TWCCC ⋆ Texas – Wisconsin – California Control Consortium

Technical report number 2018-02

Inherent robustness of discontinuous MPC: Even (u, u^3) is robust

Douglas A. Allan[a,*] and James B. Rawlings[b]

December 31, 2018

Abstract

In order to be useful in applications, model predictive control (MPC) must be robust to small disturbances. Continuity of the regulator's optimal value function ensures a measure of robustness, but, in general, there is no guarantee that this function is continuous. Here we present an inherent robustness result for a broad class of systems that have a (not necessarily robust) positive invariant set that is disjoint from the set of discontinuities of the optimal value function. If this set attracts every set in the domain of robustness in a single control move, the system is robustly stable. We demonstrate this result with the notorious (u, u^3) example, which has an optimal value function that is discontinuous on every neighborhood of its target, and establish that MPC is still robust.

1 Introduction

The robustness of model predictive control (MPC) is a subject that has received considerable attention. The paper Grimm, Messina, Tuna, and Teel (2004) demonstrates that there are nominally stabilizing examples of MPC that are destabilized by arbitrarily small disturbances. Three major approaches have appeared: robust MPC, stochastic MPC, and inherently robust MPC. The first two explicitly take disturbances into account, whereas the third designs the nominal optimal control problem such that it has a measure of robustness to sufficiently small disturbances.

*Corresponding Author

[a]University of Wisconsin–Madison. <dallan@wisc.edu>

[b]University of California–Santa Barbara. <jbraw@ucsb.edu>

Here, we present a new result on inherently robust MPC that applies to systems with discontinuous feedback. Some results on the inherent robustness of MPC, such as those about suboptimal MPC in Pannocchia, Rawlings, and Wright (2011), rely on assumptions strong enough to guarantee the continuity of the optimal value function. Continuity of this function is, as noted in Grimm et al. (2004), strong enough to guarantee continuity of optimal MPC. Later results, such as that in Yu, Reble, Chen, and Allgöwer (2014), weakened these assumptions, but it was not clear that those assumptions were weak enough to permit discontinuous value functions because no example of such a function was provided. In Allan, Bates, Risbeck, and Rawlings (2017), an improved version of a result from Pannocchia et al. (2011) concerning the inherent robustness of suboptimal MPC is provided, with an example that has a provably discontinuous optimal value function yet is still robust. However, in that example, the discontinuity in the optimal value function is a finite distance away from a robust positive invariant set.

In this paper, we provide a result that applies to systems that have discontinuity sets that are disjoint from some (not necessarily robust) positive invariant set, but whose distance to that positive invariant set shrinks near the system's control target. Because the optimal value function is a Lyapunov function, it must be continuous at the target, and thus the size of the discontinuity shrinks to zero as the state approaches the target. Therefore, violations of the discontinuity set for states near the target are tolerable so long as the state is not disturbed over the discontinuity set for states far from the target.

To motivate this result, we present the notorious (u, u^3) system, taken from Meadows, Henson, Eaton, and Rawlings (1995), that can be stabilized only by a discontinuous control law. Although discontinuous control laws need not give rise to discontinuous optimal value functions, the authors of Meadows et al. (1995) found that the optimal value function was discontinuous when it was calculated numerically and plotted. Here, we show that the discontinuity arises from the continuity properties of the set of feasible controls. This system is then proven to be inherently robust. The result in, for example, Picasso, Desiderio, and Scattolini (2012) on the inherent robustness of potentially discontinuous systems cannot be applied to this system, both because it requires fairly tight bounds on the Lyapunov function, which we suspect cannot be derived for this system, and because the system's linearization about its steady state is not stabilizable, and as a result the usual method to generate a terminal control law, applying the linear quadratic regulator to the linearized system, fails.

Notation. The set of nonnegative integers is denoted $\mathbb{I}_{\geq 0}$. We denote the euclidean norm of a vector x by $|x|$. Sequences are denoted $\mathbf{w} := (w(0), w(1), \dots)$, and the symbol $\|\mathbf{w}\| := \sup_{k\in\mathbb{I}_{\geq 0}} |w(k)|$. For a set $\mathbb{D}$, the distance from a point x to the set $\mathbb{D}$ is denoted by $|x|_{\mathbb{D}} := \inf_{y\in\mathbb{D}} |x - y|$. The operator $\oplus$ denotes the Minkowski sum of two sets. We denote the unit ball centered at the origin by $\mathbb{B}$, and, for some scalar λ, we define $\lambda\mathbb{X} := \{\lambda x \mid x \in \mathbb{X}\}$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be in class $\mathcal{K}$ if it is continuous, strictly increasing, and $\alpha(0) = 0$. It is said to be in class $\mathcal{K}_\infty$ if it is in class $\mathcal{K}$ and $\lim_{s\to\infty} \alpha(s) = \infty$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{I}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be in class $\mathcal{KL}$ if $\beta(\cdot, k) \in \mathcal{K}$ for all fixed $k \in \mathbb{I}_{\geq 0}$, and $\beta(s, \cdot)$ is nonincreasing and $\lim_{t\to\infty} \beta(s, t) = 0$ for all fixed $s \in \mathbb{R}_{\geq 0}$.

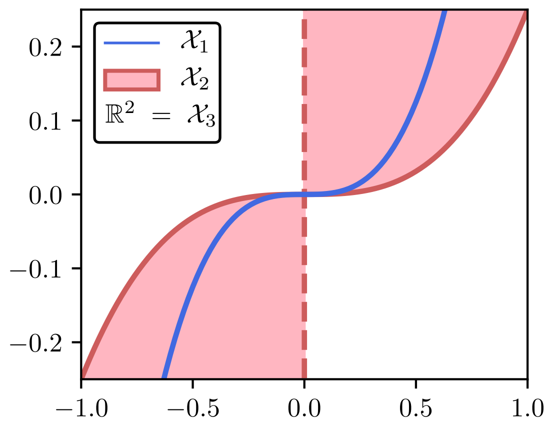

Figure 1: Feasibility sets $\mathcal{X}_1$, $\mathcal{X}_2$, and $\mathcal{X}_3$. Note that, besides the origin, the line $x_1 = 0$ is not included in $\mathcal{X}_2$

2 Motivating Example

Consider the nonlinear system defined by

$$\begin{aligned} x_1^+ &= x_1 + u \\ x_2^+ &= x_2 + u^3 \end{aligned}$$

with stage cost $\ell(\cdot)$ given by

$$\ell(x, u) \coloneqq |x|^2 + u^2 \ .$$

For a horizon length N, we define the objective function

$$V_N(x, \mathbf{u}) \coloneqq \sum_{k=0}^{N-1} \ell(x(k), u(k))$$

in which $\mathbf{u} \coloneqq (u(0), \ldots, u(N-1))$. There are no state and control constraints, but we do include a terminal region $\mathbb{X}_f \coloneqq \{0\}$. Let $\mathcal{Z}_N \coloneqq \{(x, \mathbf{u}) \mid x(N) = 0\}$, $\mathcal{U}_N(x) \coloneqq \{\mathbf{u} \mid (x, \mathbf{u}) \in \mathcal{Z}_N\}$, and $\mathcal{X}_N \coloneqq \{x \mid \exists \mathbf{u} \in \mathcal{U}_N(x)\}$. For simplicity, we denote $x_i(0) \coloneqq x_i$ and $u(k) \coloneqq u_k$.

