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On the equivalence between statements with ϵ-δ and K-functions

James B. Rawlings* and Michael J. Risbeck†
Department of Chemical and Biological Engineering
University of Wisconsin-Madison

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1 Introduction and Motivating Examples

The purpose of this note is to establish some simple results enabling direct translation between classic ϵ-δ statements and K-function statements. The definition of K-function is standard

Definition 1 (K-function)**.** A K-function is a function defined on a nonempty interval $[0, b]$ with $b > 0$, $\gamma : \mathbb{R}_{[0:b]} \to \mathbb{R}_{\geq 0}$ that is continuous, strictly increasing, and zero at zero.

Note that we require $\gamma(\cdot)$ to be defined only on some nonzero interval, not $[0, \infty)$.

As a motivating example, consider the standard ϵ-δ definition of continuity of a function $f(\cdot)$ at a point x.

Definition 2 (Continuity: ϵ-δ)**.** A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous at x if for every $\epsilon > 0$, there exits $\delta(\epsilon) > 0$ (note that $\delta(\epsilon)$ may depend on x) such that

$$|f(x + p) - f(x)| \leq \epsilon \quad \text{for all } |p| \leq \delta(\epsilon) \tag{1}$$

The equivalent definition of continuity in the language of K-functions is the following.

Definition 3 (Continuity: K-function)**.** A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous at x if there exists a K-function $\gamma(\cdot)$ (note that the function $\gamma(\cdot)$ may depend on x) such that

$$|f(x + p) - f(x)| \leq \gamma(|p|) \quad \text{for all } |p| \in \text{Dom}(\gamma) \tag{2}$$

*james.rawlings@wisc.edu
†risbeck@wisc.edu

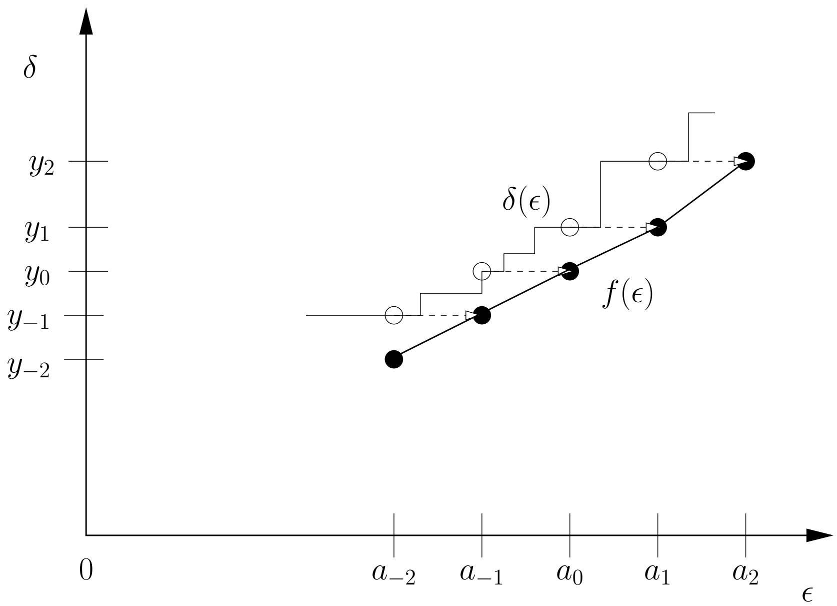

Figure 1: The right shift trick to construct a continuous, increasing underbounding function from samples of a possibly discontinuous, increasing function. A similar *left* shift trick in conjunction with the supremum is useful when creating a continuous, increasing *overbounding* function.

To establish the equivalence of these definitions, we require the following result establishing a connection between the (possibly discontinuous) function $\delta(\epsilon)$[1] and existence of a K-function underbound.

