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An input/output-to-state stability converse theorem for closed positive invariant sets

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1 Introduction

1.1 Prior results

The authors in Cai and Teel (2008) prove that if a system is IOSS on $\mathbb{R}^n$, then it admits an IOSS Lyapunov function on $\mathbb{R}^n$. Many models are not defined on all of $\mathbb{R}^n$, however, but on a closed positive invariant set $\mathbb{X}$. Furthermore, inputs may not always take on values in all of $\mathbb{R}^m$, but only on a closed subset $\mathbb{U}$. It is therefore advantageous to produce a converse theorem for systems on closed positive invariant sets.

Most of the ideas present in Cai and Teel (2008) can be used directly with minor modifications. In particular, arguments based on "large" values of the state compared to either the input or output implying some sort of reduction in the size of an upper bound for the state can usually be directly applied. There are several places, however, where $\mathcal{K}$ functions are defined by maximization over compact subsets of $\mathbb{R}^n$. Because the domain of maximization varies continuously with some parameter s, the value function is therefore continuous. However, when we intersect these sets with an arbitrary closed set $\mathbb{X}$, we lose the continuity of the value function. Rather than try to repair this approach, we instead follow the lead of Grüne and Kellett (2014), whose results on implication-form input-to-state stability (ISS) Lyapunov functions for discontinuous systems can be adapted to this context.

The biggest, problem, however, is that the result to obtain a smooth Lypaunov function presented in Kellett and Teel (2004) or its generalization in Kellett and Teel (2005) require the state evolution equation to be defined on open sets. When working on an open set, one can average the value of a function on a neighborhood of every point to create a smoother function. However, it's not obvious how to generalize this procedure to arbitrary closed sets. Integrating over a neighborhood probably would still work for sets that are closures of some open set, but would not work for systems defined on a set of measure zero, such as the surface of a sphere.

We solve this problem instead by extending the state evolution equation to $\mathbb{R}^n \times \mathbb{R}^m$ as an outer semicontinuous set-valued map. This procedure is used in Cai and Teel (2008) as part of a converse theorem for output-to-state stability.

Notation. We denote the set of nonnegative reals as $\mathbb{R}_{\geq 0}$, nonnegative integers as $\mathbb{I}_{\geq 0}$, and integers from j to k as $\mathbb{I}_{j:k}$. Sequences are denoted $(x(j))$ when given by their individual elements or $\mathbf{x}$ when taken as a whole. We denote the euclidean norm of a vector x as $|x|$, define $\|\mathbf{x}\| := \sup_{k \in \mathbb{I}_{\geq 0}} |x(k)|$, and define $\|\mathbf{x}\|_{j:k}$ as $\max_{i \in \mathbb{I}_{j:k}} |x(i)|$.

A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{CPD}$ if $\alpha(0) = 0$, $\alpha(s) > 0$ for all $s > 0$, and it is continuous. It is said to be of class $\mathcal{K}$ if it is of class $\mathcal{CPD}$ and is strictly increasing. It is said to be of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and $\lim_{s \to \infty} \alpha(s) = \infty$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{I}_{\geq 0}$ is said to be of class $\mathcal{KL}$ if, for constant $k \in \mathbb{I}_{\geq 0}$, we have that $\beta(\cdot, k)$ is of class $\mathcal{K}$ and, for constant $s \in \mathbb{R}_{\geq 0}$, the function $\beta(s, \cdot)$ is nonincreasing and $\lim_{k \to \infty} \beta(s, k) = 0$.

Let $\mathcal{N}(x)$ denote the set of neighborhoods of x, i.e., the set of sets that contain x in their interior. For a set-valued map $S : \mathbb{X} \rightsquigarrow \mathbb{R}^m$, let $S^{-1}(\cdot)$ denote its inverse image, i.e., for a set $W \subseteq \mathbb{R}^m$ we have that $S^{-1}(W) := \{x \in \mathbb{X} \mid S(x) \cap W \neq \emptyset\}$. Furthermore, let $\operatorname{gph} S := \{(x, u) \mid u \in S(x)\}$ denote the graph of $S(\cdot)$.

Inspired by the max-plus algebra, we define $x_1 \oplus x_2 := \max(x_1, x_2)$. The operator $\oplus$ is both associative and commutative, and for a non-decreasing function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ we have that $\alpha(x_1 \oplus x_2) = \alpha(x_1) \oplus \alpha(x_2)$. This distributive property is useful for the manipulation of $\mathcal{K}$ functions.

1.2 Concerning set-valued maps

In order to use converse theorem results for difference inclusions such as that in Kellett and Teel (2005), regularity conditions are important. In particular, Kellett and Teel (2005) prove that $\mathcal{KL}$ stability is robust for difference inclusions that are upper semicontinuous (USC) in the common sense of the map of a δ-neighborhood of a point being contained in an ϵ-neighborhood of that point's map, and, as a result, that those difference inclusions admit continuous Lyapunov functions. However, the notion of upper semicontinuity for set-valued maps has fallen out of favor since that paper was published; Rockafellar and Wets (1998), for example, find that it does not adequately capture the continuity properties of unbounded sets or infinite-dimensional sets. As a result, Rockafellar and Wets (1998) took a competing regularity condition (sometimes also called "upper semicontinuity") and named it outer semicontinuity (OSC). To avoid introducing notions of the limits of set-valued maps, we provide an equivalent condition from (Rockafellar and Wets, 1998, Exercise 5.6.a).

Definition 1 (Outer semicontinuity)**.** Consider a mapping $S : \mathbb{R}^n \rightsquigarrow \mathbb{R}^m$, a set $\mathbb{X} \subseteq \mathbb{R}^n$, and any point $\overline{x} \in \mathbb{X}$. $S(\cdot)$ is outer semicontinuous at $\overline{x}$ if for every $u \notin S(\overline{x})$ there are neighborhoods $W \in \mathcal{N}(u)$ and $V \in \mathcal{N}(\overline{x})$ such that $\mathbb{X} \cap V \cap S^{-1}(W) = \emptyset$

Because the issues with USC occur with either unbounded or infinite dimensional sets, we should expect that it is equivalent to OSC under some boundedness criterion. Indeed, as noted in (Rockafellar and Wets, 1998, pp. 193-194), Theorem 5.19 implies that for a locally-bounded (LB), closed-valued map, USC and OSC are equivalent.

Definition 2 (Local boundedness (Rockafellar and Wets, 1998, Definition 5.14))**.** A mapping $S : \mathbb{R}^n \rightsquigarrow \mathbb{R}^m$ is locally bounded at a point $\overline{x}$ if for some neighborhood $V \in \mathcal{N}(\overline{x})$ the set $S(V)$ is bounded. It is called locally bounded on $\mathbb{R}^n$ if this property holds at every $\overline{x} \in \mathbb{R}^n$.

Because Kellett and Teel (2005) considers only compact-valued USC difference inclusions (which, in conjunction with USC, implies local boundedness), there is no loss of generality in using the more modern notion of continuity.

2 Output-to-state Stability and OSS Lyapunov functions

We consider a difference inclusion of the form

$$x^+ \in f(x) \qquad y = h(x) \tag{1}$$

in which $x \in \mathbb{X} \subseteq \mathbb{R}^n$ is the system's state and $y \in \mathbb{R}^p$ is the system's output. We denote particular solutions to this difference inclusion as $(x(0), x(1), \dots) := \mathbf{x}$, in which $x(1) \in f(x(0))$, $x(2) \in f(x(1))$, etc. The corresponding sequence of outputs is denoted $(y(0), y(1), \dots) := \mathbf{y}$. The entire set of solutions to the difference inclusion starting from some $x \in \mathbb{X}$ is denoted $S(x)$.

Assumption 3 (Continuity)**.** The set-valued map $f : \mathbb{X} \rightsquigarrow \mathbb{X}$ is compact-valued, outer semicontinuous, and locally bounded, the function $h : \mathbb{X} \to \mathbb{R}^p$ is continuous, and the set $\mathbb{X}$ is closed

Assumption 4 (Steady State)**.** We have that $0 \in \mathbb{X}$, $f(0) = \{0\}$, and $h(0) = 0$.

Definition 5 (Output-to-state stability)**.** A system is output-to-state stable (OSS) if there exist $\beta \in \mathcal{KL}$ and $\gamma_y \in \mathcal{K}$ such that along all solutions to the difference inclusion $\mathbf{x} \in S(x)$ the following bound holds

$$|x(k)| = \beta(|x(0)|, k) \oplus \gamma_y(\|\mathbf{y}\|_{0:k-1})$$

for all $k \in \mathbb{I}_{\geq 0}$.

Remark 6. If we were formulating ISS for difference inclusions, we could just take the supremum along all possible trajectories. However, here we may have different output sequences along different trajectories.

Definition 7 (Strong global asymptotic stability modulo outputs)**.** A system is strongly globally asymptotically stable modulo outputs (GASMO) if there exist $\rho \in \mathcal{KL}$ and $\varphi_y, \phi_y \in \mathcal{K}_\infty$ such following implications hold along all solutions

$$|x(j)| \geq \varphi_y(|y(j)|) \quad \forall j \in \mathbb{I}_{0:k-1} \implies |x(j)| \leq \rho(|x(0)|, j) \quad \forall j \in \mathbb{I}_{0:k} \tag{2}$$

$$|x(k)| < \varphi_y(|y(k)|) \implies |x(k+1)| \leq \phi_y(|y(k)|) \tag{3}$$

for all $k \in \mathbb{I}_{\geq 0}$.

