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Suboptimal MPC and partial enumeration: robust stability and computational performance

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Abstract

We revise in this paper the Partial Enumeration (PE) method for the fast computation of a suboptimal solution to linear MPC problems. We derive novel robust exponential stability results for difference inclusions to show that the existence of a continuous Lyapunov function implies *Strong Robust Exponential Stability* (SRES), i.e. for *any* sufficiently small perturbation. Given the fact that the suboptimal PE-based control law is non-unique and discontinuous, i.e. a set-valued map, we treat the closed-loop system, appropriately augmented, as a difference inclusion. Such approach allows us to show SRES of the closed-loop system under PE-based MPC. Application to a simulated open-loop unstable CSTR is presented to show the performance and timing results of PE-based MPC, as well as to highlight its robustness to process/model mismatch, disturbances and measurement noise.

Keywords

Partial Enumeration MPC, Explicit MPC, Robust stability, Lyapunov functions

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Notation.

Given a vector $x \in \mathbb{R}^n$, $|x|$ denotes the 2-norm; given a positive scalar r, we define $\mathbb{B}_r = \{x \in \mathbb{R}^n,\ |x| \leq r\}$; given a sequence of vectors $\mathbf{x}_k = \{x(j)\}_{j=0}^{k-1}$, we define $\|\mathbf{x}_k\| = \sup_j |x(j)|$; if A is a subset of $\mathbb{R}^n$ and $b \in \mathbb{R}^n$, we denote the set $A + b = \{c = a + b|\ a \in A\}$. The superscript $'$ indicates the transpose operator; given an integer j, the symbol $\mathbb{I}_{\geq j}$ indicates the set of integers greater than or equal to j; the symbols I and 0 indicate the identity and zero matrices of appropriate dimensions.

1 Introduction

A significant amount of research activity of the last decade (or so) in the field of Model Predictive Control (MPC) has been devoted to the implementation of efficient methods for solving the associated constrained optimal control problem, which for linear systems subject to linear constraints and quadratic performance function can be casted as a Quadratic Program (QP). Several methods rely *exclusively* on efficient on-line calculations [11, 5, 4], whereas so-called Explicit MPC methods [1, 3] move the most expensive calculations offline and the online computations are *limited* to a table lookup involving simple matrix/vector multiplications and inequality checks. However, due to the exponential *explosion* of the required number of table entries with respect to the problem size (number of inputs, states and constraints), Explicit MPC is limited to *small* systems. A method that can be considered in the middle field between Explicit MPC and online optimization is Partial Enumeration [9, 10], in which a table (with entries equivalent to those of Explicit MPC) of fixed size is scanned online to find the optimal control input. If none of the entries is optimal, a quick suboptimal input is computed, but the table is updated to include the new optimal entry for future decision times (while the least recently optimal entry is discarded). In this way, as the time goes on, the table adapts to the new operating conditions and contains only the entries that are currently more likely to be optimal.

The objective of this paper is to revise the Partial Enumeration (PE) approach and to make appropriate modifications with the goal of showing its robust stability properties. To this aim we develop novel tools for robust stability of suboptimal MPC, and this represents the second (perhaps the most relevant) contribution of this paper. The rest of this paper is organized as follows. In Section 2 we revise the PE-MPC method, and in Section 3 we derive from scratch novel robust stability tools and apply such results to PE-MPC. A simulated application to the control of an unstable CSTR is presented in Section 4, conclusions are drawn in Section 5 and additional derivations and proofs are reported in Appendix A.

2 Partial Enumeration MPC

2.1 Control problem and main assumptions

We consider LTI systems with input constraints:

$$x^+ = Ax + Bu, \quad u \in \mathbb{U}\,,$$

in which $u \in \mathbb{R}^m$ is the input, $x \in \mathbb{R}^n$ and $x^+ \in \mathbb{R}^n$ are the state and the successor state, respectively. $\mathbb{U} = \{u | Du \leq b_u\}$ is compact and contains the origin in its interior (i.e. $b_u > 0$). We consider the problem of steering the state to a given target $\bar{x}$ that satisfies $\bar{x} = A\bar{x} + B\bar{u}$, with $\bar{u} \in \mathbb{U}$. To this aim, we define deviation state and input: $\tilde{x} = x - \bar{x}$, $\tilde{u} = u - \bar{u}$, and consequently the deviation input admissible space is $\tilde{\mathbb{U}} = \{\tilde{u} | D\tilde{u} \leq b_u - D\bar{u}\}$. In the sake of notation simplicity, we omit the dependance of $\tilde{\mathbb{U}}$ (and its derived sets) on $\bar{u}$. Given a deviation input sequence vector $\tilde{\mathbf{u}} = [\tilde{u}(0)', \tilde{u}(1)', \ldots, \tilde{u}(N-1)']'$, we define the following cost function:

$$V_N(\tilde{x}, \tilde{\mathbf{u}}) = \tfrac{1}{2} \sum_{k=0}^{N-1} \tilde{x}(k)'Q\tilde{x}(k) + \tilde{u}(k)'R\tilde{u}(k) + \tfrac{1}{2}\tilde{x}(N)'P\tilde{x}(N),$$
$$\text{s.t. } \tilde{x}^+ = A\tilde{x} + B\tilde{u},\ \tilde{x}(0) = \tilde{x}, \quad (1)$$

and the following constrained optimal control problem:

$$\min_{\tilde{\mathbf{u}}} V_N(\tilde{x}, \tilde{\mathbf{u}}), \quad \text{s.t. } \tilde{\mathbf{u}} \in \tilde{\mathbb{U}}^N,\ S_u'\tilde{x}(N) = 0\,, \quad (2)$$

where $\tilde{\mathbb{U}}^N := \underbrace{\tilde{\mathbb{U}} \times \cdots \times \tilde{\mathbb{U}}}_{N \text{ times}}$, and S_u is defined in Assumption 1.

Assumption 1. *The pair (A, B) is stabilizable, (Q, R) are positive definite, $P = S_s'\Pi S_s$ with Π solution to the Lyapunov equation $\Pi = A_s'\Pi A_s + S_s'QS_s$ in which (A_s, S_s) come from the real Schur decomposition: $A = \begin{bmatrix} S_s & S_u \end{bmatrix} \begin{bmatrix} A_s & A_{su} \\ 0 & A_u \end{bmatrix} \begin{bmatrix} S_s' \\ S_u' \end{bmatrix}$, and A_s contains all stable eigenvalues of A.*

Remark 1. *The constraint $S_u'\tilde{x}(N) = 0$ zeroes the unstable modes at time N. Thus, $V_N(\cdot)$ represents an infinite-horizon cost, under the control $\tilde{u}(k) = 0$ (i.e. $u(k) = \bar{u}$) for all $k \geq N$.*

We define $\tilde{\mathbb{X}}$ as the set of $\tilde{x}(0)$ for which problem (2) has a solution, i.e., $\tilde{\mathbb{X}} = \{\tilde{x} \,|\, \exists \tilde{\mathbf{u}} : D\tilde{u}(k) \leq b_u - D\bar{u}$ for all $k = 0, 1, \ldots, N-1$ and $S_u'(A^N\tilde{x} + A^{N-1}B\tilde{u}(0) + \cdots + B\tilde{u}(N-1)) = 0\}$, and we make the following assumption on the input set and target.

Assumption 2. *The set $\mathbb{U}$ is compact, and $\bar{u} \in \operatorname{int} \mathbb{U}$[1].*

[1] The main robust stability results presented in this paper can be extended to case in which $\bar{u}$ lies on the boundary of $\mathbb{U}$, but several technical issues arise by doing so. For this reason such case is not treated in this paper.

