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Stochastic Lyapunov Functions and Asymptotic Stability in Probability

Robert D. McAllister and James B. Rawlings

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Abstract

The following report consists of definitions and results from multiple sources, complied and modified for the specific problem of interest. Articles published by Teel and coworkers (Teel, 2013; Teel, Hespanha, and Subbaraman, 2014) are the primary source of results and definitions for Sections 2-4. However, Teel and coworkers considered difference inclusions with random stopping times creating extra complications in analysis. We are interested in difference equations without random stopping times so the analysis can be streamlined. In Section 5, original results are presented analogous to the results for deterministic systems in Lin, Sontag, and Wang (1996) and Jiang and Wang (2002). In Section 6, we establish a slightly weaker stability property that is, nonetheless, frequently used. The main result of this work is summarized as follows: If a time-varying, stochastic, system admits a stochastic Lyapunov function then the origin is uniformly asymptotically stable in probability under three different definitions (A classical version, an equivalent $\mathcal{KL}$ version, and a slightly weaker $\mathcal{KL}$ version).

Basic Notation: Let $\mathbb{R}_{\geq 0}$ denote the non-negative real numbers; $\mathbb{I}_{\geq 0}$ denote the non-negative integers. For a closed set $S \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, $|x|_S := \inf_{y \in S} |x-y|$ is the Euclidean distance to S. The set $\mathbb{B}$ (resp., $\mathbb{B}^o$) denotes the closed (resp., open) unit ball in $\mathbb{R}^n$. Given a closed set $S \subset \mathbb{R}^n$ and $\varepsilon > 0$, $S + \varepsilon\mathbb{B}$ (resp., $S + \varepsilon\mathbb{B}^o$) represents the set $\{x \in \mathbb{R}^n : |x|_S \leq \varepsilon\}$ (resp., $\{x \in \mathbb{R}^n : |x|_S < \varepsilon\}$). Let $I_S(x)$ denote the indicator function for the set S, i.e., $I_S(x) = 1$ for all $x \in S$ and $I_S(x) = 0$ otherwise. The function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{PD}$ if it is continuous, $\alpha(s) > 0$ for all $s > 0$, and $\alpha(0) = 0$. The function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ if it is continuous, strictly increasing, and $\alpha(0) = 0$. The function $\alpha(\cdot)$ is of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{I}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if for every fixed k the function $\beta(\cdot, k)$ is of class $\mathcal{K}$ and for fixed s the function $\beta(s, \cdot)$ is nonincreasing and $\lim_{k\to\infty} \beta(s, k) = 0$.

Probability Notation: Let $\mathbf{B}(\mathbb{R}^m)$ denote the Borel field, the subsets of $\mathbb{R}^m$ generate from open subsets of $\mathbb{R}^m$ through complements and finite and countable unions. A set $F \subset \mathbb{R}^m$ is measurable if $F \in \mathbf{B}(\mathbb{R}^m)$. A function $f : \mathbb{R}^p \to \mathbb{R}^n$ is measurable if for each open set $\mathcal{O} \subset \mathbb{R}^n$, the set $f^{-1}(\mathcal{O}) := \{v \in \mathbb{R}^p : f(v) \in \mathcal{O}\} \subset \mathbb{R}^m$.

1 Stochastic System and Preliminary Definitions

Consider a function $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{I}_{\geq 0} \to \mathbb{R}^n$ and a stochastic, time-varying, discrete-time system

$$x^+ = f(x, v, i) \tag{1}$$

in which $x \in \mathbb{R}^n$ is the current state at time $i \in \mathbb{I}_{\geq 0}$, $v \in \mathbb{V} \subset \mathbb{R}^m$ is the (random) input, and $x^+ \in \mathbb{R}^n$ is the successor state at time $i + 1$. We assume that the system starts at some state $x \in \mathbb{R}^n$ at time $t \in \mathbb{I}_{\geq 0}$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space for the inputs in (1). In particular, we have the probability measure $\mathbb{P}(v_i \in F) := \mathbb{P}(\{\omega \in \Omega : v_i(\omega) \in F\})$, i.e., the probability that v_i is in the measurable set F. For $i \in \mathbb{I}_{\geq t}$, let $v_i : \Omega \to \mathbb{V}$ be a sequence of independent, identically distributed (i.i.d.) random variables. Thus, $v_i^{-1}(F) := \{\omega \in \Omega : v_i(\omega) \in F\} \in \mathcal{F}$ for each $F \in \mathbf{B}(\mathbb{V})$. Let $(\mathcal{F}_t, \mathcal{F}_{t+1}, \dots)$ denote the natural filtration of the sequence $(v_t, v_{t+1}, \dots)$. That is, $\mathcal{F}_i \subset \mathcal{F}$ is all sets of the form $\{\omega \in \Omega : (v_t(\omega), \dots, v_i(\omega)) \in F\}$, $F \in \mathbf{B}((\mathbb{V})^{i+1})$. Each random variable has the same probability measure $\mu : \mathbf{B}(\mathbb{V}) \to [0, 1]$ defined as $\mu(F) := \mathbb{P}\{\omega \in \Omega : v_i(\omega) \in F\}$ and, for all $\omega \in \Omega$, we define conditional expected value as

$$\mathbb{E}\left[g(v_t, \dots, v_i, v_{i+1}) \mid \mathcal{F}_i\right](\omega) = \int_{\mathbb{V}} g(v_t(\omega), \dots, v_i(\omega), v)\mu(dv)$$

for each $i \in \mathbb{I}_{\geq t}$ and measurable $g : \mathbb{V}^{i+2} \to \mathbb{R}$.

We define a random process $\mathbf{x}$ as the sequence of random variables $x_i : \text{dom } x_i \subset \Omega \to \mathbb{R}^n$ and $\text{dom } x_{i+1} \subset \text{dom } x_i$ for all $i \in \mathbb{I}_{\geq t}$. A random process $\mathbf{x}$ is adapted to the natural filtration of $\mathbf{v}$ if x_{i+1} is $\mathcal{F}_i$-measurable for each $i \in \mathbb{I}_{\geq t}$. That is, $x_{i+1}^{-1} \subset \mathcal{F}_i$ for each $F \in \mathbf{B}(\mathbb{R}^n)$. A random process $\mathbf{x}$ starting at the initial condition x_t that is adapted to the natural filtration of $\mathbf{v}$, is a random solution of (1) if $x_{i+1}(\omega) = f(x_i(\omega), v_i(\omega), i)$ for all $\omega \in \text{dom } x_{i+1}$ and $i \in \mathbb{I}_{\geq t}$. We use $\Phi(x, t) = \mathbf{x}$ to denote the random solution of (1) given the initial condition x (i.e., $x_t(\omega) = x$ for all $\omega \in \Omega$) and $\phi(i; x, t)$ to denote the element of $\Phi(x, t)$ at time i. For $\mathbf{x} = \Phi(x, t)$ we use the convention that $I_s(\phi(i; x, t)(\omega)) = 0$ for $\omega \notin \text{dom } \phi(i; x, t)$ and we define

$$\text{graph}(\mathbf{x}(\omega)) := \bigcup_{i \in \mathbb{I}_{\geq t}} \{(i, x_i(\omega))\} = \bigcup_{i \in \mathbb{I}_{\geq t}} \{(i, \phi(i; x, t)(\omega))\}$$

We impose the following standing assumption for the rest of this paper to ensure that probabilistic notions of stability are well-defined.

Assumption 1. Either the function $f(\cdot)$ is continuous or $\mathbb{V}$ is a countable set.

If $f(\cdot)$ is continuous, then compositions of $f(\cdot)$ with itself are also continuous and therefore measurable. Thus, $\Phi(x,t)$ is measurable with respect to $\mathbf{v}$. Alternatively, if $\mathbb{V}$ is a countable set, then any integral over all $v \in \mathbb{V}$ may be evaluated as a summation and continuity of $f(\cdot)$ or any function is not required.

Remark 1. To extend these results to discontinuous $f(\cdot)$ with $\mathbb{V}$ an uncountable set, we need to utilize methods detailed in Grammatico, Subbaraman, and Teel (2013). However, to streamline the subsequent analysis, which is already sufficiently complex, we exclude these potential complications with Assumption 1.

Finally, we define robust sequential positive invariance for a sequence of sets $(\mathcal{X}(i))_{i\geq 0}$ relative to the stochastic system.

Definition 1 (Robust sequential positive invariance)**.** A sequence of closed sets $(\mathcal{X}(i))_{i\geq 0}$ is robustly sequentially positive invariant relative to (1) if $x \in \mathcal{X}(i)$ implies $x^+ = f(x,v,i) \in \mathcal{X}(i+1)$ for all $v \in \mathbb{V}$ and $i \in \mathbb{I}_{\geq 0}$.

2 Definitions of Stochastic Stability

We begin with the definition a stochastic Lyapunov function for (1), from which we establish all the following forms of stability.

Definition 2 (Stochastic Lyapunov function)**.** A measurable function $V : \mathbb{R}^n \times \mathbb{I}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a stochastic Lyapunov function on the closed sets $(\mathcal{X}(i))_{i\geq 0}$ for (1) if $(\mathcal{X}(i))_{i\geq 0}$ is robustly sequentially forward invariant for (1) and there exists $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\alpha_3 \in \mathcal{PD}$ such that

$$\alpha_1(|x|) \leq V(x,i) \leq \alpha_2(|x|) \tag{2}$$

$$\mathbb{E}\left[V(f(x,v,i),i+1)\right] := \int_{\mathbb{V}} V(f(x,v,i),i+1)\mu(dv) \leq V(x,i) - \alpha_3(|x|) \tag{3}$$

for all $x \in \mathcal{X}(i)$, $i \in \mathbb{I}_{\geq 0}$, and the integral is well defined.

Remark 2. The stochastic Lyapunov function is the one element of stochastic stability theory that remains the most consistent across literature. Despite small variations in the exact nature of α_3 ($\mathcal{PD}$, $\mathcal{K}$, or some other similar form), the use of expected value of the Lypaunov function, i.e., $\mathbb{E}\left[V(f(x,v,i),i+1)\right]$, on the left-hand side of (3) is nearly universal.

2.1 Asymptotic stability in probability ($\mathcal{KL}$-version)

We begin by defining asymptotic stability in probability using the modern convention of comparison functions (specifically $\mathcal{KL}$ functions). The primary goal of this work is to demonstrate that systems admitting a stochastic Lyapunov function satisfy this definition.

Definition 3 (Asymptotic stability in probability - $\mathcal{KL}$)**.** Suppose that the sequence $(\mathcal{X}(i))_{i\geq 0}$ is robustly sequentially positive invariant for (1). The origin is asymptotically stable in probability, $\mathcal{KL}$ version, (ASiP-$\mathcal{KL}$) for (1) on the sets $(\mathcal{X}(i))_{i\in\mathbb{I}_{\geq 0}}$ if for each $\varrho > 0$, there exists $\beta_\varrho \in \mathcal{KL}$ such that

$$\mathbb{P}\left((\mathrm{graph}(\Phi(x,t)) \cap (\mathbb{I}_{\geq t+k} \times \mathbb{R}^n)) \subset (\mathbb{I}_{\geq t+k} \times \beta_\varrho(|x|,k)\mathbb{B})\right) \geq 1-\varrho \tag{4}$$

for all $k \in \mathbb{I}_{\geq 0}$, $x \in \mathcal{X}(t)$, and $t \in \mathbb{I}_{\geq 0}$.

Note that the condition in (4) is equivalent to $|\phi(t+i;x,t)(\omega)| \leq \beta_\varrho(|x|,k)$ for all $i \in \mathbb{I}_{\geq k}$. In contrast, consider the following weaker definition.

Definition 4 (Weak asymptotic stability in probability - $\mathcal{KL}$)**.** Suppose that the sequence $(\mathcal{X}(i))_{i\geq 0}$ is robustly sequentially positive invariant for (1). The origin is weakly asymptotically stable in probability, $\mathcal{KL}$ version, (weakly ASiP-$\mathcal{KL}$) for (1) on the sets $(\mathcal{X}(i))_{i\geq 0}$ if for each $\varrho > 0$, there exists $\beta_\varrho \in \mathcal{KL}$ such that

$$\mathbb{P}\left(|\phi(t+k;x,t)| \leq \beta_\varrho(|x|,k)\right) \geq 1-\varrho \tag{5}$$

for all $k \in \mathbb{I}_{\geq 0}$, $x \in \mathcal{X}(t)$, and $t \in \mathbb{I}_{\geq 0}$.