Proposition 4 (A K-function underbound of $\delta(\epsilon)$)**.** *Let $\delta : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ be an increasing, i.e., nondecreasing, function. Then there exists a K-function $\alpha(\cdot)$ such that for all $\epsilon > 0$*

$$\alpha(\epsilon) \leq \delta(\epsilon)$$

Proof. In this proof, we construct the K-function $\alpha(\cdot)$ from the given function $\delta(\cdot)$. Figures 1–3 shows the techniques we employ. Start by taking an arbitrary $a_0 > 0$, and create a doubly infinite sequence, a_i with $i = 0, \pm 1, \pm 2, ...$, such that a_i is strictly increasing and tends to infinity and a_{-i} is strictly decreasing and tends to zero as i tends to infinity. We have that the a_i sequence is strictly increasing. Now define the sequence y_i by

$$y_i = \delta(a_{i-1}) \qquad i = 0, \pm 1, \pm 2, \ldots$$

Note this right shift trick, depicted in Figure 1, is useful when creating an underbounding function. Since $\delta(\cdot)$ is a positive, increasing function, we have that $y_i = \delta(a_{i-1}) > 0$ and y_i is an increasing sequence. Next define the continuous function $f(\cdot)$ by connecting the

[1]Note that we can assume $\delta(\epsilon)$ is an increasing function. See Proposition 11.

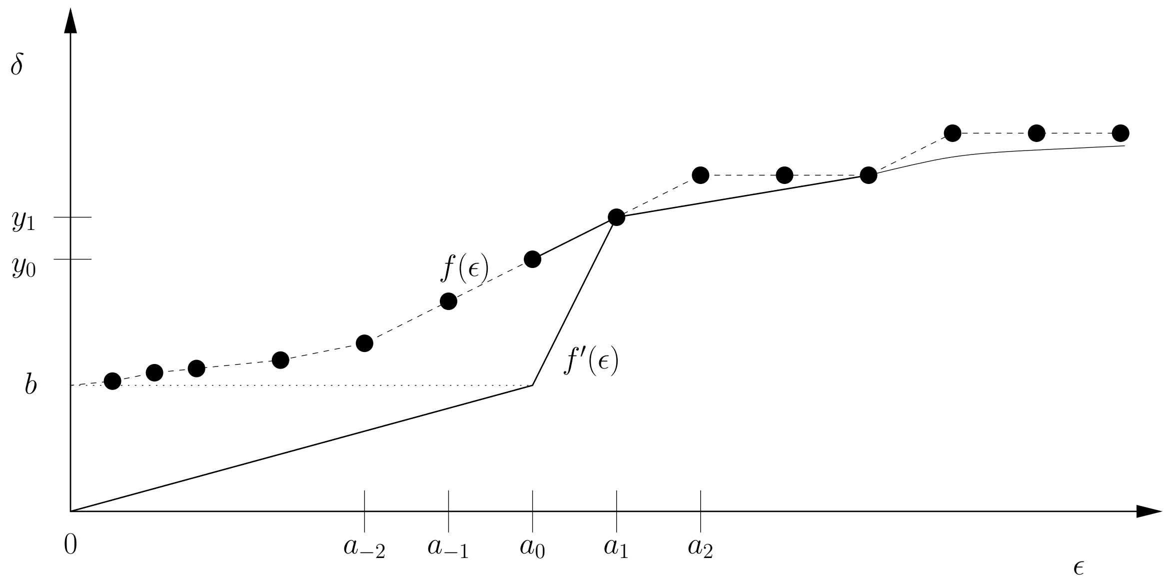

Figure 2: Making an $f'(\epsilon)$ function (solid line) that is strictly increasing with $f'(0) = 0$ from an increasing function $f(\epsilon)$ (dashed line).

points (a_i, y_i) with linear functions

$$f(\epsilon) = \left(\frac{a_{i+1} - \epsilon}{a_{i+1} - a_i}\right) y_i + \left(\frac{\epsilon - a_i}{a_{i+1} - a_i}\right) y_{i+1}, \qquad \epsilon \in [a_i, a_{i+1}], \qquad i = 0, \pm 1, \pm 2, \ldots$$

So far we have a function $f(\cdot)$ defined[2] on $[0, \infty)$ that is continuous (and piecewise linear), increasing, and satisfies $f(\cdot) < \delta(\cdot)$. But $f(\cdot)$ may not be a K-function because $f(0)$ may not be zero, and $f(\cdot)$ may not be strictly increasing. We next create a function with these properties. See also Figure 2.