Remark 2. *S_u is vacuous for stable systems, thus for any $\bar{u} \in \operatorname{int} \mathbb{U}$ it follows that $\tilde{\mathbb{X}} = \mathbb{R}^n$, and hence $\tilde{\mathbb{X}}$ contains the origin in its interior. For unstable systems S_u is a full rank matrix, and for any $\bar{u} \in \operatorname{int}(\mathbb{U})$ it is straightforward to show that $\tilde{\mathbb{X}}$ is nonempty and contains the origin in its interior.*

2.2 Partial Enumeration: main definitions

As shown in Appendix A.1, problem (2) can be written as the following convex parametric QP:

$$\min_{\tilde{\mathbf{u}}} V_N(\tilde{x}, \tilde{\mathbf{u}}) = \tfrac{1}{2}\tilde{\mathbf{u}}'\mathbf{H}\tilde{\mathbf{u}} + \tilde{\mathbf{u}}'\mathbf{G}\tilde{x} + \tfrac{1}{2}\tilde{x}'\mathbf{P}\tilde{x}, \tag{3a}$$

$$\text{s.t. } \mathbf{D}\tilde{\mathbf{u}} + \mathbf{C}\bar{u} \leq \mathbf{d}, \quad \mathbf{E}\tilde{\mathbf{u}} + \mathbf{F}\tilde{x} = 0\,. \tag{3b}$$

Given the optimal point of this problem, $\tilde{\mathbf{u}}^0$, we denote with $(\mathbf{D}_a, \mathbf{C}_a, \mathbf{d}_a)$ the stacked rows of $(\mathbf{D}, \mathbf{C}, \mathbf{d})$ such that $\mathbf{D}_a\tilde{\mathbf{u}}^0 + \mathbf{C}_a\bar{u} = \mathbf{d}_a$ (i.e. the *active* constraints). We also denote with $(\mathbf{D}_i, \mathbf{C}_i, \mathbf{d}_i)$ the stacked complementary rows, i.e. such that $\mathbf{D}_i\tilde{\mathbf{u}}^0 + \mathbf{C}_i\bar{u} < \mathbf{d}_i$ (i.e. the *inactive* constraints). Since $\tilde{\mathbf{u}}^0$ is optimal for (3), the following first-order optimality KKT conditions hold:

$$\mathbf{H}\tilde{\mathbf{u}}^0 + \mathbf{G}\tilde{x} + \mathbf{D}_a'\lambda_a^0 + \mathbf{E}'\mu^0 = 0\,, \tag{4a}$$

$$\mathbf{D}_a\tilde{\mathbf{u}}^0 + \mathbf{C}_a\bar{u} = \mathbf{d}_a\,, \tag{4b}$$

$$\mathbf{E}\tilde{\mathbf{u}}^0 + \mathbf{F}\tilde{x} = 0, \tag{4c}$$

$$\lambda_a^0 \geq 0, \tag{4d}$$

$$\mathbf{D}_i\tilde{\mathbf{u}}^0 + \mathbf{C}_i\bar{u} \leq \mathbf{d}_i\,. \tag{4e}$$

In Appendix A.2, we derive the following equivalent set of conditions for $\tilde{\mathbf{u}}^0$ satisfying (4):

$$\tilde{\mathbf{u}}^0 = \mathbf{\Gamma}_u(\mathbf{d}_a - \mathbf{C}_a\bar{u}) + \mathbf{\Gamma}_x\tilde{x}, \quad \text{where} \tag{5a}$$

$$\begin{bmatrix} \mathbf{\Psi}_P \\ \mathbf{\Psi}_D \end{bmatrix} z \leq \begin{bmatrix} \psi_P \\ \psi_D \end{bmatrix}, \quad z = \begin{bmatrix} \bar{u} \\ \tilde{x} \end{bmatrix}, \tag{5b}$$

Thus, we can express the optimal cost as follows:

$$V_N^*(\bar{u}, \tilde{x}) = \tfrac{1}{2}z'V_2z + v_1'z + v_0\,. \tag{6}$$

Explicit MPC [3] partitions (offline) the space of z in a number of *regions*, each defined by the tuple:

$$(\mathbf{\Psi}_P, \mathbf{\Psi}_D, \mathbf{\Gamma}_u, \mathbf{\Gamma}_x, \psi_P, \psi_D, V_2, v_1, v_0)\,. \tag{7}$$

The on-line evaluation consists in finding the region for which (5b) holds, and then computing $\tilde{\mathbf{u}}^0$ from (5a) and the optimal objective value from (6). Several enhancements can be made to reduce the storage requirements and also the online computations [2, 1]. Still, Explicit MPC can be effectively implemented for small dimensional systems, as the number of regions grows exponentially with the problem size. On the other hand, in Partial

Enumeration, PE, [9] we store the tuples (7) for a fixed number of active sets that were optimal at the most recent decision time points. Online, we scan the table to check if, for given parameters $(\bar{u}, \tilde{x})$, the optimality conditions (5b) are satisfied, and in such case we compute the optimal solution from (5a). However, given the fact that not all possible optimal active sets are stored, it is possible that no table entry is optimal. In such case we compute a suboptimal solution for closed-loop control. Nonetheless, a QP solver is called afterwards to compute the optimal solution $\tilde{\mathbf{u}}^0$, and thus derive the optimal missing tuple (7). Whenever, this table entry becomes available, it is inserted into the table. If after this insertion, the table would exceed its maximum size (user defined), we delete the entry that was optimal least recently. In this way, the table size is fixed and hence the table lookup process is fast, but the table entries are updated to keep track of new operating conditions for the systems. In [1] a table with fixed number of entries is also proposed for fast evaluation, but differently from PE the table is not updated during online operation.

2.3 Partial Enumeration algorithms

In order to compute quickly a suboptimal input sequence when the table does not include the optimal active set for the current parameters $(\bar{u}, \tilde{x})$, several options can be considered. To this end, a procedure based on violations of optimality conditions in (5) is developed in [1], and closed-loop nominal stability is checked *a posteriori*. In [9] we used the previous shifted optimal sequence in nominal conditions, thus guaranteeing nominal stability, or the solution to a short-horizon MPC problem in the presence of disturbances. Here, instead, we propose a slightly different procedure that allows us to prove *robust exponential stability* of the closed-loop under PE-MPC. The procedure requires two points, the first one of which needs to be feasible and its computation is discussed later in Algorithm 1. The second point, instead, is the minimizer of (3a) subject to the equality constraint (if present). More specifically, we define $\tilde{\mathbf{u}}^*$ as the solution to:

$$\min_{\tilde{\mathbf{u}}} V_N(\tilde{x}, \tilde{\mathbf{u}}) \quad \text{s.t. } \mathbf{E}\tilde{\mathbf{u}} + \mathbf{F}\tilde{x} = 0\,. \tag{8}$$

Following similar reasoning to Appendix A.2, we have that

$$\tilde{\mathbf{u}}^* = \hat{\mathbf{\Gamma}}\tilde{x}, \tag{9}$$

where the matrix $\hat{\mathbf{\Gamma}}$ can be computed offline.

Next, we denote by $\tilde{\mathbf{u}}_+ = [(u^*(1)-\bar{u})', \ldots, (u^*(N-1)-\bar{u})', 0]'$ the previous shifted optimal sequence vector, where the inputs $u(1)^*, \ldots, u^*(N-1)$ were computed at the previous decision time, while $\bar{u}$ is the current input target. We now present the PE algorithm.

Algorithm 1 (General purpose PE)**.** ***Require:*** *Table with M entries, each a tuple of the form (7); current parameters $(\bar{u}, \tilde{x})$; candidate sequence $\tilde{\mathbf{u}}_+$, its cost V_N^+ if feasible (otherwise $V_N^+ = \infty$); maximum table size $M_{\max}$.*

1: {%Initialize} *Set opt_found=**false**.*
2: ***while*** *($j \leq M$ & opt_found=**false**)* ***do***
3: *Extract the $j-th$ tuple from the table.*

4: **if** $\boldsymbol{\Psi}_P z \le \psi_P$ **then** {%Entry is feasible}
5: **if** $\boldsymbol{\Psi}_D z \le \psi_D$ **then** {%Entry is optimal}
6: *Compute optimal solution* $\tilde{\mathbf{u}}$ *from* (5a)*. Put tuple* j *in first position of the table. Set opt_found=***true***.*
7: **else** {%Entry is suboptimal}
8: *Compute cost* V_N *from* (6)*.*
9: **if** $V_N < V_N^+$ **then**
10: *Set* $\tilde{\mathbf{u}}_+ = \boldsymbol{\Gamma}_u(\mathbf{d}_a - \mathbf{C}_a\bar{u}) + \boldsymbol{\Gamma}_x\tilde{x}$.
11: **end if**
12: **end if**
13: **end if**
14: **end while**
15: **if** *opt_found=***false then** {%No optimal entry found}
16: **if** $V_N^+ = \infty$ **then** {%$\tilde{\mathbf{u}}_+$ is infeasible}
17: *Solve the LP:*

$$\min_{\mathbf{q},\mathbf{s}} \mathbf{1}'(\mathbf{q}+\mathbf{s}) \quad s.t.\ \mathbf{D}(\mathbf{q}-\mathbf{s}) \le \mathbf{r}_1,\ \mathbf{E}(\mathbf{q}-\mathbf{s}) = \mathbf{r}_2,\ \mathbf{q} \ge 0,\ \mathbf{s} \ge 0$$