2.1 Feasibility Sets

We consider the feasibility sets $\mathcal{X}_N$ for $N \geq 1$. For $N = 1$, the terminal constraint $x(N) = 0$ gives

$$x_1(1) = x_1 + u_0 = 0 \qquad x_2(1) = x_2 + u_0^3 = 0 \ .$$

These equations have a solution only for $x_2 = x_1^3$, which defines the feasibility set $\mathcal{X}_1$, depicted in Figure 1. Next consider $N = 2$. We express the terminal constraint as

$$x_1(2) = x_1 + u_0 + u_1 = 0 \qquad x_2(2) = x_2 + u_0^3 + u_1^3 = 0 \ .$$

Solving the first equation for u_1 and substituting into the second equation gives

$$0 = 3x_1u_0^2 + 3x_1^2u_0 + x_1^3 - x_2 \; . \tag{1}$$

For u_0 to be real, we require the discriminant of this quadratic equation to be nonnegative, which reduces to

$$-3x_1^4 + 12x_1x_2 \geq 0 \; .$$

Note that if $x_1 = 0$, the quadratic equation is degenerate. The equation then has a solution only if $x_1^3 = x_2 = 0$. The feasibility region $\mathcal{X}_2$ is thus defined by the inequalities

$$\mathcal{X}_2 = \left\{ (x_1, x_2) : \begin{cases} x_2 \geq (1/4)x_1^3, & x_1 > 0 \\ x_2 = 0, & x_1 = 0 \\ x_2 \leq (1/4)x_1^3, & x_1 < 0 \end{cases} \right\}$$

which is also depicted as the shaded region in Figure 1.

Remark 1. Because $\mathcal{X}_2$ does not contain any points $(x_1, 0)$ for nonzero x_1, the set $\mathcal{X}_2$ is not closed. However, the magnitude of control necessary to steer the system to the origin increases to infinity as x_1 approaches zero for fixed nonzero x_2. The set of points that can be controlled to the origin for less than some fixed cost is compact.

We next show that $\mathcal{X}_3 = \mathbb{R}^2$. Solving the equation

$$\begin{aligned} x_1(3) &= x_1 + u_0 + u_1 + u_2 = 0 \\ x_2(3) &= x_2 + u_0^3 + u_1^3 + u_2^3 = 0 \end{aligned}$$

for u_1 and u_2 in terms of u_0 and (x_1, x_2) yields

$$u_1 = -x_1/2 - u_0/2 \pm \sqrt{b(u_0; x)} \tag{2}$$

$$u_2 = -x_1/2 - u_0/2 \mp \sqrt{b(u_0; x)} \tag{3}$$

in which

$$b(u_0; x) = \frac{3u_0^3 - 3u_0^2x_1 - 3u_0x_1^2 - x_1^3 + 4x_2}{12(u_0 + x_1)} \tag{4}$$

so long as $u_0 \neq -x_1$.

A real solution exists only if $b(u_0; x)$ is positive, i.e., if both the numerator and denominator in the expression for $b(\cdot)$ have the same sign. Note that the leading order terms of u_0 are positive in both the numerator and denominator. As a result, for fixed x, $b(u_0; x)$ is asymptotically positive. Once u_0 is fixed, u_1 and u_2 are determined up to the sign of $\sqrt{b(u_0; x)}$. Therefore, for all $x \in \mathbb{R}^2$, there exists a feasible sequence of controls that steers the state to the origin in three moves, and thus $\mathcal{X}_3 = \mathbb{R}^2$.

2.2 Continuity of optimal value function $V_3^0(x)$

The optimal control problem is defined by

$$V_3^0(x) = \min_{\mathbf{u}} \{V_3(x, \mathbf{u}) \mid x(3) = 0\} \tag{5}$$

Because the terminal equality constraint can be viewed as constraining $\mathbf{u}$ to reside on the nonempty level set of a continuous nonlinear function, a radially unbounded function is being optimized over a closed domain. Thus by the Weierstrass Theorem, this minimization is well-defined for all $x \in \mathbb{R}^2$.

Existence is one thing—continuity is another. The value function for a parametric optimization problem is continuous at a point if its feasible set is a continuous compact set-valued map of its parameters (Polak, 1997, pp. 684-685). Informally, a set-valued map is continuous if its boundaries deform continuously as a function of its parameter. It can neither suddenly gain new points far from its existing points nor suddenly lose a finite volume with an infinitesimal change in parameter. Detailed treatment of the continuity of set-valued maps and its characterization in terms of inner and outer semicontinuity are given in (Rockafellar and Wets, 1998, p. 144) and (Polak, 1997, pp. 676-682), and an application of them in the context of MPC is given in (Rawlings, Mayne, and Diehl, 2017, Appendix C.3.1).

In order to apply this result to this problem, we need to examine the behavior of $\mathcal{U}_3(x)$. It is sufficient to study the behavior of $u_1(u_0; x)$ and $u_2(u_0; x)$, because once u_0 is determined, the only remaining degree of freedom is the choice of sign of $\sqrt{b(u_0; x)}$, and this degree of freedom remains so long as $b(\cdot)$ takes on a real value. Because the leading-order coefficients of the numerator and denominator of $b(u_0; x)$ are not functions of the parameter x, its roots and poles are continuous functions in $\mathbb{C}$ of x (Marden, 1989, p. 3). The only places where u_0 can suddenly take on new values are when the number of roots of the cubic numerator of $b(\cdot)$ changes from one to three or vice-versa. Points where there is zero-pole cancellation should also be inspected.

First, we determine where the cubic numerator

$$p(u_0; x) \coloneqq 3u(0)^3 - 3u(0)^2 x_1 - 3u(0)x_1^2 - x_1^3 + 4x_2 \tag{6}$$

changes in number of roots.

From the theory of cubic equations, we know that changes in the number of real roots to the equation $au^3 + bu^2 + cu + d = 0$ are determined by the sign of the discriminant

$$\Delta = 18abcd - 4b^3 d + b^2 c^2 - 4ac^3 - 27a^2 d^2$$

For $\Delta > 0$, the equation has three distinct real roots, and when $\Delta < 0$, one root is real and the other two are complex conjugates. When $\Delta = 0$, there is a double root. Setting $a = 3$, $b = -3x_1$, $c = -3x_1^2$, and $d = -x_1^3 + 4x_2$ gives

$$\begin{aligned} \Delta &= 432(-x_1^6 + 10x_1^3 x_2 - 9x_2^2) \\ &= -432(x_1^3 - 9x_2)(x_1^3 - x_2) \end{aligned}$$

Setting $\Delta = 0$ gives two (x_1, x_2) curves at which the number of real roots changes. We analyze the behavior of $V_3^0(x)$ at each of these curves in turn.