Remark 8. In Cai and Teel (2008), the condition (2) is the definition of GASMO. Note, however, that we use the range $0:k-1$, rather than $0:k$, to bring it into conformity with our definition of OSS, which similarly uses the range $0:k-1$ for maximization over measurements. Following the results in Grüne and Kellett (2014) about ISS Lyapunov functions for discontinuous systems, however, we can derive the regularity condition (3) in a simpler way directly from OSS, rather than from system continuity.

We require the following proposition to establish an equivalence between IOSS and strong GASMO.

Proposition 9. (Krichman, Sontag, and Wang, 2001, Lemma 2.11) *Let* $\rho : (\mathbb{R}_{\geq 0})^2 \to \mathbb{R}_{\geq 0}$ *be a function such that*

1. *For all* $\epsilon > 0$ *and for all* $R > 0$ *there exists* $T(R, \epsilon)$ *such that* $\rho(r, t) < \epsilon$ *for all* $0 \leq r \leq R$ *and all* $t \geq T$.
2. *For all* $\epsilon > 0$ *there exists* $\delta > 0$ *such that if* $r \leq \delta$ *then* $\rho(r, t) \leq \epsilon$ *for all* $t > 0$.

Then there exists $\beta \in \mathcal{KL}$ *such that* $\rho(r, t) \leq \beta(r, t)$ *for all* $r \geq 0$ *and* $t \geq 0$.

The following proposition is a modified version of Proposition 2.3 in Cai and Teel (2008), which in turn is the discrete-time counterpart to Proposition 2.10 in Krichman et al. (2001).

Proposition 10. *A system is strong GASMO if and only if there exists* $\varphi_y \in \mathcal{K}_\infty$ *such that the following implications hold along all solutions:*

1. *($\mathcal{K}$-stability modulo outputs) There exists* $\nu \in \mathcal{K}$ *such that we have the implication*
$$|x(j)| \geq \varphi_y(|y(j)|) \quad \forall j \in \mathbb{I}_{0:k-1} \quad \implies |x(j)| \leq \nu(|x(0)|) \quad \forall j \in \mathbb{I}_{0:k}$$
for all $k \in \mathbb{I}_{\geq 0}$.
2. *(Locally uniform global attractivity modulo outputs) For every* $\epsilon > 0$ *and* $r > 0$, *there exists* $J(r, \epsilon) \in \mathbb{I}_{\geq 0}$ *such that for all* $x(0) \in \mathbb{X} \cap r\mathbb{B}$ *we have the following implication*
$$|x(j)| \geq \varphi_y(|y(j)|) \quad \forall j \in \mathbb{I}_{0:k-1} \quad \implies |x(j)| \leq \epsilon \quad \forall j \in \mathbb{I}_{J:k}$$
for every $k \in \mathbb{I}_{\geq 0}$.
3. *($\mathcal{K}$-boundedness) There exists* $\phi_y \in \mathcal{K}$ *such that we have the implication*
$$|x(k)| < \varphi_y(|y(k)|) \implies |x(k+1)| \leq \phi_y(|y(k)|) \tag{4}$$

The proof of this proposition largely consists of verifying that the proof of Proposition 2.3 in Cai and Teel (2008) works in the new context of a difference inclusion rather than a difference equation. The addition of requirement 3 strengthens the conclusion of GASMO to strong GASMO.

Proof. The fact that strong GASMO implies these conditions follows immediately from the properties of $\mathcal{KL}$ functions. To prove that these conditions imply strong GASMO, define the function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{I}_{\geq 0} : \mathbb{R}_{\geq 0}$ such that:

$$\begin{aligned} \beta(r,k) &:= \sup_{\mathbf{x} \in S(x(0))} |x(k)| \\ \text{s.t. } & x(0) \in \mathbb{X} \\ & |x(0)| \leq r \\ & |x(j)| \geq \varphi_y(|y(j)|) \quad \forall j \in \mathbb{I}_{0:k-1} \end{aligned}$$

Because the origin is a steady-state for the system and $h(0) = 0$, the set of feasible $\mathbf{x} \in S(x(0))$ is always nonempty. Furthermore, by construction of the constraints, Condition 1 applies so we have that $|x(k)| \leq \nu(|x(0)|)$ for all feasible $x(k)$ and $k \in \mathbb{I}_{\geq 0}$. Therefore, $\beta(r,k) < \infty$ for all $r \in \mathbb{R}_{\geq 0}$ and $k \in \mathbb{I}_{\geq 0}$, and thus it is a well-defined function.

We now seek to use Proposition 9 in order to find a $\mathcal{KL}$ function upper bound for $\beta(\cdot)$. First extend $\beta(r,k)$ from $\mathbb{R}_{\geq 0} \times \mathbb{I}_{\geq 0}$ to $(\mathbb{R}_{\geq 0})^2$ by defining $\tilde{\beta}(r,t) = \beta(r, \lceil t \rceil)$. Condition 1 implies that the second condition of Proposition 9 is fulfilled. Condition 2 implies that the first condition of Proposition 9 is fulfilled. Therefore there exists $\check{\beta} \in \mathcal{KL}$ such that $\beta(r,t) \leq \check{\beta}(r,t)$, which by construction fulfills the conditions for GASMO. Finally note that Condition 3 is sufficient for strong GASMO. □

Proposition 11. *If a difference inclusion is OSS, it is also strong GASMO.*

Note that this proposition is not trivial. Output-to-state stability directly implies that a state is bounded above by a $\mathcal{KL}$ function of its initial state if the state is large compared to all prior outputs. GASMO says that if each state has been small compared to its corresponding output, then the state is bounded above by a $\mathcal{KL}$ function of the initial state.

Proof. The same argument used in Lemma 2.4 in Cai and Teel (2008) to prove that OSS implies GASMO for continuous difference equations also applies in this more general context of difference inclusions. The change from indexing over the range $0:k$ to $0:k-1$ requires some minor changes in the proof, but the argument is not changed in substance, so we do not repeat it here.

All that remains is to establish (3). Note that we defined OSS in terms of $\|\mathbf{y}\|_{0:k-1}$ rather than $\|\mathbf{y}\|_{0:k}$. Therefore, if we have that $|x(k)| < \varphi_y(|y(k)|)$, then we have that

$$\begin{aligned} |x(k+1)| &\leq \beta(|x(k)|, 1) \oplus \gamma_y(|y(k)|) \\ &\leq \beta(\varphi_y(|y(k)|), 1) \oplus \gamma_y(|y(k)|) \end{aligned}$$

Let $\phi_y(s) := \beta(\varphi_y(s), 1) \oplus \gamma_y(s)$ and note that $\phi_y \in \mathcal{K}$. This function fulfills (3) and thus OSS implies strong GASMO. □

Remark 12. Lemma 2.4 in Cai and Teel (2008) goes on to directly show that GASMO implies OSS. We instead first show that GASMO implies that there exists an OSS Lyapunov function, and then go on to show that an OSS Lyapunov function implies OSS.

Definition 13 (OSS Lyapunov Function (strong implication-form))**.** A function $V : \mathbb{X} \to \mathbb{R}_{\geq 0}$ is called a strong implication form OSS Lyapunov function if there exist $\alpha_i, \nu_y, \varphi_y \in \mathcal{K}_\infty$ such that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \tag{5}$$
$$\sup_{x^+ \in f(x)} V(x^+) \leq V(x) - \alpha_3(|x|) \quad \text{if } x \geq \varphi_y(|y|) \tag{6}$$
$$\sup_{x^+ \in f(x)} V(x^+) \leq \nu_y(|y|) \quad \text{if } x < \varphi_y(|y|) \tag{7}$$

for all $x \in \mathbb{X}$.

In line with the distinction between "normal" implication-form and strong implication-form ISS Lyapunov functions introduced by Grüne and Kellett (2014), the GASMO Lyapunov function from Cai and Teel (2008) is a "normal" implication-form OSS Lyapunov function. The condition (7), which is absent from Cai and Teel (2008), is what makes this implication-form OSS Lyapunov function strong.

We first extend $f(\cdot)$ from $\mathbb{X}$ to $\mathbb{R}^n$. Define a new map $F : \mathbb{R}^n \rightsquigarrow \mathbb{X}$ such that

$$F(x) \coloneqq \begin{cases} f(x) & \text{if } x \in \text{int}(\mathbb{X}) \text{ and } |x| > \varphi_y(|h(x)|) \qquad (8) \\ f(x) \cup \{0\} & \begin{cases} \text{if both } x \in \text{int}(\mathbb{X}) \text{ and } \\ |x| = \varphi_y(|h(x)|) \text{ or both } |x| > \\ \varphi_y(|h(x)|) \text{ and } x \in \partial\mathbb{X} \end{cases} \qquad (9) \\ \{0\} & \text{if } |x| < \varphi_y(|h(x)|) \text{ or } x \notin \mathbb{X} \qquad (10) \end{cases}$$

in which $\partial\mathbb{X} \coloneqq \mathbb{X} \setminus \text{int}(\mathbb{X})$ is the boundary of $\mathbb{X}$. Note that because $F(x)$ maps all points in which $|x| < \varphi_y(h(x))$ to zero, $F(\cdot)$ is also a $\mathcal{KL}$ stable difference inclusion with $\mathcal{KL}$ function $\rho(\cdot)$ from the definition of GASMO.

Proposition 14. *If $f(\cdot)$ is outer semicontinuous on $\mathbb{X}$, locally bounded, and compact-valued, we have that $F(\cdot)$ is outer semicontinuous on $\mathbb{R}^n$, locally bounded, and compact-valued.*

Proof. We first prove that $F(\cdot)$ is OSC. Fix a point $\overline{x} \in \mathbb{R}^n$ and $u \notin F(x)$.