If the function $f(\cdot)$ is a constant function with value $y_0 > 0$, define $\alpha(\epsilon)$ as any strictly increasing function starting at zero that underbounds y_0. For example

$$\alpha(\epsilon) = y_0(1 - e^{-\epsilon})$$

If $f(\cdot)$ is *not* constant, take any index i_0 such that $y_{i_0} < y_{i_0+1}$. For simplicity, relabel the a_i, y_i sequences such that $i_0 = 0$. Starting at $i = 1$, find the first set of indices (if any) $i \in [i_1, i_2]$ where y_i is constant and $y_{i_2} < y_{i_2+1}$. On such intervals define $f'(\epsilon)$ to be the linear function

$$f'(\epsilon) = \left(\frac{a_{i_2} - \epsilon}{a_{i_2} - a_{i_1}}\right) y_{i_1} + \left(\frac{\epsilon - a_{i_1}}{a_{i_2} - a_{i_1}}\right) y_{i_2}, \qquad \epsilon \in [a_{i_1}, a_{i_2}]$$

Note that $f'(\cdot)$ is continuous, strictly increasing, and underbounds $f(\cdot)$ on the interval $[a_{i_1}, a_{i_2}]$. Continue to the next interval of indices over which y_i is constant and repeat.

[2]We define $f(0)$ as $\lim_{\epsilon \searrow 0} f(\epsilon)$, which exists because $f(\cdot)$ is monotone.

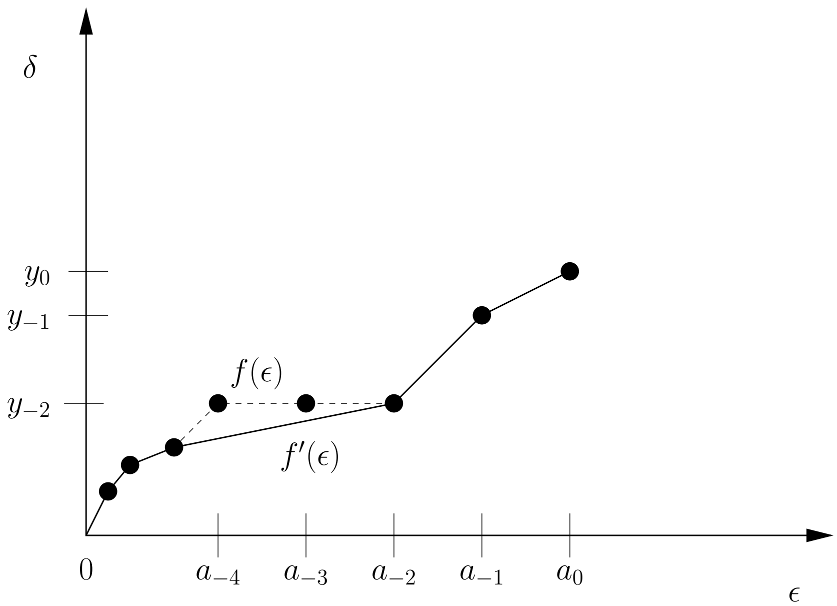

Figure 3: Treating the (usual) case when $f(\epsilon)$ converges to zero as ϵ converges to zero.

While increasing i, if y_i becomes constant on an interval $[i_3, \infty)$ with $y_{i_3-1} < y_{i_3}$, then create the underbound

$$f'(\epsilon) = (y_{i_3} - y_{i_3-1})(1 - e^{-\epsilon_{i_3-1}}), \qquad \epsilon \geq a_{i_3-1}$$

In this fashion we have constructed an $f'(\cdot)$ that is strictly increasing on $[a_0, \infty)$ and is an underbound of $\delta(\cdot)$ on this interval.

Next we turn attention to the interval $[0, a_0]$. If $f(\epsilon)$ converges to some $b > 0$ as $\epsilon \to 0$, then define $f'(\epsilon)$ on $[0, a_1]$ as the linear function connecting the point $(0, 0)$ to (a_0, b) and then join the function $f'(\epsilon)$ for $\epsilon \geq a_1$ as shown in Figure 2.[3] Setting $\alpha(\cdot) = f'(\cdot)$, we then have a K-function underbound on $[0, \infty)$ for this case.