with $\mathbf{r}_1 = \mathbf{d} - \mathbf{C}\bar{u} - \mathbf{D}\tilde{\mathbf{u}}_+$, $\mathbf{r}_2 = -\mathbf{F}\tilde{x} - \mathbf{E}\tilde{\mathbf{u}}_+$, $\mathbf{1}$ *vector of ones. Redefine* $\tilde{\mathbf{u}}_+ \leftarrow \tilde{\mathbf{u}}_+ + \mathbf{q} - \mathbf{s}$*, compute its cost* V_N^+.
18: **end if**
19: *Evaluate* $\tilde{\mathbf{u}}^*$ *from (9), and compute the largest* $t \in [0,1]$ *such that* $\mathbf{D}(\tilde{\mathbf{u}}^* - \tilde{\mathbf{u}}_+)t \le \mathbf{d} - \mathbf{C}\bar{u} - \mathbf{D}\tilde{\mathbf{u}}_+$*. Set* $\tilde{\mathbf{u}} = \tilde{\mathbf{u}}_+(1-t) + t\tilde{\mathbf{u}}^*$.
20: {%Table update, performed after returning $\tilde{\mathbf{u}}$} *Solve the QP* (3)*, and find the optimal tuple (7). If* $M = M_{\max}$*, delete the entry that was optimal least recently (hence* $M \leftarrow M-1$*). Insert the new entry in first position of the table, set* $M \leftarrow M+1$.
21: **end if**
22: **return** *(Sub)optimal sequence* $\tilde{\mathbf{u}}$*, updated table.*

Remark 3. *The "feasibility recovery" step (Line 17) is required* only if *the system is open-loop unstable* and *either the target has changed from the previous decision time or a disturbance occurred. In the nominal case without target change, such step is not performed because* $\tilde{\mathbf{u}}_+$ *is feasible. Line 17 is the only "expensive" computation in Algorithm 1 and is justified by closed-loop stability reasons of an open-loop unstable system. Also notice that Line 19 computes the largest feasible step from* $\tilde{\mathbf{u}}_+$ *to* $\tilde{\mathbf{u}}^*$*; if* $\tilde{\mathbf{u}}^*$ *is feasible it follows that* $t = 1$.

If the input constraints are: $u_{\min} \le u \le u_{\max}$, i.e. the constraint matrix/vector are given by $D = [I\ \ -I]'$, $d = [u'_{\max}\ \ -u'_{\min}]'$, we propose a tailored enhanced strategy.

Algorithm 2 (Enhanced PE for box constraints)**.** *Same as Algorithm 1, with Lines 15–19 replaced by the following.*

1: *Set* $(\hat{\mathbf{D}}, \hat{\mathbf{d}})$ *empty matrix/vector. Set feas_found=***false***.*
2: **while** *feas_found=***false do**
3: *Solve the following (equality-constrained) QP:*

$$\min_{\tilde{\mathbf{u}}} V_N(\tilde{x}, \tilde{\mathbf{u}}) \qquad s.t.\ \hat{\mathbf{D}}\tilde{\mathbf{u}} = \hat{\mathbf{d}},\ \mathbf{E}\tilde{\mathbf{u}} + \mathbf{F}\tilde{x} = 0\,, \tag{10}$$

Let $(\tilde{\mathbf{u}}^*, \lambda^*)$ *be the associated minimizer and Lagrange multipliers for* $\hat{\mathbf{D}}\tilde{\mathbf{u}} = \hat{\mathbf{d}}$*, respectively.*
if $\mathbf{D}\tilde{\mathbf{u}}^* \leq \mathbf{d}$ ***then*** {*%Feasible solution found*}
  Define $V_N^* = V_N(\tilde{x}, \tilde{\mathbf{u}}^*)$*. Set feas_found=**true**.*
  if $V_N^* \leq V_N^+$ ***then***
    Set $\tilde{\mathbf{u}} = \tilde{\mathbf{u}}^*$.
  else
    Set $\tilde{\mathbf{u}} = \tilde{\mathbf{u}}_+$.
  end if
else {*%Infeasible solution: add/remove constraints*}
  Define $(\hat{\mathbf{D}}, \hat{\mathbf{d}})$ *as the rows of* $(\mathbf{D}, \mathbf{d})$ *for the violated inequalities plus the rows of the previous* $(\hat{\mathbf{D}}, \hat{\mathbf{d}})$ *with nonnegative multipliers.*
end if
end while

Remark 4. *In Algorithm 2, when* $\tilde{\mathbf{u}}^*$ *violates any inequality such constraints are regarded as equalities, and* $\tilde{\mathbf{u}}^*$ *is recomputed. When a given constraint (say an upper bound) is included in* $(\hat{\mathbf{D}}, \hat{\mathbf{d}})$*, there is no need to check feasibility with respect to the parallel (say lower bound) constraint. Also notice that* (10) reduces *to solving a square linear system.*

3 Robust stability results

In [6], Teel and coworkers showed that under standard assumptions, the origin of a *linear* closed-loop system $\xi^+ = A\xi + B\kappa_N(\xi)$, with $\kappa_N(\xi)$ being a nominally stabilizing MPC-generated control law, is *robustly asymptotically stable.* This result relies on continuity of $\kappa_N(\xi)$, which however holds *only* when the *optimal* solution to the MPC problem is attained. Unfortunately, the *suboptimal* MPC law is not continuous, even for *linear* systems, and furthermore it is not a unique function of the state ξ as it also depends on the initial guess input sequence. These facts prevent us from establishing readily even *nominal* stability. This point was discussed in [13] in the context of nonlinear MPC to show that suboptimal nonlinear MPC, under appropriate restrictions, is *nominally asymptotically stabilizing.* We note that the suboptimal input computed by PE also depends on the entries contained in the working table and hence the outcome for the same state and initial guess input sequence may be different with different working tables. Thus, in this paper we treat the suboptimal MPC law as a *set-valued* map, and we derive from scratch novel results for *robust exponential stability* of difference inclusions [7] and show that such results apply to PE-MPC.

3.1 General stability results for difference inclusions

Let $F(\cdot)$ be a set-valued map from $\Xi \subseteq \mathbb{R}^n$ to subsets of $\mathbb{R}^n$, with 0 being the equilibrium point, i.e. $F(0) = \{0\}$; let $\phi(\xi, k)$ be *a solution* at time k of the difference inclusion $\xi^+ \in F(\xi)$ starting from an initial condition $\xi(0) = \xi \in \Xi$. Rawlings and Mayne [12, pp.196–203] provide an introduction to MPC with difference inclusions. We also consider a perturbed difference inclusion $\xi^+ \in F(\xi + e) + p$, and we denote with $\phi_{ep}(\xi, k)$ *a solution* to the perturbed difference inclusion at time k with initial condition $\xi(0) = \xi$ for given state and additive disturbance sequences $\mathbf{e}_k = \{e(j)\}_{j=0}^{k-1}$, $\mathbf{p}_k = \{p(j)\}_{j=0}^{k-1}$.

Definition 1 (Exponential Lyapunov function)**.** *A function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is an* exponential Lyapunov *function in the set Ξ for the difference inclusion $\xi^+ \in F(\xi)$ if there exist positive scalars a, a_1, a_2, a_3 such that $\xi \in \Xi$ implies that:*

$$a_1|\xi|^a \leq V(\xi) \leq a_2|\xi|^a, \quad \max_{\xi^+ \in F(\xi)} V(\xi^+) \leq V(\xi) - a_3|\xi|^a .$$

Definition 2 (Exponential Stability)**.** *The origin of the difference inclusion $\xi^+ = F(\xi)$ is said to be* exponentially stable *(ES) on Ξ, $0 \in \Xi$, if there exist positive scalars b and λ, $\lambda < 1$, such that for any $\xi \in \Xi$ all solutions $\phi(\xi, \cdot)$ satisfy:*

$$\phi(\xi, k) \in \Xi, \quad |\phi(\xi, k)| \leq b\lambda^k|\xi| \quad \text{for all } k \in \mathbb{I}_{\geq 0} .$$

We have the following result.

