Homework 6 — Multiple Choice

ChE 132A

Exercise 5 (Eigenfunctions of d^2/dx^2 under non-Dirichlet boundary conditions).

Adapted from Graham and Rawlings (2022).

The handout works out the eigenpairs of $L = d^2/dx^2$ on $[0, 1]$ under homogeneous Dirichlet boundary conditions $u(0) = u(1) = 0$, recovering the sine basis. Here we change the boundary conditions and discover two more bases.

(a) **Neumann–Neumann.** For $u'' + \lambda u = 0$ with $\lambda > 0$, the general solution is $u = c_1 \sin(\sqrt{\lambda}\, x) + c_2 \cos(\sqrt{\lambda}\, x)$. The condition $u'(0) = 0$ forces

- (A) $c_2 = 0$
- (B) $c_1 = 0$
- (C) $c_1 = c_2$
- (D) $\sqrt{\lambda} = 0$

(b) The remaining condition $u'(1) = 0$ requires $\sin(\sqrt{\lambda}) = 0$, i.e., $\sqrt{\lambda} = n\pi$. Including the $\lambda = 0$ case (where $u = \text{const}$ satisfies both Neumann conditions), the Neumann–Neumann eigenpairs are

- (A) $\lambda_n = (n\pi)^2,\ u_n = \sin(n\pi x),\ n = 1, 2, 3, \ldots$
- (B) $\lambda_n = (n\pi)^2,\ u_n = \cos(n\pi x),\ n = 0, 1, 2, \ldots$
- (C) $\lambda_n = ((n + \frac{1}{2})\pi)^2,\ u_n = \cos((n + \frac{1}{2})\pi x),\ n = 0, 1, 2, \ldots$
- (D) $\lambda_n = (n\pi)^2,\ u_n = \cos(n\pi x),\ n = 1, 2, 3, \ldots$

Note that $n = 0$ now contributes a nontrivial eigenfunction (the constant), in contrast to the Dirichlet case.

(c) **Dirichlet–Neumann (mixed).** For $u'' + \lambda u = 0$ with $u(0) = 0$ and $u'(1) = 0$, the condition $u(0) = 0$ kills the cosine, leaving $u = c_1 \sin(\sqrt{\lambda}\, x)$. The condition $u'(1) = 0$ then requires

- (A) $\sin(\sqrt{\lambda}) = 0$, so $\sqrt{\lambda} = n\pi$
- (B) $\cos(\sqrt{\lambda}) = 0$, so $\sqrt{\lambda} = (n + \frac{1}{2})\pi$
- (C) $\sqrt{\lambda} = 1$
- (D) $\tan(\sqrt{\lambda}) = 1$

(d) Comparing the three problems — Dirichlet–Dirichlet, Neumann–Neumann, and Dirichlet–Neumann — we conclude that

- (A) The basis is determined by the operator L alone
- (B) The basis depends on both the operator and the boundary conditions
- (C) All three problems share the basis $\sin(n\pi x)$
- (D) Only Dirichlet boundary conditions yield orthogonal bases

Exercise 11 (Plane Poiseuille flow).

A Newtonian fluid of viscosity μ flows steadily between two parallel no-slip walls at $y = 0$ and $y = h$ under constant streamwise pressure gradient $G = -dP/dx$. In fully-developed flow the streamwise momentum balance reduces to

$$\mu\, u''(y) = -G, \qquad u(0) = 0,\ u(h) = 0$$

(a) Integrating $\mu u'' = -G$ twice and applying the no-slip BCs gives

(A) $u(y) = \dfrac{G}{2\mu}\, y^2$

(B) $u(y) = \dfrac{G}{2\mu}\, y(h - y)$

(C) $u(y) = \dfrac{G}{\mu}\, y(h - y)$

(D) $u(y) = \dfrac{G}{2\mu}\, (h - y)^2$

(b) The maximum velocity occurs at the centerline $y = h/2$, with value

(A) $u_{\text{max}} = \dfrac{Gh^2}{2\mu}$

(B) $u_{\text{max}} = \dfrac{Gh^2}{4\mu}$

(C) $u_{\text{max}} = \dfrac{Gh^2}{8\mu}$

(D) $u_{\text{max}} = \dfrac{Gh}{2\mu}$

(c) The mean velocity $\bar{u} = (1/h) \int_0^h u(y)\ \mathrm{d}y$ relates to the maximum by