Finally, if $f(\epsilon)$ converges to zero as $\epsilon \to 0$ (the usual case), proceed as in the previous part and replace intervals of constant values by their linear underbounds as shown in Figure 3.[4] In this case also, setting $\alpha(\cdot) = f'(\cdot)$, we have constructed a K-function underbound on $[0, \infty)$ and the proof is complete. ■

Note that the K-function $\alpha(\cdot)$ is defined on $[0, \infty)$ and the K-function $\alpha^{-1}(\cdot)$ is defined on $[0, \bar{\delta}]$ in which $\bar{\delta} > 0$ is any value satisfying $\bar{\delta} < \sup_{\epsilon > 0} \delta(\epsilon)$.

Proposition 5 (Equivalence of two continuity definitions)**.** *The classic ϵ-δ definition and K-function definition of continuity are equivalent.*

Proof.

[3]Note that we have now redefined $f'(\epsilon)$ on the interval $[a_0, a_1]$.

[4]Note that in this last case, unlike when treating the increasing a_i values, there is no interval $[0, a_{i_4}]$ on which $f(\epsilon)$ can be constant because $f(0) = 0$ but $f(a_{i_4}) > 0$ for all i_4.

K definition implies ϵ-δ definition. Given the K-function $\gamma(\cdot)$ satisfying (2) choose $\delta(\epsilon) := \gamma^{-1}(\epsilon)$. We then have $|p| \leq \delta(\epsilon) = \gamma^{-1}(\epsilon)$ implies that $|f(x+p) - f(x)| \leq \gamma(|p|) \leq \gamma(\gamma^{-1}(\epsilon)) = \epsilon$ and the ϵ-δ definition of continuity is established.

ϵ-δ definition implies K definition. Since $\alpha(\cdot)$ defined in Proposition 4 is defined on $[0, \infty)$, for any $\epsilon > 0$ choose p so that $|p| = \alpha(\epsilon)$. Since $|p| = \alpha(\epsilon) \leq \delta(\epsilon)$, by ϵ-δ continuity, we have that $|f(x+p) - f(x)| \leq \epsilon = \alpha^{-1}(|p|)$. Note that $\alpha^{-1}(\cdot)$ is a K-function defined on $[0, \bar{\delta}]$ and the K-function definition of continuity is established. ■

As a second example, consider the definition of Lyapunov stability.

Definition 6 (Lyapunov stability: ϵ-δ)**.** Consider the dynamic system $x^+ = f(x)$ with $f(0) = 0$. The origin is Lyapunov stable if for every $\epsilon > 0$ the exists $\delta(\epsilon) > 0$ such that for $|x| \leq \delta$, the solution satisfies for all $k \geq 0$

$$|\phi(k; x)| \leq \epsilon$$

The equivalent definition with a K-function is the following.

Definition 7 (Lyapunov stability: K-function)**.** Consider the dynamic system $x^+ = f(x)$ satisfying $f(0) = 0$. The origin is Lyapunov stable if there exists a scalar $\rho > 0$ and a K-function $\gamma(\cdot)$ such that for all $|x| \leq \rho$ and $k \geq 0$

$$|\phi(k; x)| \leq \gamma(|x|)$$

As a third example, consider the definition of robust global asymptotic stability in ϵ-δ language.

Definition 8 (Robust global asymptotic stability: ϵ-δ)**.** Consider a nominal system $x^+ = f(x)$ with $f(0) = 0$ in which the origin is globally asymptotically stable. The origin of the perturbed system $x^+ = f(x) + w$ is robustly globally asymptotically stable if there exists a KL-function $\beta(\cdot)$ and for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that for all $\|\mathbf{w}\| \leq \delta$, $x \in \mathbb{R}^n$, and $k \geq 0$

$$|x(k; x, \mathbf{w})| \leq \beta(|x|, k) + \epsilon$$

The equivalent K-function definition is the following.