We illustrate the differences between these two definitions in Figure 1. We consider a random process that produces two equally likely trajectories ($\omega = 1, 2$). We choose $\varrho = 0.5$ and select a function $\beta_\varrho(\cdot)$ such that the random process satisfies (5). Note that at any given $k \in \mathbb{I}_{\geq 0}$ at least one of the two trajectories satisfies this bound and therefore this random process satisfies (5). However, both of these trajectories violate the bound $\beta_\varrho(\cdot)$ at some value of $k \in \mathbb{I}_{\geq 0}$. Thus, they do not satisfy (4) with this function $\beta_\varrho(\cdot)$, for any value of $\varrho \in (0, 1)$.

The first (stronger) definition is often more desirable, but this property may be unrealistic for many systems of interest. Nonetheless, we establish that a stochastic Lyapunov function for a system is sufficient to guarantee the stronger version of ASiP-$\mathcal{KL}$. Furthermore, we establish that the first definition is, in fact, stronger than the second definition.

Remark 3. The distinction between the stronger and weaker definitions of ASiP-$\mathcal{KL}$ are important. The desirable and achievable properties of a system must be carefully considered before endeavoring to prove a specific type of stability. Consider a chemical process in which the variable we are attempting to stabilize is a measure of product quality. Manufacturing of a specific product is performed in one month long campaigns.

1. If any deviation of product quality beyond specified bound (i.e., β_ϱ) ruins the entire production campaign, then the stronger definition of ASiP is desired. This definition ensures that a campaign is successful at least $1-\varrho$ of the time. If the weaker definition is used, we allow for the possibility that all potential trajectories for the campaign eventually violate this bound and all the campaigns fail.

2. If instead, we are able to blend the products or remove bad product at some cost without ruining the entire campaign, then the stronger definition of ASiP is not

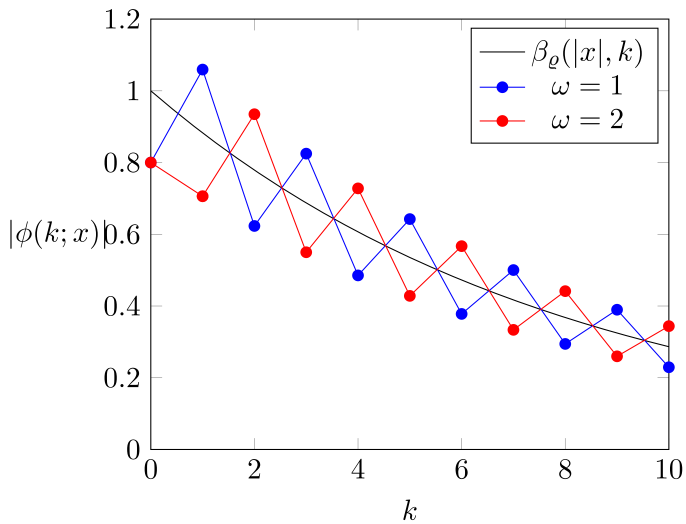

Figure 1: For β_ϱ as shown, the random process $\phi(k;x)$, which produces two equally likely trajectories ($\omega = 1, 2$), satisfies (5) for $\varrho = 0.5$. However, for this function β_ϱ, the random process does not satisfy (4) for *any* $\varrho > 0$.

required. The weaker definition of ASiP ensures that we respect the bound at least $1 - \varrho$ of the time and over multiple campaigns we expect that $1 - \varrho$ of product manufactured meets specifications.

We must also consider what is an achievable form of ASiP for the system and disturbance of interest. For zero mean disturbances with small variability that *vanish* for $x = 0$, we expect that the stronger definition is achievable. If the disturbances become large, the stronger result may be impossible to guarantee; eventually, a sequence of large disturbances occurs and violates any quality bound with probability one.

Remark 4. In the literature, the weaker definition is often used even if the system of interest may admit the stronger result. We trace this tendency back to Krstic and Deng's initial work that focused on the weaker definition (Krstic and Deng, 1998).

2.2 Uniform asymptotic stability in probability

For the deterministic analog, we are able to directly construct the $\mathcal{KL}$ bound from the Lyapunov function. In the stochastic setting, this approach does not appear tractable. Therefore, we first establish *uniform* asymptotic stability in probability (UASiP) from the stochastic Lyapunov function and use this result to establish ASiP-$\mathcal{KL}$. We stress that uniformity is essential to construct a valid $\mathcal{KL}$ bound. Here we provide definitions for uniform stability in probability, uniform attractivity in probability, and uniform asymptotic stability in probability. We define these properties exclusively in graph notation.

Definition 5 (Uniform stability in probability)**.** Suppose that the sequence $(\mathcal{X}(i))_{i\geq 0}$ is robustly sequentially positive invariant for (1). The origin is said to be uniformly stable in probability (USiP) on the sets $(\mathcal{X}(i))_{i\geq 0}$ for (1) if for each $\varrho > 0$ there exists $\gamma_\varrho \in \mathcal{K}_\infty$ such that

$$\mathbb{P}\left(\operatorname{graph}(\Phi(x,t)) \subset (\mathbb{I}_{\geq t} \times \gamma_\varrho(|x|)\mathbb{B})\right) \geq 1 - \varrho \tag{6}$$

for all $x \in \mathcal{X}(t)$ and $t \in \mathbb{I}_{\geq 0}$.

Remark 5. The condition in (6) is equivalent to $|\phi(i;x,t)(\omega)| \leq \gamma_\varrho(|x|)$ for all $i \in \mathbb{I}_{\geq t}$, $x \in \mathcal{X}(t)$, and $t \in \mathbb{I}_{\geq 0}$.

Definition 6 (Uniformly attractive in probability)**.** Suppose that the sequence $(\mathcal{X}(i))_{i\geq 0}$ is robustly sequentially positive invariant for (1). The origin is uniformly attractive in probability (UAiP) on the sets $(\mathcal{X}(i))_{i\geq 0}$ for (1) if for each $\varrho > 0$, $\varepsilon > 0$, and $\Delta > 0$ there exists $J \in \mathbb{I}_{\geq 0}$ such that

$$\mathbb{P}\left((\operatorname{graph}(\Phi(x,t)) \cap (\mathbb{I}_{\geq J+t} \times \mathbb{R}^n)) \subset (\mathbb{I}_{\geq J+t} \times \varepsilon\mathbb{B}^o)\right) \geq 1 - \varrho \tag{7}$$

for all $x \in \Delta\mathbb{B} \cap \mathcal{X}(t)$ and $t \in \mathbb{I}_{\geq 0}$.

Remark 6. The condition in (6) is equivalent to $|\phi(i;x,t)(\omega)| < \varepsilon$ for all $i \in \mathbb{I}_{\geq J+t}$.

Definition 7 (Uniformly asymptotically stable in probability)**.** The origin is uniformly asymptotically stable (UASiP) for (1) if it is USiP and UAiP for (1).

A Weaker Form of ASiP

In some cases, a weaker (non-uniform) definition of attraction and asymptotic stability is used.

Definition 8 (Attraction in probability)**.** Suppose that the sequence $(\mathcal{X}(i))_{i\geq 0}$ is robustly sequentially positive invariant for (1). The origin is attractive in probability (AiP) on the sets $(\mathcal{X}(i))_{i\geq 0}$ for (1) if

$$\mathbb{P}\left(\lim_{i\to\infty} |\phi(i;x,t)| = 0\right) = 1 \tag{8}$$

for all $x \in \mathcal{X}(t)$ and $t \in \mathbb{I}_{\geq 0}$.

Definition 9 (Asymptotic stability in probability)**.** The origin is asymptotically stable (ASiP) on the sets $(\mathcal{X}(i))_{\geq 0}$ for (1) if it is USiP and AiP for (1).

Establishing that a stochastic Lyapunov function guarantees (non-uniform) attraction in probability is easier than UAiP (Kushner, 1965). We also note that for continuous, time-invariant $f(\cdot)$, AiP (combined with USiP) may be sufficient to establish UASiP. In the deterministic analogs of these definitions, this result holds (Jiang and Wang, 2002). In the stochastic setting, however, we are unaware of any proofs that rigorously establish this fact. Theorem 2.3 in Krstic and Deng (1998) states that AiP combined with USiP is sufficient to establish the existence of a $\mathcal{KL}$ upper bound (i.e., weak ASiP-$\mathcal{KL}$) for continuous, time-invariant $f(\cdot)$, but the proof appears incomplete.

In Appendix A, we provide a more extensive discussion of the subtleties concerning uniform and non-uniform asymptotic stability results in the deterministic setting.

2.3 Uniform recurrence

We also define uniform recurrence for some open set $\mathcal{O}$ and note its similarity to UAiP. The recurrence condition requires that the solution reaches the set $\mathcal{O}$ at least once within J steps.

Definition 10 (Uniformly recurrent)**.** Suppose that the sequence $(\mathcal{X}(i))_{i\geq 0}$ is robustly sequentially positive invariant for (1). The open set $\mathcal{O}$ is uniformly recurrent on the sets $(\mathcal{X}(i))_{i\geq 0}$ for (1) if for each $\varrho > 0$ and $\Delta > 0$ there exists $J \in \mathbb{I}_{\geq 0}$ such that

$$\mathbb{P}\left((\mathrm{graph}(\Phi(x,t)) \cap (\mathbb{I}_{\leq J+t} \times \mathcal{O})) \neq \emptyset\right) \geq 1 - \varrho \tag{9}$$

for all $x \in \Delta\mathbb{B} \cap \mathcal{X}(t)$ and $t \in \mathbb{I}_{\geq 0}$.[1]

3 Reachability and viability probabilities

3.1 Uniform reachability

Reachability probabilities for a closed set $S \subset \mathbb{R}^n$ are defined as

$$\begin{aligned} m_{\cap S}(0,x,t) &:= 0 \\ m_{\cap S}(j+1,x,t) &:= \int_{\mathbb{V}} \max\{I_S(f(x,v,t)), m_{\cap S}(j, f(x,v,t), t+1)\}\mu(dv) \end{aligned}$$

for all $j \in \mathbb{I}_{\geq 0}$, $x \in \mathcal{X}(t)$, and $t \in \mathbb{I}_{\geq 0}$.

The reachability probabilities $m_{\cap S}(j,x,t)$ define the probability that the random solution starting from $x \notin S$ reaches the set S, at least once, within j time steps.

Lemma 1. *If $S \subset \mathbb{R}^n$ is closed then $m_{\cap S}(j,x,t) \in [0,1]$ is well defined and $m_{\cap S}(j+1,x,t) \geq m_{\cap S}(j,x,t)$ for all $j \in \mathbb{I}_{\geq 0}$, $x \in \mathcal{X}(t)$, and $t \in \mathbb{I}_{\geq 0}$.*

Proposition 2. *Let $S \subset \mathbb{R}^n$ be closed. For each $j \in \mathbb{I}_{\geq 0}$, $x \in \mathcal{X}(t)$, and $t \in \mathbb{I}_{\geq 0}$, we have that*

$$m_{\cap S}(j,x,t) = \mathbb{E}\left[\max_{i\in\{1,\dots,j\}} I_s(\phi(t+i;x,t))\right]$$

3.2 Uniform viability

The viability probabilities for a closed set $S \subset \mathbb{R}^n$ are defined as

$$\begin{aligned} m_{\subset S}(0,x,t) &:= 1 \\ m_{\subset S}(j+1,x,t) &:= \int_{\mathbb{V}} I_S(f(x,v,t)) m_{\subset S}(j, f(x,v,t), t+1)\mu(dv) \end{aligned}$$

for all $j \in \mathbb{I}_{\geq 0}$, $x \in \mathcal{X}(t)$, and $t \in \mathbb{I}_{\geq 0}$.

The viability probabilities $m_{\subset S}(j,x,t)$ define the probability that the random solution starting at $x \in S$ remains in S for j time steps.

[1] If $x \in \mathcal{O}$, we satisfy (9) with $J = 0$ so we need establish this property only for $x \in \Delta\mathbb{B} \cap \mathcal{X}(t) \cap (\mathbb{R}^n \setminus \mathcal{O})$.

