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Addendum to the paper "On computing solutions to the continuous time constrained linear quadratic regulator"

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Abstract

In this report we present detailed proofs of the results presented in the paper "On Computing Solutions to the Continuous Time Constrained Linear Quadratic Regulator" [1], which were omitted in the final published paper due to space limitations.

Keywords
Constrained Linear Quadratic Regulation, Continuous Time Systems, Model Predictive Control, Optimal Control

1 Proofs

1.1 Lemma 2

Proof: We first define $\bar{t} = t - t_k$ and we observe from (4) that: $x(t) = e^{A\bar{t}}x_k + \int_0^{\bar{t}} e^{A(\bar{t}-\tau)}B(u_k + s_k\tau)d\tau = e^{A\bar{t}}x_k + I_0(\bar{t})Bu_k + I_1(\bar{t})Bs_k$ with $I_0(t) = \int_0^t e^{A\tau}d\tau$ and $I_1(t) =$

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$\int_0^t e^{A(t-\tau)}\tau d\tau$. We can now write:

$$\int_{t_k}^{t_{k+1}}(x'Qx+u'Ru)dt = \int_0^{\Delta_k}(u_k+s_k\bar{t})'R(u_k+s_k\bar{t})d\bar{t}+$$
$$\int_0^{\Delta_k}(e^{A\bar{t}}x_k+I_0(\bar{t})Bu_k+I_1(\bar{t})Bs_k)'Q(e^{A\bar{t}}x_k+I_0(\bar{t})Bu_k+I_1(\bar{t})Bs_k)d\bar{t}=$$
$$x_k'\left(\int_0^{\Delta_k}(e^{A\bar{t}})'Qe^{A\bar{t}}d\bar{t}\right)x_k+2x_k'\left(\int_0^{\Delta_k}(e^{A\bar{t}})'Q(I_0(\bar{t})B)d\bar{t}\right)u_k+$$
$$2x_k'\left(\int_0^{\Delta_k}(e^{A\bar{t}})'Q(I_1(\bar{t})B)d\bar{t}\right)s_k+u_k'\left(\int_0^{\Delta_k}R+(I_0(\bar{t})B)'Q(I_0(\bar{t})B)d\bar{t}\right)u_k+$$
$$2u_k'\left(\int_0^{\Delta_k}R\bar{t}+(I_0(\bar{t})B)'Q(I_1(\bar{t})B)d\bar{t}\right)s_k+s_k'\left(\int_0^{\Delta_k}R\bar{t}^2+(I_1(\bar{t})B)'Q(I_1(\bar{t})B)d\bar{t}\right)s_k$$

from the final result immediately follows. □

1.2 Lemma 3

Proof: We proceed as in the proof of Lemma 2 to write:

$$x(t)=e^{A\bar{t}}x_k+\left(I_0-\tfrac{I_1}{\Delta_k}\right)Bu_k+\left(\tfrac{I_1}{\Delta_k}\right)u_{k+1}$$

where for simplicity we omit the argument $\bar{t}$ in I_0 and I_1. We can now write:

$$\int_{t_k}^{t_{k+1}}(x'Qx+u'Ru)dt=$$
$$\int_0^{\Delta_k}\left(u_k\left(1-\tfrac{\bar{t}}{\Delta_k}\right)+u_{k+1}\left(\tfrac{\bar{t}}{\Delta_k}\right)\right)'R\left(u_k\left(1-\tfrac{\bar{t}}{\Delta_k}\right)+u_{k+1}\left(\tfrac{\bar{t}}{\Delta_k}\right)\right)d\bar{t}+$$
$$\int_0^{\Delta_k}\left(e^{A\bar{t}}x_k+\left(I_0-\tfrac{I_1}{\Delta_k}\right)Bu_k+\left(\tfrac{I_1}{\Delta_k}\right)u_{k+1}\right)'Q\left(e^{A\bar{t}}x_k+\left(I_0-\tfrac{I_1}{\Delta_k}\right)Bu_k+\left(\tfrac{I_1}{\Delta_k}\right)u_{k+1}\right)d\bar{t}$$

from which we obtain the final result by expanding the products and moving outside the integral sign the constant terms x_k, u_k and u_{k+1} as in the proof of Lemma 2. □

1.3 Theorem 1

The convergence result for ZOH is proved here in, to illustrate the proposed idea, while the proofs for other holds are not shown but follow the same pattern and may benefit from the use of symbolic manipulation software.