Definition 9 (Robust global asymptotic stability: K-function)**.** Consider a nominal system $x^+ = f(x)$ with $f(0) = 0$ in which the origin is globally asymptotically stable. The origin of the perturbed system $x^+ = f(x) + w$ is robustly globally asymptotically stable if there exists a scalar $\rho > 0$, K-function $\gamma(\cdot)$, and KL-function $\beta(\cdot)$ such that for all $x \in \mathbb{R}^n$, $\|\mathbf{w}\| \leq \rho$, and $k \geq 0$

$$|x(k; x, \mathbf{w})| \leq \beta(|x|, k) + \gamma(\|\mathbf{w}\|)$$

Note that we could write the final inequality equivalently as

$$|x(k; x, \mathbf{w})| \leq \beta(|x|, k) + \gamma(\|\mathbf{w}\|_{0:k-1})$$

because $x(k; x, \mathbf{w})$ depends on $\mathbf{w}$ only up to time $k-1$. The last statement is equivalent to the statement that the origin of the system $x^+ = f(x) + w$ is input-to-state stable (ISS) for small disturbances ($\|\mathbf{w}\| \leq \rho$) considering the disturbance w as the input.

2 Generalization

The following definitions and theorem generalize the previous examples. Let X be any normed space.

Definition 10 (Property P)**.** A system with testable condition $C : X \to \mathbb{R}_{\geq 0}$ satisfying $C(0) = 0$ has property P if for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$C(x) \leq \epsilon \qquad \text{for every } x \in X \text{ satisfying } |x| \leq \delta(\epsilon) \tag{3}$$

We note that the function $\delta(\epsilon)$ in Definition 10 can be made increasing, as shown in the following proposition.

Proposition 11 ($\delta(\epsilon)$ can be made increasing)**.** *Suppose a system has property P as in Definition 10. Then, without loss of generality, the function $\delta(\epsilon)$ can be assumed to be a nondecreasing function.*

Proof. Suppose (3) holds for $\hat{\delta}(\epsilon)$ which is possibly *not* nondecreasing. Next, define $\overline{\delta}(\epsilon) := \min(\hat{\delta}(\epsilon), 1)$. We note that (3) holds also for $\overline{\delta}(\epsilon)$ because $(0, \overline{\delta}(\epsilon)) \subseteq (0, \hat{\delta}(\epsilon))$, and thus (3) holding for $\overline{\delta}$ is weaker than for $\hat{\delta}$. Then, define

$$\delta(\epsilon) := \frac{1}{2} \sup_{s \in (0,\epsilon]} \overline{\delta}(s)$$

which is well-defined because $\overline{\delta}(s) \in (0, 1]$ for all $s > 0$, and all bounded sets of real numbers have suprema. Furthermore, $\delta(\epsilon)$ is clearly nondecreasing. To show that (3) holds for $\delta(\epsilon)$, let $\epsilon_1 > 0$ be arbitrary. By definition, there exists positive $\epsilon_0 < \epsilon_1$ such that $\overline{\delta}(\epsilon_0) \geq \delta(\epsilon_1)$.[5] Thus, from (3) and these two inequalities, we know that for arbitrary $x \in X$,

$$|x| \leq \delta(\epsilon_1) \implies |x| \leq \overline{\delta}(\epsilon_0) \implies C(x) \leq \epsilon_0 \implies C(x) \leq \epsilon_1$$

which means (3) holds for $\delta(\epsilon)$ and the statement is proved. ■

In the language of K-functions, we have the following definition of property P_K.

Definition 12 (Property P_K)**.** A system with testable condition $C : X \to \mathbb{R}_{\geq 0}$ satisfying $C(0) = 0$ has property P_K if there exists $b > 0$ and K-function $\gamma(\cdot)$ defined on $[0, b]$, such that for all $x \in X$ satisfying $|x| \leq b$

$$C(x) \leq \gamma(|x|)$$

Proposition 13 (Equivalence of P and P_K)**.** *A system has property P if and only if it has property P_K. The constant b defined in Property P_K can be chosen as any positive value satisfying $b < \sup_{\epsilon > 0} \delta(\epsilon)$ with $\delta(\epsilon)$ defined in Property P.*

[5] Suppose not. Then, for all $s \in (0, \epsilon_1]$, we have $\overline{\delta}(s) < \delta(s) < \sup_{s \in (0,\epsilon_1]} \overline{\delta}(s)$, which is a contradiction because we have found an upper bound strictly less than the supremum.

3 Extensions

Here we show how a *global* K-function can be found for a locally bounded function.

