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Stochastic Model Predictive Control: Existence and Measurability

Robert D. McAllister and James B. Rawlings

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Abstract

In this report, we establish a selection of critical and underlying properties for stochastic model predictive control (SMPC) that are necessary to ensure that stochastic properties of the closed-loop system are indeed well-defined. We begin by introducing a stochastic, discrete-time system and presenting a typical SMPC formulation with suitable regularity assumptions. We then establish, under these assumptions, that the stochastic optimization problem is well-defined and the minimum is attained. Next, we establish that the optimal control law defined by the SMPC optimization problem is Borel measurable. Thus, we guarantee the resulting closed-loop system is also Borel measurable and all stochastic properties of interest (e.g., expected value) for the closed-loop system are well-defined. We acknowledge the work of Bertsekas and Shreve (1978) for providing multiple technical results on optimization and measurability that were essential in establishing the main results of this report.

Notation and basic definitions

Let $\mathbb{I}$ and $\mathbb{R}$ denote the integers and reals, respectively. Let superscripts on these sets denote dimension and subscripts on these sets denote restrictions (e.g., $\mathbb{R}^n_{\geq 0}$ denotes nonnegative reals of dimension n). We use $|\cdot|$ to denote Euclidean norm. For a closed set $S \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, $|x|_S := \min_{y\in S} |x-y|$ denotes the Euclidean distance from the point x to the set S. Let $I_S(x)$ denote the indicator function for a set S, i.e., $I_S(x) = 1$ is $x \in S$ and zero otherwise. A function $f : \mathbb{R}^n \to \mathbb{R}$ is lower semicontinuous if and only if the set $\{x \in \mathbb{R}^n : f(x) \leq y\}$ is closed for all $y \in \mathbb{R}$ or, equivalently, if $\liminf_{t\to x} f(t) \geq f(x)$ for all $x \in \mathbb{R}^n$.

Let $\mathcal{P}(\Omega)$ denote the power set of some set Ω, i.e., all the subsets of Ω. Let $\mathcal{B}(\Omega)$ denote the Borel field of some set Ω, i.e., the subsets of Ω generated through relative complements and countable unions of all open subsets of Ω. A set $F \subseteq \mathbb{R}^n$ is Borel measurable if $F \in \mathcal{B}(\mathbb{R}^n)$. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is Borel measurable if for each open set $O \subseteq \mathbb{R}^m$, the set $f^{-1}(O) := \{x \in \mathbb{R}^n : f(x) \in O\}$ is Borel measurable, i.e., $f^{-1}(O) \in \mathcal{B}(\mathbb{R}^n)$. For two metric spaces X, Y, a set-valued mapping $S : X \rightrightarrows Y$ is Borel measurable if for every open set $O \subseteq Y$, the set

$$S^{-1}(O) := \{x \in X : S(x) \cap O \neq \emptyset\}$$

is Borel measurable, i.e., $S^{-1}(O) \in \mathcal{B}(X)$ (Rockafellar and Wets, 1998).

1 Problem Formulation and Preliminaries

1.1 The stochastic system

We consider the discrete-time, stochastic system

$$x^+ = f(x, u, w) \quad f : \mathbb{X} \times \mathbb{U} \times \mathbb{R}^p \to \mathbb{X} \tag{1}$$

in which $x \in \mathbb{X} \subseteq \mathbb{R}^n$ is the state, $u \in \mathbb{U} \subseteq \mathbb{R}^m$ is the controlled input, $w \in \mathbb{R}^p$ is a disturbance (random variable), and x^+ is the successor state.

Let $\mathbb{W} \subseteq \mathbb{R}^p$ be the (potentially unbounded) support for each disturbance, i.e., $w \in \mathbb{W}$. Let $(\Omega, \mathcal{F}, P)$ be a probability space for the sequence $\mathbf{w}_\infty : \Omega \to \mathbb{W}^\infty$ of random variables, i.e., $\mathbf{w}_\infty := \{w_i\}_{i=0}^\infty$ for $w_i : \Omega \to \mathbb{W}$.[1] In particular, we have the probability measure $\Pr(w_i \in F) := P(\{\omega \in \Omega : w_i(\omega) \in F\})$ for all $F \in \mathcal{B}(\mathbb{W})$, i.e., the probability that w_i is in the Borel measurable set F. We define the subsequence $\mathbf{w}_i : \Omega \to \mathbb{W}^i$ as $\mathbf{w}_i := (w_0, \dots, w_{i-1})$. We define the probability of the event $g(\mathbf{w}_i) \in S$ as

$$\Pr(g(\mathbf{w}_i) \in S) := \int_\Omega I_S(g(\mathbf{w}_i)) dP(\omega)$$

for a Borel measurable function $g : \mathbb{W}^i \to \mathbb{R}^n$ and a Borel measurable set $S \subseteq \mathbb{R}^n$. We also define expected value of the Borel measurable function $g : \mathbb{W}^i \to \mathbb{R}$ as

$$\mathbb{E}[g(\mathbf{w}_i)] := \int_\Omega g(\mathbf{w}_i(\omega)) dP(\omega)$$