Lemma 3. *If $S \subset \mathbb{R}^n$ is closed, then $x \to m_{\subset S}(k,x,t) \in [0,1]$ is well-defined and $m_{\subset S}(k+1,x,t) \leq m_{\subset S}(k,x,t)$ for all $k \in \mathbb{I}_{\geq t}$, $x \in \mathcal{X}(t)$, and $t \in \mathbb{I}_{\geq 0}$.*

Proposition 4. *Let $S \subset \mathbb{R}^n$ be closed. For each $j \in \mathbb{I}_{\geq 0}$, $x \in \mathcal{X}(t)$, and $t \in \mathbb{I}_{\geq 0}$ we have that*

$$m_{\subset S}(j,x,t) = \mathbb{E}\left[\prod_{i=1}^{j} I_S(\phi(t+i;x,t))\right]$$

4 Main Results

4.1 USiP

In this subsection, we establish that the existence of a stochastic Lyapunov function is sufficient for uniform stability in probability for (1). The proof uses reachability probabilities to define the probability that the system reaches some set S at least once. This set S is taken to be the complement of the stability region $\varepsilon\mathbb{B}^o$. We proceed by defining ε sufficiently large that the probability of reaching this set S is less than ϱ for all $k \in \mathbb{I}_{\geq t}$. Thus, the probability that we stay in the complement of S for all $k \in \mathbb{I}_{\geq t}$ is greater than $1-\varrho$.

Theorem 5. *If there exists a stochastic Lyapunov function on the sets $(\mathcal{X}(i))_{i\geq 0}$ for (1), then the origin is USiP on the sets $(\mathcal{X}(i))_{i\geq 0}$ for (1).*

Proof. Let $\varepsilon > 0$, $\mathbf{x} = \Phi(x,t)$ for some $x \in \mathcal{X}(t)$, $t \in \mathbb{I}_{\geq 0}$, and define $S := \mathbb{R}^n \setminus \varepsilon\mathbb{B}^o$. We claim that $\alpha_1(\varepsilon)m_{\cap S}(j,x,t) \leq V(x,t)$ for all $(j,x) \in \mathbb{I}_{\geq 0} \times \mathbb{R}^n$. The bound holds for $j = 0$ since $m_{\cap S}(0,x,t) = 0 \leq V(x,t)$ for all $x \in \mathcal{X}(t)$, $t \in \mathbb{I}_{\geq 0}$. Suppose the bound holds for some $j \in \mathbb{I}_{\geq 0}$. Then using the definitions of I_S and $m_{\cap S}$ and $\alpha_1(|x|) \leq V(x,t)$ for all $x \in \mathcal{X}(t)$ and $t \in \mathbb{I}_{\geq 0}$, we have

$$\begin{aligned}
\alpha_1(\varepsilon)m_{\cap S}(j+1,x,t) &\leq \int_{\mathbb{V}} \max\{\alpha_1(\varepsilon)I_S(f(x,v,t)), \alpha_1(\varepsilon)m_{\cap S}(j,f(x,v,t),t+1)\}\mu(dv) \\
&\leq \int_{\mathbb{V}} \max\{V(f(x,v,t),t+1), V(f(x,v,t),t+1)\}\mu(dv) \\
&\leq \int_{\mathbb{V}} V(f(x,v,t),t+1)\mu(dv) \leq V(x)
\end{aligned}$$

The final inequality is due to the expected cost decrease of the stochastic Lyapunov function. Then, by induction, the desired bound holds for all $j \in \mathbb{I}_{\geq 0}$.

Using $V(x,t) \leq \alpha_2(|x|)$, it follows that $m_{\cap S}(j,x,t) \leq \alpha_2(|x|)/\alpha_1(\varepsilon)$ for all $(j,x) \in \mathbb{I}_{\geq 0} \times \mathcal{X}(t)$, $t \in \mathbb{I}_{\geq 0}$. Choose $\varrho > 0$ and define $\gamma_\varrho(|x|) := \alpha_1^{-1}(\alpha_2(|x|)/\varrho)$. Note that $\gamma_\varrho(\cdot)$ is a composition of $\mathcal{K}_\infty$ functions and, thus, $\gamma_\varrho \in \mathcal{K}_\infty$.

For any $x \in \mathcal{X}(t)\setminus\{0\}$, $t \in \mathbb{I}_{\geq 0}$ define $S := \mathbb{R}^n \setminus \gamma_\varrho(|x|)\mathbb{B}^o$ and we have that $m_{\cap S}(j,x,t) \leq \varrho$ for all $j \in \mathbb{I}_{\geq 0}$. From Proposition 2 we have

$$\begin{aligned}
\mathbb{P}\left(\mathrm{graph}(\mathbf{x}) \cap (\mathbb{I}_{[t,t+j]} \times S) \neq \emptyset\right) &= \mathbb{E}\left[\max_{i\in\{1,\ldots,j\}} I_S(\phi(t+i;x,t))\right] \\
&= m_{\cap S}(j,x,t) \leq \varrho
\end{aligned}$$

Letting $j \to \infty$ and noting that the above probability statement is the complement of (6), we have that

$$\mathbb{P}\left(\text{graph}(\mathbf{x}) \subset (\mathbb{I}_{\geq t} \times \gamma_\varrho(|x|)\mathbb{B}^o)\right) \geq 1 - \varrho \tag{10}$$

This statement holds for any $x \in \mathcal{X}(t) \setminus \{0\}$ and $t \in \mathbb{I}_{\geq 0}$. Note that we can replace $\mathbb{B}^o$ with $\mathbb{B}$ since $\mathbb{B}^o \subset \mathbb{B}$.

Finally, if $x = 0$, we know that $V(x,t) = 0$ and

$$\mathbb{E}\left[V(\phi(t+1;x,t), t+1)\right] := \int_{\mathbb{V}} V(f(x,v,t), t+1)\mu(dv) \leq V(x,t) = 0$$

Thus, $V(\phi(t+1;x,t)) = 0$ with probability one. Furthermore,

$$\begin{aligned}\mathbb{E}\left[V(\phi(k+1;x,t), k+1) \mid \mathcal{F}_k\right](\omega) &:= \int_{\mathbb{V}} V(f(\phi(k;x,t)(\omega), v, k), k+1)\mu(dv) \\ &\leq V(\phi(k;x,t)(\omega), k)\end{aligned}$$

and by induction $V(\phi(k;x,t),t) = 0$ with probability one for all $k \in \mathbb{I}_{\geq t}$, $t \in \mathbb{I}_{\geq 0}$. Using the $\mathcal{K}_\infty$ bounds on $V(\cdot)$, we also have that $\phi(k;x,t) = 0$ with probability one, i.e.,

$$\mathbb{P}\left(\text{graph}(\mathbf{x}) \subset (\mathbb{I}_{\geq t} \times \{0\}\right) = 1 \geq 1 - \varrho$$

for $x = 0$ and all $k \in \mathbb{I}_{\geq t}$, $t \in \mathbb{I}_{\geq 0}$. Combine this equation with (10) to give

$$\mathbb{P}\left(\text{graph}(\mathbf{x}) \subset (\mathbb{I}_{\geq t} \times \gamma_\varrho(|x|)\mathbb{B})\right) \geq 1 - \varrho$$

and the proof is complete. □

4.2 Uniform Recurrence

In this subsection, we establish that a stochastic Lyapunov function is sufficient to guarantee that for any $r > 0$ the set $\mathcal{O} := r\mathbb{B}^o$ is uniformly recurrent on the sets $(\mathcal{X}(i))_{i \geq 0}$ for (1). We achieve this through two propositions.

We begin by demonstrating that for each $r > 0$ and any $R > r$ uniform recurrence of $\mathcal{O} := r\mathbb{B}^o \cup (\mathbb{R}^n \setminus R\mathbb{B})$, combined with uniform stability in probability implies uniform recurrence of $r\mathbb{B}^o$. We establish this by using uniform stability in probability to bound the probability of entering $\mathcal{O}$ by means of $|x|$ exceeding R. Therefore, the remaining probability of recurrence for $\mathcal{O}$ must be a consequence of the uniform recurrence of the bounded set $r\mathbb{B}^o$.

Proposition 6. *Let $r > 0$. The set $r\mathbb{B}^o$ is uniformly recurrent on the sets $(\mathcal{X}(i))_{i \geq 0}$ for (1) if the origin is uniformly stable in probability on the sets $(\mathcal{X}(i))_{i \geq 0}$ for (1) and, for each $R > r$, the open set $\mathcal{O} := r\mathbb{B}^o \cup (\mathbb{R}^n \setminus R\mathbb{B})$ is uniformly recurrent on the sets $(\mathcal{X}(i))_{i \geq 0}$ for (1).*

Proof. Choose some $x \in \mathcal{X}(t)$, $t \in \mathbb{I}_{\geq 0}$. Let $p_{\cap\mathcal{O}}(x,t,J)$ represent the probability in (9), and $p_{\subset\varepsilon\mathbb{B}^0}(x,t)$ represent the probability in (6). Given a closed set $\mathcal{C} \subset \mathbb{R}^n$, let

$$p_{\cap\mathcal{C}}(x,t) := \mathbb{P}\left(\text{graph}(\Phi(x,t)) \cap (\mathbb{I}_{\geq t} \times \mathcal{C}) \neq \emptyset\right)$$

Define open sets $\mathcal{O}_r := r\mathbb{B}^o$ and $\mathcal{O}_R := \mathbb{R}^n \setminus R\mathbb{B}$ and the closed set $\mathcal{C}_R := \mathbb{R}^n \setminus R\mathbb{B}^o$. Then

$$\begin{aligned} p_{\cap(\mathcal{O}_r\cup\mathcal{O}_R)}(x,t,J) &\leq p_{\cap\mathcal{O}_r}(x,t,J) + p_{\cap\mathcal{C}_R}(x,t) \\ p_{\cap\mathcal{C}_R}(x,t) + p_{\subset R\mathbb{B}^o}(x,t) &= 1 \end{aligned} \tag{11}$$

Let $\Delta > 0$ and $\varrho > 0$. Using uniform stability in probability, choose $R > 0$ large so that $p_{\subset R\mathbb{B}^o}(x,t) \geq 1 - \varrho/2$ for all $x \in \Delta\mathbb{B} \cap \mathcal{X}(t)$, $t \in \mathbb{I}_{\geq 0}$. Then, using uniform recurrence of $\mathcal{O}_r \cup \mathcal{O}_R$, pick $J \in \mathbb{I}_{\geq t}$ such that $p_{\cap(\mathcal{O}_r\cup\mathcal{O}_R)}(x,t,J) \leq 1 - \varrho/2$ for all $x \in \Delta\mathbb{B} \cap \mathcal{X}(t)$ and $t \in \mathbb{I}_{\geq 0}$. Using (11) we have

$$\begin{aligned} 2 - \varrho &\leq p_{\cap(\mathcal{O}_r\cup\mathcal{O}_R)}(x,t,J) + p_{\subset R\mathbb{B}^o}(x,t) \\ &\leq p_{\cap\mathcal{O}_r}(x,t,J) + p_{\cap\mathcal{C}_R}(x,t) + p_{\subset R\mathbb{B}^o}(x,t) \\ &= p_{\cap\mathcal{O}_r}(x,t,J) + 1 \end{aligned}$$

for all $x \in \Delta\mathbb{B} \cap \mathcal{X}(t)$ and $t \in \mathbb{I}_{\geq 0}$. Subtracting 1 from both sides of this inequality establishes uniform recurrence of $\mathcal{O}_r$. $\square$

Next, we demonstrate that the existence of a stochastic Lyapunov function guarantees that for each $r > 0$ and any $R > r$, the set $\mathcal{O} := r\mathbb{B}^o \cup (\mathbb{R}^n \setminus R\mathbb{B})$ is uniformly recurrent. We establish this result by using viability probabilities for the set S defined as the complement of $\mathcal{O}$. Using this construction, we demonstrate that there exists a value J for which the probability of remaining in S is less than ϱ. Therefore, the probability of entering $\mathcal{O}$ at least once before $k = J$ is greater than $1 - \varrho$. We then invoke Theorem 5 to establish USiP from the stochastic Lyapunov function and Proposition 6 to complete the proof.

