On the equivalence between statements with $\epsilon$-$\delta$ and $K$-functions

James B. Rawlings* and Michael J. Risbeck†
Department of Chemical and Biological Engineering
University of Wisconsin-Madison
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1 Introduction and Motivating Examples

The purpose of this note is to establish some simple results enabling direct translation between classic $\epsilon$-$\delta$ statements and $K$-function statements. The definition of $K$-function is standard

Definition 1 (K-function). A $K$-function is a function defined on a nonempty interval $[0,b]$ with $b > 0$, $\gamma : [0,b] \to \mathbb{R}_{\geq 0}$ that is continuous, strictly increasing, and zero at zero.

Note that we require $\gamma(\cdot)$ to be defined only on some nonzero interval, not $[0,\infty)$.

As a motivating example, consider the standard $\epsilon$-$\delta$ definition of continuity of a function $f(\cdot)$ at a point $x$.

Definition 2 (Continuity: $\epsilon$-$\delta$). A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $x$ if for every $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ (note that $\delta(\epsilon)$ may depend on $x$) such that

$$|f(x + p) - f(x)| \leq \epsilon \quad \text{for all } |p| \leq \delta(\epsilon) \quad (1)$$

The equivalent definition of continuity in the language of $K$-functions is the following.

Definition 3 (Continuity: $K$-function). A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $x$ if there exists a $K$-function $\gamma(\cdot)$ (note that the function $\gamma(\cdot)$ may depend on $x$) such that

$$|f(x + p) - f(x)| \leq \gamma(|p|) \quad \text{for all } |p| \in \text{Dom}(\gamma) \quad (2)$$

\*james.rawlings@wisc.edu
†risbeck@wisc.edu
Figure 1: The right shift trick to construct a continuous, increasing underbounding function from samples of a possibly discontinuous, increasing function. A similar left shift trick in conjunction with the supremum is useful when creating a continuous, increasing overbounding function.

To establish the equivalence of these definitions, we require the following result establishing a connection between the (possibly discontinuous) function $\delta(\epsilon)^1$ and existence of a $K$-function underbound.

**Proposition 4 (A $K$-function underbound of $\delta(\epsilon)$).** Let $\delta : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ be an increasing, i.e., nondecreasing, function. Then there exists a $K$-function $\alpha(\cdot)$ such that for all $\epsilon > 0$

$$\alpha(\epsilon) \leq \delta(\epsilon)$$

**Proof.** In this proof, we construct the $K$-function $\alpha(\cdot)$ from the given function $\delta(\cdot)$. Figures 1–3 shows the techniques we employ. Start by taking an arbitrary $a_0 > 0$, and create a doubly infinite sequence, $a_i$ with $i = 0, \pm 1, \pm 2, \ldots$, such that $a_i$ is strictly increasing and tends to infinity and $a_{-i}$ is strictly decreasing and tends to zero as $i$ tends to infinity. We have that the $a_i$ sequence is strictly increasing. Now define the sequence $y_i$ by

$$y_i \;=\; \delta(a_{i-1}) \quad i = 0, \pm 1, \pm 2, \ldots$$

Note this right shift trick, depicted in Figure 1, is useful when creating an underbounding function. Since $\delta(\cdot)$ is a positive, increasing function, we have that $y_i = \delta(a_{i-1}) > 0$ and $y_i$ is an increasing sequence. Next define the continuous function $f(\cdot)$ by connecting the

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1Note that we can assume $\delta(\epsilon)$ is an increasing function. See Proposition 11.
Figure 2: Making an $f'(\epsilon)$ function (solid line) that is strictly increasing with $f'(0) = 0$ from an increasing function $f(\epsilon)$ (dashed line).

points $(a_i, y_i)$ with linear functions

$$f(\epsilon) = \left( \frac{a_{i+1} - \epsilon}{a_{i+1} - a_i} \right) y_i + \left( \frac{\epsilon - a_i}{a_{i+1} - a_i} \right) y_{i+1}, \quad \epsilon \in [a_i, a_{i+1}], \quad i = 0, \pm 1, \pm 2, \ldots$$

So far we have a function $f(\cdot)$ defined on $[0, \infty)$ that is continuous (and piecewise linear), increasing, and satisfies $f(\cdot) < \delta(\cdot)$. But $f(\cdot)$ may not be a $K$-function because $f(0)$ may not be zero, and $f(\cdot)$ may not be strictly increasing. We next create a function with these properties. See also Figure 2.