Lemma 1. *If the set Ξ is positively invariant for difference inclusion $\xi^+ = F(\xi)$ and there exists an exponential Lyapunov function V in Ξ, $0 \in \Xi$, then the origin is ES on Ξ.*

Proof: From the definition of V, we have that $\xi \in \Xi$ implies:

$$\max_{\xi^+ \in F(\xi)} V(\xi^+) \leq V(\xi) - a_3|\xi|^a \leq V(\xi) - a_3/a_2 V(\xi) \leq \gamma V(\xi) ,$$

with $\gamma = 1 - a_3/(2a_2)$. Notice that $a_2 \geq a_3$, hence $0 < \gamma < 1$. Since $\phi(\xi, k) \in \Xi$ for all k, we can write: $|\phi(\xi, k)|^a \leq \frac{V(\phi(\xi,k))}{a_1} \leq \frac{\gamma^k V(\xi)}{a_1} \leq \frac{\gamma^k a_2 |\xi|^a}{a_1}$. From this, we obtain: $|\phi(\xi, k)| \leq b\lambda^k|\xi|$ in which $\lambda = \gamma^{1/a}$ and $b = \left(\frac{a_2}{a_1}\right)^{1/a}$, and we observe that $0 < \lambda < 1$. □ □

Definition 3 (Robust Exponential Stability)**.** *The origin of the difference inclusion $\xi^+ = F(\xi)$ is said to be* robustly exponentially stable *(RES) on Ξ, $0 \in \Xi$, with respect to state and additive disturbances if there exist positive scalars b and λ, $\lambda < 1$, and for each $\epsilon > 0$ there exists $\delta > 0$ such that for all $\xi \in \Xi$ and all disturbance sequences $\mathbf{e}_k, \mathbf{p}_k$ satisfying:*

$$0 < \max\{\|\mathbf{e}_k\|, \|\mathbf{p}_k\|\} \leq \delta, \quad \phi_{ep}(\xi, k) \in \Xi \text{ for all } k \ ,$$

the perturbed solutions $\phi_{ep}(\xi, \cdot)$ satisfy $|\phi_{ep}(\xi, k)| \leq b\lambda^k|\xi| + \epsilon$.

Lemma 2. *If there exists a* continuous *exponential Lyapunov function V on Ξ, then the origin of the difference inclusion $\xi^+ \in F(\xi)$ is RES on Ξ w.r.t. state and additive disturbances.*

Proof: From the proof of Lemma 1, we have that for all $\xi \in \Xi \setminus 0$ the following strict inequality holds: $\max_{\xi^+ \in F(\xi)} V(\xi^+) < \gamma V(\xi)$ for some $\gamma < 1$. We now require the following Lemma proved in Appendix A.3.

Lemma 3. *For every $\mu > 0$, there exists $\delta > 0$ such that, for all $(\xi, e, p) \in \Xi \times \delta\mathbb{B} \times \delta\mathbb{B}$ that satisfy $\xi + e \in \Xi$ and $F(\xi + e) + p \subseteq \Xi$, the following condition holds: $V(\xi^+) \leq \max\{\gamma V(\xi), \mu\}$ for all $\xi^+ \in F(\xi + e) + p$.*

Now, assume that $\phi_{ep}(\xi, k) \in \Xi$ for all $k \in \mathbb{I}_{\geq 0}$. Then, by induction we can write: $a_1|\phi_{ep}(\xi, k)|^a \leq V(\phi_{ep}(\xi, k)) \leq \max\{\gamma^k V(\xi), \mu\} \leq \max\{\gamma^k a_2|\xi|^a, \mu\}$, from which we obtain: $|\phi_{ep}(\xi, k)| \leq \max\{b\lambda^k|\xi|, (\mu/a_1)^{1/a}\}$ in which $\lambda = \gamma^{1/a}$ and $b = (a_2/a_1)^{1/a}$. Finally, we define $\epsilon = (\mu/a_1)^{1/a}$ and write: $|\phi_{ep}(\xi, k)| \leq \max\{b\lambda^k|\xi|, \epsilon\} \leq b\lambda^k|\xi| + \epsilon$, which completes the proof by noticing, as in the proof of Lemma 1, that $\lambda < 1$. □ □

Remark 5. *The definition of RES and also that of Robust Asymptotic Stability (RAS) in [6] assume that there exist "nice" and small perturbation sequences $\mathbf{e}_k, \mathbf{p}_k$ such that the perturbed solutions $\phi_{ep}(\xi, \cdot)$ remain in Ξ at all times. The next definition only assumes that the perturbations are sufficiently small, while feasibility of $\phi_{ep}(\xi, \cdot)$ is implied.*

Definition 4 (Strong Robust Exponential Stability)**.** *The origin of the difference inclusion $\xi^+ \in F(\xi)$ is said to be* strongly robustly exponentially stable *(SRES) on Ξ, $0 \in \Xi$, with respect to state and additive disturbances if there exist a compact set $\mathcal{C} \subseteq \Xi$ and positive scalars b and λ, $\lambda < 1$, and for each $\epsilon > 0$ there exists $\delta > 0$ such that for all $\mathbf{e}_k, \mathbf{p}_k$ satisfying:*

$$\max\{\|\mathbf{e}_k\|, \|\mathbf{p}_k\|\} \leq \delta\,,$$

for all $\xi \in \mathcal{C}$ the perturbed solutions $\phi_{ep}(\xi, \cdot)$ satisfy:

$$\phi_{ep}(\xi, k) \in \mathcal{C} \text{ for all } k, \quad |\phi_{ep}(\xi, k)| \leq b\lambda^k|\xi| + \epsilon\,.$$

Theorem 1 (Strong Robust Exponential Stability)**.** *If $0 \in \operatorname{int}\Xi$ and there exists a* continuous *exponential Lyapunov function V on Ξ, then the origin of the difference inclusion $\xi^+ \in F(\xi)$ is SRES on Ξ with respect to state and additive disturbances.*

Proof: Let c be a positive scalar such that $\mathcal{C} = \{\xi \in \mathbb{R}^n : V(\xi) \leq c\} \subseteq \Xi$, i.e. $\mathcal{C}$ is a sublevel set contained in Ξ. Notice that Ξ is compact and that $0 \in \operatorname{int}\mathcal{C}$. From the proof of Lemma 1 and from Lemma 3 we have that for each $\mu > 0$, there exists a $\delta_1 > 0$ such that the following condition holds:

$$\max_{\xi^+ \in F(\xi+e)+p} V(\xi^+) \leq \max\{\gamma V(\xi), \mu\} \leq \gamma V(\xi) + \mu$$

for all $(\xi, e, p) \in \Xi \times \delta_1\mathbb{B} \times \delta_1\mathbb{B}$ that satisfy $\xi + e \in \Xi$ and $F(\xi + e) + p \in \Xi$, in which $0 < \gamma < 1$. We also have that for each $\rho > 0$, the condition $\max_{\xi^+ \in F(\xi+e)+p} V(\xi^+) < V(\xi) - \rho$ holds for all $(\xi, e, p) \in \Xi \times \delta_1\mathbb{B} \times \delta_1\mathbb{B}$ that also satisfy $V(\xi) > r^* = \frac{\mu+\rho}{1-\gamma}$. Define the sublevel set $\mathcal{R} = \{\xi : V(\xi) \leq r^*\}$ and choose μ and ρ sufficiently small that $\mathcal{R} \subset \mathcal{C}$ (hence $\mathcal{R} \subset \Xi$). We now observe that for all $\mathbf{e}_k, \mathbf{p}_k$ satisfying $\max\{\|\mathbf{e}_k\|, \|\mathbf{p}_k\|\} \leq \delta_1$, the following conditions hold: (i) if $\xi \in \mathcal{R}$, it follows that $\max_{\xi^+ \in F(\xi+e)+p} V(\xi^+) \leq \gamma r^* + \mu \leq \frac{\mu+\gamma\rho}{1-\gamma} \leq r^*$, and hence the solutions $\phi_{ep}(\xi, k)$ remain in $\mathcal{R}$ for all $k \in \mathbb{I}_{\geq 0}$, (ii) if $\xi \in \mathcal{C} \setminus \mathcal{R}$, the solutions $\phi_{ep}(\xi, k)$ remain in $\mathcal{C}$ for all $k \in \mathbb{I}_{\geq 0}$ and enter $\mathcal{R}$ in finite time. This proves that there exists a compact set $\mathcal{C} \subseteq \Xi$ such that if $\xi \in \mathcal{C}$ the solutions $\phi_{ep}(\xi, k)$ remain in $\mathcal{C}$ at all times. We can apply Lemma 2 to show that there exist positive scalars b and λ, $\lambda < 1$, and for each $\epsilon > 0$ there exists a δ, $0 < \delta \leq \delta_1$ such $|\phi_{ep}(\xi, k)| \leq b\lambda^k|\xi| + \epsilon$ for all $\mathbf{e}_k, \mathbf{p}_k$ satisfying $\max\{\|\mathbf{e}_k\|, \|\mathbf{p}_k\|\} \leq \delta$. □ □

3.2 Nominal and robust stability of Partial Enumeration MPC

We now prove the main (nominal and robust) stability results of the closed-loop using Partial Enumeration MPC. In the following results we denote by $\tilde{\mathbf{u}} = [\tilde{u}(0)', \tilde{u}(1)', \dots, \tilde{u}(N-1)']'$ the suboptimal control sequence returned by Algorithm 1 (or 2) for a given initial state $\tilde{x}(0) = \tilde{x}$. Furthermore the target pair $(\bar{x}, \bar{u})$ is regarded as fixed, for the results in this section.