1. Suppose that (8) holds at $\overline{x}$. Then we have that $\overline{x} \in \text{int}(\mathbb{X})$ and $|\overline{x}| > \varphi_y(|h(\overline{x})|)$. Then there exists $V_1 \in \mathcal{N}(\overline{x})$ such that $V_1 \in \text{int}(\mathbb{X})$ and $|x| > \varphi_y(|x|)$ for all $x \in V_1$. Thus for all $x \in V_1$, we have that $f(x) = F(x)$. Because $f(\cdot)$ is OSC, there exists some $V_2 \in \mathcal{N}(\overline{x})$ and $W \in \mathcal{N}(u)$ such that $\mathbb{X} \cap V_2 \cap f^{-1}(W)$. Let $V \coloneqq V_1 \cap V_2$. We then have that $V \cap F^{-1}(W) = \emptyset$, and, as a result, $F(\cdot)$ is OSC at $\overline{x}$.

2. Suppose next that (9) holds at $\overline{x}$. There exists some $V \in \mathcal{N}(\overline{x})$ and $W \in \mathcal{N}(u)$ such that $\mathbb{X} \cap V \cap f^{-1}(W) = \emptyset$. Because $u \neq 0$, we can assume, without loss of generality, that $0 \notin W$. As a result $F^{-1}(W) = f^{-1}(W) \subseteq \mathbb{X}$. Thus we have that $V \cap F^{-1}(W) = \emptyset$, and thus $F(\cdot)$ is OSC at $\overline{x}$.

3. Finally, suppose that (10) holds at $\overline{x}$. Then either $|\overline{x}| < \varphi_x(|h(\overline{x})|)$ or $x \notin \mathbb{X}$. Note that because $\mathbb{X}$ is closed, $\mathbb{R}^n \backslash \mathbb{X}$ is open. No matter which condition is true, there exists some $V \in \mathcal{N}(\overline{x})$ such that $F(V) = \{0\}$. Then, because $u \neq 0$, there exists some $W \in \mathcal{N}(u)$ such that $0 \notin W$. Thus $V \cap F^{-1}(W) = \emptyset$ and $F(\cdot)$ is OSC at $\overline{x}$.

Therefore we have that $F(\cdot)$ is outer semicontinuous on all $\mathbb{R}^n$. We have both that $F(\cdot)$ is locally bounded and compact valued because it has at most a single point added to the image of $f(\cdot)$, which is both locally bounded and compact-valued. □

Proposition 15. *If a system* (1) *defined on the positive-invariant set* $\mathbb{X}$ *both satisfies Assumptions 3 and 4 and is strong GASMO, then it admits a smooth strong implication-form OSS Lyapunov function.*

The proof of this proposition is largely the same as that in Cai and Teel (2008). The major difference is that we achieve a strong implication-form IOSS Lyapunov function by using strong GASMO.

Proof. We first prove that $F(\cdot)$ is $\mathcal{KL}$ stable. Note that the expression for $F(\cdot)$ has been designed such that if either (8) or (9) applies, then we have that (2) applies for $f(\cdot)$. Furthermore, the $k = 0$ case of (2) implies that $\rho(s, 0) \geq s$ for all $s \in \mathbb{R}_{\geq 0}$. Thus we have that

$$|x(k)| \leq \rho(|x(0)|, k)$$

along all solutions of $F(\cdot)$.

Because $F(\cdot)$ is outer semicontinuous, locally bounded, compact-valued (by Proposition 14), and $\mathcal{KL}$ stable, by Theorem 12 in Kellett and Teel (2004) it is robustly $\mathcal{KL}$ stable, and thus by Theorem 10 in Kellett and Teel (2004), the system $x^+ \in F(x)$ admits some smooth Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$. Consider $x \in \mathbb{X}$ such that $|x| \geq \varphi_y(h(|x|))$. Because $f(x) \subseteq F(x)$, we have that

$$\sup_{x^+ \in f(x)} V(x^+) \leq V(x) - \alpha_3(|x|)$$

Now suppose that $x \in \mathbb{X}$ and $|x| < \varphi_y(|h(x)|)$. By (3), we have that $|x^+| \leq \phi_y(|y|)$. Because we have that $V(x^+) \leq \alpha_2(|x^+|)$, we have that

$$V(x^+) \leq \alpha_2(\phi_y(|y|))$$

Let $\nu_y(s) := \alpha_2(\phi_y(s))$ and note that $\nu_y \in \mathcal{K}$. Therefore, we have that $V(\cdot)$ is a strong implication-form OSS Lyapunov function. □

The conventional type of OSS Lyapunov function is of dissipation-form.

Definition 16 (Dissipation-form OSS Lyapunov function)**.** A function $V : \mathbb{X} \to \mathbb{R}_{\geq 0}$ is a dissipation-form Lyapunov function if there exists $\alpha_i \in \mathcal{K}_\infty$ and $\sigma_y \in \mathcal{K}$ such that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$$
$$\sup_{x^+ \in f(x)} V(x^+) \leq V(x) - \alpha_3(|x|) + \sigma_y(|y|)$$

for all $x \in \mathbb{X}$.

Proposition 17. *If a function $V : \mathbb{X} \to \mathbb{R}_{\geq 0}$ is a strong implication-form OSS Lyapunov function, then it is also a dissipation-form OSS Lyapunov function.*

This proof is an adaptation of Theorem 4.4 part (a) in Grüne and Kellett (2014) to the context of OSS.

Proof. Without loss of generality, assume that $\alpha_3(s) \leq \alpha_1(s)$ for all $s \in \mathbb{R}_{\geq 0}$. Suppose that $x < \varphi_y(|y|)$. Then we have that

$$\sup_{x^+ \in f(x)} V(x^+) \leq \nu_y(|y|) \leq V(x) - \alpha_3(|x|) + \nu_y(|y|)$$

because $V(x) \geq \alpha_1(|x|) \geq \alpha_3(|x|)$ for all $x \in \mathbb{X}$. Now suppose that $x \geq \varphi_y(|y|)$. Then we have that

$$\sup_{x^+ \in f(x)} V(x^+) \leq V(x) - \alpha_3(|x|) \leq V(x) - \alpha_3(|x|) + \nu_y(|y|)$$

Thus we have that $V(\cdot)$ is a dissipation-form OSS Lyapunov function. □

Finally, we prove that a dissipation-form OSS Lyapunov function implies OSS.

Proposition 18. *If a system* (1) *admits an OSS Lyapunov function, then it is OSS.*

Proof. Suppose that $(1/2)\alpha_3(|x|) \leq \sigma_y(|y|)$. In that case, we have that $|x| \leq \alpha_3^{-1}(2\sigma_y(|x|))$, and so

$$V(x) \leq \alpha_2(|x|) \leq \alpha_2(\alpha_3^{-1}(2\sigma_y(|y|)))$$

Thus we have that

$$\begin{aligned} V(x^+) &\leq V(x) - \alpha_3(|x|) + \sigma_y(|y|) \\ &\leq \alpha_2(\alpha_3^{-1}(2\sigma_y(|y|))) + \sigma_y(|y|) \\ &:= \nu_y(|y|) \end{aligned}$$

Now suppose that $(1/2)\alpha_3(|x|) > \sigma_y(|y|)$. We then have that

$$\begin{aligned} V(x^+) &\leq V(x) - \alpha_3(|x|) + \sigma_y(|y|) \\ &\leq V(x) - (1/2)\alpha_3(|x|) \\ &\leq V(x) - (1/2)\alpha_3(\alpha_2^{-1}(V(x))) \end{aligned}$$

Let $\tilde{\sigma}(s) := s - (1/2)\alpha_3(\alpha_2^{-1}(s))$. We have that $\tilde{\sigma}$ is continuous and $\tilde{\sigma}(s) < s$ for all $s > 0$. By the same construction as used in Theorem 12 in Rawlings and Mayne (2011), we can construct a function $\sigma \in \mathcal{K}_\infty$ such that $\sigma(s) < s$ for all $s > 0$. Furthermore, the function $\tilde{\beta}(s, k) := \sigma^k(s)$, the kth composition of σ with itself, is a $\mathcal{KL}$ function.

Note that whatever value $|x|$ and $|y|$ have, we have that

$$V(x^+) \leq \sigma(V(x)) \oplus \nu_y(|y|)$$

We can then recursively apply this equation. For example, we have that

$$V(x(2)) \leq \sigma^2(V(x(0))) \oplus \sigma \circ \nu_y(|y(0)|) \oplus \nu_y(|y(1)|)$$

and so on. Therefore, we have that

$$V(x(k)) \leq \tilde{\beta}(V(x(0)), k) \oplus \max_{j \in \mathbb{I}_{0:k-1}} \tilde{\beta}(\nu_y(|y(j)|), k-j-1)$$

for all $k \in \mathbb{I}$. Noting that $|x| \leq \alpha_1^{-1}(V(x))$ and that $V(x(0)) \leq \alpha_2(|x(0)|)$, we then have that

$$|x(k)| \leq \alpha_1^{-1}\left(\tilde{\beta}(\alpha_2(|x(0)|), k) \oplus \max_{j \in \mathbb{I}_{0:k-1}} \tilde{\beta}(\nu_y(|y(j)|), k-j-1)\right)$$

Let $\beta(s,k) \coloneqq \alpha_1^{-1}(\tilde{\beta}(\alpha_2(s), k))$ and $\rho_y(s,k) \coloneqq \alpha_1^{-1}(\tilde{\beta}(\nu_y(s), k))$. We have that

$$|x(k)| \leq \beta(|x(0)|, k) \oplus \max_{j \in \mathbb{I}_{0:k-1}} \rho_y(|y(j)|, k-j-1)$$

We can finally define $\gamma_y(s) \coloneqq \rho_y(s, 0)$ to complete the OSS bound

$$|x(k)| \leq \beta(|x(0)|, k) \oplus \gamma_y(\|\mathbf{y}\|_{0:k-1})$$

which completes the proof. □

Remark 19. This proof suggests an alternate characterization of ISS-like properties. Instead of using a constant gain function on the input or output, one can instead maximize over time-discounted inputs and outputs. This characterization makes converging-inputs/outputs implies converging states results much more obvious.