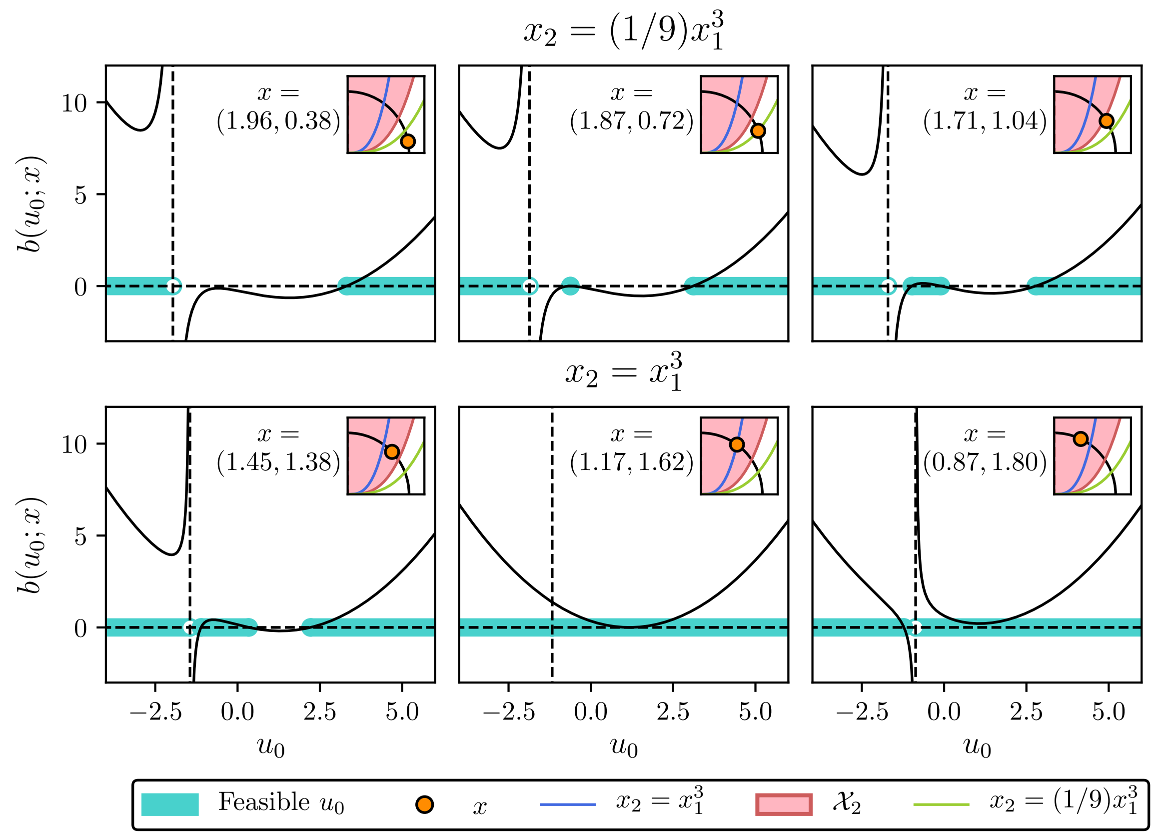

Figure 2: Plots of $b(u_0; x)$ at several different x on a radius 2 circle centered at the origin. *(Top)* When passing over the curve $x_2 = (1/9)x_1^3$, an isolated feasible point, $u_0 = -x_1/3$, emerges. *(Bottom)* When passing over $x_2 = x_1^3$, the set of feasible u_0 continuously deforms as a function of x.

2.3 Minimum Energy Control ($x_2 = (1/9)x_1^3$)

We can calculate the two roots of $p(u_0; x)$ along $x_2 = (1/9)x_1^3$ to obtain

$$r_d = (-1/3)x_1 \qquad r_s = (5/3)x_1$$

in which r_d and r_s are the double and single roots, respectively. Note that choosing $u_0 = (-1/3)x_1$ results in $u_1 = u_2 = (-1/3)x_1$. Because we have $-x_1 = u_0 + u_1 + u_2$, this choice of solution can be interpreted as minimum energy control. Note also that the double root is located between the single root and the pole.

Assume without loss of generality that $x_1 > 0$. For this value of x, then, we have $u_0 < -x_1$, $u_0 = -(1/3)x_1$, or $u_0 \geq (5/3)x_1$. By developing lower bounds for the cost function $V_3(x, \mathbf{u})$ for choices of u_0 below the pole or above the single root, we can prove that the double root is strictly optimal.

Because the terms of $V_3(\cdot)$ corresponding to $x(0)$ are independent of u_0, we can ignore them. Let $\tilde{V}_3(x, \mathbf{u}) := V_3(x, \mathbf{u}) - |x(0)|^2$. Evaluating $\tilde{V}_3(r_d, x)$, we obtain

$$\tilde{V}_3(r_d, x) = 8(x_1/3)^2 + 5(x_1/3)^6$$

If we choose $u_0 \geq 5/3x_1$, we have that

$$\begin{aligned}
\tilde{V}_3(u_0, x) &\geq u_0^2 + x_2(1)^2 \\
&= u_0^2 + ((1/9)x_1^3 + u_0^3)^2 \\
&\geq 25(x_1/3)^2 + ((1/9)x_1^3 + ((5/3)x_1)^3)^2 \\
&= 25(x_1/3)^2 + 16384(x_1/3)^6
\end{aligned}$$

As a result, u_0 cannot be greater than $5/3x_1$.

The other case, requires more development. If $u_0 < -x_1$, then we have that $x_2(1) < -(8/9)x_1^3$. As a result, we have that $u_1^3 + u_2^3 > (8/9)x_1^3$. For any three numbers a, b, c such that $a^3 + b^3 = c$ and $c \geq 0$, by considering the constrained minimization

$$\min_{a,b} a^2 + b^2 \quad \text{such that } a^3 + b^3 \geq c$$

it can be shown that $a^2 + b^2 \geq c^{2/3}$. As a result, we have that $u_1^2 + u_2^2 \geq (8/9)^{2/3}x_1^2$. Thus, by considering the $x_2(1)^2$ and u_k terms of $\tilde{V}_3(\cdot)$, we have that

$$\begin{aligned}
\tilde{V}_3(u_0, x) &\geq u_0^2 + u_1^2 + u_2^2 + ((1/9)x_1^3 + u_0^3)^2 \\
&> (1 + (8/9)^{2/3})x_1^2 + (-(8/9)x_1^3)^2 \\
&\approx 1.92x_1^2 + 0.79x_1^6
\end{aligned}$$

which is strictly greater than the cost from $u_0 = r_d$. Thus, so long as $x_2 = (1/9)x_1^3$, we have that $u_0^0 = r_d$. Now, if $x_2 < (1/9)x_1^3$, the discriminant becomes negative and the double root disappears. Furthermore, because it is located between the single root and the pole, there are no feasible points near the double root's location. Thus the objective function is discontinuous along $x_2 = (1/9)x_1^3$, so long as $x_1 \neq 0$. We show that $V_3^0(x)$ is continuous at $(0, 0)$ by showing that it is a Lyapunov function.