If the function $f(\cdot)$ is a constant function with value $y_0 > 0$, define $\alpha(\epsilon)$ as any strictly increasing function starting at zero that underbounds $y_0$. For example

$$\alpha(\epsilon) = y_0(1 - e^{-\epsilon})$$

If $f(\cdot)$ is not constant, take any index $i_0$ such that $y_{i_0} < y_{i_0+1}$. For simplicity, relabel the $a_i, y_i$ sequences such that $i_0 = 0$. Starting at $i = 1$, find the first set of indices (if any) $i \in [i_1, i_2]$ where $y_i$ is constant and $y_{i_2} < y_{i_2+1}$. On such intervals define $f'(\epsilon)$ to be the linear function

$$f'(\epsilon) = \left( \frac{a_{i_2} - \epsilon}{a_{i_2} - a_{i_1}} \right) y_{i_1} + \left( \frac{\epsilon - a_{i_1}}{a_{i_2} - a_{i_1}} \right) y_{i_2}, \quad \epsilon \in [a_{i_1}, a_{i_2}]$$

Note that $f'(\cdot)$ is continuous, strictly increasing, and underbounds $f(\cdot)$ on the interval $[a_{i_1}, a_{i_2}]$. Continue to the next interval of indices over which $y_i$ is constant and repeat.

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2We define $f(0)$ as $\lim_{\epsilon \searrow 0} f(\epsilon)$, which exists because $f(\cdot)$ is monotone.
While increasing $i$, if $y_i$ becomes constant on an interval $[i_3, \infty)$ with $y_{i_3} < y_{i_3-1}$, then create the underbound

$$f'(\epsilon) = (y_{i_3} - y_{i_3-1})(1 - e^{-\epsilon_{i_3-1}}), \quad \epsilon \geq a_{i_3-1}$$

In this fashion we have constructed an $f'(\cdot)$ that is strictly increasing on $[a_0, \infty)$ and is an underbound of $\delta(\cdot)$ on this interval.

Next we turn attention to the interval $[0, a_0]$. If $f(\epsilon)$ converges to some $b > 0$ as $\epsilon \to 0$, then define $f'(\epsilon)$ on $[0, a_1]$ as the linear function connecting the point $(0, 0)$ to $(a_0, b)$ and then join the function $f'(\epsilon)$ for $\epsilon \geq a_1$ as shown in Figure 2.\(^3\) Setting $\alpha(\cdot) = f'(\cdot)$, we then have a $K$-function underbound on $[0, \infty)$ for this case.

Finally, if $f(\epsilon)$ converges to zero as $\epsilon \to 0$ (the usual case), proceed as in the previous part and replace intervals of constant values by their linear underbounds as shown in Figure 3.\(^4\) In this case also, setting $\alpha(\cdot) = f'(\cdot)$, we have constructed a $K$-function underbound on $[0, \infty)$ and the proof is complete.

Note that the $K$-function $\alpha(\cdot)$ is defined on $[0, \infty)$ and the $K$-function $\alpha^{-1}(\cdot)$ is defined on $[0, \hat{\delta}]$ in which $\hat{\delta} > 0$ is any value satisfying $\hat{\delta} < \sup_{\epsilon > 0} \delta(\epsilon)$.

**Proposition 5** (Equivalence of two continuity definitions). The classic $\epsilon$-$\delta$ definition and $K$-function definition of continuity are equivalent.

**Proof.**

\(^3\)Note that we have now redefined $f'(\epsilon)$ on the interval $[a_0, a_1]$.

\(^4\)Note that in this last case, unlike when treating the increasing $a_i$ values, there is no interval $[0, a_{i_4}]$ on which $f(\epsilon)$ can be constant because $f(0) = 0$ but $f(a_{i_4}) > 0$ for all $i_4$. 

Figure 3: Treating the (usual) case when $f(\epsilon)$ converges to zero as $\epsilon$ converges to zero.
K definition implies \(\epsilon\)-\(\delta\) definition. Given the \(K\)-function \(\gamma(\cdot)\) satisfying (2) choose \(\delta(\epsilon) := \gamma^{-1}(\epsilon)\). We then have \(|p| \leq \delta(\epsilon) = \gamma^{-1}(\epsilon)\) implies that \(|f(x + p) - f(x)| \leq \gamma(|p|) \leq \gamma(\gamma^{-1}(\epsilon)) = \epsilon\) and the \(\epsilon\)-\(\delta\) definition of continuity is established.

\(\epsilon\)-\(\delta\) definition implies \(K\) definition. Since \(\alpha(\cdot)\) defined in Proposition 4 is defined on \([0, \infty)\), for any \(\epsilon > 0\) choose \(p\) so that \(|p| = \alpha(\epsilon)|\). Since \(|p| = \alpha(\epsilon) \leq \delta(\epsilon)|\), by \(\epsilon\)-\(\delta\) continuity, we have that \(|f(x + p) - f(x)| \leq \epsilon = \alpha^{-1}(|p|)|\). Note that \(\alpha^{-1}(\cdot)\) is a \(K\)-function defined on \([0, \delta]\) and the \(K\)-function definition of continuity is established.