Now that we've returned to OSS, we summarize these results.

Theorem 20. *For a system* (1) *satisfying Assumptions 3 and 4, the following are equivalent:*

1. *Output-to-state stability (OSS)*
2. *Strong global asymptotic stabiltiy modulo outputs (strong GASMO)*
3. *Admitting a strong implication-form OSS Lyapunov function*
4. *Admitting a dissipation-form OSS Lyapunov function*

3 Input/output-to-state stability and IOSS Lyapunov functions

Now, we start with a system that has both inputs and outputs.

$$x^+ \in f(x,u) \qquad y = h(x) \tag{11}$$

We again start by assuming that both $f(\cdot)$ and $h(\cdot)$ have sufficient regularity properties and that the origin is a steady-state.

Assumption 21 (Continuity)**.** The sets $\mathbb{X}$ and $\mathbb{U}$ are both closed, the set-valued map $f : \mathbb{X} \times \mathbb{U} \rightsquigarrow \mathbb{X}$ is outer semicontinuous, locally bounded, and compact-valued, and the function $h : \mathbb{X} \to \mathbb{R}^p$ is continuous.

Assumption 22 (Steady states)**.** We have that $0 \in \mathbb{X}$, $0 \in \mathbb{U}$, $f(0,0) = \{0\}$, and $h(0) = 0$.

Definition 23 (Input/output-to-state stability)**.** A system (11) is input/output-to-state stable (IOSS) if there exists $\beta \in \mathcal{KL}$ and $\gamma_u, \gamma_y \in \mathcal{K}$ such that

$$|x(k)| \leq \beta(|x(0)|, k) \oplus \gamma_u(\|\mathbf{u}\|_{0:k-1}) \oplus \gamma_y(\|\mathbf{y}\|_{0:k-1}) \tag{12}$$

for all solutions $\mathbf{x} \in S(x(0))$, for all initial states $x(0) \in \mathbb{X}$, and all controls $u \in \mathbb{U}$.

Again, note that we maximize the outputs over $0:k-1$ rather than $0:k$. In order to produce an IOSS Lyapunov function, we first reduce the IOSS system to an OSS system by scaling the inputs to be proportional to the size of $|x|$. We then demonstrate that the resulting OSS Lyapunov function for the modified system is also an IOSS Lyapunov function for the original system.

Definition 24 (Strong robust output-to-state stability)**.** A difference inclusion $x^+ \in f(x,u)$ with measurement $y = h(x)$ is strongly robustly output-to-state stable (ROSS) if there exists $\varphi_u \in \mathcal{K}_\infty$ such that the system

$$x^+ \in f(x, (\varphi_u^{-1}(|x|)\mathbb{B}) \cap \mathbb{U}) \qquad y = h(x) \tag{13}$$

is output-to-state stable, and if for some $\phi_u, \phi_y \in \mathcal{K}$ we have the implication

$$\begin{aligned} &|x| \leq \varphi_u(|u|) \\ \implies &\left|x^+\right| \leq \phi_u(|u|) \oplus \phi_y(|y|) \end{aligned} \tag{14}$$

for all $u \in \mathbb{U}$, $x \in \mathbb{X}$, and $x^+ \in f(x,u)$.

In line with the distinction between GASMO and strong GASMO, the fact that the system (13) is OSS is implied in the definition of ROSS used by Cai and Teel (2008), while the implication (14) is what makes ROSS strong.

Proposition 25. *If the system* (11) *is IOSS, then is is also strong ROSS.*

Proof. The proof that the system (13) is OSS requires little modification from that of Lemma 3.6 in Cai and Teel (2008). Minor differences occur because of indexing from $0 : k-1$ rather than $0 : k$. The proofs of Claims B.1, B.2, and B.3 within the proof of Lemma 3.6 in Cai and Teel (2008) hold within this context of a difference inclusion, and can be modified in a straightforward way to deal with maximizing $|y(k)|$ between 0 and $k-1$ rather than 0 and k. We therefore do not replicate their proofs here, but restate their (modified) results here.

Claim 26. There exists some $\varphi_u \in \mathcal{K}_\infty$ such that for any initial condition $x(0)$ and any input sequence $\mathbf{u}$, we have the following implication for any $J \in \mathbb{I}_{\geq 0}$:

$$\begin{aligned} &|x(j)| \geq \varphi_u(|u(j)|) \quad \forall j \in \mathbb{I}_{0:J-1} \\ &\implies |x(j)| \leq \beta(|x(0)|, j) \oplus \gamma_y(\|\mathbf{y}\|_{0:j-1}) \oplus |x(0)|/2 \quad \forall j \in \mathbb{I}_{0:J}. \end{aligned}$$

Claim 27. There exist $\varphi_y, \nu \in \mathcal{K}_\infty$ such that for any initial condition $x(0)$, any input sequence $\mathbf{u}$, and any $J \in \mathbb{I}_{\geq 0}$ the following implication holds

$$\begin{aligned} &|x(j)| \geq \varphi_u(|u(j)|) \oplus \varphi_y(|y(j)|) \quad \forall j \in \mathbb{I}_{0:J-1} \\ &\implies x(j) \leq \nu(|x(0)|) \quad \forall j \in \mathbb{I}_{0:J}. \end{aligned}$$

Claim 28. For any $\epsilon > 0$ and $r \geq 0$, there exists $T(\epsilon, r) \in \mathbb{I}_{\geq 0}$ such that, for any initial condition $x(0) \in \mathbb{X} \cap r\mathbb{B}$, input sequence $\mathbf{u}$, and any $J \in \mathbb{I}_{\geq 0}$, the following implication holds

$$\begin{aligned} &|x(j)| \geq \varphi_u(|u(j)|) \oplus \varphi_y(|y(j)|) \quad \forall j \in \mathbb{I}_{0:J-1} \\ &\implies x(j) \leq \epsilon \quad \forall j \in \mathbb{I}_{T:J}. \end{aligned}$$

Now we define a new difference inclusion with the inputs removed.

$$x^+ \in f(x, (\varphi_u^{-1}(|x|)\mathbb{B}) \cap \mathbb{U}) \coloneqq G(x)$$

Note that we have that $\varphi_u(|u|) \leq x$ in this formulation, and as a result the input condition for these claims is satisfied. Because the input condition is satisfied, Claims 27 and 28 show that $G(\cdot)$ satisfies Conditions 1 and 2 in Proposition 10. Therefore we need show only that Condition 3 holds to show that $G(\cdot)$ is strong GASMO and thus OSS.

By applying the implication from Claim 26 for $J = 1$, we have that

$$\left|x^+\right| \leq \beta(|x(0)|, 1) \oplus \gamma_y(|y(0)|) \oplus |x(0)|/2$$

for all $x^+ \in G(x)$. Assume without loss of generality that $\beta(s, 0) \geq s$. Then we have that

$$\left|x^+\right| \leq \beta(|x|, 0) \oplus \gamma_y(|y|).$$

If we have that $|x| < \varphi_y(|y|)$, then we have that

$$\left|x^+\right| \leq \beta(\varphi_y(|y|), 0) \oplus \gamma_y(|y|)$$

Let $\phi_y(s) \coloneqq \beta(\varphi_y(s), 0) \oplus \gamma_y(s)$. We have that $\phi_y \in \mathcal{K}_\infty$, and thus Condition 3 of Proposition 10 is fulfilled. Therefore $x^+ \in G(x)$ is strong GASMO and thereby OSS.

Now we need only find a function upper bound (14) in order to show that the system (11) is strong ROSS. Suppose that $|x| \leq \varphi_u(|u|)$. Then by applying (12) for $k = 1$, we obtain

$$\begin{aligned} \left|x^+\right| &\leq \beta(|x|, 1) \oplus \gamma_u(|u|) \oplus \gamma_y(|y|) \\ &\leq \beta(\varphi_u(|u|), 1) \oplus \gamma_u(|u|) \oplus \gamma_y(|y|) \\ &= \phi_u(|u|) \oplus \gamma_y(|y|) \end{aligned}$$

in which $\phi_u(s) \coloneqq \beta(s, 1) \oplus \gamma_u(s)$ is of class $\mathcal{K}_\infty$. Thus the system (11) is strong ROSS. $\square$

Proposition 29. *The set-valued map* $f(x, (\varphi_u^{-1}(|x|)\mathbb{B}) \cap \mathbb{U}) \coloneqq G(x)$ *is outer semicontinuous, locally bounded, and compact-valued.*

Proof. See appendix. □

Now that we have that $x^+ \in G(x)$ is OSS and satisfies sufficient regularity conditions, we know that it admits an OSS Lyapunov function. We next prove that this OSS Lyapunov function also serves as an IOSS Lyapunov function for the original system (11).