2.4 Deadbeat Control ($x_2 = x_1^3$)

If $x_2 = x_1^3$, the polynomial $p(u_0; x)$ can be solved to obtain

$$r_d = x_1 \qquad r_s = -x_1$$

Furthermore, note that $b(u_0; x)$ has a pole at $u_0 = -x_1$. Thus there is zero-pole cancellation. The function $b(u_0; x)$ can be brought by polynomial long division into an alternate form that provides more clarity in this case:

$$b(u_0; x) = \left(\frac{u_0 - x_1}{2}\right)^2 - \frac{x_1^3 - x_2}{3(u_0 + x_1)}$$

This form of $b(\cdot)$ shows that $x_2 = x_1^3$ is the only place where there can exist zero-pole cancellation.

The emergence or disappearance of the double root does not affect the continuity properties of $V_3^0(x)$ because it emerges at a place strictly greater than both the single root and the pole. It begins to divide the feasible region into two regions, but there are feasible point close to the the point of emergence for small perturbations in x.

The effects of the zero-pole cancellation are more difficult to analyze. The pole exists only because we have projected the feasible region $\mathcal{U}_3(x)$ into a single dimension, but the behavior of that set is harder to visualize as a function of x. However, close inspection of $u_0^0(x)$ allows us to conclude that $V_3^0(x)$ is continuous.

First, we can simplify the expressions for u_2 and u_3 in terms of u_0 along the curve $x_2 = x_1^3$. We have that

$$\begin{aligned} u_1 &= -\frac{u_0 + x_1}{2} \pm \frac{u_0 - x_1}{2} = -u_0 \text{ or } -x_1 \\ u_1 &= -\frac{u_0 + x_1}{2} \mp \frac{u_0 - x_1}{2} = -x_1 \text{ or } -u_0 \end{aligned}$$

which indicates that one of these control moves must cancel out u_0 and the other must bring the system to the origin. Recall, however, that this expression was valid only if $u_0 \neq -x_1$. Symmetry shows that there is a third valid combination of u_k:

$$u_0 = -x_1 \quad u_1 = -u_2$$

Since at least one of these controls must be equal to $-x_1$ and $u_0 = -x_1$ brings the system to the origin in a single move, the optimal control moves are

$$u_0 = -x_1 \quad u_1 = u_2 = 0$$

and deadbeat control is the result.

Because we have that $V_3^0(x) \leq V_2^0(x)$, and $u_0 = -x_1$ is feasible for the two step-controller, the two step controller is also deadbeat along $x_2 = x_1^3$. Examination of the continuity of $V_2^0(x)$ allows us to conclude that $V_3^0(x)$ is upper semicontinuous.

For the two-step controller, it is required that (1)

$$0 = 3x_1u_0^2 + 3x_1^2u_0 + x_1^3 - x_2$$

be satisfied. Recall that the region for which this system has a real solution is $\mathcal{X}_2$. The cubic $x_2 = x_1^3$ is in the interior of $\mathcal{X}_2$ if $x_1 \neq 0$. As a result, if $x_1 \neq 0$, we can solve this equation to obtain

$$u_0 = \frac{-(x_1) \pm \sqrt{x_1^2 - \frac{4(x_1^3 - x_2)}{x_1}}}{2}$$

Note that so long as $x_1 \neq 0$, there is no pole-zero cancellation and for all $x \in \text{int}(\mathcal{X})_2$, we have that $\mathcal{U}_2(x)$ is a continuous set-valued map of x. Thus $V_2^0(x)$ is continuous around the curve $x_2 = x_1^3$. Because we have that $V_3^0(x) \leq V_2^0(x)$ everywhere in $\mathbb{X}_2$, and we have that $V_3^0(x) = V_2^0(x)$ along $x_2 = x_1^3$, we have that $V_3^0(x)$ is upper semicontinuous along $x_2 = x_1^3$ (possibly excluding the origin).

Now, we need only show that $V_3^0(x)$ is lower semicontinuous along $x_2 = x_1^3$ in order to obtain full continuity. Fortunately, under mild conditions, optimal value functions of parametric nonlinear programs are *always* lower semicontinuous functions. In Rockafellar and Wets (1998, Example 5.8, p.154), it is shown that the set-valued map

$$T(w) := \{x \in \mathcal{X} \mid f_i(x, w) \leq 0 \text{ for } i \in I_1 \text{ and } f_i(x, w) = 0 \text{ for } i \in I_2\}$$

is outer semicontinuous if $\mathcal{X}$ is closed and $f_i(x, w)$ are continuous. Furthermore, Polak (1997, Theorem 5.4.1, p. 682) implies that the function

$$\phi^0(w) := \min_{x \in T(w)} \phi(x, w)$$

is lower semicontinuous if $T(\cdot)$ is outer semicontinuous and $\phi(\cdot)$ is both continuous and bounded below. The optimal control problem (5) can be formulated in this framework, and thus $V_3^0(\cdot)$ is lower semicontinuous everywhere in $\mathbb{R}^2$.

As a result, $V_3^0(x)$ is both upper and lower semicontinuous along $x_2 = x_1^3$, and as such is continuous along that curve (possibly excluding the origin). Because both $x_2 = x_1^3$ and $x_2 = (1/9)x_1^3$ meet at the origin, the behavior of $V_3^0(\cdot)$ at the origin is unclear. Fortunately, all that is necessary to prove that $V_3^0(\cdot)$ is continuous at the origin is to prove that it is a Lyapunov function.

2.5 $V_3^0(\cdot)$ is a Lyapunov function on $\mathbb{R}^2$

Because, in general, the optimal control law may be set-valued we provide the definition of a Lyapunov function for difference inclusions.

Definition 2 (Lyapunov Function)**.** A function $V : \mathbb{X} \to \mathbb{R}_{\geq 0}$ for a difference inclusion $x^+ \in H(x)$ is called a *Lyapunov function* if there exist functions $\alpha_i \in \mathcal{K}_\infty$ such that

$$\begin{aligned} \alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \\ V(x^+) \leq V(x) - \alpha_3(|x|) \end{aligned}$$

for all $x^+ \in H(x)$ and $x \in \mathbb{X}$.

If a system admits a Lyapunov function on some positive invariant set $\mathbb{X}$, then it is asymptotically stable (Allan et al., 2017, Proposition 13).