Lemma 4. *The set $\tilde{\mathbb{X}}$ is positively invariant for the closed-loop system $\tilde{x}^+ = A\tilde{x} + B\tilde{u}(0)$.*

Proof: Let $\tilde{\mathbf{u}} = (\tilde{u}(0), \tilde{u}(1), \dots, \tilde{u}(N-1))$ be the solution computed by Algorithm 1 for the initial state $\tilde{x}(0) = \tilde{x}$, and assume that $\tilde{x} \in \tilde{\mathbb{X}}$. Then, for $\tilde{x}^+ = A\tilde{x} + B\tilde{u}(0)$ we consider the sequence $\tilde{\mathbf{u}}_+ = (\tilde{u}(1), \tilde{u}(2), \dots, \tilde{u}(N-1), 0)$, and observe that it is feasible w.r.t. to $D\tilde{u} \le b_u - D\bar{u}$ and such that $S_u'(A^N\tilde{x}^+ + A^{N-1}B\tilde{u}(0) + \cdots + B\tilde{u}(N-1)) = 0$. Thus, $\tilde{x}^+ \in \tilde{\mathbb{X}}$. □ □

Lemma 5. *There exists a constant $c > 0$ such that the (sub)optimal solution $\tilde{\mathbf{u}}$ computed by Algorithm 1 for the initial state $\tilde{x}(0) = \tilde{x}$ satisfies $|\tilde{\mathbf{u}}| \le c|\tilde{x}|$ for all $\tilde{x} \in \tilde{\mathbb{X}}$.*

Proof: We first consider any $\tilde{x} \in \mathbb{B}_r \subset \tilde{\mathbb{X}}$ and show that the result holds provided that $r > 0$ is sufficiently small. If the optimal solution is found in the table, i.e. from Line 6 and when $\tilde{x}$ is sufficiently small, the optimal solution $\tilde{\mathbf{u}}$ coincides with the "unconstrained" minimizer $\tilde{\mathbf{u}}^* = \hat{\mathbf{\Gamma}}\tilde{x}$ because such point gives the lowest cost and it is feasible w.r.t. to the constraint: $\mathbf{D}\tilde{\mathbf{u}} + \mathbf{C}\bar{u} \le \mathbf{d}$ because $0 < \mathbf{d} - \mathbf{C}\bar{u}$. If instead a *suboptimal* solution is computed from Line 19, since $\tilde{\mathbf{u}}^*$ is feasible for $\tilde{x}$ sufficiently small we have that $t = 1$, i.e. the computed solution is again the "unconstrained" minimizer. Therefore, there exist positive constants c' and r such that $|\tilde{\mathbf{u}}| \le c'|\tilde{x}|$ for all $\tilde{x} \in \mathbb{B}_r$.
To complete the proof, we take account of the remaining case in which $|\tilde{x}| \ge r$. We define $\eta = \max_{\tilde{\mathbf{u}} \in \tilde{\mathbb{U}}^N} |\tilde{\mathbf{u}}|$ (which is finite because $\mathbb{U}$ is compact and so are $\tilde{\mathbb{U}}$ and $\tilde{\mathbb{U}}^N$) and set $c = \max(c', \eta/r)$. We then have that $\tilde{\mathbf{u}}$ computed by Algorithm 1 satisfies the required inequality $|\tilde{\mathbf{u}}| \le c|\tilde{x}|$ because: (i) for $|\tilde{x}| < r$ we have $|\tilde{\mathbf{u}}| \le c'|\tilde{x}| \le c|\tilde{x}|$ as discussed above; (ii) for $|\tilde{x}| \ge r$, we have $|\tilde{\mathbf{u}}| \le \eta \le \eta(|\tilde{x}|/r) \le c|\tilde{x}|$. □ □

Theorem 2 (ES under PE-MPC)**.** *The origin of the closed-loop system $\tilde{x}^+ = A\tilde{x} + B\tilde{u}(0)$, is ES on $\tilde{\mathbb{X}}$.*

Proof: We denote with $\tilde{\mathbf{u}}^+$ the suboptimal solution computed for the successor state $\tilde{x}^+$, and we observe that: $\tilde{x}^+ = A\tilde{x} + B[I\ 0]\tilde{\mathbf{u}}$ and $\tilde{\mathbf{u}}^+ \in G(\tilde{x}, \tilde{\mathbf{u}})$ in which $G(\cdot)$ is a set-valued map, because $\tilde{\mathbf{u}}^+$ depends on: the state $\tilde{x}^+ = A\tilde{x} + B[I\ 0]\tilde{\mathbf{u}}$, the initial guess obtained by shifting $\tilde{\mathbf{u}}$ but also on the current working table. The following conditions hold. i) $V_N(\tilde{x}, \tilde{\mathbf{u}}) \ge a_1|(\tilde{x}, \tilde{\mathbf{u}})|^2$ for some $a_1 > 0$ because $V_N(\cdot)$ is quadratic and strictly convex in its arguments, as discussed at the end of Appendix A.1. Similarly, there exists some $a_2 > 0$ such that $V_N(\tilde{x}, \tilde{\mathbf{u}}) \le a_2|(\tilde{x}, \tilde{\mathbf{u}})|^2$. ii) There exists a $c > 0$ such that $|\tilde{\mathbf{u}}| \le c|\tilde{x}|$ for all $\tilde{x} \in \tilde{\mathbb{X}}$, from Lemma 5. iii) $V_N(\tilde{x}^+, \tilde{\mathbf{u}}^+) - V_N(\tilde{x}, \tilde{\mathbf{u}}) \le -\frac{1}{2}(\tilde{x}'Q\tilde{x} + \tilde{u}(0)'R\tilde{u}(0)) \le -\tilde{a}_3|(\tilde{x}, \tilde{u}(0))|^2 \le -\tilde{a}_3|\tilde{x}|^2 \le -a_3|(\tilde{x}, \tilde{\mathbf{u}})|^2$ for $a_3 = \frac{1}{2}\tilde{a}_3 \min\{1, 1/c^2\}$ for some $\tilde{a}_3 > 0$, where in the last inequality we have used that $|\tilde{x}| \ge \frac{1}{c}|\tilde{\mathbf{u}}|$. Thus, we have that $V_N(\tilde{x}, \tilde{\mathbf{u}})$ is an exponential Lyapunov function for the *extended* closed-loop system, expressed as a difference inclusion: $(\tilde{x}^+, \tilde{\mathbf{u}}^+) \in F(\tilde{x}, \tilde{\mathbf{u}})$. We also have that $\tilde{\mathbb{X}} \times \tilde{\mathbb{U}}^N$ is forward invariant for the extended closed-loop difference inclusion. Hence, there exist positive scalars $\tilde{b}$ and λ, $\lambda < 1$, such that for all initial extended state $(\tilde{x}, \tilde{\mathbf{u}}) \in \tilde{\mathbb{X}} \times \tilde{\mathbb{U}}^N$ the following condition holds for all $k \in \mathbb{I}_{\geq 0}$: $|(\tilde{x}(k), \tilde{u}(k))| \leq \tilde{b}\lambda^k |(\tilde{x}, \tilde{u}(0))|$. If we denote $\phi(\tilde{x}, k) = \tilde{x}(k)$, we can now write: $|\phi(\tilde{x}, k)| \leq |(\tilde{x}(k), \tilde{u}(k))| \leq \tilde{b}\lambda^k |(\tilde{x}, \tilde{u}(0))| \leq b\lambda^k |\tilde{x}|$, with $b = \tilde{b}(1 + c)$. □ □

Theorem 3 (SRES under PE-MPC)**.** *The origin of the closed-loop system* $\tilde{x}^+ = A\tilde{x} + B\tilde{u}(0)$*, is SRES on* $\tilde{\mathbb{X}}$ *with respect to state and additive disturbances.*