Definition 30 (IOSS Lyapunov function (strong implication-form))**.** A function $V : \mathbb{X} \to \mathbb{R}_{\geq 0}$ is a strong implication-form IOSS Lyapunov function if there exist $\alpha_i \in \mathcal{K}_\infty$ and $\nu_u, \nu_y, \varphi_u, \varphi_y \in \mathcal{K}_\infty$ such that

$$\begin{aligned} \alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \\ \sup_{x^+ \in f(x)} V(x^+) \leq V(x) - \alpha_3(|x|) \quad \text{if } x \geq \varphi_u(|u|) \oplus \varphi_y(|y|) \\ \sup_{x^+ \in f(x)} V(x^+) \leq \nu_u(|u|) \oplus \nu_y(|y|) \quad \text{if } x < \varphi_u(|u|) \oplus \varphi_y(|y|) \end{aligned}$$

for all $x \in \mathbb{X}$.

Proposition 31. *If a system* (11) *satisfying Assumptions 21 and 22 is ROSS, then it admits a strong implication-form IOSS Lyapunov function.*

Proof. Because we have that $x^+ \in f(x,u)$ is strong ROSS, the system $x^+ \in G(x)$ is OSS. Therefore, there exists some function $V : \mathbb{X} \to \mathbb{R}_{\geq 0}$ and $\alpha_i, \varphi_y, \tilde{\nu}_y \in \mathcal{K}_\infty$ such that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \tag{15}$$

$$\sup_{x^+ \in G(x)} V(x^+) \leq V(x) - \alpha_3(|x|) \quad \text{if } x \geq \varphi_y(|y|) \tag{16}$$

$$\sup_{x^+ \in G(x)} V(x^+) \leq \tilde{\nu}_y(|y|) \quad \text{if } x < \varphi_y(|y|) \tag{17}$$

because $G(\cdot)$ is sufficiently regular (Proposition 29). By the definition of $G(x)$, we have that the same conditions hold for $x^+ \in f(x,u)$ if $|x| \geq \varphi_u(|u|)$. If we have that $|x| \leq \varphi_u(|u|)$, then by (14) we have that

$$\left|x^+\right| \leq \phi_u(|u|) \oplus \phi_y(|y|).$$

As a result, we have that

$$\begin{aligned} \sup_{x^+ \in f(x,u)} V(x^+) \leq \alpha_2(\left|x^+\right|) &\leq \alpha_2(\phi_u(|u|) \oplus \phi_y(|y|)) \\ &= \alpha_2 \circ \phi_u(|u|) \oplus \alpha_2 \circ \phi_y(|y|) \end{aligned}$$

Now define $\nu_y(s) \coloneqq \tilde{\nu}_y(s) \oplus \alpha_2 \circ \phi_y(s)$ and note that it is $\mathcal{K}_\infty$. Furthermore, define $\nu_u(s) \coloneqq \alpha_2 \circ \phi_u(s)$ and note that it also is $\mathcal{K}_\infty$. Thus we have that if $|x| \geq \varphi_u(|u|) \oplus \varphi_y(|y|)$ then we have that

$$\sup_{x^+ \in f(x,u)} V(x^+) \leq V(x) - \alpha_3(|x|)$$

If we have that $|x| < \varphi_y(|y|)$ but $|x| \geq \varphi_u(|u|)$, we have that

$$\sup_{x^+ \in f(x,u)} V(x^+) \leq \tilde{\nu}_y(|y|) \leq \nu_y(|y|)$$

Finally, if $|x| < \varphi_u(|u|)$, we have that

$$\sup_{x^+ \in f(x,u)} V(x^+) \leq \nu_u(|u|) \oplus \nu_y(|y|)$$

Thus we have that $V(\cdot)$ is a strong implication-form IOSS Lyapunov function. □

Next, we show that a strong implication-form IOSS Lyapunov function is also a dissipation-form IOSS Lyapunov function.

Definition 32 (IOSS Lyapunov function (dissipation-form))**.** A function $V : \mathbb{X} \to \mathbb{R}_{\geq 0}$ is a dissipation-form IOSS Lyapunov function if there exist $\alpha_i \in \mathcal{K}_\infty$ and $\sigma_u, \sigma_y \in \mathcal{K}$ such that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$$
$$\sup_{x^+ \in f(x,u)} V(x^+) \leq V(x) - \alpha_3(|x|) + \sigma_u(|u|) + \sigma_y(|y|)$$

for all $u \in \mathbb{U}$.

Proposition 33. *A function $V : \mathbb{X} \to \mathbb{R}_{\geq 0}$ is a strong implication-form IOSS Lyapunov function for a system* (11) *if and only if it also a dissipation-form IOSS Lyapunov function.*

Strong implication-form implies dissipation-form. Without loss of generality, assume that $\alpha_3(s) \geq \alpha_1(s)$ for all $s \in \mathbb{R}_{\geq 0}$. Then if $x \geq \varphi_u(|u|) \oplus \varphi_y(|y|)$, then we have that

$$\begin{aligned}\sup_{x^+ \in f(x,u)} V(x^+) &\leq V(x) - \alpha_3(|x|) \\ &\leq V(x) - \alpha_3(|x|) + \nu_u(|u|) + \nu_y(|y|)\end{aligned}$$

because $\nu_u(|u|) + \nu_y(|y|) \geq 0$. On the other hand, if $x < \varphi_u(|u|) \oplus \varphi_y(|y|)$, we have that

$$\begin{aligned}\sup_{x^+ \in f(x,u)} V(x^+) &\leq \nu_u(|u|) \oplus \nu_y(|y|) \\ &\leq V(x) - \alpha_3(|x|) + \nu_u(|u|) + \nu_y(|y|)\end{aligned}$$

because $V(x) \geq \alpha_1(|x|) \geq \alpha_3(|x|)$ for all $x \in \mathbb{X}$. Therefore, no matter the values of $|x|$, $|u|$, and $|y|$, this dissipation inequality holds. Therefore $V(\cdot)$ is a dissipation-form IOSS Lyapunov function. □

Dissipation-form implies strong implication-form. We want to design $\varphi_y(\cdot)$ and $\varphi_u(\cdot)$ such that $|x| \geq \varphi_u(|u|) \oplus \varphi_y(|y|)$ implies that $\sigma_u(|u|) + \sigma_y(|y|) \leq (1/2)\alpha_3(|x|)$, and thus

$$\sup_{x^+ \in f(x,u)} V(x^+) \leq V(x) - (1/2)\alpha_3(|x|)$$

Therefore, let $\varphi_u(s) := \alpha_3^{-1}(4\sigma_u(s))$ and $\varphi_y(s) := \alpha_3^{-1}(4\sigma_y(s))$. Thus we have that

$$\begin{aligned} |x| &\geq \varphi_u(|u|) \oplus \varphi_y(|y|) \\ (1/2)\alpha_3(|x|) &\geq 2\sigma_u(s) \oplus 2\sigma_y(s) \\ &\geq \sigma_u(s) + \sigma_y(s) \end{aligned}$$

which gives us the desired cost decrease. If, on the other hand, we have that $|x| < \varphi_u(|u|) \oplus \varphi_y(|y|)$, then we have that

$$\begin{aligned} \sup_{x^+ \in f(x,u)} V(x^+) &\leq V(x) + \sigma_u(|u|) + \sigma_y(|y|) \\ &\leq \alpha_2(|x|) + \sigma_u(|u|) + \sigma_y(|y|) \\ &< \alpha_2(\varphi_u(|u|) \oplus \varphi_y(|y|)) + \sigma_u(|u|) + \sigma_y(|y|) \\ &\leq \alpha_2(\varphi_u(|u|) \oplus \varphi_y(|y|)) + (2\sigma_u(|u|) \oplus 2\sigma_y(|y|)) \\ &= (\alpha_2 \circ \varphi_u(|u|) \oplus \alpha_2 \circ \varphi_y(|y|)) + (2\sigma_u(|u|) \oplus 2\sigma_y(|y|)) \\ &\leq 2\alpha_2 \circ \varphi_u(|u|) \oplus 2\alpha_2 \circ \varphi_y(|y|) \oplus 4\sigma_u(|u|) \oplus 4\sigma_y(|y|) \\ &= \nu_u(|u|) \oplus \nu_y(|y|) \end{aligned}$$

in which $\nu_u(s) := 2\alpha_2 \circ \varphi_u(s) \oplus 4\sigma_u(s)$ and $\nu_y(s) := 2\alpha_2 \circ \varphi_y(s) \oplus 4\sigma_y(s)$. Thus we have that $V(\cdot)$ is a strong implication-form IOSS Lyapunov function. □

Finally, we complete the equivalence relationship by demonstrating that a strong implication-form IOSS Lyapunov function implies IOSS.