Definition 3 (Asymptotic Stability)**.** The origin of the difference inclusion $x^+ \in H(x)$ is asymptotically stable in a positive invariant set $\mathbb{X}$ if there exists a function $\beta(\cdot) \in \mathcal{KL}$ such that for any $x \in \mathbb{X}$ and for all $k \in \mathbb{I}_{\geq 0}$, all solutions $x(k)$ satisfy

$$|x(k)| \leq \beta(|x|, k)$$

Proposition 4. *If the set $\mathbb{X}$ contains the origin, is positive invariant under the difference inclusion $x^+ \in H(x)$, $H(0) = \{0\}$, and it admits a Lyapunov function $V(\cdot)$ in $\mathbb{X}$, then the origin is asymptotically stable in $\mathbb{X}$.*

It is simple enough to find $\alpha_1(\cdot)$ and $\alpha_3(\cdot)$ for this system. We have that $V_3^0(x) \geq \ell(x,u) \geq |x|^2$, and because $x(3) = 0$, it follows that

$$V_3^0(x^+) \leq V_3^0(x) - \ell(x,u) \leq V_3^0(x) - |x|^2$$

so $\alpha_1(s) \coloneqq \alpha_3(s) \coloneqq s^2$. The cost upper bound is more difficult. In Meadows et al. (1995), it is stated that $u_0 = |x| + 2\sqrt[3]{|x|}$ ensures that $b(\cdot)$ is nonnegative for all $x \in \mathbb{R}^2$. However, this choice of does in fact lead to negative $b(\cdot)$ and thus infeasibility along the line $x_2 = 0$ for large values of x_1. We instead take the approach of choosing u_0 such that $b(u_0; x)$ is zero, which leads to simple expressions for u_1 and u_2.

The magnitude of the roots of a polynomial $a_n z^n + \cdots + a_1 z + a_0$ can be bounded by (Marden, 1989, p. 137)

$$|z| \leq 2 \max\left(\left|\frac{a_{n-1}}{a_n}\right|, \left|\frac{a_{n-2}}{a_n}\right|^{\frac{1}{2}}, \ldots, \left|\frac{a_1}{a_n}\right|^{\frac{1}{n-1}}, \left|\frac{a_0}{2a_n}\right|^{\frac{1}{n}}\right)$$

By applying this bound to the polynomial $p(u_0; x)$, applying the triangle inequality, and noting that $|x_i| \leq |x|$ we obtain

$$\begin{aligned}
|u_0| &\leq 2 \max\left(|x_1|, |x_1|, \left|\frac{4x_2 - x_1^3}{6}\right|^{\frac{1}{3}}\right) \\
&\leq 2 \max\left(|x_1|, (4|x_2| + |x_1|^3)^{\frac{1}{3}}\right) \\
&\leq 2(4|x| + |x|^3)^{\frac{1}{3}} \coloneqq \alpha_{u_0}(|x|)
\end{aligned}$$

As a result, we have that

$$\begin{aligned}
|u_1| \leq |x|/2 + \alpha_{u_0}(|x|)/2 \coloneqq \alpha_u(|x|) \\
|u_2| \leq \alpha_u(|x|)
\end{aligned}$$

Note that this strategy may break down as the result of zero-pole cancellation. However, in that case, we have that $u_0 = -x_1$ and $u_1 = u_2 = 0$ is optimal, and those control actions obey these bounds.

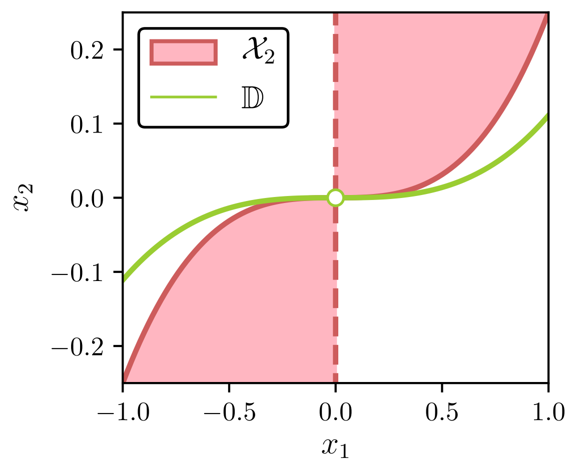

Figure 3: The set $\mathbb{D}$, points of discontinuity in cost $V_3^0(\cdot)$, and invariant set $\mathcal{X}_2$. Note that the two sets do not intersect (the origin is not an element of $\mathbb{D}$).

From these estimates, then, both $|x(1)|$ and $|x(2)|$ can be bounded above by some $\mathcal{K}_\infty$ function of $|x|$, and thus there exists $\alpha_2 \in \mathcal{K}_\infty$ such that

$$V_3^0(x) \leq \alpha_2(|x|)$$

Thus $V_3^0(x)$ is a Lyapunov function and MPC stabilizes this system with a horizon length of three. Furthermore, because $V_3^0(0) = 0$ and $V_3^0(\cdot)$ is nonnegative, by Proposition 4 in Rawlings and Risbeck (2015) we have that $V_3^0(\cdot)$ is continuous at the origin.

2.6 Summary of Continuity Properties of $V_3^0(x)$

So we have that the function $V_3^0(x)$ is continuous everywhere except the curve

$$x_2 = (1/9)x_1^3$$

where it has a discontinuity for all $x_1 \neq 0$. It is not discontinuous at the origin because $V_3^0(x)$ on both sides of the curve of discontinuity go to zero as x approaches 0. Let the set of points of discontinuity be denoted by

$$\mathbb{D} := \{(x_1, x_2) \mid x_2 = (1/9)x_1^3, \quad x_1 \neq 0\}$$

The set $\mathbb{D}$ is shown along with the invariant set $\mathcal{X}_2$ in Figure 3. Note that these two sets do not intersect, which is the key reason why the closed-loop system is *robustly* stable.

3 Inherent Robustness

3.1 $\mathcal{K}_\infty$ function bound for the minimum distance to the discontinuity set

In order to demonstrate that MPC control of this system is inherently robust, we rely on the fact that $\mathcal{X}_2$ is disjoint from $\mathbb{D}$. Furthermore, the distance between points in $\mathcal{X}_2$ and $\mathbb{D}$ grows with increasing size of the state $|x|$. In order to demonstrate this feature and create a $\mathcal{K}_\infty$ function $\alpha_d(\cdot)$ such that $|x|_{\mathbb{D}} \geq \alpha_d(|x|)$ for all $x \in \mathcal{X}_2$.

Let $\overline{\mathcal{X}}_2$ and $\overline{\mathbb{D}}$ denote the closures of $\mathcal{X}_2$ and $\mathbb{D}$, respectively. Consider the optimization

$$\min_{y \in \overline{\mathcal{X}}_2, z \in \overline{\mathbb{D}}} |y - z| \quad \text{subject to } |y| \geq r \tag{7}$$

in which $r > 0$ is a parameter.

Without loss of generality, we can assume $y_1, y_2 \geq 0$ by the symmetry of $\overline{\mathcal{X}}_2$ and $\overline{\mathbb{D}}$.