Proposition 7. *If there exists a stochastic Lyapunov function on the sets $(\mathcal{X}(i))_{i\geq 0}$ for (1), then for each $r > 0$ the set $r\mathbb{B}^o$ is uniformly recurrent on the sets $(\mathcal{X}(i))_{i\geq 0}$ for (1).*

Proof. We first choose $R > r > 0$ and define $\mathcal{O} := r\mathbb{B}^o \cup (\mathbb{R}^n \setminus R\mathbb{B})$ and $S := \mathbb{R}^n \setminus \mathcal{O}$. Let $\mathbf{x} = \Phi(x,t)$ for some $x \in \mathcal{X}(t)$, $t \in \mathbb{I}_{\geq 0}$. We define the event in (9) and its complement for some $J \in \mathbb{I}_{\geq 0}$, respectively, as

$$\begin{aligned} R &:= \mathrm{graph}(\mathbf{x}) \cap (\mathbb{I}_{\leq J+t} \times \mathcal{O}) \neq \emptyset \\ R^c &:= (\mathrm{graph}(\mathbf{x}) \cap (\mathbb{I}_{\leq J+t} \times \mathbb{R}^n)) \subset (\mathbb{I}_{\leq J} \times S) \end{aligned}$$

Note that R^c requires that the trajectory never leaves S (which is the complement of $\mathcal{O}$ in $\mathbb{R}^n$). Using Proposition 4 we can write the probability of R^c, given $x \in S$, as follows.

$$\mathbb{P}(R^c) = \mathbb{E}\left[\prod_{i=t+1}^{t+J} I_S(\phi(i;x,t))\right] = m_{\subset S}(J,x,t) \tag{12}$$

For all $x \in S$ we know that $\alpha_3(|x|) \geq \min_{y\in S} \alpha_3(|y|) := \nu$. Note that the minimum is well defined because S is compact and $\alpha_3(\cdot)$ is *continuous*.[2] Furthermore, $\nu > 0$ because $\alpha_3 \in \mathcal{PD}$ and S does not contain the origin. Then for all $x \in S \cap \mathcal{X}(t)$ and $t \in \mathbb{I}_{\geq 0}$,

$$\int_{\mathbb{V}} V(f(x,v,t), t+1)\mu(dv) \leq V(x,t) - \nu$$

[2]This part of the proof requires $\alpha_3(\cdot)$ to be continuous, i.e., the reason that $\mathcal{PD}$ is restricted to continuous functions.

We claim that $(\nu i) m_{\subset S}(i,x,t) \leq V(x,t)$ for all $i \in \mathbb{I}_{\geq 0}$, $x \in S \cap \mathcal{X}(t)$, and $t \in \mathbb{I}_{\geq 0}$. This bound holds for $i=0$ since $0 \leq V(x,t)$ for all $x \in \mathcal{X}(t)$. Suppose the bound holds for some $i \in \mathbb{I}_{\geq 0}$ and let $x \in S \cap \mathcal{X}(t)$ and $t \in \mathbb{I}_{\geq 0}$. Then

$$\begin{aligned} \nu(i+1) m_{\subset S}(i+1,x,t) &= (\nu + i\nu) \int_{\mathbb{V}} I_S(f(x,v,t)) m_{\subset S}(i, f(x,v,t), t+1) \mu(dv) \\ &\leq \nu + \int_{\mathbb{V}} V(f(x,v,t),t+1)\mu(dv) \leq V(x) \end{aligned}$$

Thus, by induction, this bound holds for all $i \in \mathbb{I}_{\geq 0}$. Let $\varrho > 0$ and $\Delta > 0$. Pick $J \in \mathbb{I}_{\geq 0}$ large enough so that $\alpha_2(\Delta) \leq \nu J \varrho$. Thus, we have

$$m_{\subset S}(J,x,t) \leq V(x,t)/(\nu J) \leq \alpha_2(\Delta)/(\nu J) \leq \varrho$$

for all $x \in \Delta\mathbb{B} \cap S \cap \mathcal{X}(t)$ and $t \in \mathbb{I}_{\geq 0}$. Using (12) and recalling that R is the complement of R^c, we have that for any $\varrho > 0$, $\Delta > 0$, there exists $J \in \mathbb{I}_{\geq 0}$ such that

$$\mathbb{P}(R) = 1 - \mathbb{P}(R^c) = 1 - m_{\subset S}(J,x,t) \geq 1 - \varrho \tag{13}$$

for all $x \in \Delta\mathbb{B} \cap S \cap \mathcal{X}(t)$ and $t \in \mathbb{I}_{\geq 0}$. Furthermore, if $x \notin S$, we immediately have that the $\mathbb{P}(R) = 1$ for all $J \geq 0$. Therefore, for all $R > r > 0$ the set $\mathcal{O}$ is uniformly recurrent on the sets $(\mathcal{X}(i))_{i \geq 0}$ for (1).

From the stochastic Lyapunov function and Theorem 5, we also know that the origin is uniformly stable in probability on the sets $(\mathcal{X}(i))_{i\geq 0}$ for (1). Therefore, by Proposition 6, for each $r > 0$ the set $r\mathbb{B}^o$ is uniformly recurrent on the sets $(\mathcal{X}(i))_{i \geq 0}$ for (1) and the proof is complete. ☐

4.3 UASiP

Here we establish that uniform recurrence combined with USiP implies UASiP. For this results, we need to prove only that the system is also UAiP. Uniform recurrence ensures that we reach the set $r\mathbb{B}^o$ in no more than J steps with probability of at least $1-\varrho/2$. Then USiP ensures that after reaching this set, we stay within the larger set $\varepsilon\mathbb{B}^o$ with probability of at least $1-\varrho/2$. Finally, we establish that the probability of both these events occurring is at least $1-\varrho$ and therefore the system is UAiP.

Proposition 8. *If the origin is USiP on the sets $(\mathcal{X}(i))_{i\geq 0}$ for (1) and for each $r > 0$, $r\mathbb{B}^o$ is uniformly recurrent on the sets $(\mathcal{X}(i))_{i\geq 0}$ for (1), then the origin is UASiP on the sets $(\mathcal{X}(t))_{t \geq 0}$ for (1).*

Proof. We need demonstrate only that the origin is uniformly attractive in probability. Let $\varepsilon > 0$, $\tilde{\varrho}$, and $\Delta > 0$ be given. For $\varrho = \varrho_s := \tilde{\varrho}/2$ and $\gamma_\varrho(\cdot)$ defined such that (6) holds, let $r = \gamma_\varrho^{-1}(\varepsilon)$. For $\varrho = \varrho_r := \tilde{\varrho}/2$, $\Delta = \Delta$, and $\mathcal{O} = \mathcal{O}_r := r\mathbb{B}^o$, let $J \in \mathbb{I}_{\geq 0}$ be such that (9) holds. We must establish that (7) holds with $\varrho = \tilde{\varrho}$. Let $x \in \Delta\mathbb{B} \cap \mathcal{X}(t)$, $t \in \mathbb{I}_{\geq 0}$, and $\mathbf{x} = \Phi(x,t)$. With the following definition of the event Ω_a we must show that $\mathbb{P}(\Omega_a) \geq 1 - \tilde{\varrho}$.

$$\Omega_a := \{\omega \in \Omega : (\mathrm{graph}(\mathbf{x}(\omega)) \cap (\mathbb{I}_{\geq t+J} \times \mathbb{R}^n)) \subset (\mathbb{I}_{\geq t+J} \times (\varepsilon\mathbb{B}^o))\}$$

First we define

$$\Omega_r := \{\omega \in \Omega : \mathrm{graph}(\mathbf{x}(\omega)) \cap (\mathbb{I}_{\leq t+J} \times \mathcal{O}_r) \neq \emptyset\}$$

and note that $\mathbb{P}(\Omega_r) \geq 1 - \varrho_r$ from the definition of uniformly recurrent. Moreover, $\mathbb{P}(\Omega_a) \geq \mathbb{P}(\Omega_a \cap \Omega_r)$.

Next, we establish that

$$\mathbb{P}(\Omega_a \cap \Omega_r) \geq (1 - \varrho_s)\mathbb{P}(\Omega_r) \tag{14}$$

Let $\hat{J}(\omega) := \inf\{j \in \mathbb{I}_{\geq t} : x_j(\omega) \in \mathcal{O}_r\}$ and $p(\omega) = x_{\hat{J}(\omega)}(\omega)$ for all $\omega \in \mathrm{dom}\ \hat{J}$, i.e., the time j and the state x_j for which $\mathbf{x}(\omega)$ first enters the set $\mathcal{O}_r$. These functions are $\mathcal{F}_i$-measurable because x_i is $\mathcal{F}_i$-measurable. We further define $E_i := \{\omega \in \Omega : \hat{J}(\omega) = i\}$ for al $i \in \{t, \ldots, t+J\}$, i.e., the event that $\mathbf{x}(\omega)$ first enters the set $\mathcal{O}_r$ at time i. Observe that $\Omega_r = \bigcup_{i=t}^{t+J} E_i$ and the sets E_i are disjoint (the trajectory can enter the set $\mathcal{O}_r$ for the first time only once). Since E_i are disjoint, we need only to show that

$$\mathbb{P}\left(\Omega_a \cap E_i\right) \geq (1 - \varrho_s)\mathbb{P}(E_i) \quad \forall\ i \in \{t, \ldots, t+J\} \tag{15}$$

Let $i \in \{t, \ldots, t+J\}$ be given. Then we have that $p_i(\omega) \in r\mathbb{B}$ for all $\omega \in E_i$. We fix $(v_t(\omega), \ldots, v_{i-1}(\omega))$ so that $i = \hat{J}(\omega)$ and we define $\mathbf{z}$ as the random solution starting from $p_i(\omega)$, i.e., $\mathbf{z} = \Phi(p_i(\omega), i)$. From the definitions of $p_i(\omega)$, Ω_a, E_i, and (6), it follows that for all $\omega \in E_i$,

$$\mathbb{P}\left(\Omega_a \mid \mathcal{F}_i\right)(\omega) = \mathbb{P}\left(\mathrm{graph}(\mathbf{z}) \subset (\mathbb{I}_{\geq i} \times \varepsilon\mathbb{B}^o)\right) \geq 1 - \varrho_s$$

I.e., given we enter $\mathcal{O}_r$ ($\omega \in E_i$), the probability that we remain in $\varepsilon\mathbb{B}^o$ is greater than or equal to the probability of stability $(1 - \varrho_s)$. Therefore, according to Proposition 2 in Fristedt and Gray (1997, s. 21.1),

$$\begin{aligned}\mathbb{P}(\Omega_a \cap E_i) &= \mathbb{E}\left[\mathbb{P}(\Omega_a \mid \mathcal{F}_i) I_{E_i}\right] \\ &\geq (1 - \varrho_s)\mathbb{P}(E_i)\end{aligned}$$

which is (15). Since E_i are disjoint, we also establish (14).

Finally, we use (14) to conclude

$$\begin{aligned}\mathbb{P}\left(\Omega_a\right) &\geq \mathbb{P}\left(\Omega_a \cap \Omega_r\right) \\ &\geq (1 - \varrho_s)\mathbb{P}\left(\Omega_r\right) \\ &\geq (1 - \varrho_s)(1 - \varrho_r) \\ &\geq 1 - \varrho_s - \varrho_r = 1 - \tilde{\varrho}\end{aligned}$$

and the proof is complete. □

Theorem 9. *If there exists a stochastic Lyapunov function on the sets* $(\mathcal{X}(i))_{i\geq 0}$ *for (1), then the set origin is UASiP on the sets* $(\mathcal{X}(i))_{i\geq 0}$ *for (1).*

Proof. From Theorem 5, the origin is USiP on the sets $(\mathcal{X}(i))_{i\geq 0}$ for (1). From Proposition 7, for each $r > 0$, $r\mathbb{B}^o$ is uniformly recurrent on the sets $(\mathcal{X}(i))_{i\geq 0}$ for (1). Finally, from Proposition 8, we have that the origin is UASiP on the sets $(\mathcal{X}(i))_{i\geq 0}$ for (1). □

5 ASiP-$\mathcal{KL}$

Lemma 10. *The origin is uniformly attractive in probability on the set* $(\mathcal{X}(i))_{i\geq 0}$ *if and only if for each* $\varrho > 0$, *there exists a family of mapping* $(J_r)_{r>0}$ *satisfying:*

1. *for each fixed* $r > 0$, $J_r : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ *is continuous and is strictly decreasing;*
2. *for each* $\varepsilon > 0$, $J_r(\varepsilon)$ *is (strictly) increasing as* r *increases and* $\lim_{r\to\infty} J_r(\varepsilon) = \infty$.

such that the random solution $\mathbf{x} = \Phi(x, t)$ *satisfies*

$$\mathbb{P}\left((\mathrm{graph}(\mathbf{x}) \cap (\mathbb{I}_{\geq t+J} \times \mathbb{R}^n)) \subset (\mathbb{I}_{\geq t+J} \times (\varepsilon\mathbb{B}^o))\right) \geq 1 - \varrho \tag{16}$$

for all $x \in r\mathbb{B} \cap \mathcal{X}(t)$, $t \in \mathbb{I}_{\geq 0}$, *and* $J \geq J_r(\varepsilon)$.