Proposition 34. *If a system* (11) *admits a strong implication-form IOSS Lyapunov function, then it is IOSS.*

Proof. If $x \geq \varphi_u(|u|) \oplus \varphi_y(|y|)$, we have that

$$\begin{aligned} \sup_{x^+ \in f(x,u)} V(x^+) &\leq V(x) - \alpha_3(|x|) \\ &\leq V(x) - \alpha_3 \circ \alpha_2^{-1}(V(x)) \end{aligned}$$

Let $\tilde{\sigma}(s) := s - \alpha_3 \circ \alpha_2^{-1}(s)$. Note that $\tilde{\sigma}(s) < s$ for all $s > 0$. By the same argument used in the proof of Theorem 12 in Rawlings and Mayne (2011), we can construct $\sigma \in \mathcal{K}_\infty$ such that $\tilde{\sigma}(s) \leq \sigma(s) < s$ for all $s > 0$. Therefore, we have that

$$\sup_{x^+ \in f(x,u)} V(x^+) \leq \sigma(V(x))$$

if $x \geq \varphi_u(|u|) \oplus \varphi_y(|y|)$. Now suppose that $x < \varphi_u(|u|) \oplus \varphi_y(|y|)$. Then we have that

$$\sup_{x^+ \in f(x,u)} V(x^+) \leq \nu_u(|u|) \oplus \nu_y(|y|)$$

Regardless of the values of $|x|$, $|u|$, and $|y|$, we have that

$$\sup_{x^+ \in f(x,u)} V(x^+) \leq \sigma(V(x)) \oplus \phi_u(|u|) \oplus \phi_y(|y|)$$

for all $x \in \mathbb{X}$ and all $u \in \mathbb{U}$. We can apply this inequality twice to obtain

$$\begin{aligned} V(x(2)) \leq \sigma^2(V(x(0))) \oplus \sigma \circ \nu_u(|u(0)|) \oplus \sigma \circ \nu_y(|y(0)|) \\ \oplus \nu_u(|u(1)|) \oplus \nu_y(|y(1)|) \end{aligned}$$

and so on. Let $\beta_V(s,k) := \sigma^k(s)$ and as noted in the proof of Theorem 12 in Rawlings and Mayne (2011), we have that $\beta_V \in \mathcal{KL}$. We then have

$$\begin{aligned} V(x(k)) \leq \beta_V(V(x(0)),k) &\oplus \max_{j \in \mathbb{I}_{0:k-1}} \beta_V(\nu_u(|u(j)|), k-j-1) \\ &\oplus \max_{j \in \mathbb{I}_{0:k-1}} \beta_V(\nu_y(|y(j)|), k-j-1) \end{aligned}$$

Because we have that $\alpha_1(|x(k)|) \leq V(x(k))$ and that $V(x(0)) \leq \alpha_2(|x(0)|)$, we have that

$$\begin{aligned} |x(k)| \leq &\alpha_1^{-1}(\beta_V(\alpha_2(|x(0)|),k)) \oplus \max_{j \in \mathbb{I}_{0:k-1}} \alpha_1^{-1}(\beta_V(\nu_u(|u(j)|), k-j-1)) \\ &\oplus \max_{j \in \mathbb{I}_{0:k-1}} \alpha_1^{-1}(\beta_V(\nu_y(|y(j)|), k-j-1)) \\ =&\beta_x(|x(0)|,k) \oplus \max_{j \in \mathbb{I}_{0:k-1}} \beta_u(|u(j)|, k-j-1) \\ &\oplus \max_{j \in \mathbb{I}_{0:k-1}} \beta_y(|y(j)|, k-j-1) \end{aligned}$$

in which $\beta_x(s) := \alpha_1^{-1}(\beta_V(\alpha_2(s),k))$, $\beta_u(s) := \alpha_1^{-1}(\beta_V(\nu_u(s),k))$, and $\beta_y(s) := \alpha_1^{-1}(\beta_V(\nu_y(s),k))$. Note that all these functions are $\mathcal{KL}$. Finally, we have that

$$\max_{j \in \mathbb{I}_{0:k-1}} \beta_u(|u(j)|, k-j-1) \leq \max_{j \in \mathbb{I}_{0:k-1}} \beta_u(|u(j)|, 0) = \beta_u(\|\mathbf{u}\|_{0:k-1}, 0)$$

and similarly

$$\max_{j \in \mathbb{I}_{0:k-1}} \beta_y(|y(j)|, k-j-1) \leq \max_{j \in \mathbb{I}_{0:k-1}} \beta_y(|y(j)|, 0) = \beta_y(\|\mathbf{y}\|_{0:k-1}, 0)$$

Let $\gamma_u(s) := \beta_u(s,0)$ and $\gamma_y(s) := \beta_y(s,0)$. We thus have that

$$|x(k)| \leq \beta_x(|x(0)|,k) \oplus \gamma_u(\|\mathbf{u}\|_{0:k-1}) \oplus \gamma_y(\|\mathbf{y}\|_{0:k-1})$$

which is the desired IOSS property. □

Remark 35. Note that this result shows that IOSS also admits a characterization in terms of time-discounted inputs and outputs, rather than just the largest inputs and outputs.

We summarize these results in the following.

Theorem 36. *For a system* (11) *satisfying Assumptions 21 and 22 the following statements are equivalent:*

1. *The system is input/output-to-state stable*
2. *The system is strongly robust output-to-state stable*
3. *The system admits a strong implication-form IOSS Lyapunov function*
4. *The system admits a dissipation-form IOSS Lyapunov function*

4 Changing supply rates

It is straightforward to provide a Lyapunov proof of stability for model predictive control (MPC) with a stage cost bounded below by a $\mathcal{K}_\infty$ function of $|(x,u)|$; under appropriate conditions (e.g., a terminal region with a control Lyapunov function as a terminal cost) the controller's optimal cost function is a Lyapunov function for the system. When a semidefinite stage cost is used instead, the controller's optimal cost function cannot be used as a Lyapunov function alone. Grimm, Messina, Tuna, and Teel (2005) define stage cost detectability as the existence of a continuous, strictly dissipative storage function bounded below by a $\mathcal{K}_\infty$ function of $|x|$ that uses some $\mathcal{K}$ function of the stage cost $\ell(x,u)$ as a supply rate. When a system's outputs are penalized, as in (Rawlings, Mayne, and Diehl, 2017, Section 2.4.4), this storage function is an IOSS Lyapunov function.

The construction of a Lyapunov function for the closed-loop MPC system depends on the way the stage cost relates to the storage function. If the stage cost can be used directly as a supply rate, then the sum of the optimal cost and the storage function is a Lyapunov function. If the stage cost cannot be used directly, but a $\mathcal{K}$ function of the stage cost can be used instead, a Lyapunov function can be constructed, as is done in Grimm et al. (2005), but its construction is much more complex.

We provide a proposition showing that we can change the supply rate of such a storage function to use the stage cost directly, but at a cost; although the original storage function has a dissipation rate given by a $\mathcal{K}_\infty$ function of $|x|$, the modified storage function may have a dissipation rate given by only a continuous positive definite function of $|x|$.

We first require a technical proposition.

Proposition 37. *Suppose the function $V : \mathbb{X} \to \mathbb{R}_{\geq 0}$ for some closed domain $\mathbb{X}$ is continuous and positive definite, i.e, $V(0) = 0$ and $V(x) > 0$ if $|x| > 0$. Then there exists $\alpha \in \mathcal{CPD}$ such that $V(x) \geq \alpha(|x|)$ for all $x \in \mathbb{X}$.*

Proof. Provided in the appendix. □

Now, we can state a useful result about changing supply rates for IOSS Lyapunov functions.

Theorem 38. *Suppose we have some measure of input and output size $\ell : \mathbb{R}^p \times \mathbb{U} \to \mathbb{R}_{\geq 0}$ and a function $V : \mathbb{X} \to \mathbb{R}_{\geq 0}$ such that*

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$$
$$\sup_{x^+ \in f(x,u)} V(x^+) \leq V(x) - \alpha_3(|x|) + \sigma(\ell(y,u))$$

for some $\alpha_1, \alpha_2, \alpha_3, \sigma \in \mathcal{K}_\infty$. Then for every $\bar{\sigma} \in \mathcal{K}_\infty$ there exists $\rho \in \mathcal{K}_\infty$ such that $\Lambda(x) := \rho(V(x))$ satisfies

$$\bar{\alpha}_1(|x|) \leq \Lambda(x) \leq \bar{\alpha}_2(|x|)$$
$$\sup_{x^+ \in f(x,u)} \Lambda(x^+) \leq \Lambda(x) - \bar{\alpha}_3(|x|) + \bar{\sigma}(\ell(y,u))$$

for some $\bar{\alpha}_1, \bar{\alpha}_2 \in \mathcal{K}_\infty$ and $\bar{\alpha}_3 \in \mathcal{CPD}$.

Proof. We first demonstrate that the dissipation-form storage function $V(\cdot)$ is also a strong implication-form storage function. We then design the scaling function $\rho(\cdot)$ such that the supply rate term is of the required form when the system is transformed back into a dissipation-form storage function.

Suppose that $\sigma(\ell(y,u)) \leq (1/2)\alpha_3(|x|)$ for some $x \in \mathbb{X}$ and $u \in \mathbb{U}$. Then we have that

$$\sup_{x^+ \in f(x,u)} V(x^+) \leq V(x) - (1/2)\bar{\alpha}_3(|x|).$$

Let $\varphi(s) \coloneqq \alpha_3^{-1}(2\sigma(s))$. Now suppose that $\sigma(\ell(y,u)) > (1/2)\alpha_3(|x|)$ (and thus $\varphi(\ell(y,u)) > |x|$) for some $x \in \mathbb{X}$ and $u \in \mathbb{U}$. Then we have that

$$\begin{aligned}\sup_{x^+ \in f(x,u)} V(x^+) &\leq V(x) + \sigma(\ell(y,u)) \\ &\leq \alpha_2(|x|) + \sigma(\ell(y,u)) \\ &< \alpha_2(\varphi(\ell(y,u))) + \sigma(\ell(y,u)).\end{aligned}$$

