Homework 8 — Multiple Choice

ChE 132A

Exercise 8 (Conservation of energy and the heat equation).

A solid (density ρ, heat capacity $\hat{C}_P$, conductivity k) has volumetric heat-generation rate $\dot{q}$ and no fluid motion. The integral energy balance for a fixed control volume V with boundary S is

$$\frac{d}{dt}\int_V \rho\hat{C}_P T\,dV = -\int_S \boldsymbol{n}\cdot\boldsymbol{q}\,dS + \int_V \dot{q}\,dV$$

(a) Because V is fixed, the time derivative on the LHS commutes with the volume integral, giving

- (A) $\int_S \rho\hat{C}_P\,(\partial T/\partial t)\,dS$
- (B) $\int_V \rho\hat{C}_P\,(\partial T/\partial t)\,dV$
- (C) $\rho\hat{C}_P\,(\partial T/\partial t)$ (a single number, not an integral)
- (D) $\int_V(\partial/\partial t)(T\,\rho\hat{C}_P\,dV)$, with dV also differentiated

(b) Applying the divergence theorem to the surface integral converts it to

- (A) $\int_V \boldsymbol{n}\cdot\boldsymbol{q}\,dV$
- (B) $\int_V \nabla\cdot\boldsymbol{q}\,dV$
- (C) $\int_S \nabla\cdot\boldsymbol{q}\,dS$
- (D) $\int_S \boldsymbol{q}\,dS$ (no change)

(c) Substituting Fourier's law $\boldsymbol{q} = -k\nabla T$ replaces $-\nabla\cdot\boldsymbol{q}$ by $\nabla\cdot(k\nabla T) = k\nabla^2 T$ (constant k). The integral identity holds for arbitrary V, so the integrand vanishes pointwise, giving the heat equation

- (A) $\rho\hat{C}_P\,\partial T/\partial t = -k\nabla^2 T + \dot{q}$
- (B) $\rho\hat{C}_P\,\partial T/\partial t = k\nabla^2 T + \dot{q}$
- (C) $\rho\hat{C}_P\,(\partial T/\partial t)^2 = k\nabla^2 T$
- (D) $\rho\hat{C}_P\,\nabla T = k\,\partial T/\partial t$

(d) Dividing by $\rho\hat{C}_P$ and comparing with the species mass-balance PDE $\partial c_A/\partial t = D_A\nabla^2 c_A + R_A$ (no convection), the role of the diffusivity D_A is played by

- (A) k
- (B) $\rho\hat{C}_P$
- (C) the thermal diffusivity $\alpha = k/(\rho\hat{C}_P)$
- (D) $\dot{q}/(\rho\hat{C}_P)$

(e) For unidirectional flow of an incompressible Newtonian fluid the streamwise momentum equation reduces to $\partial u/\partial t = \nu\,\partial^2 u/\partial y^2$. The kinematic viscosity $\nu = \mu/\rho$ here plays the role of

- (A) the source term $\dot{q}/(\rho \hat{C}_P)$
- (B) the thermal diffusivity α — the momentum-diffusion timescale is set by ν exactly as the heat-diffusion timescale is set by α
- (C) the temperature T
- (D) the surface flux $\boldsymbol{n} \cdot \boldsymbol{q}$

Exercise 9 (Flux heating of a semi-infinite slab).

A semi-infinite solid ($x > 0$) at zero initial temperature is heated by applying a constant flux q_0 at the surface $x = 0$. In the dimensionless variables $\Theta = kT/q_0$, $\tau = \alpha t$, the problem is

$$\partial_\tau \Theta = \partial_x^2 \Theta, \quad -\partial_x \Theta\big|_{x=0} = 1, \quad \Theta(x, 0) = 0, \quad \Theta \to 0 \ (x \to \infty)$$

(a) The Laplace transform (in τ) of the PDE plus the bounded-as-$x \to \infty$ condition gives $\bar{\Theta}(x, s) = A(s)\, e^{-x\sqrt{s}}$. Applying the transformed flux BC $-d\bar{\Theta}/dx|_0 = 1/s$ determines $A(s)$ as

- (A) $A(s) = 1/s$
- (B) $A(s) = 1/\sqrt{s}$
- (C) $A(s) = 1/s^{3/2}$
- (D) $A(s) = 1/s^2$

(b) Differentiating $\bar{\Theta} = e^{-x\sqrt{s}}/s^{3/2}$ once in x gives $-\partial\bar{\Theta}/\partial x = e^{-x\sqrt{s}}/s$, which inverts to

- (A) $-\partial\Theta/\partial x = e^{-x^2/(4\tau)}$
- (B) $-\partial\Theta/\partial x = \mathrm{erfc}(x/(2\sqrt{\tau}))$
- (C) $-\partial\Theta/\partial x = 1 - \mathrm{erf}(x\sqrt{\tau})$
- (D) $-\partial\Theta/\partial x = 1$

(c) Integrating $-\partial\Theta/\partial x = \mathrm{erfc}(x/2\sqrt{\tau})$ from x to ∞ (using $\Theta \to 0$ at infinity) and integrating by parts gives