Proof. Sufficiency is clear as the requirements for $(J_r)_{r>0}$ imply UGAiP. Now we show necessity. Let $\varrho > 0$. For any $r, \varepsilon > 0$, let

$$\begin{aligned}A_{r,\varepsilon} := \{J \in \mathbb{I}_{\geq 0} :\ & \\ & \mathbb{P}\left((\mathrm{graph}(\Phi(x,t)) \cap (\mathbb{I}_{\geq t+J} \times \mathbb{R}^n)) \subset (\mathbb{I}_{\geq t+J} \times (\varepsilon\mathbb{B}^o))\right) \geq 1 - \varrho, \\ & \qquad\qquad\qquad\qquad \forall\ x \in r\mathbb{B} \cap \mathcal{X}(t),\ t \in \mathbb{I}_{\geq 0}\}\end{aligned}$$

From UAiP, we know that $A_{r,\varepsilon} \neq \emptyset$ for any $r, \varepsilon > 0$. Moreover, $A_{r,\varepsilon_1} \subset A_{r,\varepsilon_2}$ if $\varepsilon_1 \leq \varepsilon_2$ and $A_{r_1,\varepsilon} \supset A_{r_2,\varepsilon}$ if $r_1 \leq r_2$.

Now define $\bar{J}_r(\varepsilon) := \inf A_{r,\varepsilon}$. Then $\bar{J}_r(\varepsilon) < \infty$, for any $r, \varepsilon > 0$, and it satisfies $\bar{J}_r(\varepsilon_1) \geq \bar{J}_r(\varepsilon_2)$ if $\varepsilon_1 \leq \varepsilon_2$ and $\bar{J}_{r_1}(\varepsilon) \leq \bar{J}_{r_2}(\varepsilon)$ if $r_1 \leq r_2$. So we can define for any $r, \varepsilon > 0$,

$$\tilde{J}_r(\varepsilon) := \frac{2}{\varepsilon} \int_{\varepsilon/2}^{\varepsilon} \bar{J}_r(s) ds$$

Since $\bar{J}_r(\cdot)$ is decreasing, $\tilde{J}_r(\cdot)$ is well defined and is locally absolutely continuous. Also

$$\tilde{J}_r(\varepsilon) \geq \frac{\varepsilon}{2} \bar{J}_r(\varepsilon) \int_{\varepsilon/2}^{\varepsilon} ds = \bar{J}_r(\varepsilon)$$

Furthermore,

$$\begin{aligned}\frac{d\tilde{J}_r(\varepsilon)}{d\varepsilon} &= -\frac{2}{\varepsilon^2} \int_{\varepsilon/2}^{\varepsilon} \bar{J}_r(s) ds + \frac{2}{\varepsilon}\left(\bar{J}_r(\varepsilon) - \frac{1}{2}\bar{J}_r(\varepsilon/2)\right) \\ &= \frac{1}{\varepsilon}\left(\bar{J}_r(\varepsilon) - \frac{2}{\varepsilon}\int_{\varepsilon/2}^{\varepsilon} \bar{J}_r(s) ds\right) + \frac{1}{\varepsilon}\left(\bar{J}_r(\varepsilon) - \bar{J}_r(\varepsilon/2)\right) \\ &= \frac{1}{\varepsilon}\left(\bar{J}_r(\varepsilon) - \tilde{J}_r(\varepsilon)\right) + \frac{1}{\varepsilon}\left(\bar{J}_r(\varepsilon) - \bar{J}_r(\varepsilon/2)\right) \leq 0\end{aligned}$$

hence, $\tilde{J}_r(\cdot)$ decreases. Since $\bar{J}_{(\cdot)}(\varepsilon)$ increases, from the definition, $\tilde{J}_{(\cdot)}(\varepsilon)$ also increases. Finally, define

$$J_r(\varepsilon) := \tilde{J}_r(\varepsilon) + \frac{r}{\varepsilon} \tag{17}$$

Then it follows that for any ϱ we have defined $J_r(\varepsilon)$ such that

- for any fixed $r > 0$, $J_r(\cdot)$ is continuous, maps $\mathbb{R}_{>0} \to \mathbb{R}_{>0}$, and is strictly decreasing.
- for any fixed $\varepsilon > 0$, $J_r(\varepsilon)$ is increasing as r increases, and $\lim_{r\to\infty} J_r(\varepsilon) = \infty$

Finally, we show that $J_r(\varepsilon)$ defined in (17) satisfies (16). Choose any $x \in \mathbb{R}^n$ and J such that $|x| \le r$ and $J \ge J_r(\varepsilon)$. Then

$$J \ge J_r(\varepsilon) > \tilde{J}_r(\varepsilon) \ge \bar{J}_r(\varepsilon)$$

By the definition of $\bar{J}_r(\varepsilon)$, (16) holds as claimed. □

Proposition 11. *The origin is ASiP-$\mathcal{KL}$ on the sets $(\mathcal{X}(i))_{i\ge0}$ for (1) if and only if the origin is UASiP on the sets $(\mathcal{X}(i))_{i\ge0}$ for (1).*

Proof. [$\Longrightarrow$] Assume that the origin is ASiP-$\mathcal{KL}$ on the sets $(\mathcal{X}(i))_{i\ge0}$ for (1). Choose $\varrho > 0$. Let $k = t$ and $\gamma_\varrho(s) := \beta_\varrho(s, 0)$. Thus, $\gamma_\varrho \in \mathcal{K}$ and satisfies (6) for all $x \in \mathcal{X}(t)$, $t \in \mathbb{I}_{\ge0}$. UAiP follows from the fact that for any $\Delta > 0$, $\lim_{k\to\infty} \beta_\varrho(\Delta, k) = 0$. So for any $\Delta > 0$, $\varepsilon > 0$, we can always find a $J \in \mathbb{I}_{\ge0}$ such that (7) holds.

[$\Longleftarrow$] Assume that the origin is UASiP on the sets $(\mathcal{X}(i))_{i\ge0}$ for (1). Choose $\varrho > 0$. In the subsequent steps, we reference the following probabilistic statement for some $\varepsilon > 0$.

$$\mathbb{P}\left((\mathrm{graph}(\Phi(x,t)) \cap (\mathbb{I}_{\ge t+k} \times \mathbb{R}^n)) \subset (\mathbb{I}_{\ge t+k} \times \varepsilon\mathbb{B})\right) \ge 1 - \varrho \tag{18}$$

To establish this part of the proof, we must demonstrate that there exists $\beta_\varrho \in \mathcal{KL}$ such that (18) holds for all $k \in \mathbb{I}_{\ge0}$, $x \in \mathcal{X}(t)$, and $t \in \mathbb{I}_{\ge0}$, in which $\varepsilon := \beta_\varrho(|x|, k)$.

Let $\gamma_\varrho(\cdot)$ be defined according to the definition of USiP. It follows that (18) holds for $\varepsilon \ge \gamma_\varrho(|x|)$ and all $k \in \mathbb{I}_{\ge0}$, $x \in \mathcal{X}(t)$, and $t \in \mathbb{I}_{\ge0}$.

Let $(J_r)_{r>0}$ be as defined in Lemma 10. For each $r > 0$ denote $\psi_r := J_r^{-1}$. Then, for each $r > 0$, $\psi_r : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is again continuous and strictly decreasing. We also write that $\psi_r(0) = \infty$.

We claim that for any $r > 0$, (18) holds for all $k \in \mathbb{I}_{\ge0}$, $x \in r\mathbb{B} \cap \mathcal{X}(t)$, and $t \in \mathbb{I}_{\ge0}$ with $\varepsilon \ge \psi_r(k)$. It follows from the definition of the maps J_r that, for any $r, \varepsilon > 0$, if $x \in r\mathbb{B} \cap \mathcal{X}(t)$, $k \ge J_r(\varepsilon)$ then (18) is satisfied with this ε for all $t \in \mathbb{I}_{\ge0}$. Since $k = J_r(\psi_r(k))$ if $k > 0$, we have for any $x \in r\mathbb{B} \cap \mathcal{X}(t)$ that (18) is satisfied with $\varepsilon \ge \psi_r(k)$ for all $k > 0$ and $t \in \mathbb{I}_{\ge0}$. Since $\psi_r(0) = \infty$, this condition is satisfies for $k = 0$ as well and the claim is true.

For any $s \in \mathbb{R}_{\ge0}$ and $k \in \mathbb{I}_{\ge0}$, let

$$\bar{\psi}(s,k) := \min\left\{\gamma_\varrho(s), \inf_{r\in(s,\infty)} \psi_r(k)\right\}$$

By definitions of γ_ϱ and ψ_r, we have that for all $x \in \mathcal{X}(t)$, $t \in \mathbb{I}_{\ge0}$ and $k \in \mathbb{I}_{\ge0}$, (18) is satisfied for $\varepsilon \ge \bar{\psi}(|x|, k)$. Moreover, for any fixed k, $\bar{\psi}(\cdot, k)$ is increasing. For any fixed r, $\psi_r(k)$ decreases to 0 as $k \to \infty$. Therefore, for any fixed s, $\bar{\psi}(s,k)$ decreases to 0 as $k \to \infty$.

Note that $\bar{\psi}$ satisfies *almost* all of the conditions to be a $\mathcal{KL}$ function. Unfortunately, we do not know that it is continuous in s so we upper bound $\bar{\psi}$ by such a function. Define

$$\hat{\psi}(s,k) := \int_s^{s+1} \bar{\psi}(s',k)ds' \tag{19}$$

Then $\hat{\psi}(\cdot,k)$ is a continuous function and it satisfies

$$\hat{\psi}(s,k) \geq \bar{\psi}(s,k)\int_s^{s+1} ds' = \bar{\psi}(s,k)$$

It follows that

$$\frac{\partial\hat{\psi}(s,k)}{\partial s} = \bar{\psi}(s+1,k) - \bar{\psi}(s,k) \geq 0$$

and hence $\hat{\psi}$ is increasing. Since for any fixed s, $\bar{\psi}(s,\cdot)$ decreases, so does $\hat{\psi}(s,\cdot)$. Note that

$$\bar{\psi}(s,t) \leq \bar{\psi}(s,0) \leq \gamma_\varrho(s)$$

so by the Lebesgue-dominated convergence theorem, for any fixed $s \geq 0$,

$$\lim_{k\to\infty}\hat{\psi}(s,k) = \int_s^{s+1}\lim_{k\to\infty}\bar{\psi}(s',k)ds' = 0$$

Next, we define

$$\tilde{\psi}(s,k) = \hat{\psi}(s,k) + \frac{s}{(s+1)(k+1)}$$

and note that

- for any fixed k, $\tilde{\psi}(\cdot,k)$ is continuous and strictly increasing
- for any fixed $s \geq 0$, $\lim_{k\to\infty}\tilde{\psi}(s,k) = 0$

We can not guarantee, however, that $\tilde{\psi}(0,k) = 0$ for all $k \in \mathbb{I}_{\geq 0}$. Thus, we define

$$\beta_\varrho(s,k) := \sqrt{\gamma_\varrho(s)}\sqrt{\tilde{\psi}(s,k)}$$

By definition, $\bar{\psi}(s,k) \leq \gamma_\varrho(s)$ and we have

$$\beta_\varrho(s,k) \geq \sqrt{\gamma_\varrho(s)}\sqrt{\bar{\psi}(s,k)} \geq \bar{\psi}(s,k)$$

for all $s \in \mathbb{R}_{\geq 0}$ and $k \in \mathbb{I}_{\geq 0}$. Therefore, (18) holds for all $k \in \mathbb{I}_{\geq 0}$, $x \in \mathcal{X}(t)$, and $t \in \mathbb{I}_{\geq 0}$ in which $\varepsilon := \beta_\varrho(|x|,k)$.