Now let $\phi(s) \coloneqq \alpha_2(\varphi(s)) + \sigma(s)$. We thus have that

$$\begin{aligned}\alpha_1(|x|) \leq V(x) &\leq \alpha_2(|x|) \\ \sup_{x^+ \in f(x,u)} V(x^+) &\leq V(x) - (1/2)\alpha_3(|x|) \quad \text{if } |x| \geq \varphi_u(\ell(y,u)) \\ \sup_{x^+ \in f(x,u)} V(x^+) &\leq \phi(\ell(y,u)) \quad \text{if } |x| < \varphi_u(\ell(y,u))\end{aligned}$$

for all $x \in \mathbb{X}$ and $u \in \mathbb{U}$. Let $\rho(s) \coloneqq \bar{\sigma}(\phi^{-1}(s))$ and recall $\Lambda(x) \coloneqq \rho(V(x))$. We immediately have that

$$\bar{\alpha}_1(|x|) \leq \Lambda(x) \leq \bar{\alpha}_2(|x|)$$

for $\bar{\alpha}_1(s) \coloneqq \rho(\alpha_1(s))$ and $\bar{\alpha}_2(s) \coloneqq \rho(\alpha_2(s))$. Now, examining the cost-decrease condition, we have that

$$\begin{aligned}\sup_{x^+ \in f(x,u)} \Lambda(x^+) &\leq \rho(V(x) - (1/2)\alpha_3(|x|)) \quad \text{if } |x| \geq \varphi_u(\ell(y,u)) \\ &= \Lambda(x) - \big(\Lambda(x) - \rho(\rho^{-1}(\Lambda(x)) - (1/2)\alpha_3(|x|))\big)\end{aligned}$$

Consider the function

$$G(x) \coloneqq \Lambda(x) - \rho(\rho^{-1}(\Lambda(x)) - (1/2)\alpha_3(|x|))$$

Assume without loss of generality that $\alpha_1(s) \geq \alpha_3(s)$. Because $V(x) \geq \alpha_1(|x|) \geq \alpha_3(|x|)$, we have that the quantity $\rho^{-1}(\Lambda(x)) - (1/2)\alpha_3(|x|)$ is nonnegative for all $x \in \mathbb{X}$. Furthermore, for $|x| > 0$, we have that $\rho(\rho^{-1}(\Lambda(x)) - (1/2)\alpha_3(|x|)) < \Lambda(x)$ because $\rho \in \mathcal{K}_\infty$. Thus we have that $G(x)$ is continuous and positive definite. Thus, there exists $\bar{\alpha}_3 \in \mathcal{CPD}$ such that $G(x) \geq \bar{\alpha}_3(|x|)$ for all $x \in \mathbb{X}$. Thus we have the implication

$$\sup_{x^+ \in f(x)} \Lambda(x^+) \leq \Lambda(x) - \bar{\alpha}_3(|x|) \quad \text{if } |x| \geq \varphi_u(\ell(y,u))$$

for all $x \in \mathbb{X}$. Finally, we have the implication that

$$\sup_{x^+ \in f(x)} \Lambda(x^+) \leq \rho(\phi(\ell(y,u))) \quad \text{if } |x| < \varphi_u(\ell(y,u))$$
$$= \bar{\sigma}(\ell(y,u))$$

In either case, we have that

$$\sup_{x^+ \in f(x)} \Lambda(x^+) \leq \Lambda(x) - \bar{\alpha}_3(|x|) + \bar{\sigma}(|x|)$$

because $\Lambda(x) - \bar{\alpha}_3(|x|)$ is nonnegative for all $x \in \mathbb{X}$. Thus $\Lambda(\cdot)$ satisfies the required conditions. □

5 Appendix

5.1 Minor technical results

Lemma 39. *If $S : \mathbb{X} \rightsquigarrow \mathbb{R}^m$ is outer semicontinuous with respect to $\mathbb{X} \subseteq \mathbb{R}^n$, in which $\mathbb{X}$ is closed, then it is outer semicontinuous with respect to $\mathbb{R}^n$ when extended in such a way that $S(x) = \emptyset$ if $x \notin \mathbb{X}$.*

Proof. Note that because $S(x) = \emptyset$ if $x \notin \mathbb{X}$, we have that $S^{-1}(W) \subseteq \mathbb{X}$ for all $W \subseteq \mathbb{R}^m$. Consider some point $\overline{x} \in \mathbb{R}^n$ and $u \notin S(x)$.

1. Suppose that $\overline{x} \in \mathbb{X}$. By Definition 1, there exist $W \in \mathcal{N}(u)$ and $V \in \mathcal{N}(x)$ such that $\mathbb{X} \cap V \cap S^{-1}(W) = \emptyset$. However, we have that $S^{-1}(W) \subseteq \mathbb{X}$. Thus $\mathbb{X} \cap S^{-1}(W) = S^{-1}(W)$ and so $V \cap S^{-1}(W) = \emptyset$. Thus $S(\cdot)$ is OSC with respect to all $\mathbb{R}^n$ at $\overline{x}$.

2. Suppose that $\overline{x} \notin \mathbb{X}$. Because $\mathbb{X}$ is closed, $\mathbb{R}^n \backslash \mathbb{X}$ is open, and thus there exists some $V \in \mathcal{N}(\overline{x})$ such that $V \cap \mathbb{X} = \emptyset$. Thus, for any $W \in \mathcal{N}(u)$, we have that $V \cap S^{-1}(W) = \emptyset$. Thus $S(\cdot)$ is OSC with respect to all $\mathbb{R}^n$ at $\overline{x}$.

□

Lemma 40. *Let $f : \mathbb{R}^n \rightsquigarrow \mathbb{R}^m$ and $g : \mathbb{R}^n \rightsquigarrow \mathbb{R}^p$ be outer semicontinuous set-valued maps. Then $h(x) := f(x) \times g(x)$, in which $\times$ denotes the Cartesian product, is an outer semicontinuous set-valued map from $\mathbb{R}^n$ to $\mathbb{R}^{m+p}$.*

Proof. By Definition 1, we know that a set-valued map $S(\cdot)$ is OSC at a point $\overline{x}$ if and only if for every $\overline{u} \notin S(\overline{x})$ there exist $W \in \mathcal{N}(\overline{u})$ and $V \in \mathcal{N}(\overline{x})$ such that $V \cap S^{-1}(W) = \emptyset$.

Fix $\overline{x} \in \mathbb{R}^n$ and $\overline{u} \notin h(\overline{x})$. Let $(\overline{u}_f, \overline{u}_g) := \overline{u}$ such that $\overline{u}_f \in \mathbb{R}^m$ and $\overline{u}_g \in \mathbb{R}^p$. Because both $f(\cdot)$ and $g(\cdot)$ are outer semicontinuous, there exist neighborhoods $V_f, V_g \in \mathcal{N}(\overline{x})$, $W_f \in \mathcal{N}(\overline{u}_f)$, and $W_g \in \mathcal{N}(\overline{u}_g)$ such that $V_f \cap f^{-1}(W_f) = \emptyset$ and $V_g \cap g^{-1}(W_f) = \emptyset$. We have that

$$\begin{aligned} h^{-1}(W_f \times W_g) &= \{x \mid h(x) \cap (W_f \times W_g) \neq \emptyset\} \\ &= \{x \mid (f(x) \times g(x)) \cap (W_f \times W_g) \neq \emptyset\} \\ &= \{x \mid (f(x) \cap W_f) \times (g(x) \cap W_g) \neq \emptyset\} \\ &= \{x \mid f(x) \cap W_f \neq \emptyset \text{ and } g(x) \cap W_g \neq \emptyset\} \\ &= f^{-1}(W_f) \cap g^{-1}(W_g). \end{aligned}$$

Let $V_h \coloneqq V_f \cap V_g$. We thus have that

$$V_h \cap h^{-1}(W_f \times W_g) = (V_f \cap f^{-1}(W_f)) \cap (V_g \cap g^{-1}(W_g)) = \emptyset$$

Furthermore, note that $W_f \times W_g \in \mathcal{N}(\overline{u})$ and $V_h \in \mathcal{N}(\overline{x})$. Thus we have that $h(\cdot)$ is OSC at an arbitrary point $\overline{x}$, and thus is OSC over all of $\mathbb{R}^n$. □

Lemma 41. *Let $f : \mathbb{R}^n \rightsquigarrow \mathbb{R}^m$ and $g : \mathbb{R}^n \rightsquigarrow \mathbb{R}^m$ be outer semicontinuous maps. Then $h(x) \coloneqq f(x) \cap g(x)$ is an outer semicontinuous map.*

Proof. By Rockafellar and Wets (1998, Theorem 5.7), a set-valued map is OSC on $\mathbb{R}^n$ if and only if its graph is closed. Furthermore, we have that

$$\begin{aligned} \operatorname{gph}(f \cap g) &= \{(x, u) \mid u \in (f(x) \cap g(x))\} \\ &= \{(x, u) \mid u \in f(x)\} \cap \{(x, u) \mid u \in g(x)\} \\ &= \operatorname{gph} f \cap \operatorname{gph} g. \end{aligned}$$

Because the intersection of two closed sets is closed, we have that $h(\cdot)$ is OSC. □

5.2 Propositions with deferred proofs

Proposition 29. *The function $f(x, (\varphi_u^{-1}(|x|)\mathbb{B}) \cap \mathbb{U}) \coloneqq G(x)$ is outer semicontinuous, locally bounded, and compact-valued.*

Proof. We have that $f(\cdot)$ has a domain of $\mathbb{X} \times \mathbb{U}$ on which it is OSC. Because $\mathbb{X}$ and $\mathbb{U}$ are both closed, by Lemma 39 it can be extended to $\mathbb{R}^n \times \mathbb{R}^m$ by setting $f(x) = \emptyset$ for $x \notin \mathbb{X}$ while remaining OSC. Let $S(x) \coloneqq \{x\} \times ((\varphi^{-1}(|x|)\mathbb{B}) \cap \mathbb{U})$. We have that $G(x) = (f \circ S)(x)$. Both the maps $x \to \{x\}$ and $x \to \varphi^{-1}(|x|)\mathbb{B}$ are continuous. Furthermore, because closed constant maps are continuous, by Lemma 41 we have that $(\varphi^{-1}(|x|)\mathbb{B}) \cap \mathbb{U}$ is OSC. Thus, because $S(\cdot)$ is the Cartesian product of two OSC maps, by Lemma 40 it is OSC. Furthermore, both $S(\cdot)$ and $f(\cdot)$ are locally bounded. Therefore by Rockafellar and Wets (1998, Proposition 5.52), $G(\cdot)$ is both OSC and locally bounded. Finally, because $G(\cdot)$ is OSC on $\mathbb{R}^n \times \mathbb{R}^m$, it must be closed-valued (Rockafellar and Wets, 1998, p. 154), and, because it is also locally-bounded, it is compact-valued. □