1.2 SMPC formulation

We make the following standard assumption for the disturbance model used in the optimization problem. Note that we distinguish between the true probability distribution of the disturbance given by the probability space $(\Omega, \mathcal{F}, P)$ and the *stochastic model* of the

[1] We may construct a probability space $(\Omega, \mathcal{F}, P)$ for an infinite sequence of independently distributed random variables (Fristedt and Gray, 1997, Section 9.6, Theorem 16).

disturbance used in the SMPC problem. We emphasize that the following results hold regardless of whether the stochastic model is *accurate*, i.e., the true probability distribution and the model are equivalent.[2]

Assumption 1 (Disturbance model)**.** We have a stochastic model of the disturbance w for the system that includes the probability measure $\mu : \mathcal{B}(\hat{\mathbb{W}}) \to [0,1]$ and support $\hat{\mathbb{W}}$. The support $\hat{\mathbb{W}}$ is compact and contains the origin. Furthermore, the stochastic model assumes that the underlying disturbance is independent and identically distributed (i.i.d.).

For the i.i.d. random variables $(w_i, w_{i+1}, \dots, w_{i+N-1})$ and $N \in \mathbb{I}_{\geq 1}$, their joint distribution measure $\mu^N : \mathcal{B}(\hat{\mathbb{W}}^N) \to [0,1]$ is defined $\mu^N(F) = \mu(F_i)\mu(F_{i+1})\dots\mu(F_{i+N-1})$ for all $F = (F_i, F_{i+1}, \dots, F_{i+N-1}) \in \mathcal{B}(\hat{\mathbb{W}}^N)$.

Instead of selecting a trajectory of specific inputs u, in SMPC we solve for a trajectory of control policies. To formulate a computationally tractable optimization problem, however, these control policies are typically parameterized a-priori. We define the policy $\pi : \mathbb{X} \times \mathbb{V} \to \mathbb{U}$ in which $x \in \mathbb{X}$ is the current state of the system and $v \in \mathbb{V} \subseteq \mathbb{R}^p$ are the parameters in the control policy. The system of interest is then redefined as

$$x^+ = f(x, \pi(x,v), w) \tag{2}$$

We denote the solution to (2) at time k, given the initial condition x, the trajectory of parameters $\mathbf{v} = (v(0), v(1), \dots, v(N-1))$, and the disturbances $\mathbf{w} \in \hat{\mathbb{W}}^N$, as $\hat{\phi}(k; x, \mathbf{v}, \mathbf{w})$.

We consider hard input and state constraints, i.e., $(x,u) \in \mathbb{Z}_h \subseteq \mathbb{X} \times \mathbb{U}$. In addition, we allow probabilistic constraints on the state defined as

$$\Pr\left(f(x,u,w) \in \tilde{\mathbb{X}}\right) = \int_{\hat{\mathbb{W}}} I_{\tilde{\mathbb{X}}}(f(x,u,w))d\mu(w) \geq 1 - \varepsilon \tag{3}$$

for a set $\tilde{\mathbb{X}} \subseteq \mathbb{R}^n$ and constant $\varepsilon \in [0,1]$. We observe, however, that this method to represent probabilistic constraints appears to be inconsistent with other constraints typically treated in MPC. Thus, we reformulate the probabilistic constraint using the function

$$G(x,u) := 1 - \int_{\hat{\mathbb{W}}} I_{\tilde{\mathbb{X}}}(f(x,u,w))d\mu(w) \tag{4}$$

and the constraint set $\tilde{\mathbb{Z}} := \{(x,u) : G(x,u) \leq \varepsilon\}$. Note that (x,u) satisfy (3) if and only if $(x,u) \in \tilde{\mathbb{Z}}$. Then, we combined hard and probabilistic constraints as

$$(x,u) \in \mathbb{Z} := \mathbb{Z}_h \cap \tilde{\mathbb{Z}}$$

We note that calculating or approximating $\tilde{\mathbb{Z}}$ is a difficult and important research problem that we obscure with this formulation. We find, however, that this reformulation is very useful in subsequent analysis of the SMPC problem. Additional stochastic properties, such as expected value of the state, can also be reformulated in this manner, but we omit this discussion in the interest of brevity.

