Homework 7 — Multiple Choice

ChE 132A

Exercise 1 (Computing area of a rectangle in polar coordinates).

Consider the rectangle of sides a (along x) and b (along y), centered at the origin. The two coordinate axes and the two diagonals partition it into eight congruent right triangles; we compute the area of one and multiply by 8.

(a) For the shaded triangle with vertices at the origin, $(a/2, 0)$, and $(a/2, b/2)$, the apex angle (measured from the x-axis to the diagonal) is

- (A) $\theta_0 = \tan^{-1}(a/b)$
- (B) $\theta_0 = \tan^{-1}(b/a)$
- (C) $\theta_0 = \pi/4$ for any a, b
- (D) $\theta_0 = \sin^{-1}(b/a)$

(b) The right edge of the triangle is the vertical line $x = a/2$, i.e., $r\cos\theta = a/2$. Solving for r gives

- (A) $r_{\max}(\theta) = a/(2\sin\theta)$
- (B) $r_{\max}(\theta) = a/(2\cos\theta)$
- (C) $r_{\max}(\theta) = a/2$
- (D) $r_{\max}(\theta) = (a\cos\theta)/2$

(c) Applying the polar area formula $A_{1/8} = \frac{1}{2}\int_0^{\theta_0} r_{\max}^2(\theta)\,d\theta$ and using $\int \sec^2\theta\,d\theta = \tan\theta$, the one-eighth area evaluates to

- (A) $a^2/8$
- (B) $ab/8$
- (C) $\pi ab/8$
- (D) $ab/(4\pi)$

(d) Multiplying by 8 gives the total area

- (A) $a^2 + b^2$
- (B) πab
- (C) ab
- (D) $2ab$

recovering the elementary result without ever transforming back to Cartesian variables.

Exercise 2 (Derivatives of unit vectors in polar coordinates).

The polar unit vectors $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$ depend only on the polar angle θ, not on the radial coordinate r.

(a) Moving along the r-direction with θ fixed leaves both unit vectors unchanged, so

  (A) $\partial \boldsymbol{e}_r/\partial r = \boldsymbol{e}_\theta$
  (B) $\partial \boldsymbol{e}_\theta/\partial r = \boldsymbol{e}_r$
  (C) $\partial \boldsymbol{e}_r/\partial r = \partial \boldsymbol{e}_\theta/\partial r = \mathbf{0}$
  (D) $\partial \boldsymbol{e}_r/\partial r = -\boldsymbol{e}_\theta$

(b) Advancing θ by $d\theta$ rotates $\boldsymbol{e}_r$ by that angle, so to first order $\boldsymbol{e}_r(\theta + d\theta) = \boldsymbol{e}_r(\theta) + d\theta\, \boldsymbol{e}_\theta$. Forming the limit gives $\partial \boldsymbol{e}_r/\partial\theta =$

  (A) $-\boldsymbol{e}_\theta$
  (B) $\boldsymbol{e}_\theta$
  (C) $\boldsymbol{e}_r$
  (D) $\mathbf{0}$

(c) Differentiating the orthogonality condition $\boldsymbol{e}_r \cdot \boldsymbol{e}_\theta = 0$ with respect to θ and using $\partial \boldsymbol{e}_r/\partial\theta = \boldsymbol{e}_\theta$ yields $\boldsymbol{e}_\theta \cdot \boldsymbol{e}_\theta + \boldsymbol{e}_r \cdot (\partial \boldsymbol{e}_\theta/\partial\theta) = 0$, so the component of $\partial \boldsymbol{e}_\theta/\partial\theta$ along $\boldsymbol{e}_r$ is

  (A) $+1$
  (B) -1
  (C) 0
  (D) $\sin\theta$

(d) Differentiating $\boldsymbol{e}_\theta \cdot \boldsymbol{e}_\theta = 1$ shows $\partial \boldsymbol{e}_\theta/\partial\theta$ is also perpendicular to $\boldsymbol{e}_\theta$. In two dimensions the only direction left is along $\boldsymbol{e}_r$, so

  (A) $\partial \boldsymbol{e}_\theta/\partial\theta = \boldsymbol{e}_\theta$
  (B) $\partial \boldsymbol{e}_\theta/\partial\theta = -\boldsymbol{e}_r$
  (C) $\partial \boldsymbol{e}_\theta/\partial\theta = \boldsymbol{e}_r$
  (D) $\partial \boldsymbol{e}_\theta/\partial\theta = \mathbf{0}$

Exercise 3 (Divergence of the flux in polar coordinates).

For a vector field $\boldsymbol{q} = q_r \boldsymbol{e}_r + q_\theta \boldsymbol{e}_\theta$ expressed in polar coordinates, the divergence is computed via $\nabla \cdot \boldsymbol{q}$ with the polar gradient $\nabla = \boldsymbol{e}_r\, \partial/\partial r + (\boldsymbol{e}_\theta/r)\, \partial/\partial\theta$.