- (A) $\Theta(x, \tau) = \mathrm{erfc}(x/2\sqrt{\tau})$
- (B) $\Theta(x, \tau) = 2\sqrt{\tau/\pi}\, e^{-x^2/(4\tau)} - x\, \mathrm{erfc}(x/2\sqrt{\tau})$
- (C) $\Theta(x, \tau) = \sqrt{\tau/\pi}$
- (D) $\Theta(x, \tau) = x\ \mathrm{erf}(x/2\sqrt{\tau})$

(d) Setting $x = 0$ in the closed-form solution shows that the surface temperature grows as

- (A) a constant (independent of τ)
- (B) $\Theta(0, \tau) = 2\sqrt{\tau/\pi} \propto \sqrt{\tau}$
- (C) $\Theta(0, \tau) \propto \tau$
- (D) $\Theta(0, \tau) \propto e^\tau$

The $\sqrt{\tau}$ growth is the hallmark of diffusive penetration with constant-flux forcing.

(e) The problem has no intrinsic length scale (the slab is semi-infinite). The only length appearing in the solution is therefore

- (A) a fixed length built into the geometry
- (B) the diffusion length $\sqrt{\tau} = \sqrt{\alpha t}$, which grows in time
- (C) $1/\sqrt{q_0}$
- (D) the wavelength of the boundary forcing

Exercise 11 (Startup of plane Couette flow).

A Newtonian fluid of kinematic viscosity ν fills the gap between two parallel plates at $y = 0$ (stationary) and $y = L$. At $t = 0$ the upper plate is set in motion at constant velocity U_0. In the dimensionless variables $\eta = y/L$, $\tau = \nu t/L^2$, $\hat{u} = u/U_0$,

$$\hat{u}_\tau = \hat{u}_{\eta\eta}, \quad \hat{u}(0,\tau) = 0,\ \hat{u}(1,\tau) = 1, \quad \hat{u}(\eta, 0) = 0$$

The inhomogeneous BC at $\eta = 1$ blocks a direct sine-series attack; we split off the steady piece first.

(a) The steady particular solution $\hat{u}_p(\eta)$ satisfies $\hat{u}_p'' = 0$ with $\hat{u}_p(0) = 0$ and $\hat{u}_p(1) = 1$. This is

- (A) $\hat{u}_p = 1$
- (B) $\hat{u}_p = \eta$
- (C) $\hat{u}_p = \frac{1}{2}\eta(1-\eta)$
- (D) $\hat{u}_p = \sin(\pi\eta/2)$

the linear Couette profile.

(b) Writing $\hat{u} = \hat{u}_p + w$ and substituting, the transient correction $w(\eta, \tau)$ satisfies

- (A) $w_\tau = w_{\eta\eta}$ with $w(0,\tau) = w(1,\tau) = 0$ and $w(\eta, 0) = -\eta$
- (B) $w_\tau = w_{\eta\eta}$ with $w(0,\tau) = w(1,\tau) = 0$ and $w(\eta, 0) = +\eta$
- (C) $w_\tau = w_{\eta\eta}$ with $w(0,\tau) = 0$, $w(1,\tau) = 1$
- (D) $w_\tau = -w_{\eta\eta}$ (a backwards heat equation)

(c) Expanding $w(\eta, 0) = -\eta$ in the sine basis on $[0, 1]$, the coefficients $a_n = -2\int_0^1 \eta\ \sin(n\pi\eta)\ \mathrm{d}\eta$ evaluate to

- (A) $a_n = -2/(n\pi)$
- (B) $a_n = -2$ for all n
- (C) $a_n = 2(-1)^n/(n\pi)$
- (D) $a_n = 4/(n\pi)$ for odd n, 0 for even n

(d) Assembling, the full solution is

- (A) $\hat{u}(\eta,\tau) = \eta\, e^{-\pi^2\tau}$
- (B) $\hat{u}(\eta,\tau) = \dfrac{4}{\pi} \sum_{n \text{ odd}} \dfrac{1}{n}\, e^{-n^2\pi^2\tau}\ \sin(n\pi\eta)$
- (C) $\hat{u}(\eta,\tau) = \eta + \dfrac{2}{\pi} \sum_{n\geq 1} \dfrac{(-1)^n}{n}\, e^{-n^2\pi^2\tau}\ \sin(n\pi\eta)$
- (D) $\hat{u}(\eta,\tau) = \eta\big(1 - e^{-\pi^2\tau}\big)$

(e) As $\tau \to \infty$ every transient mode decays and the velocity profile approaches

- (A) $\hat{u} \to 0$ everywhere
- (B) $\hat{u} \to 1$ everywhere
- (C) $\hat{u} \to \eta$, the linear Couette profile
- (D) $\hat{u} \to (4/\pi)\,\eta$

The slowest-decaying transient mode has timescale $\tau_1 = 1/\pi^2 \approx 0.10$, so steady state is reached on the momentum-diffusion time L^2/ν.

References