[2] Although, we note that inaccuracies in the disturbance support $\mathbb{W}$ may create complications with Assumption 4. Robust recursive feasibility, however, is not the topic of this report and this complication is therefore ignored.

Remark 1. If we wish to consider multiple probabilistic constraints (e.g., multiple sets $\tilde{\mathbb{X}}_j$ with varying values of ε_j), we simply define a set $\tilde{\mathbb{Z}}_j$ for each of these constraints and define $\mathbb{Z}$ as the intersection of all of these sets, i.e., $\mathbb{Z} := \mathbb{Z}_h \cap \left(\bigcap_j \tilde{\mathbb{Z}}_j\right)$. All subsequent results extend to multiple probabilistic constraints as well.

For SMPC with a horizon of $N \in \mathbb{I}_{\geq 1}$, the constraint $\mathbb{Z}$, and an additional terminal constraint $\mathbb{X}_f \subseteq \mathbb{X}$, we have the set of admissible $(x, \mathbf{v})$ pairs defined as

$$\begin{aligned}\mathcal{Z}_N := \{(x, \mathbf{v}) \in \mathbb{X} \times \mathbb{V}^N :\ & \\ & (x(k), \pi(x(k), v(k))) \in \mathbb{Z} \quad \forall \mathbf{w} \in \hat{\mathbb{W}}^N,\ k \in \mathbb{I}_{[0,N-1]} \\ & x(N) \in \mathbb{X}_f \quad \forall \mathbf{w} \in \hat{\mathbb{W}}^N\}\end{aligned}$$

in which $x(k) = \hat{\phi}(k; x, \mathbf{v}, \mathbf{w})$. From this set, we define the set of admissible parameter trajectories given $x \in \mathbb{X}$ and the set of admissible initial states, respectively, as follows.

$$\mathcal{V}_N(x) := \{\mathbf{v} \in \mathbb{V}^N : (x, \mathbf{v}) \in \mathcal{Z}_N\}$$
$$\mathcal{X}_N := \{x \in \mathbb{X} : \exists \mathbf{v} \in \mathcal{V}_N(x)\}$$

We define a stage costs $\ell : \mathbb{X} \times \mathbb{U} \to \mathbb{R}$ and terminal cost $V_f : \mathbb{X} \to \mathbb{R}$ for the SMPC problem. With these costs, we define the function

$$J_N(x, \mathbf{v}, \mathbf{w}) = \sum_{k=0}^{N-1} \ell(x(k), \pi(x(k), v(k))) + V_f(x(N))$$

in which $x(k) := \hat{\phi}(k; x, \mathbf{v}, \mathbf{w})$, $(x, \mathbf{v}) \in \mathcal{Z}_N$, and $\mathbf{w} \in \hat{\mathbb{W}}^N$.

With the function $J_N(\cdot)$, we may define the cost function in several ways, differentiated by their treatment of the random variable $\mathbf{w}$. In nominal MPC, we choose $\mathbf{w} = \mathbf{0}$ and define $V_N(x, \mathbf{v}) = J_N(x, \mathbf{v}, \mathbf{0})$. In robust MPC, we assume the disturbance always takes the worst value possible and define $V_N(x, \mathbf{v}) = \max_{\mathbf{w} \in \hat{\mathbb{W}}^N} J_N(x, \mathbf{v}, \mathbf{w})$. In SMPC, we use a stochastic model of the disturbance and define our cost function based on the expected value of $J_N(\cdot)$, i.e.,

$$V_N(x, \mathbf{v}) := \int_{\hat{\mathbb{W}}^N} J_N(x, \mathbf{v}, \mathbf{w}) d\mu^N(\mathbf{w})$$

With this cost function, the SMPC problem for any $x \in \mathcal{X}_N$ is defined as

$$\mathbb{P}_N(x) : V_N^0(x) = \min_{\mathbf{v} \in \mathcal{V}_N(x)} V_N(x, \mathbf{v})$$

and the optimal solution(s) for a given initial state are defined by the set-valued mapping $\mathbf{v}^0 : \mathcal{X}_N \rightrightarrows \mathbb{V}^N$ such that

$$\mathbf{v}^0(x) := \arg \min_{\mathbf{v} \in \mathcal{V}_N(x)} V_N(x, \mathbf{v})$$

Note that $\mathbf{v}^0(x)$ is a set-valued mapping because there may be multiple solutions to $\mathbb{P}_N(x)$ for a single $x \in \mathcal{X}_N$.