In order to satisfy the optimality conditions for this problem, the line connecting y^0 and z^0 must exist in the normal cone of both $\{x \in \overline{\mathcal{X}}_2 : \ |x| \geq r\}$ and $\overline{\mathbb{D}}$. As a result, this line must be normal to the graph of $\overline{\mathbb{D}}$, and either must be normal to an edge of $\{x \in \overline{\mathcal{X}}_2 : \ |x| \geq r\}$ or y^0 must be a corner of $\{x \in \overline{\mathcal{X}}_2 : \ |x| \geq r\}$. As a result, y^0 cannot be along the edge given by the line $x_1 = 0$, because no line points outward from that edge to $\overline{\mathbb{D}}$. The corner $(0, r)$ is connected to $(0, 0) \in \overline{\mathbb{D}}$ by a line orthogonal to both the arc $\mathcal{A} := \{x \in \overline{\mathcal{X}}_2 \mid x_1, x_2 \geq 0 \text{ and } |x| = r\}$ and $\overline{\mathbb{D}}$, but any point along $\mathcal{A}$ has an equivalent distance, and none of those points connect to $\overline{\mathbb{D}}$ orthogonally. As a result, we have that y^0 exists along the curve $\mathcal{B} := \{x \mid x_2 = (1/4)x_1^3\}$.

We expect y^0 to be the corner generated by the intersection of $\mathcal{B}$ and the arc $|x|$, but solving the optimality conditions explicitly is intractible. We can, however, find a lower bound for the distance from a point on $\mathcal{B}$ to $\overline{\mathbb{D}}$ that is monotone in $|x|$. This bound, evaluated at the corner, gives a lower bound for $|y^0|_{\mathbb{D}}$.

Let $a := (x_1, (1/4)x_1^3)$. Let $b := (x_1, (1/9)x_1^3)$ denote the point on $\overline{\mathbb{D}}$ directly beneath a. Similarly, let $c := ((9/4)^{1/3}x_1, (1/4)x_1^3)$ denote the point on $\overline{\mathbb{D}}$ directly to the right of a. Let $m := (\tilde{x}_1, (1/9)\tilde{x}_1^3)$ denote the point on $\overline{\mathbb{D}}$ that is closest to a. Suppose $\tilde{x}_1 < x_1$. Then m would be a greater distance away from a than b. Likewise, if $\tilde{x}_1 > (9/4)^{1/3}x_1$, then m would be further away from a than c. Therefore, we have that $\tilde{x}_1 \in [x_1, (9/4)^{1/3}x_1]$. In the first quadrant, the function $x_2 = (1/9)x_1^3$ is strictly convex. Therefore, any segment $\overline{am}$ must pass through $\overline{bc}$. Therefore, the minimum length from a to $\overline{\mathbb{D}}$ is bounded below by the minimum length ℓ from a to $\overline{bc}$. This length can be found through geometry. Note that $\triangle abc$ is a right triangle. The altitude to the hypotenuse of a right triangle forms two smaller right triangles similar to the larger one. Let the length of $\overline{ab}$ be denoted as Δx_2, the length of $\overline{ac}$ be denoted as Δx_1, and the length of $\overline{bc}$ be denoted as h. Then, by similarity, we have that $\ell/\Delta x_1 = \Delta x_2/h$. We also have that $h = \sqrt{\Delta x_1^2 + \Delta x_2^2}$. Therefore,

$$\ell = \frac{\Delta x_1 \Delta x_2}{\sqrt{\Delta x_1^2 + \Delta x_2^2}} = \frac{(((9/4)^{1/3} - 1)x_1)((5/36)x_1^3)}{\sqrt{(((9/4)^{1/3} - 1)x_1)^2 + ((5/36)x_1^3)^2}}$$

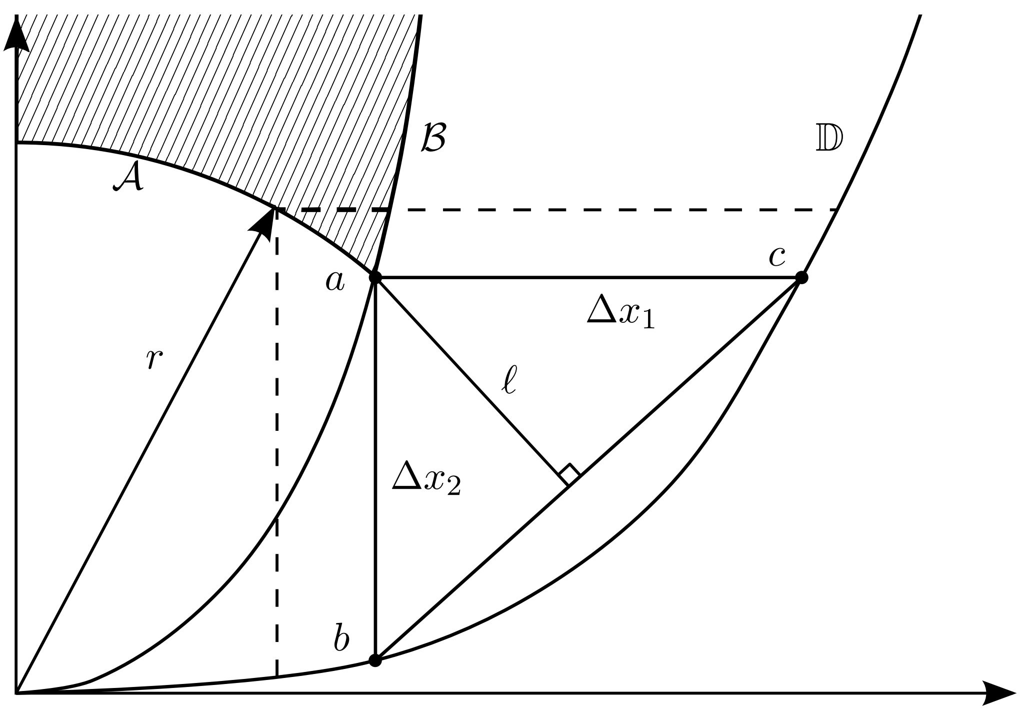

Figure 4: A sketch of the argument which produces $\alpha_d(\cdot)$. Note that neither $\mathcal{B}$ nor $\mathbb{D}$ are precisely plotted and that some features are exaggerated for clarity. The hatched region indicates the set of y feasible for the optimization problem (7).

because $\Delta x_1 = (9/4)^{1/3}x_1 - x_1$ and $\Delta x_2 = (1/4)x_1^3 - (1/9)x_1^3$. Letting $C_1 := (9/4)^{1/3} - 1$, $C_2 = 5/36$, and rearranging, we have that

$$\ell = \frac{C_1 C_2 x_1^3}{\sqrt{C_1^2 + C_2^2 x_1^4}}$$

It can be shown that $\ell(\cdot)$ is a strictly increasing function of x_1. Therefore, because $\ell(x_1)$ is increasing, zero at zero, continuous, and unbounded, $\ell(s) := \tilde{\alpha}_d(s) \in \mathcal{K}_\infty$.

Finally, to create a $\mathcal{K}_\infty$ function in terms of r, note that

$$\alpha_{x_1}(s) := \sqrt{(1/16)s^6 + s^2}$$

is a $\mathcal{K}_\infty$ function that maps x_1 to r. As a result, it is invertible. Thus define $\alpha_d(s) := \tilde{\alpha}_d(\alpha_{x_1}^{-1}(s))$ to obtain a $\mathcal{K}_\infty$ function in terms of r.