Finally, we establish that $\beta_\varrho \in \mathcal{KL}$. For fixed k, $\tilde{\psi}(\cdot,k)$ is continuous and strictly increasing, $\gamma_\varrho(\cdot)$ is continuous and strictly increasing, and therefore the product of these two functions is also continuous and strictly increasing. Furthermore, we have that

$$\beta_\varrho(0,k) \leq \sqrt{\gamma_\varrho(0)}\sqrt{\tilde{\psi}(0,0)} = 0\sqrt{\tilde{\psi}(0,0)} = 0$$

Thus, for fixed $k \in \mathbb{I}_{\geq 0}$, $\beta_\varrho(\cdot, k) \in \mathcal{K}$. For fixed $s \in \mathbb{R}_{\geq 0}$,

$$\lim_{k\to\infty} \beta_\varrho(s,k) = \sqrt{\gamma_\varrho(s)}\sqrt{\lim_{k\to\infty} \tilde{\psi}(s,k)} = \sqrt{\gamma_\varrho(s)} \cdot 0 = 0 \tag{20}$$

Therefore, $\beta_\varrho \in \mathcal{KL}$ and the proof is complete. □

Theorem 12. *If there exists a stochastic Lyapunov function relative to the origin on the sets $(\mathcal{X}(i))_{i\geq 0}$ for (1), then the origin is ASiP-$\mathcal{KL}$ on the sets $(\mathcal{X}(i))_{i\geq 0}$ for (1).*

Proof. We use Theorem 9 and then Proposition 11 to complete the proof. □

6 A weaker version of ASiP

Proposition 13. *If the origin is ASiP-$\mathcal{KL}$ on the sets $(\mathcal{X}(i))_{i\geq 0}$ for (1), then the origin is weakly ASiP-$\mathcal{KL}$ on the sets $(\mathcal{X}(t))_{t\geq 0}$ for (1).*

Proof. Choose $\varrho > 0$ and set $\mathbf{x} := \Phi(x,t)$ for some $x \in \mathcal{X}(t)$ and $t \in \mathbb{I}_{\geq 0}$. Using the definition of ASiP-$\mathcal{KL}$ what have that there exists $\beta_\varrho \in \mathcal{KL}$ such that

$$\mathbb{P}\left((\mathrm{graph}(\mathbf{x}) \cap (\mathbb{I}_{\geq t+k} \times \mathbb{R}^n)) \subset (\mathbb{I}_{\geq t+k} \times \beta_\varrho(|x|,k)\mathbb{B})\right) \geq 1 - \varrho$$

for all $k \in \mathbb{I}_{\geq 0}$, $x \in \mathcal{X}(t)$, and $t \in \mathbb{I}_{\geq 0}$. We note that

$$\begin{aligned}&\{(t+k, \phi(t+k;x,t)(\omega))\}\\ &\qquad = \mathrm{graph}(\mathbf{x}(\omega)) \cap (\{t+k\} \times \mathbb{R}^n) \subset \mathrm{graph}(\mathbf{x}(\omega)) \cap (\mathbb{I}_{\geq t+k} \times \mathbb{R}^n)\end{aligned}$$

Therefore,

$$\begin{aligned}&(\mathrm{graph}(\mathbf{x}(\omega)) \cap (\mathbb{I}_{\geq t+k} \times \mathbb{R}^n)) \subset (\mathbb{I}_{\geq t+k} \times \beta_\varrho(|x|,k)\mathbb{B})\\ &\implies \{(t+k, \phi(t+k;x,t)(\omega))\} \subset (\mathbb{I}_{\geq t+k} \times \beta_\varrho(|x|,k)\mathbb{B})\\ &\implies |\phi(t+k;x,t)(\omega)| \leq \beta_\varrho(|x|,k)\end{aligned}$$

for all $\omega \in \mathrm{dom}\ \phi(t+k;x,t)$. Finally, we have

$$\begin{aligned}&\mathbb{P}\left(|\phi(t+k;x,t)| \leq \beta_\varrho(|x|,k)\right)\\ &\quad \geq \mathbb{P}\left((\mathrm{graph}(\mathbf{x}) \cap (\mathbb{I}_{\geq t+k} \times \mathbb{R}^n)) \subset (\mathbb{I}_{\geq t+k} \times \beta_\varrho(|x|,k)\mathbb{B})\right)\\ &\quad \geq 1 - \varrho\end{aligned}$$

for all $k \in \mathbb{I}_{\geq 0}$, $x \in \mathcal{X}(t)$, and $t \in \mathbb{I}_{\geq 0}$ which completes the proof. □

Corollary 14. *If there exists a stochastic Lyapunov function relative to the origin on the sets $(\mathcal{X}(i))_{i\geq 0}$ for (1), then the origin is weakly ASiP-$\mathcal{KL}$ the sets $(\mathcal{X}(i))_{i\geq 0}$ for (1).*

Proof. We use Theorem 12 and then Proposition 13 to complete the proof. □

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A Deterministic Stability Parallels

In this section of the appendix, we provide deterministic parallels for some of the definitions and results in the body of the report. We consider a function $f : \mathbb{R}^n \to \mathbb{R}^n$ and a discrete-time system with state $x \in \mathbb{R}^n$ written as

$$x^+ = f(x) \tag{21}$$

in which x^+ is the subsequent state of the system. We define the solution to (21) at time step k starting at the initial state x as $\phi(k; x)$. We assume for the rest of this section that $f(0) = 0$.

A.1 Stability

We begin by defining a few versions of stability for a deterministic system.

Definition 11 (Local Stability)**.** The origin is said to be stable for (21) if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $|\phi(k;x)| \leq \varepsilon$ for all $k \in \mathbb{I}_{\geq 0}$ and $|x| \leq \delta$.

Lemma 15. *Local stability of the origin for (21) is equivalent to the existence of $\alpha \in \mathcal{K}$ such that for any $\varepsilon \geq 0$, $|\phi(k;x)| \leq \varepsilon$ for all $|x| \leq \alpha(\varepsilon)$ and $k \in \mathbb{I}_{\geq 0}$.*

Proof. Define $\bar{\delta} : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that for any $\varepsilon > 0$ and corresponding $\delta > 0$, $\bar{\delta}(\varepsilon) = \delta$. Note that $\bar{\delta}(\cdot)$ is nondecreasing. Then from Proposition 4 in Rawlings and Risbeck (2015), there exists $\alpha \in \mathcal{K}$ such that $\alpha(\varepsilon) \leq \bar{\delta}(\varepsilon)$. For the opposite direction, simply choose $\delta = \alpha(\varepsilon)$. □

Lemma 16. *Local stability of the origin for (21) is equivalent to the existence of $\rho > 0$ and $\gamma \in \mathcal{K}$ such that $|\phi(k;x)| \leq \gamma(|x|)$ for all $k \in \mathbb{I}_{\geq 0}$ and $|x| \leq \rho$.*

Proof. Start from Lemma 15 and define $\rho > 0$ such that $\rho < \lim_{\varepsilon\to\infty} \alpha(\varepsilon)$. Then, we define $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $\gamma(s) = \alpha^{-1}(s)$ for all $s \in [0, \rho]$ and note that $\gamma \in \mathcal{K}$. Thus, for any $x \in \mathbb{R}^n$ such that $|x| \leq \rho$, choose $\varepsilon = \alpha^{-1}(|x|)$ and we have

$$|\phi(k;x)| \leq \varepsilon = \alpha^{-1}(|x|) = \gamma(|x|)$$

for all $k \in \mathbb{I}_{\geq 0}$.

For the opposite direction, define $\bar{\alpha}(s) := \min\{\gamma^{-1}(s), \rho\}$ for all $s \in \mathbb{R}_{\geq 0}$ and note that $\bar{\alpha}$ is nondecreasing. By Proposition 4 in Rawlings and Risbeck (2015), there exists $\alpha \in \mathcal{K}$ such that $\alpha(s) \leq \bar{\alpha}(s)$. Thus, for $\varepsilon > 0$, choose $|x| \leq \alpha(\varepsilon) < \rho$ and we have

$$|\phi(k;x)| \leq \gamma(|x|) \leq \gamma(\alpha(\varepsilon)) = \varepsilon$$

for all $k \in \mathbb{I}_{\geq 0}$. □

Definition 12 (Uniform Global Stability)**.** The origin is said to be uniformly globally stable for (21) if there exists $\gamma \in \mathcal{K}$ such that $|\phi(k;x)| \leq \gamma(|x|)$ for all $k \in \mathbb{I}_{\geq 0}$ and $x \in \mathbb{R}^n$.

Remark 7. From Lemma 16, we immediately conclude that uniform global stability implies local stability.

Lemma 17. *Uniform global stability of the origin is equivalent to the existence of $\alpha \in \mathcal{K}_\infty$ such that for any $\varepsilon \geq 0$, $|\phi(k;x)| \leq \varepsilon$ for all $|x| \leq \alpha(\varepsilon)$ and $k \in \mathbb{I}_{\geq 0}$.*

Proof. We define $\hat{\gamma}(s) = \gamma(s) + s$ and $\alpha := \hat{\gamma}^{-1}$. Note that $\alpha \in \mathcal{K}_\infty$. Thus, for any $\varepsilon > 0$, choose $|x| \leq \alpha(\varepsilon)$ and we have

$$|\phi(k;x)| \leq \gamma(|x|) \leq \hat{\gamma}(|x|) \leq \hat{\gamma}(\alpha(\varepsilon)) = \varepsilon$$

for all $k \in \mathbb{I}_{\geq 0}$. For the opposite direction, let $\gamma := \alpha^{-1}$ and note that $\gamma \in \mathcal{K}$. Thus, for any $\varepsilon > 0$, choose $|x| = \alpha(\varepsilon)$ and we have

$$|\phi(k;x)| \leq \varepsilon = \gamma(\alpha(\varepsilon)) = \gamma(|x|)$$

□

The main difference between local stability in Lemma 15 and uniform global stability in Lemma 17 is that α must be a $\mathcal{K}_\infty$ function instead of just a $\mathcal{K}$ function. This strengthened requirement allows α to be inverted to construct the more familiar definition of uniform global stability.

A.2 Attraction

In this subsection, we define general and uniform attraction for a system.

Definition 13 (Global Attraction)**.** The origin is globally attractive for (21) if $\lim_{k\to\infty} |\phi(k;x)| = 0$ for all $x \in \mathbb{R}^n$. Or, equivalently, for each $\varepsilon > 0$ and $x \in \mathbb{R}^n$, there exists $J \in \mathbb{I}_{\geq 0}$ such that $|\phi(k;x)| < \varepsilon$ for all $k \in \mathbb{I}_{\geq J}$.

Definition 14 (Uniform Global Attraction)**.** The origin is uniformly globally attractive for (21) if for each $\varepsilon > 0$ and $\Delta > 0$, there exists $J \in \mathbb{I}_{\geq 0}$ such that $|\phi(k;x)| < \varepsilon$ for all $k \in \mathbb{I}_{\geq J}$ and $|x| \leq \Delta$.

The distinction from general and uniform attraction is more nuanced than local and uniform global stability. The main difference is that for global attraction, we may select a different $J \in \mathbb{I}_{\geq 0}$ for each specific $x \in \mathbb{R}^n$. For uniform global attraction, we must find a single value of $J \in \mathbb{I}_{\geq 0}$ that works for all $|x| \leq \Delta$.

A.3 Global asymptotic stability

Here we present three different forms of global asymptotic stability and note their similarities and differences.

Definition 15 (Global asymptotic stability)**.** The origin is globally asymptotically stable (GAS) for (21) if it is both locally stable and globally attractive.

Definition 16 (Uniform global asymptotic stability)**.** The origin is uniformly globally asymptotically stable (UGAS) for (21) if it is both uniformly globally stable and uniformly globally attractive.

Definition 17 (Global asymptotic stability-$\mathcal{KL}$)**.** The origin is globally asymptotically stable - $\mathcal{KL}$ (GAS-$\mathcal{KL}$) if there exists $\beta \in \mathcal{KL}$ such that

$$|\phi(k;x)| \leq \beta(|x|, k)$$

for all $k \in \mathbb{I}_{\geq 0}$ and $x \in \mathbb{R}^n$.

To compare UGAS and GAS-$\mathcal{KL}$ we use the following lemma.