We require the following regularity assumptions for the SMPC problem.

Assumption 2 (Continuity of system and cost)**.** The function $f : \mathbb{X} \times \mathbb{U} \times \mathbb{R}^p \to \mathbb{X}$, $\pi : \mathbb{X} \times \mathbb{V} \to \mathbb{U}$, $\ell : \mathbb{X} \times \mathbb{U} \to \mathbb{R}$, and $V_f : \mathbb{X}_f \to \mathbb{R}$ are continuous. The functions $\ell(x, u)$ and $V_f(x)$ are lower bounded for all $(x, u) \in \mathbb{Z}_h$ and $x \in \mathbb{X}_f$, respectively. Furthermore, we have that $f(0, 0, 0) = 0$, $\ell(0, 0) = 0$, and $V_f(0) = 0$ (without loss of generality).

Assumption 3 (Properties of constraint sets)**.** The sets $\mathbb{Z}_h$ and $\tilde{\mathbb{X}}$ are closed and contain the origin. The sets $\mathbb{U}$, $\mathbb{V}$, and $\mathbb{X}_f \subseteq \mathbb{X}$ are compact and contain the origin.

Note that we require $\mathbb{V}$ in addition to $\mathbb{U}$ to be compact. Since we intend to optimize over $\mathbf{v} \in \mathbb{V}^N$ for a nonlinear (potentially non-coercive) function, compactness of $\mathbb{V}$ is required to ensure the $\mathbb{P}_N(x)$ is well-defined.

Remark 2. We can relax the requirement that $\mathbb{V}$ is compact if the function $V_N(x, \mathbf{v})$ is coercive, i.e., $V_N(x, \mathbf{v}) \to \infty$ as $|\mathbf{v}| \to \infty$ for all $x \in \mathbb{X}$. But, to streamline the subsequent presentation, we omit this discussion.

1.3 Preliminaries on Integration and Measurability

We begin with a few important results for Borel measurable functions. First, we note that continuous and upper/lower semicontinuous functions are Borel measurable. Furthermore, if $f : X \to Y$ and $g : Y \to Z$ are Borel measurable functions, then their composition $f \circ g : X \to Z$ is also Borel measurable. Note that this property of *closure under composition* is the reason we consider Borel measurable functions instead of the more general class of Lebesgue measurable functions; Compositions of Lebesgue measurable functions are not necessarily Lebesgue measurable.[3]

Throughout this report, we find the following result useful.

Lemma 1. *Let $f : X \times S \to \mathbb{R}$ be a Borel measurable function defined for $X \subseteq \mathbb{R}^n$ and the probability space (S, Σ, μ). Then the function $F : X \to \mathbb{R}$ defined by the Lebesgue integral*

$$F(x) := \int_S f(x, s) d\mu(s)$$

satisfies the following:

1. *If $f(x, s)$ is lower bounded and lower semicontinuous w.r.t. $x \in X$, then $F(x)$ is lower semicontinuous.*

2. *If $f(x, s)$ is continuous w.r.t. $x \in X$ and uniformly bounded for all $(x, s) \in X \times S$, then $F(x)$ is finite and continuous.*

[3] We can, if needed, expand the class of functions to *universally measurable* functions (a slightly more general class than Borel measurable functions, but less general than Lebesgue measurable functions) that also preserve universal measurability under composition (Bertsekas and Shreve, 1978, Proposition 7.44). This extension is, however, unnecessary for the SMPC problem as all functions of interest are indeed Borel measurable under mild regularity assumptions.

Proof. Fix $x \in X$ and let $(x_n)_{n=1}^{\infty}$ be any sequence of real numbers that converges to x, i.e., $\lim_{n\to\infty} x_n = x$. We define the corresponding sequence of functions $(f_n)_{n=1}^{\infty}$ such that $f_n(s) := f(x_n, s)$ for all $s \in S$. If $f(x,s)$ is lower semicontinuous w.r.t. x, we have that $\liminf_{n\to\infty} f_n(s) \geq f(x,s)$.

If $f(\cdot)$ is nonnegative, we apply Fatou's Lemma to give

$$\begin{aligned}\liminf_{n\to\infty} F(x_n) &= \liminf_{n\to\infty} \int_S f_n(s) d\mu(s) \\ &\geq \int_S \liminf_{n\to\infty} f_n(s) d\mu(s) \\ &\geq \int_S f(x,s) d\mu(s) = F(x)\end{aligned}$$

Since the choice of $x \in X$ and the sequence $(x_n)_{n=1}^{\infty}$ was arbitrary, we have that

$$\liminf_{t\to x} F(t) \geq F(x)$$

and therefore $F(x)$ is lower semicontinuous.