Proof: From the proof of Theorem 2, we have that $V_N(\tilde{x}, \tilde{\mathbf{u}})$ is an exponential Lyapunov function and $\tilde{\mathbb{X}} \times \tilde{\mathbb{U}}^N$ is forward invariant for the *nominal* extended difference inclusion $(\tilde{x}^+, \tilde{\mathbf{u}}^+) \in F(\tilde{x}, \tilde{\mathbf{u}})$. Also $V_N(\cdot)$ is trivially continuous (it is quadratic in $\tilde{x}$ and $\tilde{\mathbf{u}}$), and hence we can apply the result of Lemma 2 to obtain that the origin of the *extended* closed-loop difference inclusion is SRES on $\tilde{\mathbb{X}} \times \tilde{\mathbb{U}}^N$. Notice that we can express the compact set appearing in the definition of SRES as $\mathcal{C} \times \tilde{\mathbb{U}}^N$ because $\tilde{\mathbb{U}}^N$ is compact and the PE algorithm always computes a feasible solution. If we denote with $\phi_{ep}(\tilde{x}, k) = \tilde{x}(k)$, there exist positive scalars $\tilde{b}$ and λ, $\lambda < 1$, and for each $\epsilon > 0$, there exists a $\delta > 0$ such that for any $(\tilde{x}, \tilde{u}(0)) \in \mathcal{C} \times \tilde{\mathbb{U}}^N$, we can write: $|(\phi_{ep}(\tilde{x}, k), \tilde{u}(k))| \leq \tilde{b}\lambda^k |(\tilde{x}, \tilde{u}(0))| + \epsilon$ for all state and additive disturbance sequences with norm less than δ. Hence for any $\tilde{x} \in \mathcal{C}$, we can now write: $|\phi_{ep}(\tilde{x}, k)| \leq |(\phi_{ep}(\tilde{x}, k), \tilde{u}(k))| \leq \tilde{b}\lambda^k |(\tilde{x}, \tilde{u}(0))| + \epsilon \leq b\lambda^k |\tilde{x}| + \epsilon$, with $b = \tilde{b}(1 + c)$. □ □

4 Application to an unstable CSTR

4.1 Process description

We consider a Continuous Stirred Tank Reactor (CSTR), in which the irreversible exothermic reaction A → B takes place in the liquid phase and heat is removed from a cooling jacket. The system is described by [8]:

$$\begin{aligned}
\frac{dh}{dt} &= \frac{F_i - F}{S} \\
\frac{dc_A}{dt} &= \frac{F_i(c_{Ai} - c_A)}{Sh} - k_0 \exp\left(-\frac{E}{T} c_A\right) \\
\frac{dT}{dt} &= \frac{F_i(T_i - T)}{Sh} + \frac{-\Delta H_r k_0 \exp\left(-\frac{E}{T}\right) c_A}{\rho C_p} - \frac{UP(T - T_c)}{S\rho C_p}
\end{aligned} \tag{11}$$

The controlled variables are the liquid level h, and the reactor temperature T. The third (unmeasured) variable is the molar concentration of A, denoted by c_A. The manipulated variables are the outlet flow rate F and the coolant temperature T_c. It is assumed that the inlet flow rate F_i, temperature T_i and concentration c_{Ai} act as unmeasured disturbances. The model parameters are described in Table 1. In reference conditions the two manipulated variables assume the following values: $F = \bar{F} = 0.10$ m^3/min, $T_c = \bar{T}_c = 300$ K, and consequently the CSTR model admits three steady states. We consider the problem of controlling the CSTR around the middle-conversion unstable steady state, associated to the following state values: $h = \bar{h} = 0.664$ m, $c_A = \bar{c}_A = 0.50$ kmol/m^3, $T = \bar{T} = 350$ K.

Table 1: CSTR parameters – (*) nominal values.

Parameter	Symbol	Value
Inlet flow rate	F_i	0.10 m^3/min (*)
Reactor base area	S	0.151 m^2
Inlet concentration	c_{Ai}	1 kmol/m^3 (*)
Arrhenius frequency factor	k_0	$7.2 \cdot 10^{10}$ 1/min
Normalized activation energy	E	8750 K
Heat transfer coefficient	U	54.75 kJ/(min $\cdot$ m^2 $\cdot$ K)
Inlet temperature	T_i	350 K (*)
Reaction heat	ΔH_r	$-5 \cdot 10^4$ kJ/kmol
Liquid density	ρ	1000 kg/m^3
Heat capacity	C_p	0.239 kJ/(kg $\cdot$ K)
Reactor base perimeter	P	1.376 m

4.2 Offset-free MPC implementation

In the controller we use the linear discrete-time model:

$$\begin{aligned} x^+ &= \begin{bmatrix} 1.00 & -0.0730 & -0.145 \\ 0 & 0.977 & 0.0388 \\ 0 & 0 & 1.16 \end{bmatrix} x + \begin{bmatrix} -0.00806 & 0.0995 \\ 0.165 & -0.0424 \\ -0.00995 & -0.137 \end{bmatrix} u \\ y &= \begin{bmatrix} 0.0945 & -0.299 & 0.162 \\ 1.12 & 0.0215 & -0.0571 \end{bmatrix} x \end{aligned} \tag{12}$$

obtained via closed-loop identification using a sampling time of 3 sec. The (scaled) input and output vectors are:

$$u = \begin{bmatrix} \frac{F-\bar{F}}{S_1} \\ \frac{T_c-\bar{T}_c}{S_2} \end{bmatrix}, \quad y = \begin{bmatrix} \frac{h-\bar{h}}{S_3} \\ \frac{T-\bar{T}}{S_4} \end{bmatrix}$$

in which the scaling factors are: $S_1 = 0.1$ m^3/min, $S_2 = 5$ K, $S_3 = 0.5$ m, $S_4 = 5$ K. We consider: $u_{\max} = -u_{\min} = [1\ 1]'$.

Due to the inherent model error between the "true" nonlinear plant (11) and the linear model (12), and since we do not measure the states of (12), we implement an output feedback offset-free MPC as described in [8]. A steady-state Kalman filter is used to compute the state estimates of the following augmented system:

$$\begin{aligned} \hat{x}^+ &= A\hat{x} + Bu + B_d\hat{d} + w_x \\ \hat{d}^+ &= \hat{d} + w_d \\ y &= C\hat{x} + C_d\hat{d} + w_y \end{aligned} \tag{13}$$

in which $\hat{d}$ is the (fictitious) integrating disturbance state, and w_x, w_d, w_y are uncorrelated Gaussian zero mean random variables. The disturbance model matrices are: $B_d = B$, $C_d = 0$, and the Kalman filter gain for (13) is computed assuming the noise covariances: $\mathcal{E}(w_x w_x') = C'C$, $\mathcal{E}(w_d w_d') = I$, $\mathcal{E}(w_y w_y') = 5 \cdot 10^{-2} I$. Let $(\hat{x}, \hat{d})$ be the augmented state

Table 2: Performance and timing results: non-linear plant with disturbances.

Controller	J	Aver. CPU time	Max. CPU time	Opt. Rate
QP-MPC	497.2	381 ms	27800 ms	–
PE1-MPC	500.4	4.2 ms	138 ms	0.907
PE10-MPC	500.4	2.6 ms	139 ms	0.960
PE25-MPC	500.4	2.0 ms	140 ms	0.977
PE50-MPC	500.4	1.9 ms	148 ms	0.980
PE200-MPC	500.4	1.8 ms	152 ms	0.983

estimates at a given time, and let $\bar{y}$ be desired (not necessarily reachable) controlled-variable setpoint. We solve the constrained target calculation problem to compute the reachable state and input targets:

$$\min_{\bar{x},\bar{u}} (Cx_s + C_d\hat{d} - \bar{y})'\bar{Q}(Cx_s + C_d\hat{d} - \bar{y}) + \bar{u}'\bar{R}\bar{u}$$
$$\text{s.t. } \bar{x} = A\bar{x} + B\bar{u} + B_d\hat{d}, \quad u_{\min} \leq \bar{u} \leq u_{\max} \quad (14)$$

in which we use the following weighting matrices $\bar{Q} = I$, $\bar{R} = 10^{-3}I$. Finally, we define the deviation variables: $\tilde{x} = x - \bar{x}$, $\tilde{u} = u - \bar{u}$ and we readily recover the constrained control problem formulation (2) by noticing that $\tilde{x}^+ = A\tilde{x} + B\tilde{u}$. In the control problem we use: $N = 100$, $Q = C'C$, $R = 1.26I$ and we notice that the terminal constraint matrix S_u has two columns because the system (12) has two unstable modes (a marginally unstable mode associated to the tank level and purely unstable mode associated to operating point).

