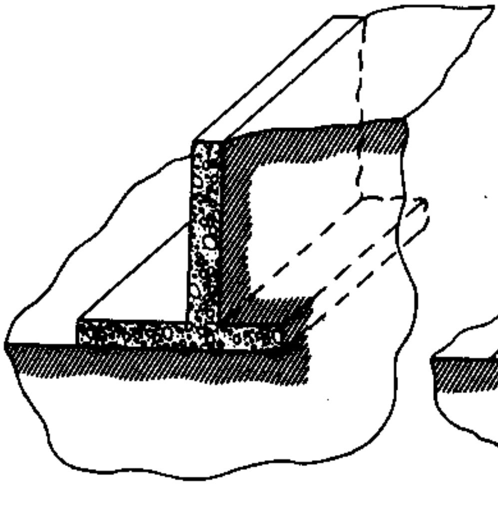
where the shear modulus G is defined as

$$G=\frac{E}{2(1+\nu)}$$

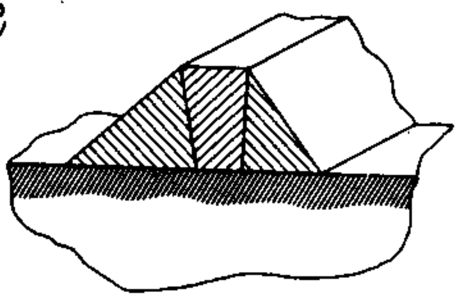
Plane Strain

Plane strain *is defined to be a state of strain in which the strain normal to the x-y plane,* ε_z *, and the shear strain* γ_{xz} *and* γ_{yz} *, are assumed to be zero.*

In plane strain, one deals with a situation in which the dimension of the structure in one direction, say the z-coordinate direction, is very large in comparison with the dimensions of the structure in the other two directions (x-and y-coordinate axes), the geometry of the body is essentially that of a prismatic cylinder with one dimension much larger than the others. The applied forces act in the x-y plane and do not vary in the z direction, i.e. the loads are uniformly distributed with respect to the large dimension and act perpendicular to it. Some important practical applications of this representation occur in the analysis of dams, tunnels, and other geotechnical works. Also such small-scale problems as bars and rollers compressed by forces normal to their cross section are amenable to analysis in this way.

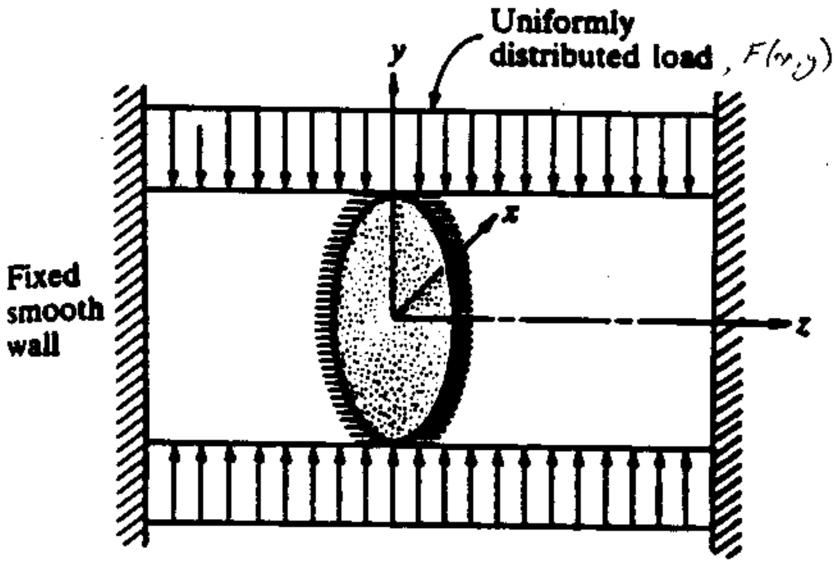


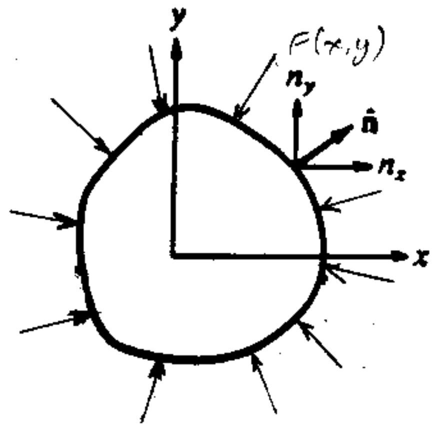
Retaining wall



Earth dam

Typical **loading** and **boundary conditions** for plane strain problems in two-dimensional elasticity.





For isotropic materials and assuming

$$\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$$

and

$$\tau_{xz} = \tau_{yz} = 0$$

yields

$$\{\sigma\} = [D]\ \{\varepsilon\}$$

$$[D] = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

with

$$\sigma_z = \frac{E}{1+\nu}\left[\frac{\nu}{1-2\nu}\left(\varepsilon_x + \varepsilon_y\right)\right]$$

The basic partial differential equations for plane strain including body and inertia forces are

$$G\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{1}{1-2\nu} G \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + X = \rho \frac{\partial^2 u}{\partial t^2},$$

$$G\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \frac{1}{1-2\nu} G \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + Y = \rho \frac{\partial^2 v}{\partial t^2}.$$

where

$$G = \frac{E}{2(1+\nu)}$$