If $f(\cdot)$ is lower bounded, we define $c \in \mathbb{R}$, such that $f(x,s) \geq c$ for all $(x,s) \in X \times S$. Next, we define $h(x,s) := f(x,s) - c$ and note that $h(\cdot)$ is nonnegative and lower semicontinuous because $f(\cdot)$ is lower semicontinuous. Thus,

$$H(x) := \int_S h(x,s) d\mu(s)$$

is lower semicontinuous and $F(x) = c + H(x)$ is also lower semicontinuous.[4]

If instead $f(x,s)$ is continuous w.r.t. x, we know that $\lim_{n\to\infty} f_n(s) = f(x,s)$. Since $f(x,s)$ is uniformly bounded, we have from the dominated convergence theorem that $F(x)$ is finite and

$$\begin{aligned}\lim_{n\to\infty} F(x_n) &= \lim_{n\to\infty} \int_S f_n(s) d\mu(s) \\ &= \int_S \lim_{n\to\infty} f_n(s) d\mu(s) \\ &= \int_S f(x,s) d\mu(s) = F(x)\end{aligned}$$

Since the choice of $x \in X$ and the sequence $(x_n)_{n=1}^{\infty}$ was arbitrary, we have that $F(x)$ is continuous. □

Presenting expected value and other stochastic properties in terms of Lebesgue integrals allows us to establish many useful properties for SMPC through the use of Lemma 1 that may remain unclear with other SMPC notation.

[4]We can define $F(x) = c + H(x)$, because (S, Σ, μ) is a probability space, i.e., $\int_S c d\mu(s) = c$.

2 Existence of Optimal Solutions

In this section we establish that the minimization problem $\mathbb{P}_N(x)$ for SMPC is well-defined for all $x \in \mathcal{X}_N$. We begin with the following results for the sets $\mathbb{Z}$ and $\mathcal{Z}_N$.

Lemma 2. *Let Assumptions 1, 2, and 3 hold. Then the set $\mathbb{Z}$ is closed.*

Proof. We begin by establishing that $G : \mathbb{X} \times \mathbb{U} \to \mathbb{R}_{\geq 0}$ is a lower semicontinuous function. Since $\tilde{\mathbb{X}}$ is closed, we know that $I_{\tilde{\mathbb{X}}} : \mathbb{X} \to \{0, 1\}$ is upper-semicontinuous. Therefore, the function $h : \mathbb{X} \to \{0, 1\}$ defined as $h(x) := 1 - I_{\tilde{\mathbb{X}}}(x)$ is lower semicontinuous.[5] Since $f(\cdot)$ is continuous, the composition $g(x, u, w) = h(f(x, u, w))$ is lower semicontinuous as well. We have that

$$G(x, u) = \int_{\hat{\mathbb{W}}} g(x, u, w) d\mu(w)$$

and since $g(\cdot)$ is lower semicontinuous (and measurable w.r.t. w) we have from Lemma 1 that $G(x, u)$ is lower semicontinuous.

Since $G(\cdot)$ is lower semicontinuous, the set $\tilde{\mathbb{Z}} = \{(x, u) : G(x, u) \leq \varepsilon\}$ is closed for all $\varepsilon \in [0, 1]$ by definition of a lower semicontinuous function. Therefore, $\mathbb{Z} := \mathbb{Z}_h \cap \tilde{\mathbb{Z}}$ is the intersection of two closed sets and is also closed. □

Lemma 3. *Let Assumptions 1, 2, and 3 hold. Then the set $\mathcal{Z}_N$ is closed.*

Proof. From Lemma 2 we know that $\mathbb{Z}$ is closed. From Assumption 3, we also know that $\mathbb{X}$ and $\mathbb{V}$ are closed sets. We define the set-valued mapping $\mathcal{Z}_N : \hat{\mathbb{W}}^N \rightrightarrows \mathbb{X} \times \mathbb{V}^N$ such that

$$\mathcal{Z}_N(\mathbf{w}) := \{(x, \mathbf{v}) \in \mathbb{X} \times \mathbb{V}^N : \eta_k(x, \mathbf{v}, \mathbf{w}) \leq 0 \quad \forall k \in \mathbb{I}_{[0,N]}\}$$

in which

$$\eta_k(x, \mathbf{v}, \mathbf{w}) := \left|(\hat{\phi}(k; x, \mathbf{v}, \mathbf{w}), \pi(\hat{\phi}(k; x, \mathbf{v}, \mathbf{w}), v(k))\right|_{\mathbb{Z}}$$