3.2 Main results

Robustness can now be demonstrated. This robustness result applies not only to this particular system, but also any system in which there is a growing separation between a (not necessarily robust) positive invariant set and the set of discontinuities of a Lyapunov function. In the case of this particular system, we have that $x^+ \in \mathcal{X}_2$ for any x, because application of u_1 and u_2 must bring the system to the origin.

For simplicity, we consider only additive state disturbances here. This result can be straightforwardly extended to non-additive state disturbances if they enter the system model continuously, and it can be extended to measurement disturbances by shifting analysis from the system's true state to the measured state, as in Roset, Heemels, Lazar, and Nijmeijer (2008) and Allan et al. (2017).

First, we define robustness as input-to-state stability (ISS) on a robust positive invariant set, then define an ISS Lyapunov function, whose existence guarantees input-to-state stability.

Definition 5. Robust asymptotic stability A system $x^+ \in H(x, w)$ is said to be robustly asymptotically stable on a robust positive invariant set $\mathbb{X}$ if there exist $\gamma \in \mathcal{K}$, $\beta \in \mathcal{KL}$, and $\delta > 0$ such that if $\|\mathbf{w}\| \leq \delta$, then

$$|x(k)| \leq \beta(|x(0)|, k) + \gamma(\|\mathbf{w}\|)$$

for all $k \in \mathbb{I}_{\geq 0}$.

Definition 6. ISS Lyapunov function A function $V : \mathbb{X} \to \mathbb{R}_{\geq 0}$ is said to be an ISS Lyapunov function in the robust positive invariant set $\mathbb{X}$ if there exist $\alpha_i \in \mathcal{K}_\infty$, $\sigma \in \mathcal{K}$, and $\delta > 0$ such that if $\|\mathbf{w}\| \leq \delta$, then

$$\begin{aligned} \alpha_1(|x|) &\leq V(x) \leq \alpha_2(|x|) \\ V(x^+) &\leq V(x) - \alpha_3(|x|) + \sigma(|w|) \end{aligned}$$

for all $x \in \mathbb{X}$ and $x^+ \in H(x, w)$.

Now we present the main result.

Theorem 7. *Suppose a system $x^+ = f(x,u)$ under a (not necessarily continuous) control law $u \in \kappa(x)$ admits, on a robust positive invariant set $\mathbb{X}$, a lower semicontinuous Lyapunov function $V : \mathbb{X} \to \mathbb{R}_{\geq 0}$ that is continuous everywhere besides some set $\mathbb{D}$. Furthermore, suppose there exists a closed (not necessarily robust) positive invariant set $\mathcal{X}$ containing the origin and $\mathcal{K}$ function $\alpha_d(\cdot)$ such that, for all $x \in \mathcal{X}$, we have that $|x|_{\mathbb{D}} \geq \alpha_d(|x|)$, and, for all $x \in \mathbb{X}$ and $u \in \kappa(x)$, we have that $f(x,u) \in \mathcal{X}$, i.e., $\mathcal{X}$ attracts all states in a single step.*

Then for every $\rho > 0$, there exists some $\delta > 0$ such that if $\|\mathbf{w}\| \leq \delta$, then the compact set $\operatorname{lev}_\rho V$ is robust positive invariant and $V(\cdot)$ is an ISS Lyapunov function on this set for the disturbed system $x^+ = f(x,u) + w$. As a result, the origin is robustly asymptotically stable.

Proof. We first seek to bound $\|\mathbf{w}\|$ such that $V(x^+) \leq \rho$ for all $x \in \tilde{\mathcal{X}}$. If $V(\cdot)$ were continuous on a robust positive invariant domain, we could divide $\operatorname{lev}_\rho V$ into arbitrary outer and inner sets and adjust the size of $\|\mathbf{w}\|$ to ensure a cost decrease in the outer set and a sufficiently small cost increase in the inner set such that all states in the inner set must at least end up in the outer set.

Because of the limited continuity properties of $V(\cdot)$, we cannot bound the size of $V(x^+) - V(f(x,u))$ in terms of w alone. However, because $V(\cdot)$ is a Lyapunov function, we can guarantee that $V(x^+) \leq \alpha_2(|f(x,u) + w|)$. As a result, we can change the size of the inner set, $\operatorname{lev}_{\lambda\rho} V$ by choosing $\lambda \in (0,1)$ such that for all sufficiently small disturbances and $x \in \operatorname{lev}_{\lambda\rho} V$ we have that $V(x^+) \leq \rho$. Because the outer set is a finite distance away from the discontinuity set $\mathbb{D}$, we can then bound $\|\mathbf{w}\|$ so that no disturbance can perturb any states in that set across the discontinuity.

Let $\tilde{x}^+ := f(x,u)$, i.e., the nominal successor state. Suppose $V(\tilde{x}^+) \leq \lambda\rho$ for some $\lambda \in (0,1)$. Then we have that

$$V(x^+) \leq \alpha_2(\left|\tilde{x}^+ + w\right|) \leq \alpha_2(\alpha_1^{-1}(\lambda\rho) + |w|)$$

We wish to choose δ_1 and λ such that

$$V(x^+) \leq \alpha_2(\alpha_1^{-1}(\lambda\rho) + \delta_1) \leq \rho$$

and as a result we choose

$$\begin{aligned}\delta_1 &= \alpha_2^{-1}(\rho)/2 \\ \lambda &= \alpha_1(\alpha_2^{-1}(\rho)/2)/\rho\end{aligned}$$

which ensures that $V(x^+) \leq \rho$.

Next, suppose that $\lambda\rho < V(\tilde{x}^+) \leq \rho$. Then we have that

$$\left|\tilde{x}^+\right|_{\mathbb{D}} \geq \alpha_d(\left|\tilde{x}^+\right|) > \alpha_d(\alpha_2^{-1}(\lambda\rho))$$

Therefore, choose

$$\delta_2 = \alpha_d(\alpha_2^{-1}(\lambda\rho))/2$$

and the state x^+ remains within the domain of continuity of $V(\cdot)$. In particular, x^+ is in the compact set $\{x \in \mathcal{X} \mid \lambda\rho \le V(x) \le \rho\} \oplus \delta_2\mathbb{B}$. By Proposition 20 of Allan et al. (2017), there exists a $\mathcal{K}_\infty$ function $\sigma_V(\cdot)$ such that

$$\left|V(x^+) - V(\tilde{x}^+)\right| \le \sigma_V(\left|x^+ - \tilde{x}^+\right|) = \sigma_V(|w|)$$

Furthermore, because $V(\cdot)$ is a Lyapunov function for the nominal system, we have that

$$V(\tilde{x}^+) \le V(x) - \alpha_3(\left|\tilde{x}^+\right|) \le V(x) - \alpha_3(\alpha_2^{-1}(\lambda\rho))$$

Combining these two bounds, we have that

$$V(x^+) \le V(x) - \alpha_3(\alpha_2^{-1}(\lambda\rho)) + \sigma_V(|w|)$$

In order to obtain a cost decrease, we require that

$$|w| \le \sigma_V^{-1}(\alpha_3(\alpha_2^{-1}(\lambda\rho))) := \delta_3$$

As a result, we have that $V(x^+) \le \rho$ and that the set $\mathrm{lev}_\rho V$ is robust positive invariant if $\|\mathbf{w}\| \le \min(\delta_1, \delta_2, \delta_3) := \delta$.