Lemma 18. *The origin is uniformly globally attractive for (21) if and only if there exists a family of mappings* $(J_r)_{r>0}$ *such that*

- *for each fixed* $r > 0$, $J_r : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ *is continuous and strictly decreasing*
- *for each fixed* $\varepsilon > 0$, $J_r(\varepsilon)$ *is strictly increasing as* r *increases and* $\lim_{r\to\infty} J_r(\varepsilon) = \infty$.

such that $|\phi(k;x)| < \varepsilon$ *for all* $|x| \le r$ *and* $k \ge J_r(\varepsilon)$.

Proof. Sufficiency is clear and we establish necessity here. For any $r, \varepsilon > 0$, let

$$A_{r,\varepsilon} := \{J \in \mathbb{I}_{\ge 0} : \ \forall |x| \le r, \ \forall k \ge J, |\phi(k;x)| < \varepsilon\}$$

Then from uniform global attraction, $A_{r,\varepsilon} \ne \emptyset$ for any $r, \varepsilon > 0$. Moreover,

$$A_{r,\varepsilon_1} \subset A_{r,\varepsilon_2} \text{ if } \varepsilon_1 \le \varepsilon_2, \text{ and } A_{r_1,\varepsilon} \supset A_{r_2,\varepsilon} \text{ if } r_1 \le r_2$$

Now define $\bar{J}_r(\varepsilon) := \inf A_{r,\varepsilon}$. Then $\bar{J}_r(\varepsilon) < \infty$ for any $r, \varepsilon > 0$, and it satisfies

$$\bar{J}_r(\varepsilon_1) \ge \bar{J}_r(\varepsilon_2) \text{ if } \varepsilon_1 \le \varepsilon_2, \text{ and } \bar{J}_{r_1}(\varepsilon) \le \bar{J}_{r_2}(\varepsilon) \text{ if } r_1 \le r_2$$

So we can define for any $r, \varepsilon > 0$,

$$\tilde{J}_r(\varepsilon) := \frac{2}{\varepsilon} \int_{\varepsilon/2}^{\varepsilon} \bar{J}_r(s) ds$$

Since $\bar{J}_r(\cdot)$ is decreasing, $\tilde{J}_r(\cdot)$ is well defined and continuous. Also

$$\tilde{J}_r(\varepsilon) \ge \frac{2}{\varepsilon} \bar{J}_r(\varepsilon) \int_{\varepsilon/2}^{\varepsilon} ds = \bar{J}_r(\varepsilon)$$

Furthermore,

$$\begin{aligned}
\frac{d\tilde{J}_r(\varepsilon)}{d\varepsilon} &= -\frac{2}{\varepsilon^2} \int_{\varepsilon/2}^{\varepsilon} \bar{J}_r(s) ds + \frac{2}{\varepsilon} \left(\bar{J}_r(\varepsilon) - \frac{1}{2} \bar{J}_r(\varepsilon/2) \right) \\
&= \frac{1}{\varepsilon} \left(\bar{J}_r(\varepsilon) - \frac{2}{\varepsilon} \int_{\varepsilon/2}^{\varepsilon} \bar{J}_r(s) ds \right) + \frac{1}{\varepsilon} \left(\bar{J}_r(\varepsilon) - \bar{J}_r(\varepsilon/2) \right) \\
&= \frac{1}{\varepsilon} \left(\bar{J}_r(\varepsilon) - \tilde{J}_r(\varepsilon) \right) + \frac{1}{\varepsilon} \left(\bar{J}_r(\varepsilon) - \bar{J}_r(\varepsilon/2) \right) \le 0
\end{aligned}$$

hence $\tilde{J}_r(\cdot)$ decreases (not necessarily strictly). Since $\bar{J}_\cdot(\varepsilon)$ increases, from the definition, $\bar{J}_\cdot(\varepsilon)$ also increases. Finally, define

$$J_r(\varepsilon) := \tilde{J}_r(\varepsilon) + \frac{r}{\varepsilon}$$

Then it follows that

- for any fixed r, $J_r(\cdot)$ is continuous, maps $\mathbb{R}_{>0} \to \mathbb{R}_{>0}$, and is strictly decreasing
- for any fixed ε, $T_r(\varepsilon)$ is strictly increasing and $\lim_{r\to\infty} T_r(\varepsilon) = \infty$

Finally, choose any $x \in \mathbb{R}^n$ and $k \in \mathbb{I}_{\ge 0}$ with $|x| \le r$ and $k \ge J_r(\varepsilon)$. Then

$$k \ge J_r(\varepsilon) > \tilde{J}_r(\varepsilon) \ge \bar{J}_r(\varepsilon)$$

and by the definition of $\bar{J}_r(\varepsilon)$, we have $|\phi(k;x)| < \varepsilon$ as claimed. ∎

Proposition 19. *The origin is UGAS for (21) if and only if the origin is GAS-$\mathcal{KL}$.*

Proof. [$\Longleftarrow$] From GAS-$\mathcal{KL}$, we have $\beta \in \mathcal{KL}$ such that $|\phi(k;x)| \leq \beta(|x|, k)$ for all $k \in \mathbb{I}_{\geq 0}$ and $x \in \mathbb{R}^n$. Let $\gamma(s) := \beta(s, 0)$ and note that $\gamma \in \mathcal{K}$ and $|\phi(k;x)| \leq \beta(|x|, 0) = \gamma(|x|)$ for all $k \in \mathbb{I}_{\geq 0}$ and $x \in \mathbb{R}^n$. Thus, the origin is uniform globally stable. For every fixed $\Delta > 0$, $\lim_{k\to\infty} \beta(\Delta, k) = 0$ and therefore the origin is also uniformly globally attractive.

[$\Longrightarrow$] To establish this part of the proof, we must demonstrate that there exists $\beta \in \mathcal{KL}$ such that $|\phi(k;x)| \leq \beta(|x|, k)$ for all $k \in \mathbb{I}_{\geq 0}$ and $x \in \mathbb{R}^n$. Let $\gamma(\cdot)$ be defined according to the definition of uniform global stability. Let $(J_r)_{r>0}$ be defined as in Lemma 18. For each $r > 0$ denote $\psi_r := J_r^{-1}$. Then for each $r > 0$, $\psi_r : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is again continuous and strictly decreasing. We also write that $\psi_r(0) = \infty$.

We claim that for any $r > 0$, $|\phi(k;x)| \leq \psi_r(k)$ for all $k \in \mathbb{I}_{\geq 0}$ and $|x| \leq r$. It follows from the definition of J_r that, for any $r, \varepsilon > 0$ if $|x| \leq r$, $k \geq J_r(\varepsilon)$ then $|\phi(k;x)| < \varepsilon$ for all $k \in \mathbb{I}_{\geq 0}$ and $|x| \leq r$. Since $k = J_r(\psi_r(k))$ if $k > 0$, we have, for any $x \in \mathbb{R}^n$, $|\phi(k;x)| < \psi_r(k)$ for all $k > 0$. Using the fact that $\psi_r(0) = \infty$ the claim is established.

Now for any $s \geq 0$ and $k \in \mathbb{I}_{\geq 0}$, define

$$\bar{\psi}(s,k) := \min\left\{\gamma(s), \inf_{r\in(s,\infty)} \psi_r(k)\right\}$$

and note that $|x| \leq \bar{\psi}(|x|, k)$ for all $x \in \mathbb{R}^n$ and $k \in \mathbb{I}_{\geq 0}$. Next we construct a $\mathcal{KL}$ function using $\bar{\psi}$.

By definition, for any fixed k, $\bar{\psi}(\cdot, k)$ is an increasing function (not necessarily strictly increasing). Also, for any fixed $r \in (0, \infty)$, $\psi_r(k)$ decreases to 0. Thus, for any fixed $s \geq 0$, $\bar{\psi}(s, k)$ decreases to 0 as $k \to \infty$.

Note that $\bar{\psi}$ satisfies almost all the conditions of a $\mathcal{KL}$ function. However, we do not know that $\bar{\psi}(\cdot, k)$ is continuous so we upper bound $\bar{\psi}$ by such a function. Define

$$\hat{\psi}(s,k) := \int_s^{s+1} \bar{\psi}(s', k) ds'$$

Then $\hat{\psi}(\cdot, k)$ is continuous and it satisfies

$$\hat{\psi}(s,k) \geq \bar{\psi}(s,k) \int_s^{s+1} ds' = \bar{\psi}(s,k)$$

Furthermore,

$$\frac{\partial \hat{\psi}(s,k)}{\partial s} = \bar{\psi}(s+1,k) - \bar{\psi}(s,k) \geq 0$$

and, thus, $\hat{\psi}(\cdot, k)$ is increasing. For any fixed s, $\bar{\psi}(s, k\cdot)$ decreases. Therefore, $\hat{\psi}(s, \cdot)$ decreases as well. Note that

$$\bar{\psi}(s,k) \leq \bar{\psi}(s,0) = \min\left\{\gamma(s), \inf_{r\in(s,\infty)} \psi_r(0)\right\} = \gamma(s)$$

So by the Lebesgue-dominated convergence theorem, for any fixed $s \geq 0$,

$$\lim_{k\to\infty} \hat{\psi}(s,k) = \int_s^{s+1} \lim_{k\to\infty} \bar{\psi}(s', k) ds' = 0$$

Next, define

$$\tilde{\psi}(s,k) := \hat{\psi}(s,k) + \frac{s}{(s+1)(k+1)} \geq \bar{[}\psi](s,k)$$

and note that

- for any fixed $k \in \mathbb{I}_{\geq 0}$, $\tilde{\psi}(\cdot,k)$ is continuous and strictly increasing
- for any fixed $s \in \mathbb{R}_{\geq 0}$, $\tilde{\psi}(s,\cdot)$ is decreasing and $\lim_{k\to\infty} \tilde{\psi}(s,k) = 0$

Note that $\tilde{\psi}$ satisfies almost all the properties of a $\mathcal{KL}$ function. However, $\tilde{\psi}(0,k) > 0$ and therefore, for fixed $k \in \mathbb{I}_{\geq 0}$ $\tilde{\psi}(\cdot,k)$ is not a $\mathcal{K}$ function. Thus, we define

$$\beta(s,k) := \sqrt{\gamma(s)}\sqrt{\tilde{\psi}(s,k)}$$

By definition, $\bar{\psi}(s,k) \leq \gamma(s)$ and we have

$$\beta(s,k) \geq \sqrt{\gamma(s)}\sqrt{\bar{\psi}(s,k)} \geq \bar{\psi}(s,k) \tag{22}$$

for all $s \in \mathbb{R}_{\geq 0}$ and $k \in \mathbb{I}_{\geq 0}$. Therefore, $|\phi(k;x)| \leq \beta(|x|,k)$ for all $k \in \mathbb{I}_{\geq 0}$ and $x \in \mathbb{R}^n$.

Finally, we establish that $\beta \in \mathcal{KL}$. For fixed k, $\tilde{\psi}(\cdot,k)$ is continuous, nonnegative, and strictly increasing, $\gamma(\cdot)$ is continuous, nonnegative, and strictly increasing, and therefore the product of these two functions is also continuous, nonnegative, and strictly increasing. Furthermore, we have that

$$\beta(0,k) \leq \sqrt{\gamma(0)}\sqrt{\tilde{\psi}(0,0)} = 0$$

Thus, for fixed $k \in \mathbb{I}_{\geq 0}$, $\beta(\cdot,k) \in \mathcal{K}$. For fixed $s \in \mathbb{R}_{\geq 0}$,

$$\lim_{k\to\infty} \beta(s,k) = \sqrt{\gamma(s)}\sqrt{\lim_{k\to\infty} \tilde{\psi}(s,k)} = 0$$

$\square$

A.4 Examples

We illustrate the distinctions between these definitions with three separate scalar systems ($x^+ = f(x)$). The dynamics of these three systems are described by the following *discontinuous* functions.

$$f_1(x) := \begin{cases} 0.9x^2 & ;\ |x| \in [0,1) \\ 0 & ;\ |x| \in [1,\infty) \end{cases}$$

$$f_2(x) := \begin{cases} x^2 & ;\ |x| \in [0,1) \\ 0 & ;\ |x| \in [1,\infty) \end{cases}$$

$$f_3(x) := \begin{cases} 0.9|x| & ;\ |x| \in [0,0.5] \\ \frac{|x|}{1-|x|} & ;\ |x| \in (0.5,1) \\ 0 & ;\ |x| \in [1,\infty) \end{cases}$$

We note that all three of the systems generated by these functions are GAS. However, only one of these systems is also UGAS/GAS-$\mathcal{KL}$.