for all $k \in \mathbb{I}_{[0,N-1]}$ and $\eta_N(x, \mathbf{v}, \mathbf{w}) := |\hat{\phi}(N; x, \mathbf{v}, \mathbf{w})|_{\mathbb{X}_f}$. Since $f(\cdot)$ and $\pi(\cdot)$ are continuous functions, so is their composition. For each k, $\hat{\phi}(k; x, \mathbf{v}, \mathbf{w})$ is the composition of a finite number of continuous functions and is therefore continuous (Rawlings et al., 2020, Proposition 2.1). Since $\hat{\phi}(k; x, \mathbf{v}, \mathbf{w})$ and point-to-set distance for the closed sets $\mathbb{Z}$ and $\mathbb{X}_f$ are continuous functions, $\eta_k(\cdot)$ is also continuous for each $k \in \mathbb{I}_{[0,N]}$. The inequality $\eta_k(x, \mathbf{v}, \mathbf{w}) \leq 0$ therefore defines a closed set for each $k \in \mathbb{I}_{[0,N]}$. Thus, for each $\mathbf{w} \in \hat{\mathbb{W}}^N$, the set $\mathcal{Z}_N(\mathbf{w})$ is the intersection of a finite number of closed sets and is therefore closed. By the definition of $\mathcal{Z}_N$, we have that

$$\mathcal{Z}_N = \bigcap_{\mathbf{w} \in \hat{\mathbb{W}}^N} \mathcal{Z}_N(\mathbf{w})$$

Since the intersection of an arbitrary collection of closed sets is a closed set, $\mathcal{Z}_N$ is a closed set and the proof is complete. □

[5] Equivalently, we may define $h(x) = I_{\mathbb{R}^n \setminus \tilde{\mathbb{X}}}(x)$, i.e., the indicator function of an *open* set and therefore lower semicontinuous.

With these results, we can establish that the SMPC optimization problem is well-defined.

Proposition 4 (Existence of minima)**.** *Let Assumptions 1, 2, and 3 hold. Then for each* $x \in \mathcal{X}_N$*,*

1. *The function* $V_N(x, \cdot) : \mathbb{V}^N \to \mathbb{R}$ *is continuous.*
2. *The function* $\mathcal{V}_N(x)$ *is compact.*
3. *A solution to* $\mathbb{P}_N(x)$ *exists.*

Proof. From the previous proof, we know that for each k, $\hat{\phi}(k; x, \mathbf{v}, \mathbf{w})$ is continuous. Thus, $J_N(x, \mathbf{v}, \mathbf{w})$ is a continuous function since it is the composition of a finite number of continuous functions. For each $x \in \mathcal{X}_N$, we have that $J_N(x, \cdot) : \mathbb{V}^N \times \hat{\mathbb{W}}^N \to \mathbb{R}$ is continuous and uniformly bounded because $\mathbb{V}$ and $\hat{\mathbb{W}}$ are compact. Thus, from Lemma 1, we know that for each $x \in \mathcal{X}_N$ the function $V_N(x, \cdot) : \mathbb{V}^N \to \mathbb{R}$ is continuous.

From Lemma 3, we know that $\mathcal{Z}_N$ is closed and the function $|(x, \mathbf{v})|_{\mathcal{Z}_N}$ is continuous. Therefore, the set $\mathcal{V}_N(x) = \{\mathbf{v} \in \mathbb{V}^N : |(x, \mathbf{v})|_{\mathcal{Z}_N} \leq 0\}$ is closed for any $x \in \mathcal{X}_N \subseteq \mathbb{X}$. Since $\mathcal{V}_N(x) \subseteq \mathbb{V}^N$ and $\mathbb{V}$ is bounded, we know that $\mathcal{V}_N(x)$ is also bounded. Thus, $\mathcal{V}_N(x)$ is compact.

For each $x \in \mathcal{X}_N$, the function $V_N(x, \cdot)$ is continuous and $\mathcal{V}_N(x)$ is compact. By Weierstrass's theorem, a solution to $\mathbb{P}_N(x)$ exists for all $x \in \mathcal{X}_N$ (Rawlings et al., 2020, Prop. A.7). □

If $\mathcal{Z}_N$ is unbounded, however, we cannot establish that $V_N : \mathcal{Z}_N \to \mathbb{R}$ is continuous. Fortunately, lower semicontinuity is sufficient for subsequent results.