4.3 Simulation results and comments

We present the simulation results obtained by controlling the nonlinear system (11) with the MPC algorithm described in the previous paragraph over a period of 6 hours ($N_s =$ 7200 sample times). Different controllers are considered: QP-MPC uses an active set QP solver, PEM-MPC (with $M = 1$, 10, 25, 50 or 200) use the PE Algorithm 2 with a table (initially empty) of M entries . The scaled level and temperature measurements (fed to the Kalman filter) are assumed to be corrupted by uncorrelated random Gaussian noise with covariance $10^{-4}I$. Setpoint changes occur randomly on each output, with a probability of 0.005 (i.e., on average, one every 10 minutes), whereas random disturbances on F_i, c_{Ai} and T_i occur with a probability of 0.05 (i.e., on average, one every minute). To compare the performance of the different controllers we evaluate the closed-loop cost over the simulation period: $J = \frac{1}{2}\sum_{k=0}^{N_s}(y(k)-y^s(k))'(y(k)-y^s(k))+(u(k)-\bar{u}(k))'R(u(k)-\bar{u}(k))$ in which $y^s(k) = C\bar{x}(k) + C_d\hat{d}(k)$ is the reachable output target at time k and $\bar{u}(k)$ is the corresponding input target. Performance results are reported in Table 2, along with the average and maximum CPU time required to solve (2) (using Octave 3.2.3 on a 2.53 GHz MacBook Pro), and with the optimality rate. We also report in Table 3 the corresponding

Table 3: Performance and timing results: linear plant without disturbances.

Controller	J	Aver. CPU time	Max. CPU time	Opt. Rate
QP-MPC	274.5	118 ms	22800 ms	–
PE1-MPC	273.4	1.5 ms	74.2 ms	0.979
PE10-MPC	273.4	1.5 ms	76.5 ms	0.980
PE25-MPC	273.4	1.5 ms	73.5 ms	0.981
PE50-MPC	273.4	1.2 ms	76.5 ms	0.991
PE200-MPC	273.4	1.2 ms	76.8 ms	0.991

results that one would obtain in the nominal case, i.e. when the plant is exactly (12) and no disturbances and/or noise are present.

From Tables 2 and 3 we immediately observe that the difference in performance between using optimal (QP-based) and suboptimal (PE-based) is negligible[2]. Furthermore all (optimal and suboptimal) controllers show a large degree of robustness (see Table 2). It is also interesting to observe that when the table size increases, as expected, the optimality rate increases. The differences in (average and maximum) CPU times between QP-based and PE-based MPC are particularly noticeable (roughly two orders of magnitude)[3]. The effect of the table size on timing is a bit more complicated to analyze. While a larger table slightly increases the maximum CPU time, it decreases the average CPU time. To understand these results it must be kept in mind that the table scanning time is *only* a fraction of the overall CPU time, as a relevant portion of it is spent to compute a suboptimal solution (as detailed in Algorithm 2) when the optimal solution is not in the table. Thus, since the use of a larger table implies a higher optimality rate, PE resorts less frequently to the suboptimal computation steps. Nonetheless, J is essentially identical for all PE-based MPC, and this occurs because the sub-optimal steps described by Algorithm 2, which are executed when the current table does not contain the optimal tuple, often returns the optimal solution.

5 Conclusions

In this paper we revised the Partial Enumeration (PE) method for solving more efficiently the constrained optimal control problem that arises in linear MPC, especially for large-scale systems [9, 10].

We proved here that such suboptimal MPC algorithm is nominally exponentially stabilizing, and most importantly it is exponentially stabilizing in the presence of arbitrary (but sufficiently small) perturbations. Such novel results are based on considering the

[2]In the nominal case of Table 3, the fact that optimal (QP-based) MPC shows worse performance than suboptimal (PE-based) MPC occurred "by accident", due to the use of *finite* horizon MPC and because a new set-point change may occur before the variables have settled from the previous set-point change.

[3]Octave uses a null-space active set QP solver. Such solvers usually have good average performance but may experience large CPU time in some "bad" cases.

(non-unique and discontinuous) suboptimal control law $u = \kappa_N(x)$ as a set-valued map. Consequently, the closed-loop system is described by a difference inclusion, and we proved in this paper nominal and robust exponential stability for arbitrary perturbations under two main assumptions: the origin of the closed-loop system is in the interior of its region of attraction and a continuous Lyapunov function exists for the difference inclusion. Then, we showed that the closed-loop system (appropriately augmented) obtained using the proposed PE-MPC algorithm satisfies such assumptions.

We presented an application to the control of a nonlinear unstable CSTR in which the designed suboptimal (PE-based) controllers successfully faced the perturbations inherently generated by the nonlinear process vs. linear model mismatch, by process parameter disturbances and by measurement noise. Despite the presence of such upsetting disturbances closed-loop stability is maintained and the performance difference with respect to optimal (QP-based) MPC is negligible, while the CPU times are two orders of magnitude lower.

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A Supporting material and proofs

A.1 Derivation of the time invariant terms in (3)

We first compute state sequence vector as:

$$\begin{bmatrix} \tilde{x}(0) \\ \tilde{x}(1) \\ \tilde{x}(2) \\ \vdots \\ \tilde{x}(N) \end{bmatrix} = \begin{bmatrix} I \\ A \\ A^2 \\ \vdots \\ A^N \end{bmatrix} \tilde{x} + \begin{bmatrix} 0 & \cdots & & 0 \\ B & 0 & & \vdots \\ AB & B & \ddots & \\ \vdots & & \ddots & 0 \\ A^{N-1}B & A^{N-2}B & \cdots & B \end{bmatrix} \begin{bmatrix} \tilde{u}(0) \\ \tilde{u}(1) \\ \vdots \\ \tilde{u}(N-1) \end{bmatrix},$$

concisely written as $\mathbf{w} = \mathbf{A}\tilde{x} + \mathbf{B}\tilde{\mathbf{u}}$. Next, we define:

$$\mathbf{Q} = \begin{bmatrix} Q & 0 & \cdots & & 0 \\ 0 & Q & \ddots & & \vdots \\ \vdots & \ddots & \ddots & & \\ & & & Q & 0 \\ 0 & \cdots & & 0 & P \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} R & 0 & \cdots & 0 \\ 0 & R & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & R \end{bmatrix},$$

and compute the objective function as:

$$\begin{aligned} V_N(\tilde{x}, \tilde{\mathbf{u}}) &= \tfrac{1}{2}(\mathbf{w}'\mathbf{Q}\mathbf{w} + \tilde{\mathbf{u}}'\mathbf{R}\tilde{\mathbf{u}}) \\ &= \tfrac{1}{2}\left((\mathbf{A}\tilde{x} + \mathbf{B}\tilde{\mathbf{u}})'\mathbf{Q}(\mathbf{A}\tilde{x} + \mathbf{B}\tilde{\mathbf{u}}) + \tilde{\mathbf{u}}'\mathbf{R}\tilde{\mathbf{u}}\right) \\ &= \tfrac{1}{2}\tilde{\mathbf{u}}'\mathbf{H}\tilde{\mathbf{u}} + \tilde{\mathbf{u}}'\mathbf{G}\tilde{x} + \tfrac{1}{2}\tilde{x}'\mathbf{P}\tilde{x} \end{aligned}$$

in which $\mathbf{H} = \mathbf{B}'\mathbf{Q}\mathbf{B} + \mathbf{R}$, $\mathbf{G} = \mathbf{B}'\mathbf{Q}\mathbf{A}$, and $\mathbf{P} = \mathbf{A}'\mathbf{Q}\mathbf{A}$. The inequality constraint terms instead are defined as follows:

$$\mathbf{D} = \begin{bmatrix} D & 0 & \cdots & 0 \\ 0 & D & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & D \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} D & \\ D & \\ \vdots & \vdots \\ D & \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} d \\ d \\ \vdots \\ d \end{bmatrix},$$

whereas the equality constraints term are defined as:

$$\mathbf{E} = S_u' \begin{bmatrix} A^{N-1}B & A^{N-2}B & & B \end{bmatrix}, \quad \mathbf{F} = S_u' A^N.$$

We finally notice that $V_N(\tilde{x}, \tilde{\mathbf{u}}) = \frac{1}{2}\begin{bmatrix} \tilde{x} \\ \tilde{\mathbf{u}} \end{bmatrix}' \begin{bmatrix} \mathbf{H} & \mathbf{G}' \\ \mathbf{G} & \mathbf{P} \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{\mathbf{u}} \end{bmatrix}$, in which observe that the inner matrix is positive definite.