If $V(\tilde{x}^+) > \lambda\rho$, then

$$V(x^+) \le V(x) - \alpha_3(|x|) + \sigma_V(|w|)$$

All that remains to prove that $V(\cdot)$ is an ISS Lyapunov function is to derive a similar result for $V(\tilde{x}^+) \le \lambda\rho$.

Suppose $\alpha_d(|\tilde{x}^+|) \le 2\,|w|$. Because $|w| \le \delta_2$, we can invert $\alpha_d(\cdot)$ to obtain

$$\begin{aligned} V(x^+) &\le \alpha_2(\left|\tilde{x}^+\right| + |w|) \\ &\le \alpha_2(|w| + \alpha_d^{-1}(2\,|w|)) \\ &\le V(x) - \alpha_3(|x|) + \alpha_2(|w| + \alpha_d^{-1}(2\,|w|)) \end{aligned}$$

in which the last step follows because $V(x) - \alpha_3(|x|)$ is nonnegative. Define

$$\tilde{\sigma}_w := \alpha_2(|w| + \alpha_d^{-1}(2\,|w|))$$

and note that $\tilde{\sigma}_w \in \mathcal{K}$ on the interval $[0, \delta]$.

Now suppose $\alpha_d(|\tilde{x}^+|) > 2\,|w|$. Consider the set $\tilde{\mathcal{X}} := \{x \mid \exists y \in \mathcal{X}$ such that $|x - y| \le (1/2)\alpha_d(|y|)$ and $V(y) \le \rho\}$. It contains $\tilde{x}$, is disjoint from $\mathbb{D}$, and is compact. As a result, we can apply Proposition 20 of Allan et al. (2017) to obtain $\tilde{\sigma}_V \in \mathcal{K}_\infty$ such that $|V(x^+) - V(\tilde{x})| \le \tilde{\sigma}_V(|w|)$. Thus we have that

$$V(x^+) \le V(x) - \alpha_3(|x|) + \tilde{\sigma}_V(|w|)$$

Now define

$$\sigma_w(s) := \max(\sigma_V(s), \tilde{\sigma}_w(s), \tilde{\sigma}_V(s))$$

and note that $\sigma_w \in \mathcal{K}$ on the interval $[0, \delta]$. Then, if we have that $x \in \mathrm{lev}_\rho V$, we have that

$$V(x^+) \le V(x) - \alpha_3(|x|) + \sigma_w(|w|)$$

and thus is an ISS Lyapunov function on that robust positive invariant set. As a result, the origin is robustly asymptotically stable. $\square$

Remark 8. Note that for the (u, u^3) system, $\mathcal{X}_2$ is positive invariant and attracts all states in a single step, but is not closed. We can substitute its closure, $\overline{\mathcal{X}}_2$, for the role of $\mathcal{X}$ in this theorem.

4 Conclusion

A broad class of systems controlled by MPC that have discontinuous optimal cost functions, containing the (in)famous (u, u^3) example, has been demonstrated to be robustly stable. For any time-invariant MPC implementation, $\mathcal{X}_{N-1}$ is a candidate set for application of Theorem 7 because it is both positive invariant and attracts all feasible states in a single move. However, verifying that $V_N^0(\cdot)$ is continuous on it is difficult. Nevertheless, we hope that better understanding of the underlying structure of robustness results like this one can assist in deriving even simpler conditions for the inherent robustness of MPC. In particular, the result in Allan et al. (2017) implies that $V_N^0(\tilde{x}) \leq V_{N-1}^0(x) + \epsilon$ if $|x - \tilde{x}| \leq \delta(\epsilon)$ for all $\epsilon > 0$. It is possible that the satisfaction of this property, rather than continuity, is sufficient for Theorem 7 to hold, and the results of Allan et al. (2017) can be reduced to a special case of the result in this paper.

References

D. A. Allan, C. N. Bates, M. J. Risbeck, and J. B. Rawlings. On the inherent robustness of optimal and suboptimal nonlinear MPC. *Sys. Cont. Let.*, 106:68 – 78, 2017. ISSN 0167-6911. doi: https://doi.org/10.1016/j.sysconle.2017.03.005. URL `http://www.sciencedirect.com/science/article/pii/S016769111730049X`.

G. Grimm, M. J. Messina, S. E. Tuna, and A. R. Teel. Examples when nonlinear model predictive control is nonrobust. *Automatica*, 40:1729–1738, 2004.

M. Marden. *Geometry of Polynomials.* American Mathematical Society, 1989.

E. S. Meadows, M. A. Henson, J. W. Eaton, and J. B. Rawlings. Receding horizon control and discontinuous state feedback stabilization. *Int. J. Control*, 62(5):1217–1229, 1995.

G. Pannocchia, J. B. Rawlings, and S. J. Wright. Conditions under which suboptimal nonlinear MPC is inherently robust. *Sys. Cont. Let.*, 60:747–755, 2011.

B. Picasso, D. Desiderio, and R. Scattolini. Robust stability analysis of nonlinear discrete-time systems with application to MPC. *Automatic Control, IEEE Transactions on*, 57 (1):185–191, Jan 2012. ISSN 0018-9286. doi: 10.1109/TAC.2011.2163363.

E. Polak. *Optimization: Algorithms and Consistent Approximations.* Springer Verlag, New York, 1997. ISBN 0-387-94971-2.

J. B. Rawlings and M. J. Risbeck. On the equivalence between statements with epsilon-delta and K-functions. Technical Report 2015–01, TWCCC Technical Report, December 2015. URL `http://jbrwww.che.wisc.edu/tech-reports/twccc-2015-01.pdf`.

J. B. Rawlings, D. Q. Mayne, and M. M. Diehl. *Model Predictive Control: Theory, Design, and Computation.* Nob Hill Publishing, Madison, WI, 2nd edition, 2017. 770 pages, ISBN 978-0-9759377-3-0.

R. T. Rockafellar and R. J.-B. Wets. *Variational Analysis.* Springer-Verlag, 1998.

B. Roset, W. Heemels, M. Lazar, and H. Nijmeijer. On robustness of constrained discrete-time systems to state measurement errors. *Automatica*, 44(4):1161 – 1165, 2008. ISSN 0005-1098. doi: http://dx.doi.org/10.1016/j.automatica.2007.08.002.

S. Yu, M. Reble, H. Chen, and F. Allgöwer. Inherent robustness properties of quasi-infinite horizon nonlinear model predictive control. *Automatica*, 50(9):2269 – 2280, 2014. ISSN 0005-1098. doi: http://dx.doi.org/10.1016/j.automatica.2014.07.014.