Proof: The starting point is a second-order Taylor expansion (in Δ) of all terms in (11): $\bar{A}\approxeq I+\Delta A+\frac{1}{2}\Delta^2(A^2-BR^{-1}B'Q)$, $\bar{B}\approxeq\Delta B+\frac{1}{2}\Delta^2 AB$, $\bar{Q}\approxeq\Delta Q+\frac{1}{2}\Delta^2(A'Q+QA)$, $\bar{R}\approxeq\Delta R$, $\Pi^0\approxeq\Pi_0+\Delta\Pi_1+\frac{1}{2}\Delta^2\Pi_2$. After substitution of these terms into (11) and elimination of higher order terms, one can compare LHS and RHS terms in Δ. The zero-order term simply reduces to: $\Pi_0=\Pi_0$, and is not informative. The first-order term is

$$\Pi_1=\Pi_1+A'\Pi_0+\Pi_0A+Q-\Pi_0BR^{-1}B'\Pi_0\ ,$$

from which it follows that $\Pi_0 = P$. The second order term can be factorized as

$$\Pi_2 = \Pi_2 + A'(Q + A'\Pi_0 + \Pi_0 A - \Pi_0 BR^{-1}B'\Pi_0) + (Q + A'\Pi_0 + \Pi_0 A - \Pi_0 BR^{-1}B'\Pi_0)A +$$
$$-\Pi_0 BR^{-1}B'(Q+A'\Pi_0+\Pi_0 A-\Pi_0 BR^{-1}B'\Pi_0)-(Q+A'\Pi_0+\Pi_0 A-\Pi_0 BR^{-1}B'\Pi_0)BR^{-1}B'\Pi_0+$$
$$2(A'\Pi_1 + \Pi_1 A - \Pi_1 BR^{-1}B'\Pi_0 - \Pi_0 BR^{-1}B'\Pi_1)\ .$$

Since $\Pi_0 = P$, the common term $(Q + A'\Pi_0 + \Pi_0 A - \Pi_0 BR^{-1}B'\Pi_0)$ is zero, and thus the previous equation reduces to: $(A - BR^{-1}B'\Pi_0)'\Pi_1 + \Pi_1(A - BR^{-1}B'\Pi_0) = 0$, which can be rewritten as: $(A + BK)'\Pi_1 + \Pi_1(A + BK) = 0$, in which $K = -R^{-1}B'\Pi_0 = -R^{-1}B'P$ is the optimal continuous-time LQR feedback gain matrix. By noticing that $(A + BK)$ is a strictly stable matrix (in a continuous-time sense), it follows that $\Pi_1 = 0$. The proof is completed by showing that $\Pi_2 \neq 0$. □

1.4 Theorem 4

Proof: The first condition follows from the fact that ZOH in (5) is equivalent to PWLH in (6) in with the additional constraints $s_k = 0$ for all $k = 0, 1, \ldots$. Such constraints increase the cost, i.e. for any initial state x_0, we have that $\frac{1}{2}x_0'\Pi^{\mathrm{I}}x_0 \leq \frac{1}{2}x_0'\Pi^0 x_0$. The second condition can be proved in the same way by noticing that FFOH in (7) is also equivalent to PWLH (6) with the additional constraints $s_k = (u_{k+1} - u_k)/\Delta$ for all $k = 0, 1, \ldots$. □

1.5 Theorem 5

Proof: First, suppose that t_N does not change from iteration $j - 1$ to iteration j. Since the problem solved at iteration j contains the decision variables of that of iteration $j - 1$ plus the decision variables at middle points of the intervals considered at $j - 1$, then (16) follows immediately. If instead t_N at iteration j is larger than t_N at iteration $j - 1$, it follows again that the problem at iteration j has extra decision variables, namely the input and slope for the added time interval(s) as well as all decision variables at intermediate points. Thus, (16) follows. □

References

[1] G. Pannocchia, J. B. Rawlings, D. Q. Mayne, and W. Marquardt. On computing solutions to the continuous time constrained linear quadratic regulator. *IEEE Trans. Auto. Contr.*, Conditionally accepted as Technical Note, 2010.