Lemma 5. *Let Assumptions 1, 2, and 3 hold. Then* $V_N : \mathcal{Z}_N \to \mathbb{R}$ *is lower semicontinuous and lower bounded.*

Proof. From the previous proof, we know that $J_N(x, \mathbf{v}, \mathbf{w})$ is continuous and lower-bounded (because the stage and terminal costs are lower-bounded). Since continuity implies lower semicontinuity, by Lemma 1 we know that the function $V_N : \mathcal{Z}_N \to \mathbb{R}$ is lower semicontinuous and lower-bounded. □

3 Measurability of the Closed-Loop Trajectory

For SMPC, we define the control law mapping as $K_N(x) := \pi(x, v^0(0; x))$ in which $v^0(0; x)$ is the first control action in $\mathbf{v}^0(x)$. Since there may be multiple solutions to $\mathbb{P}_N(x)$ for each $x \in \mathcal{X}_N$, $K_N(x)$ may be a set-valued mapping. We typically assume there exists some selection rule that defines a single-valued control law $\kappa_N : \mathcal{X}_N \to \mathbb{U}$ such that $\kappa_N(x) \in K_N(x)$ for all $x \in \mathcal{X}_N$. With this control law, the closed-loop stochastic system is defined as

$$x^+ = f_{cl}(x, w) := f(x, \kappa_N(x), w) \tag{5}$$

We denote the solution to (5) at time $k \in \mathbb{I}_{\geq 0}$, given the initial condition x and disturbance sequence $\mathbf{w}_k = (w_0, \dots, w_{k-1})$ as $\phi(k; x, \mathbf{w}_k)$.

If we assume (or establish through properties of the SMPC formulation) that $\mathcal{X}_N$ is robustly positive invariant for the closed-loop system subject to the true set of potential disturbances ($\mathbb{W}$), we can establish that $\phi(k; x, \mathbf{w}_k)$ well-defined.

Assumption 4. The set $\mathcal{X}_N$ is robustly positive invariant for the closed-loop system $x^+ = f_{cl}(x, w)$, $w \in \mathbb{W}$, i.e., if $x \in \mathcal{X}_N$ then $x^+ \in \mathcal{X}_N$ as well.

Lemma 6. *Let Assumption 4 hold. Then the function $\phi(k; x, \mathbf{w}_k)$ is well-defined for all $x \in \mathcal{X}_N$, $\mathbf{w}_k \in \mathbb{W}^k$, and $k \in \mathbb{I}_{\geq 0}$.*

Proof. We establish this result by induction. If $x(k) = \phi(k; x, \mathbf{w}_k) \in \mathcal{X}_N$, then $u(k) = \kappa_N(x(k))$ is well-defined and $x(k+1) = f_{cl}(x(k), w_k) = f(x(k), u(k), w_k)$ is also well-defined, i.e., $\phi(k+1; x, \mathbf{w}_{k+1})$ is well-defined for all $\mathbf{w}_{k+1} \in \mathbb{W}^{k+1}$. Since $\mathcal{X}_N$ is robustly positive invariant, we also know that $\phi(k+1; x, \mathbf{w}_{k+1}) \in \mathcal{X}_N$. Since we start at $x(0) = x \in \mathcal{X}_N$, the proof is complete. □

We note that for SMPC, and indeed nominal MPC, the optimal control law $\kappa_N(x)$, and therefore the closed-loop system $f_{cl}(\cdot)$, may be discontinuous w.r.t. $x \in \mathcal{X}_N$.[6] For analysis of a deterministic closed-loop system, discontinuities in the optimal control law can be addressed with a few adjustments (Allan et al., 2017). The concern for discontinuous stochastic systems, however, is far more fundamental; A discontinuous closed-loop system may produce a non-measurable closed-loop trajectory, i.e., the function $\phi(k; x, \mathbf{w}_k)$ may not be measurable w.r.t. $\mathbf{w}_k \in \mathbb{W}^k$. If $\phi(k; x, \mathbf{w}_k)$ is not measurable w.r.t. $\mathbf{w}_k$ and Ω is uncountable, Lebesgue integrals are not well-defined and all stochastic properties of the system based on these integrals (e.g., expected value) are undefined for the closed-loop stochastic system.

Fortunately, the regularity conditions required by Assumptions 2 and 3 are sufficient to guarantee that the control law mapping $K_N(x)$ is in fact Borel measurable. We use the following result adapted from Bertsekas and Shreve (1978, Proposition 7.33).