Proposition 37. *Suppose the function $V : \mathbb{X} \to \mathbb{R}_{\geq 0}$ for some closed domain $\mathbb{X}$ is continuous and positive definite. Then there exists $\alpha \in \mathcal{CPD}$ such that $V(x) \geq \alpha(|x|)$ for all $x \in \mathbb{X}$.*

Proof. If $\mathbb{X} = \{0\}$, then any $\alpha \in \mathcal{CPD}$ is sufficient for this proposition. Now, suppose there exists $z \in \mathbb{X}$ such that $p := |z| > 0$. We first construct a continuous nondecreasing function $\nu : [0, p] \to \mathbb{R}_{\geq 0}$ such that $\nu(0) = 0$ and that $V(x) \geq \nu(|x|)$ if $|x| \leq p$, then we construct a nonincreasing function $\lambda : [p, \infty) \to \mathbb{R}_{\geq 0}$ such that $V(x) \geq \lambda(|x|)$ if $|x| > p$, then combine these functions to create $\alpha(\cdot)$.

Define

$$\nu_1(s) := \min_{x \in \mathbb{X}} V(x) \quad \text{s.t. } s \leq |x| \leq p$$

Because $\mathbb{X}$ is closed and the point $z \in \mathbb{X}$ is feasible, this function is well-defined for all $s \in [0, p]$. Furthermore, because $V(\cdot)$ is positive definite, we have that $\nu_1(0) = 0$ and $\nu_1(s) > 0$ if $s > 0$. However, $\nu_1(\cdot)$ may not be continuous.

Case I: The origin is not isolated in $\mathbb{X}$. If the origin is not isolated in $\mathbb{X}$, there exists a sequence (x_k) such that:

1. $x_0 = z$
2. $x_k \in \mathbb{X}$ for all $k \in \mathbb{I}_{\geq 0}$
3. $|x_{k+1}| < |x_k|$ for all $k \in \mathbb{I}_{\geq 0}$
4. $\lim_{k\to\infty} |x_k| = 0$

Because the function $V(\cdot)$ is continuous and zero at the origin, we have that $\lim_{k\to\infty} V(x_k) = 0$. Therefore, we have that $\lim_{k\to\infty} \nu_1(|x_k|) = 0$. Let $\theta_k := |x_k|$ and note that $\theta_0 = p$ and that (θ_k) is strictly decreasing. Because $\nu_1(\cdot)$ is nondecreasing, we have that $\nu_1(\theta_{k+1}) \leq \nu_1(\theta_k) \leq \nu_1(s)$ for all $k \in \mathbb{I}_{\geq 0}$ and $s \in [\theta_k, p]$. Define $\nu(\cdot)$ by

$$\nu(s) := \begin{cases} 0 & \text{if } s = 0 \\ \nu_1(\theta_{k+2}) + (\nu_1(\theta_{k+1}) - \nu_1(\theta_{k+2}))\dfrac{(s - \theta_{k+1})}{(\theta_k - \theta_{k+1})} & \text{if } s \in (\theta_{k+1}, \theta_k] \end{cases}$$

This function is formed by linear interpolation between the values of $\nu(\theta_k)$, but with a right-shift in the indices to ensure that the function remains a lower bound on the entire interval. This right-shift trick is used in a similar fashion in Rawlings and Risbeck (2015) to construct a lower bounding function. Note that, by construction, $\nu(s)$ is continuous for all $s \in (0, p]$. Furthermore, we have that $\nu(\theta_k) = \nu_1(\theta_{k+1})$ and that $\lim_{k\to\infty} \nu_1(\theta_k) = 0$. Therefore $\nu(s)$ is continuous at $s = 0$. Finally, because $\nu_1(s) > 0$ for all $s \in (0, p]$, we have that $\nu(s) > 0$ for all $s \in (0, p]$.

Case II: The origin is isolated in $\mathbb{X}$. If the origin is isolated in $\mathbb{X}$, we have that there exists some $\eta > 0$ such that $\mathbb{X} \cap \eta\mathbb{B} = \{0\}$. Define

$$\zeta := \min_{x \in \mathbb{X}} |x| \quad \text{subject to } \eta \leq |x| \leq p$$

This minimization is well-defined because $\mathbb{X}$ is closed. Let $\mu = \nu_1(\zeta)$, and define

$$\nu(s) := \begin{cases} \dfrac{(\mu/2)s}{\zeta} & \text{if } s \in [0, \zeta] \\ \dfrac{(\mu/2)(s-\zeta)}{p-\zeta} + (\mu/2) & \text{if } s \in (\zeta, p] \end{cases}$$

Note that this piecewise linear function is continuous, strictly increasing, and $\nu(s) \le \nu_1(s)$ for all $s \in [0, p]$, because $\nu_1(s) = \mu$ for all $s \in (0, \zeta]$. Furthermore, $\nu(s) > 0$ for all $s \in (0, p]$. Therefore we have a nondecreasing function on the interval $[0, p]$ that satisfies the required properties.

Next, we design a nonincreasing function for the interval $[p, \infty)$. Define the function

$$\lambda_1(s) := \min_{x \in \mathbb{X}} V(x) \quad \text{s.t. } p \le |x| \le s$$

Note that this function is well defined on $[p, \infty)$ because the point z is always feasible. Furthermore, this function is nonincreasing and nonzero for all $s \in [p, \infty)$, but it may be discontinuous. Finally, note that $V(x) \ge \lambda_1(|x|)$ if $|x| \in [p, \infty)$.

Case I: The set $\mathbb{X}$ is unbounded. If $\mathbb{X}$ is unbounded, there exists a sequence of points (x_k) such that:

1. $x_0 = z$
2. $x_k \in \mathbb{X}$ for all $k \in \mathbb{I}_{\ge 0}$
3. $|x_{k+1}| > |x_k|$ for all $k \in \mathbb{I}_{\ge 0}$
4. $\lim_{k \to \infty}(|x_k|) = \infty$

Again, let $\theta_k := |x_k|$ and note that $\theta_0 = p$. We define the function $\lambda(\cdot)$ as piecewise linear on the intervals $[\theta_k, \theta_k + 1]$. However, here we use a *left-shift* trick rather than a right-shift trick.

$$\lambda(s) := \lambda_1(\theta_{k+1}) + (\lambda_1(\theta_{k+2}) - \lambda_1(\theta_{k+1}))\frac{s - \theta_k}{\theta_{k+1} - \theta_k} \quad \text{if } s \in [\theta_k, \theta_{k+1})$$

Note that because $\lambda_1(\cdot)$ is nonincreasing, we have that $\lambda_1(\theta_{k+2}) \le \lambda_1(\theta_{k+1}) \le \lambda_1(s)$ for all $k \in \mathbb{I}_{\ge 0}$ and $s \in [p, \theta_{k+1}]$. Furthermore, we have that $\lambda(|x|) \le \lambda_1(|x|) \le V(x)$ for all $x \in \mathbb{X}$. Finally, by construction, we have that $\lambda(\cdot)$ is continuous.

Case II: The set $\mathbb{X}$ is bounded. Because $\mathbb{X}$ is closed and bounded, it is compact. Define

$$\zeta := \max_{x \in \mathbb{X}} |x|$$

This maximization is well-defined because $\mathbb{X}$ is compact. Let $\mu := \lambda_1(\zeta) > 0$, and define

$$\lambda(s) := \mu/2 \quad \text{for all } x \in [p, \infty)$$

This function is continuous, nonincreasing, and satisfies $\lambda(s) \leq \lambda_1(s)$ and $\lambda(s) > 0$ for all $s \in [p, \infty)$.
Therefore we have a nonincreasing function on the interval $[p, \infty)$ with the desired properties. Finally, we weld these functions together by rescaling the function larger at $s = p$. If $\nu(p) \geq \lambda(p)$, define

$$\alpha(s) = \begin{cases} \nu(s)\left(\dfrac{\lambda(p)}{\nu(p)}\right) & \text{if } x \in [0, p] \\ \lambda(s) & \text{if } x \in (p, \infty) \end{cases}$$

If, on the other hand, $\nu(p) < \lambda(p)$, define

$$\alpha(s) = \begin{cases} \nu(s) & \text{if } x \in [0, p] \\ \lambda(s)\left(\dfrac{\nu(p)}{\lambda(p)}\right) & \text{if } x \in (p, \infty) \end{cases}$$

In either case, we have that $\alpha(s) \leq \nu(s)$ on the interval $[0, p]$ and that $\alpha(s) \leq \lambda(s)$ on the interval $[p, \infty)$. Furthermore, because $\nu(s) > 0$ for $s > 0$ and $\lambda(s) > 0$ on its entire domain, we have that $\alpha(s)$ is positive definite. Furthermore, because $\nu(\cdot)$ and $\lambda(\cdot)$ are continuous, we have that $\alpha(\cdot)$ is continuous, and thus $\alpha \in \mathcal{CPD}$. Finally, by construction of $\nu(\cdot)$ and $\lambda(\cdot)$, we have that

$$V(x) \geq \alpha(|x|)$$

for all $x \in \mathbb{X}$. Thus we have constructed the desired lower bound. □

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