(A) $\bar{u} = u_{\text{max}}$

(B) $\bar{u} = \frac{1}{2}\, u_{\text{max}}$

(C) $\bar{u} = \frac{2}{3}\, u_{\text{max}}$

(D) $\bar{u} = \frac{3}{4}\, u_{\text{max}}$

(d) The volumetric flowrate per unit width is therefore $Q' = \bar{u}\, h =$

(A) $Gh^2/(2\mu)$

(B) $Gh^3/(8\mu)$

(C) $Gh^3/(12\mu)$

(D) Gh^2/μ

This is the famous lubrication-theory cubic dependence on gap thickness.

Exercise 12 (Brinkman flow in a porous channel).

A Newtonian fluid is forced through a slab of porous medium of thickness h and permeability k between two impermeable, no-slip walls by pressure gradient $G = -dP/dx$. At a length scale where the porous structure can be treated as a continuum, the streamwise momentum balance is the Brinkman equation

$$\mu_e\, u''(y) - \frac{\mu}{k}\, u(y) = -G, \qquad u(0) = 0,\; u(h) = 0$$

where μ_e is an effective viscosity capturing macroscopic diffusion of momentum across the porous medium.

(a) Define the inverse length $m = \sqrt{\mu/(k\mu_e)}$ (the reciprocal of the Brinkman screening length) and the Darcy velocity $u_D = Gk/\mu$. Dividing the ODE by μ_e rewrites it as

- (A) $u'' + m^2\, u = -m^2\, u_D$
- (B) $u'' - m^2\, u = -m^2\, u_D$
- (C) $u'' - m\, u = -u_D$
- (D) $u'' = -m^2\, u_D$

(b) This is the rectangular minus-sign equation, so the homogeneous solutions are $\cosh(my)$ and $\sinh(my)$. A particular solution of the inhomogeneous equation is the constant

- (A) $u_p = -u_D$
- (B) $u_p = u_D$
- (C) $u_p = u_D\, y$
- (D) $u_p = u_D\, y(h - y)$

(c) The problem is symmetric about the centerline $y = h/2$ (both BCs and source are unchanged under $y \to h - y$), so the homogeneous correction must be *even* about the centerline. The natural ansatz is

- (A) $A\, \sinh(my)$
- (B) $A\, \sinh(m(y - h/2))$
- (C) $A\, \cosh(m(y - h/2))$
- (D) $A\, \cosh(my) + B\, \sinh(my)$ with two unknowns

(d) Combining with the particular solution, $u(y) = u_D + A\cosh(m(y - h/2))$. Either no-slip BC (the two are equivalent by symmetry) gives $A = -u_D/\cosh(mh/2)$, so

- (A) $u(y) = u_D \left[1 + \dfrac{\cosh(m(y - h/2))}{\cosh(mh/2)}\right]$
- (B) $u(y) = u_D \left[1 - \dfrac{\cosh(m(y - h/2))}{\cosh(mh/2)}\right]$
- (C) $u(y) = u_D\, \dfrac{\sinh(m(y - h/2))}{\sinh(mh/2)}$
- (D) $u(y) = u_D \left[1 - e^{-m\, \min(y,\, h-y)}\right]$

For $mh \ll 1$ this reduces to the plane-Poiseuille parabola; for $mh \gg 1$ it approaches uniform Darcy plug flow with thin viscous boundary layers near each wall.

References

M. D. Graham and J. B. Rawlings. *Modeling and Analysis Principles for Chemical and Biological Engineers*. Nob Hill Publishing, Santa Barbara, CA, 2nd, paperback edition, 2022. 560 pages, ISBN 978-0-9759377-6-1.