A.2 Solution to the KKT system (4)

Let $\mathcal{A} = \begin{bmatrix} \mathbf{D}_a \\ \mathbf{E} \end{bmatrix}$ and let $\mathcal{Z}$ be a full rank matrix such $\mathcal{A}\mathcal{Z} = 0$. Consider the point $\tilde{\mathbf{u}}_0 = \mathcal{A}^+ \begin{bmatrix} \mathbf{d}_a - \mathbf{C}_a\bar{u} \\ -\mathbf{F}\tilde{x} \end{bmatrix} = \mathcal{A}_1(\mathbf{d}_a - \mathbf{C}_a\bar{u}) + \mathcal{A}_2\tilde{x}$, in which $\mathcal{A}^+$ is the pseudo-inverse of $\mathcal{A}$. Notice that we can express any point that is feasible for (4b)-(4c) as $\tilde{\mathbf{u}} = \tilde{\mathbf{u}}_0 + \mathcal{Z}\mathbf{p}$, and thus rewrite (4a) as follows:

$$\mathbf{H}\mathcal{Z}\mathbf{p} + \mathbf{H}\tilde{\mathbf{u}}_0 + \mathbf{G}\tilde{x} + \mathcal{A}' \begin{bmatrix} \lambda^0 \\ \mu^0 \end{bmatrix} = 0 \,.$$

Next, we multiply (on the left) by $\mathcal{Z}'$ (recalling that $\mathcal{Z}'\mathcal{A}' = 0$) and solve for $\mathbf{p}$ to obtain

$$\begin{aligned}\mathbf{p} &= -\mathcal{H}^{-1}\mathcal{Z}'\mathbf{H}\tilde{\mathbf{u}}_0 - \mathcal{H}^{-1}\mathcal{Z}'\mathbf{G}\tilde{x} \\ &= \mathcal{B}_1(\mathbf{d}_a - \mathbf{C}_a\bar{u}) + \mathcal{B}_2\tilde{x} \,,\end{aligned}$$

with $\mathcal{H} = \mathcal{Z}'\mathbf{H}\mathcal{Z}$, $\mathcal{B}_1 = -\mathcal{H}^{-1}\mathcal{Z}'\mathbf{H}\mathcal{A}_1$,$\mathcal{B}_2 = -\mathcal{H}^{-1}\mathcal{Z}'(\mathbf{H}\mathcal{A}_2 + \mathbf{G})$. Finally, we compute $\tilde{\mathbf{u}}$ as follows:

$$\begin{aligned}\tilde{\mathbf{u}} &= \tilde{\mathbf{u}}_0 + \mathcal{Z}\mathbf{p} \\ &= (\mathcal{A}_1 + \mathcal{Z}\mathcal{B}_1)(\mathbf{d}_a - \mathbf{C}_a\bar{u}) + (\mathcal{A}_2 + \mathcal{Z}\mathcal{B}_2)\tilde{x} \\ &= \mathbf{\Gamma}_u(\mathbf{d}_a - \mathbf{C}_a\bar{u}) + \mathbf{\Gamma}_x\tilde{x} \,.\end{aligned}$$

Now (4a) can be solved for (λ^0, μ^0) as follows:

$$\begin{aligned}\begin{bmatrix} \lambda^0 \\ \mu^0 \end{bmatrix} &= -(\mathcal{A}')^{+}(\mathbf{H}\tilde{\mathbf{u}} + \mathbf{G}\tilde{x}) \\ &= \mathcal{C}_1(\mathbf{d}_a - \mathbf{C}_a\bar{u}) + \mathcal{C}_2\tilde{x} \,,\end{aligned}$$

with $\mathcal{C}_1$, $\mathcal{C}_2$ suitably defined. Finally, we write the Primal and Dual inequalities (4e) and (4d) as follows:

$$\begin{bmatrix} \mathbf{C}_i - \mathbf{D}_i\mathbf{\Gamma}_u\mathbf{C}_a & \mathbf{D}_i\mathbf{\Gamma}_x \\ \begin{bmatrix} I & 0 \end{bmatrix}\mathcal{C}_1\mathbf{C}_a & -\begin{bmatrix} I & 0 \end{bmatrix}\mathcal{C}_2 \end{bmatrix} \begin{bmatrix} \bar{u} \\ \tilde{x} \end{bmatrix} \leq \begin{bmatrix} \mathbf{d}_i - \mathbf{D}_i\mathbf{\Gamma}_u\mathbf{d}_a \\ \begin{bmatrix} I & 0 \end{bmatrix}\mathcal{C}_1\mathbf{d}_a \end{bmatrix} ,$$

or more concisely as

$$\begin{bmatrix} \mathbf{\Psi}_P \\ \mathbf{\Psi}_D \end{bmatrix} z \leq \begin{bmatrix} \psi_P \\ \psi_D \end{bmatrix} .$$

A.3 Proof of Lemma 3

Proof: Given $\mu > 0$, choose π satisfying $0 < \pi < (\mu/(2a_2))^{1/a}$ and define the sets: $\Xi_1 = \{\xi \in \Xi : \ |\xi| \leq \pi\}$ and $\Xi_2 = \{\xi \in \Xi : \ |\xi| \geq \pi\}$. Choose $\delta_1 > 0$ such that $|\xi + e| \leq (\mu/(2a_2))^{1/a}$ for all $(\xi, e) \in \Xi_1 \times \delta_1\mathbb{B}$. We then have $a_2|\xi + e|^a \leq \mu/2$ and hence:

$$\max_{\xi^+ \in F(\xi+e)} V(\xi^+) \leq V(\xi + e) \leq \mu/2$$

for all $(\xi, e) \in \Xi_1 \times \delta_1\mathbb{B}$. By continuity of V, choose $\delta_2, \delta_3 > 0$ such that

$$\max_{\xi^+ \in F(\xi+e)+p} V(\xi^+) \leq \max_{\xi^+ \in F(\xi+e)} V(\xi^+) + \mu/2$$

for all $(\xi, e, p) \in \Xi_1 \times \delta_2\mathbb{B} \times \delta_3\mathbb{B}$. Adding together this inequality and the previous one, we obtain: $\max_{\xi^+ \in F(\xi+e)+p} V(\xi^+) \leq \mu$ for all $(\xi, e, p) \in \Xi_1 \times \min\{\delta_1, \delta_2\}\mathbb{B} \times \delta_3\mathbb{B}$.

If μ is chosen so large that $\Xi_1 = \Xi$ and $\Xi_2 = \emptyset$, the proof is completed. Therefore, assume that $\Xi_2 \neq \emptyset$. Since $\max_{\xi^+ \in F(\xi)} V(\xi^+) < \gamma V(\xi)$ for all $\xi \in \Xi \setminus 0$, there exist $\rho > 0, \delta_4 > 0$ such that

$$\max_{\xi^+ \in F(\xi+e)} V(\xi^+) \leq \gamma V(\xi + e) - \rho$$

for all $(\xi, e) \in \Xi_2 \times \delta_4 \mathbb{B}$. By continuity of V, choose $\delta_5, \delta_6 > 0$ such that:

$$\max_{\xi^+ \in F(\xi+e)+p} V(\xi^+) \leq \max_{\xi^+ \in F(\xi+e)} V(\xi^+) + \rho/2$$

for all $(\xi, e, p) \in \Xi_2 \times \delta_5 \mathbb{B} \times \delta_6 \mathbb{B}$. By continuity of V choose $\delta_7 > 0$ such that:

$$\gamma V(\xi + e) \leq \gamma V(\xi) + \rho/2$$

for all $(\xi, e) \in \Xi_2 \times \delta_7 \mathbb{B}$.

The last three inequalities added together lead: $\max_{\xi^+ \in F(\xi+e)+p} V(\xi^+) \leq \gamma V(\xi)$ for all $(\xi, e, p) \in \Xi_2 \times \min\{\delta_4, \delta_5, \delta_7\}\mathbb{B} \times \delta_6 \mathbb{B}$. Finally, set $\delta = \min\{\delta_1, \delta_2, \ldots, \delta_7\}$, and we have established that:

$$\max_{\xi^+ \in F(\xi+e)+p} V(\xi^+) \leq \max\{\gamma V(\xi), \mu\}$$

for all $(\xi, e, p) \in \Xi \times \delta \mathbb{B} \times \delta \mathbb{B}$. □