As a second example, consider the definition of Lyapunov stability.

**Definition 6** (Lyapunov stability: \(\epsilon\)-\(\delta\)). Consider the dynamic system \(x^+ = f(x)\) with \(f(0) = 0\). The origin is Lyapunov stable if for every \(\epsilon > 0\) the exists \(\delta(\epsilon) > 0\) such that for \(|x| \leq \delta\), the solution satisfies for all \(k \geq 0\)

\[
|\phi(k; x)| \leq \epsilon
\]

The equivalent definition with a \(K\)-function is the following.

**Definition 7** (Lyapunov stability: \(K\)-function). Consider the dynamic system \(x^+ = f(x)\) satisfying \(f(0) = 0\). The origin is Lyapunov stable if there exists a scalar \(\rho > 0\) and a \(K\)-function \(\gamma(\cdot)\) such that for all \(|x| \leq \rho\) and \(k \geq 0\)

\[
|\phi(k; x)| \leq \gamma(|x|)
\]

As a third example, consider the definition of robust global asymptotic stability in \(\epsilon\)-\(\delta\) language.

**Definition 8** (Robust global asymptotic stability: \(\epsilon\)-\(\delta\)). Consider a nominal system \(x^+ = f(x)\) with \(f(0) = 0\) in which the origin is globally asymptotically stable. The origin of the perturbed system \(x^+ = f(x) + w\) is robustly globally asymptotically stable if there exists a \(KL\)-function \(\beta(\cdot)\) and for every \(\epsilon > 0\) there exists \(\delta(\epsilon) > 0\) such that for all \(|w| \leq \delta, x \in \mathbb{R}^n\) and \(k \geq 0\)

\[
|x(k; x, w)| \leq \beta(|x|, k) + \epsilon
\]

The equivalent \(K\)-function definition is the following.

**Definition 9** (Robust global asymptotic stability: \(K\)-function). Consider a nominal system \(x^+ = f(x)\) with \(f(0) = 0\) in which the origin is globally asymptotically stable. The origin of the perturbed system \(x^+ = f(x) + w\) is robustly globally asymptotically stable if there exists a scalar \(\rho > 0\), \(K\)-function \(\gamma(\cdot)\), and \(KL\)-function \(\beta(\cdot)\) such that for all \(x \in \mathbb{R}^n, \|w\| \leq \rho\) and \(k \geq 0\)

\[
|x(k; x, w)| \leq \beta(|x|, k) + \gamma(\|w\|)
\]

Note that we could write the final inequality equivalently as

\[
|x(k; x, w)| \leq \beta(|x|, k) + \gamma(\|w\|_{0:k-1})
\]

because \(x(k; x, w)\) depends on \(w\) only up to time \(k - 1\). The last statement is equivalent to the statement that the origin of the system \(x^+ = f(x) + w\) is input-to-state stable (ISS) for small disturbances (\(\|w\| \leq \rho\)) considering the disturbance \(w\) as the input.
2 Generalization

The following definitions and theorem generalize the previous examples. Let $X$ be any normed space.

**Definition 10 (Property $P$).** A system with testable condition $C : X \rightarrow \mathbb{R}_{\geq 0}$ satisfying $C(0) = 0$ has property $P$ if for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$C(x) \leq \epsilon \quad \text{for every } x \in X \text{ satisfying } |x| \leq \delta(\epsilon) \quad (3)$$

We note that the function $\delta(\epsilon)$ in Definition 10 can be made increasing, as shown in the following proposition.

**Proposition 11 ($\delta(\epsilon)$ can be made increasing).** Suppose a system has property $P$ as in Definition 10. Then, without loss of generality, the function $\delta(\epsilon)$ can be assumed to be a nondecreasing function.

*Proof.* Suppose (3) holds for $\hat{\delta}(\epsilon)$ which is possibly not nondecreasing. Next, define $\delta(\epsilon) := \min(\hat{\delta}(\epsilon), 1)$. We note that (3) holds also for $\delta(\epsilon)$ because $(0, \delta(\epsilon)) \subseteq (0, \hat{\delta}(\epsilon))$, and thus (3) holding for $\delta(\epsilon)$ is weaker than for $\hat{\delta}(\epsilon)$. Then, define

$$\delta(\epsilon) := \frac{1}{2} \sup_{s \in (0, \epsilon]} \overline{\delta}(s)$$

which is well-defined because $\overline{\delta}(s) \in (0, 1]$ for all $s > 0$, and all bounded sets of real numbers have suprema. Furthermore, $\delta(\epsilon)$ is clearly nondecreasing. To show that (3) holds for $\delta(\epsilon)$, let $\epsilon_1 > 0$ be arbitrary. By definition, there exists positive $\epsilon_0 < \epsilon_1$ such that $\overline{\delta}(\epsilon_0) \geq \delta(\epsilon_1)$.\(^5\)