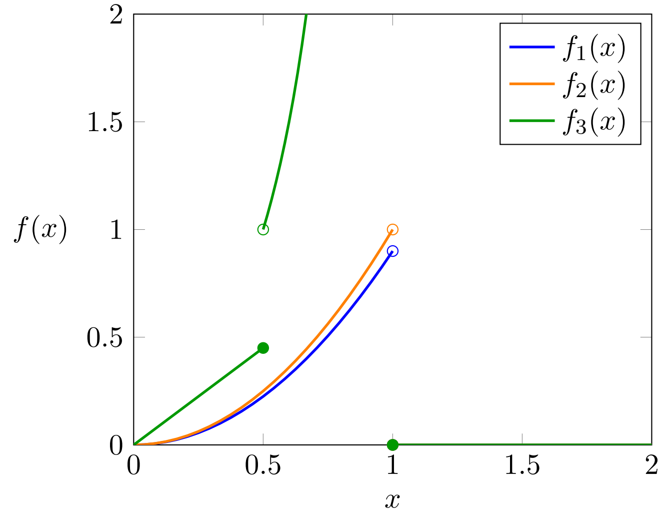

Figure 2: Three scalar functions $f(x)$, that define three scalar discrete-time systems defined by the evolution equation as $x^+ = f(x)$.

System 1 $(f_1(x))$

The origin is uniformly globally stable (choose $\gamma(s) := s$).

The origin is also uniformly globally attractive: If $|x| < 1$, we know that $|\phi(k;x)| \leq 0.9^k|x|$ so for $\varepsilon > 0$ we can choose $J \in \mathbb{I}_{\geq 0}$ such that $J > \log(\varepsilon)/\log(0.9)$. If $|x| \geq 1$, then $f(x) = 0$ and for any value of $\varepsilon > 0$ we can choose $J = 1$. So for any $\varepsilon, \Delta > 0$ we can find $J \in \mathbb{I}_{\geq 0}$ such that $|\phi(k;x)| < \varepsilon$. Thus, the system is UGAS.

System 2 $(f_2(x))$

The origin is uniformly globally stable (choose $\gamma(s) := s$).

The origin, however, is *not* uniformly globally attractive. Assume we choose $\varepsilon = 0.1$ and $\Delta \in (0, 1)$. Then we can choose $J \in \mathbb{I}_{\geq 0}$ such that $J > \log(0.1)/\log(\Delta)$. However, as $\Delta \to 1$ from $\Delta < 1$, $J \to \infty$. Thus, the origin is not uniformly globally attractive or UGAS. Nonetheless, for any specific $x \in \mathbb{R}^n$, we know that $\lim_{k\to\infty} |\phi(k;x)| = 0$ and, thus, the system is still globally attractive.

System 3 $(f_3(x))$

The origin is *not* uniformly globally stable. As $x \nearrow 1$, $f(x) \to \infty$. Thus, there does not exist a global $\mathcal{K}$ function upper bound for all $k \in \mathbb{I}_{\geq 0}$ and $x \in \mathbb{R}$. Nonetheless, the origin is locally stable: for each $\varepsilon > 0$, choose $\delta = \min\{\varepsilon, 0.5\}$. As a result, the origin is not UGAS.

The origin is, however, uniformly globally attractive. For $\Delta \in (0, 0.5]$ and $\varepsilon > 0$, we can choose $J \in \mathbb{I}_{\geq 0}$ such that $J > \log(\varepsilon/\Delta)/\log(0.9)$. For $\Delta > 0.5$, if $|x| > 0.5$, then $f_3(x) > 1$ and $f_3(f_3(x)) = 0$. Thus $J = 2$ guarantees that $|\phi(k;x)| < \varepsilon$ for any $\varepsilon > 0$ and $|x| > 0.5$. Thus, for any $\varepsilon, \Delta > 0$ we can choose $J \in \mathbb{I}_{\geq 0}$ such that $J > \min\{\log(\varepsilon/\Delta)/\log(0.9), 2\}$.

B Proofs for Section 3

Proof of Lemma 1. Under Assumption 1, if $\mathbb{V}$ is countable, integrals may be evaluated as summations and $m_{\cap S}(j,x,t)$ is necessarily well-defined. If instead $f(\cdot)$ is continuous we proceed identically to Lemma 1 in Teel (2013) to show that $m_{\cap S}(j,x,t)$ is well defined for all $x \in \mathcal{X}(t)$, $j \in \mathbb{I}_{\geq 0}$, and $t \in \mathbb{I}_{\geq 0}$.

The inequality holds for $j = 0$ and all $x \in \mathcal{X}(t)$, $t \in \mathbb{I}_{\geq 0}$. Suppose that it holds for some $j \in \mathbb{I}_{\geq 0}$. Then

$$\begin{aligned} m_{\cap S}(j+2,x,t) &= \int_{\mathbb{V}} \max\{I_S(f(x,v,t)), m_{\cap S}(j+1, f(x,v,t), t+1)\}\mu(dv) \\ &\geq \int_{\mathbb{V}} \max\{I_S(f(x,v,t)), m_{\cap S}(j, f(x,v,t), t+1)\}\mu(dv) \\ &= m_{\cap S}(j+1,x,t) \end{aligned}$$

and by induction this inequality holds for all $j \in \mathbb{I}_{\geq 0}$. □

Proof of Proposition 2. Assume without loss of generality that $t = 0$ and let $j \in \mathbb{I}_{\geq 1}$, $x \in \mathcal{X}(0)$ and $\mathbf{x} = \Phi(x, 0)$. We claim that

$$\mathbb{E}\left[\max\left\{\max_{i\in\{1,\ldots,\ell\}} I_S(x_i), m_{\cap S}(j-\ell, x_\ell, \ell)\right\}\right] = m_{\cap S}(j,x,0) \tag{23}$$

for each $\ell \in \{1,\ldots,j\}$. For $\ell = 1$, using the definition of $m_{\cap S}(j,x,0)$,

$$\begin{aligned} &\mathbb{E}\left[\max\{I_S(x_1), m_{\cap S}(j-1,x_1,1)\}\right] \\ &\quad = \int_{\mathbb{V}} \max\{I_S(f(x,v,0)), m_{\cap S}(j-1, f(x,v,0), 1)\}\mu(dv) \\ &\quad = m_{\cap S}(j,x,0) \end{aligned}$$

Let (23) hold for some $\ell \in \{1,\ldots,j-1\}$. Let $\bar{j} := j - \ell$. By definition of $m_{\cap S}$ and properties of $\mathbf{x}$,

$$\begin{aligned} m_{\cap S}(\bar{j}, x_\ell, \ell) &= \int_{\mathbb{V}} \max\{I_S(f(x_\ell,v,\ell)), m_{\cap S}(\bar{j}-1, f(x_\ell,v,\ell), \ell+1)\}\mu(dv) \\ &= \mathbb{E}\left[\max\{I_S(x_{\ell+1}), m_{\cap S}(\bar{j}-1, x_{\ell+1}, \ell+1)\} \mid \mathcal{F}_\ell\right] \end{aligned} \tag{24}$$

Using (23) and (24) we have

$$\begin{aligned} &\mathbb{E}\left[\max\left\{\max_{i\in\{1,\ldots,\ell+1\}} I_S(x_i), m_{\cap S}(\bar{j}-1, x_{\ell+1}, \ell+1)\right\}\right] \\ &\quad = \mathbb{E}\left[\max\left\{\max_{i\in\{1,\ldots,\ell\}} I_S(x_i), \mathbb{E}\left[\max\{I_S(x_{\ell+1}), m_{\cap S}(\bar{j}-1, x_{\ell+1}, \ell+1)\} \mid \mathcal{F}_\ell\right]\right\}\right] \\ &\quad = \mathbb{E}\left[\max\left\{\max_{i\in\{1,\ldots,\ell\}} I_S(x_i), m_{\cap S}(\bar{j}, x_\ell, \ell)\right\}\right] = m_{\cap S}(j,x,0) \end{aligned}$$

By induction (23) holds for all $\ell \in \{1,\ldots,j\}$. The case $\ell = j$ gives $m_{\cap S}(j,x,0) = \mathbb{E}\left[\max_{i\in\{1,\ldots,j\}} I_s(x_i)\right]$. □

Proof of Lemma 3. Under Assumption 1, if $\mathbb{V}$ is countable, integrals may be evaluated as summations and $m_{\subset S}(j,x,t)$ is necessarily well-defined. If instead $f(\cdot)$ is continuous we proceed identically to Lemma 2 in Teel (2013) to show that $m_{\subset S}(j,x,t)$ is well defined for all $x \in \mathcal{X}(t)$, $j \in \mathbb{I}_{\geq 0}$, and $t \in \mathbb{I}_{\geq 0}$.

The inequality holds for $j = 0$ and all $x \in \mathcal{X}(t)$, $t \in \mathbb{I}_{\geq 0}$. Suppose that it holds for some $j \in \mathbb{I}_{\geq 0}$. Then

$$\begin{aligned} m_{\subset S}(j+2,x,t) &= \int_{\mathbb{V}} I_S(f(x,v,t)) m_{\subset S}(j+1, f(x,v,t), t+1)\mu(dv) \\ &\leq \int_{\mathbb{V}} I_S(f(x,v,t)) m_{\subset S}(j, f(x,v,t), t+1)\mu(dv) \\ &= m_{\subset S}(j+1,x,t) \end{aligned}$$

and by induction this inequality holds for all $j \in \mathbb{I}_{\geq 0}$. □

Proof of Proposition 4. Assume without loss of generality that $t = 0$ and let $j \in \mathbb{I}_{\geq 1}$, $x \in \mathcal{X}(0)$ and $\mathbf{x} = \Phi(x, 0)$. We claim that

$$\mathbb{E}\left[\left(\prod_{i=1}^{\ell} I_S(x_i)\right) m_{\subset S}(j-\ell, x_\ell, \ell)\right] \leq m_{\subset S}(j,x,0) \tag{25}$$

for each $\ell \in \{1,\ldots,j\}$. For $\ell = 1$, using the definition of $m_{\subset S}(j,x,0)$,

$$\begin{aligned} \mathbb{E}\left[I_S(x_1) m_{\subset S}(j-1,x_1,1)\right] &= \int_{\mathbb{R}^m} I_S(f(x,v,0)) m_{\subset S}(j-1, f(x,v,0), 1)\mu(dv) \\ &= m_{\subset S}(j,x,0) \end{aligned}$$

Let (25) hold for some $\ell \in \{1,\ldots,j-1\}$. Let $\bar{j} = j - \ell$ and by definition of $m_{\subset S}$ and properties of $\mathbf{x}$, we have

$$\begin{aligned} m_{\subset S}(\bar{j}, x_\ell, \ell) &= \int_{\mathbb{R}^m} I_S(f(x_\ell,v,\ell)) m_{\subset S}(\bar{j}-1, f(x_\ell,v,\ell), \ell+1)\mu(dv)) \\ &= \mathbb{E}\left[I_S(x_{\ell+1}) m_{\subset S}(\bar{j}-1, x_{\ell+1}, \ell+1) \mid \mathcal{F}_\ell\right] \end{aligned}$$

Combining this equation with (25), we have

$$\begin{aligned} &\mathbb{E}\left[\left(\prod_{i=1}^{\ell+1} I_S(x_i)\right) m_{\subset S}(\bar{k}-1, x_{\ell+1}, \ell+1)\right] \\ &\quad = \mathbb{E}\left[\left(\prod_{i=1}^{\ell} I_S(x_i)\right) \mathbb{E}\left[I_S(x_{\ell+1}) m_{\subset S}(\bar{j}-1, x_{\ell+1}, \ell+1) \mid \mathcal{F}_\ell\right]\right] \\ &\quad = \mathbb{E}\left[\left(\prod_{i=1}^{\ell} I_S(x_i)\right) m_{\subset S}(\bar{j}, x_\ell, \ell)\right] = m_{\subset S}(j,x,0) \end{aligned}$$

Thus, by induction (25) holds for all $\ell \in \{1, \ldots, j\}$. The case $\ell = j$ in (25) gives

$$\mathbb{E}\left[\prod_{i=1}^{j} I_S(x_i)\right] = m_{\subset S}(j, x, 0) \tag{26}$$

$\square$