Proposition 14 (Global K-function overbound.)**.** *Let $X \subseteq \mathbb{R}^n$ be closed and suppose that a function $V : X \to \mathbb{R}_{\geq 0}$ is continuous at $x_0 \in X$ and locally bounded on X (i.e., bounded on every compact subset of X). Then, there exists a K-function α such that*

$$|V(x) - V(x_0)| \leq \alpha(|x - x_0|) \quad \textit{for all } x \in X$$

Proof. First, by Proposition 5, we know that there exists a local overbounding function, i.e., there exists a K-function γ and a constant $a > 0$ such that

$$|V(x) - V(x_0)| \leq \gamma(|x - x_0|) \quad \text{whenever } |x - x_0| \leq b_0$$

Note that any $b_0 \in \text{Dom}(\gamma)$ will suffice.

From here, we proceed similarly to Proposition 11 in Rawlings and Mayne (2011). Starting from b_0, choose any strictly increasing and unbounded sequence $(b_i)_{i=0}^{\infty}$. For each $i \in \mathbb{I}_{\geq 1}$[6], let $B_i = \{x \in X : |x - x_0| \leq b_i\}$. We note that each B_i is a compact subset of X and further that $X = \bigcup_{i=0}^{\infty} B_i$. Next, define a sequence $(\beta_i)_{i=0}^{\infty}$ as

$$\beta_i := \sup_{x \in B_i} |V(x) - V(x_0)| + i$$

which is well-defined by compactness of the B_i. We note also that the β_i are strictly increasing. Finally, define

$$\alpha(s) := \begin{cases} \dfrac{\beta_1}{\gamma(b_0)} \gamma(s) & s \in [0, b_0) \\ \beta_{i+1} + (\beta_{i+2} - \beta_{i+1}) \dfrac{s - b_i}{b_{i+1} - b_i} & s \in [b_i, b_{i+1}) \quad \text{for all } i \in \mathbb{I}_{\geq 0} \end{cases}$$

We illustrate this construction in Figure 4. Clearly, $\alpha(0) = 0$ and α is continuous and increasing. Furthermore, because we have shifted the β_i as before, we see that $|V(x) - V(x_0)| \leq \alpha(|x - x_0|)$. ■

We note that for the case of $V(x) \geq 0$ and $x_0 = 0$, we have

$$V(x) \leq \alpha(|x|) \quad \text{for all } x \in X$$

and thus, α gives a global overbound.

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[6] That is, positive integers greater than or equal to 1

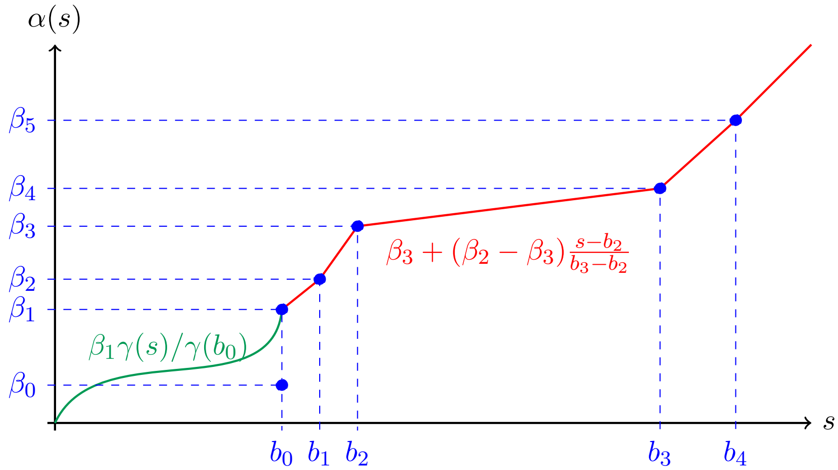

Figure 4: Construction of α. The function $\alpha(s)$ is constructed by rescaling $\gamma(s)$ on $[0, b_0]$ (green) and then linearly interpolating (red) the points (b_i, β_{i+1}) (blue).

References

J. B. Rawlings and D. Q. Mayne. Postface to Model Predictive Control: Theory and Design, 2011. URL `http://jbrwww.che.wisc.edu/home/jbraw/mpc/postface.pdf`.