Thus, from (3) and these two inequalities, we know that for arbitrary $x \in X$,

$$|x| \leq \delta(\epsilon_1) \implies |x| \leq \overline{\delta}(\epsilon_0) \implies C(x) \leq \epsilon_0 \implies C(x) \leq \epsilon_1$$

which means (3) holds for $\delta(\epsilon)$ and the statement is proved. \(\blacksquare\)

In the language of $K$-functions, we have the following definition of property $P_K$.

**Definition 12 (Property $P_K$).** A system with testable condition $C : X \rightarrow \mathbb{R}_{\geq 0}$ satisfying $C(0) = 0$ has property $P_K$ if there exists $b > 0$ and $K$-function $\gamma(\cdot)$ defined on $[0, b]$, such that for all $x \in X$ satisfying $|x| \leq b$

$$C(x) \leq \gamma(|x|)$$

**Proposition 13 (Equivalence of $P$ and $P_K$).** A system has property $P$ if and only if it has property $P_K$. The constant $b$ defined in Property $P_K$ can be chosen as any positive value satisfying $b < \sup_{\epsilon > 0} \delta(\epsilon)$ with $\delta(\epsilon)$ defined in Property $P$.

\(^5\)Suppose not. Then, for all $s \in (0, \epsilon_1]$, we have $\overline{\delta}(s) < \delta(s) < \sup_{s \in (0, \epsilon_1]} \overline{\delta}(s)$, which is a contradiction because we have found an upper bound strictly less than the supremum.
3 Extensions

Here we show how a global $K$-function can be found for a locally bounded function.

**Proposition 14** (Global $K$-function overbound.). Let $X \subseteq \mathbb{R}^n$ be closed and suppose that a function $V : X \to \mathbb{R}_{\geq 0}$ is continuous at $x_0 \in X$ and locally bounded on $X$ (i.e., bounded on every compact subset of $X$). Then, there exists a $K$-function $\alpha$ such that

$$|V(x) - V(x_0)| \leq \alpha(|x - x_0|) \text{ for all } x \in X$$

**Proof.** First, by Proposition 5, we know that there exists a local overbounding function, i.e., there exists a $K$-function $\gamma$ and a constant $a > 0$ such that

$$|V(x) - V(x_0)| \leq \gamma(|x - x_0|) \text{ whenever } |x - x_0| \leq b_0$$

Note that any $b_0 \in \text{Dom}(\gamma)$ will suffice.

From here, we proceed similarly to Proposition 11 in Rawlings and Mayne (2011). Starting from $b_0$, choose any strictly increasing and unbounded sequence $(b_i)_{i=0}^\infty$. For each $i \in \mathbb{I}_{\geq 1}$, let $B_i = \{x \in X : |x - x_0| \leq b_i\}$. We note that each $B_i$ is a compact subset of $X$ and further that $X = \bigcup_{i=0}^\infty B_i$. Next, define a sequence $(\beta_i)_{i=0}^\infty$ as

$$\beta_i := \sup_{x \in B_i} |V(x) - V(x_0)| + i$$

which is well-defined by compactness of the $B_i$. We note also that the $\beta_i$ are strictly increasing. Finally, define

$$\alpha(s) := \begin{cases} \frac{\beta_1}{\gamma(b_0)} \gamma(s) & s \in [0, b_0) \\ \beta_{i+1} + (\beta_{i+2} - \beta_{i+1}) \frac{s - b_i}{b_{i+1} - b_i} & s \in [b_i, b_{i+1}) \text{ for all } i \in \mathbb{I}_{\geq 0} \end{cases}$$

We illustrate this construction in Figure 4. Clearly, $\alpha(0) = 0$ and $\alpha$ is continuous and increasing. Furthermore, because we have shifted the $\beta_i$ as before, we see that $|V(x) - V(x_0)| \leq \alpha(|x - x_0|)$. \hfill \qed

We note that for the case of $V(x) \geq 0$ and $x_0 = 0$, we have

$$V(x) \leq \alpha(|x|) \text{ for all } x \in X$$

and thus, $\alpha$ gives a global overbound.

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That is, positive integers greater than or equal to 1
Figure 4: Construction of $\alpha$. The function $\alpha(s)$ is constructed by rescaling $\gamma(s)$ on $[0, b_0]$ (green) and then linearly interpolating (red) the points $(b_i, \beta_{i+1})$ (blue).

References