(a) When the dot product is expanded term-by-term, several terms involve unit-vector derivatives. Using $\partial \boldsymbol{e}_r/\partial r = \partial \boldsymbol{e}_\theta/\partial r = \mathbf{0}$, $\partial \boldsymbol{e}_r/\partial\theta = \boldsymbol{e}_\theta$, and $\partial \boldsymbol{e}_\theta/\partial\theta = -\boldsymbol{e}_r$, the surviving contributions are

  (A) only $\partial q_r/\partial r$
  (B) only $\partial q_r/\partial r$ and $\partial q_\theta/\partial\theta$
  (C) $\partial q_r/\partial r$, the term from $\partial \boldsymbol{e}_r/\partial\theta = \boldsymbol{e}_\theta$ acting on q_r, and $\partial q_\theta/\partial\theta$
  (D) all sixteen products of the four-by-four expansion

(b) Specifically, the surviving terms add to

  (A) $\partial q_r/\partial r + \partial q_\theta/\partial\theta$

(B) $\partial q_r/\partial r + q_r/r + (1/r)\,\partial q_\theta/\partial\theta$

(C) $(1/r)\,\partial q_r/\partial r + (1/r)\,\partial q_\theta/\partial\theta$

(D) $\partial q_r/\partial r - q_r/r + \partial q_\theta/\partial\theta$

(c) Recognising $\partial q_r/\partial r + q_r/r = (1/r)\,\partial(r q_r)/\partial r$, the polar divergence is

(A) $(1/r)\,\partial q_r/\partial r + (1/r^2)\,\partial q_\theta/\partial\theta$

(B) $(1/r)\,\partial(r q_r)/\partial r + (1/r)\,\partial q_\theta/\partial\theta$

(C) $\partial q_r/\partial r + (1/r)\,\partial q_\theta/\partial\theta$

(D) $(1/r)\,\partial(r q_r)/\partial r + \partial q_\theta/\partial\theta$

(d) The extra factor of r inside the radial derivative (compared to the Cartesian formula $\partial q_x/\partial x + \partial q_y/\partial y$) traces back to

(A) the polar unit vectors themselves depending on θ, so derivatives of $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$ contribute extra terms

(B) a sign convention chosen for the gradient

(C) the orthogonality relation $\boldsymbol{e}_r \cdot \boldsymbol{e}_\theta = 0$

(D) using Stokes' theorem rather than the divergence theorem

Exercise 4 (Gradient and Laplacian in spherical coordinates).

In spherical coordinates (r, θ, ϕ) with θ the polar angle (measured from the positive z-axis) and ϕ the azimuthal angle, the position differential is

$$d\boldsymbol{x} = dr\,\boldsymbol{e}_r + r\,d\theta\,\boldsymbol{e}_\theta + r\sin\theta\,d\phi\,\boldsymbol{e}_\phi$$

(a) Setting $df = \nabla f \cdot d\boldsymbol{x}$ and matching coefficients of dr, $d\theta$, and $d\phi$, the $\boldsymbol{e}_\theta$ component of ∇f is

(A) $\partial f/\partial\theta$

(B) $(1/r)\,\partial f/\partial\theta$

(C) $(1/r\sin\theta)\,\partial f/\partial\theta$

(D) $r\,\partial f/\partial\theta$

(b) The $\boldsymbol{e}_\phi$ component of ∇f is

(A) $\partial f/\partial\phi$

(B) $(1/r)\,\partial f/\partial\phi$

(C) $(1/r\sin\theta)\,\partial f/\partial\phi$

(D) $(1/\sin\theta)\,\partial f/\partial\phi$

The factor $1/(r\sin\theta)$ reflects that the ϕ-arclength along a parallel of latitude is $r\sin\theta\,d\phi$, not $d\phi$.

(c) The radial part of $\nabla^2 f$ comes out to $f_{rr} + 2f_r/r$. The factor of 2 (versus the cylindrical f_r/r) arises because

(A) a sign convention in the Laplacian

(B) both row 1 ($\boldsymbol{e}_r \partial/\partial r$) and the radial parts of rows 2 and 3 each contribute an f_r/r piece, and the latter two combine into a single f_r/r via the $\partial \boldsymbol{e}_r/\partial\theta = \boldsymbol{e}_\theta$-type identities, giving $2f_r/r$ in total

(C) the polar angle θ ranges over $[0, \pi]$ rather than $[0, 2\pi]$

(D) the spherical Laplacian is expressed in two extra coordinates

(d) Putting everything together, the spherical Laplacian is

(A) $\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 f_r) + \frac{1}{r^2} f_{\theta\theta} + \frac{1}{r^2} f_{\phi\phi}$

(B) $\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 f_r) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, f_\theta) + \frac{1}{r^2 \sin^2\theta} f_{\phi\phi}$

(C) $f_{rr} + \frac{1}{r} f_r + \frac{1}{r^2} f_{\theta\theta} + \frac{1}{r^2} f_{\phi\phi}$

(D) $f_{rr} + f_{\theta\theta} + f_{\phi\phi}$

(e) A function depending only on r ($f = f(r)$) has spherical Laplacian

(A) $f''(r)$

(B) $f''(r) + (2/r)\, f'(r) = (1/r^2)\,(r^2 f')'$

(C) $f''(r) + (1/r)\, f'(r) = (1/r)\,(r f')'$

(D) $r^2 f''(r)$

the spherical analogue of the cylindrical $(1/r)(rf')'$ that appears in radial heat-conduction problems.

References