Proposition 7. *Consider the closed set $X \subseteq \mathbb{R}^n$, compact set $U \subseteq \mathbb{R}^m$, closed set $Z \subseteq X \times U$, and lower semicontinuous function $V : Z \to \mathbb{R}$. Let $\mathcal{U}(x) := \{u \in U : (x, u) \in Z\}$ and*

$$V^0(x) := \min_{u \in \mathcal{U}(x)} V(x, u)$$
$$u^0(x) := \arg \min_{u \in \mathcal{U}(x)} V(x, u)$$

Then $\mathcal{X} := \{x \in X : \mathcal{U} \neq \emptyset\}$ is closed, $V^0 : \mathcal{X} \to \mathbb{R}$ is lower semicontinuous, and $u^0 : \mathcal{X} \rightrightarrows U$ is Borel measurable.

We can apply this general result for optimization problems to the SMPC problem through the following proposition.

[6] Even though we require continuous $\pi(\cdot)$, the function $v^0(0; x)$ may be discontinuous and therefore $\kappa_N(x)$ is also discontinuous.

Proposition 8. *Let Assumptions 1, 2, and 3 hold. Then the function* $V_N^0 : \mathcal{X}_N \to \mathbb{R}$ *is lower semicontinuous (Borel measurable) and the set-valued mapping* $\mathbf{v}^0 : \mathcal{X}_N \rightrightarrows \mathbb{V}^N$ *is Borel measurable. Furthermore, the optimal control law mapping* $K_N : \mathcal{X}_N \rightrightarrows \mathbb{U}$*, defined as* $K_N(x) := \pi(x, v^0(0; x))$ *is Borel measurable.*

Proof. From Assumption 3 we have that $\mathbb{X}$ is closed and $\mathbb{V}^N$ is compact. From Lemma 3 we have that $\mathcal{Z}_N$ is closed. From Lemma 5, we have that $V_N : \mathcal{Z}_N \rightrightarrows \mathbb{R}$ is lower semicontinuous. From Proposition 7, we have that $V_N^0 : \mathcal{X}_N \to \mathbb{R}$ is lower semicontinuous and the mapping $\mathbf{v}^0 : \mathcal{X}_N \rightrightarrows \mathbb{V}^N$ is Borel measurable. We define $K_N : \mathcal{X}_N \rightrightarrows \mathbb{V}$ such that $K_N(x) = \{h(x, \mathbf{v}) : \mathbf{v} \in \mathbf{v}^0(x)\}$ in which $h(x, \mathbf{v}) := \pi(x, v(0))$. Since $h(\cdot)$ is a continuous function, $K_N : \mathcal{X}_N \rightrightarrows \mathbb{U}$ is also Borel measurable. □

If $\mathbf{v}^0(x)$ is a single-valued mapping, then $\kappa_N(x) = K_N(x)$ is a single-valued, Borel measurable function. If instead, $\mathbf{v}^0(x)$ is a set-valued mapping, then we require a selection rule. By Lemma 7.18 in Bertsekas and Shreve (1978), there exists a Borel measurable function $\sigma : (\mathcal{P}(\mathbb{U}) \setminus \emptyset) \to \mathbb{U}$ such that $\sigma(A) \in A$ for every $A \in \mathcal{P}(\mathbb{U} \setminus \emptyset)$, i.e., $\sigma(\cdot)$ maps any subset of $\mathbb{U}$ to a point in $\mathbb{U}$. We therefore define the SMPC control law as $\kappa_N(x) := \sigma(K_N(x))$ for all $x \in \mathcal{X}_N$.

In theory, we can select an exotic selection rule that produces a non-measurable function $\kappa_N(x)$ from the Borel measurable set-valued mapping $K_N(x)$. We postulate, however, that accidentally constructing such a control law or selection rule is almost certainly impossible. We discuss this topic further and provide an example in Appendix A. Thus, we make the following assumption for the rest of this report.

Assumption 5. We have chosen a Borel measurable function $\sigma : (\mathcal{P}(\mathbb{U}) \setminus \emptyset) \to \mathbb{U}$ such that $\sigma(A) \in A$ for every $A \in (\mathcal{P}(\mathbb{U}) \setminus \emptyset)$ and defined the control law as $\kappa_N(x) := \sigma(K_N(x))$.

Remark 3. We emphasize that Assumption 5 does not supplant the need for Proposition 8. If $K_N(\cdot)$ is not a Borel measurable mapping, there is no guarantee that a Borel measurable selection rule $\sigma(\cdot)$ generates a Borel measurable control law $\kappa_N(\cdot)$. Indeed, if $K_N(\cdot)$ is not Borel measurable there is no guarantee that there exists a Borel measurable control law $\kappa_N(x)$ such that $\kappa_N(x) \in K_N(x)$ for all $x \in \mathcal{X}_N$. For example, if $K_N(x)$ is a single-valued mapping and not Borel measurable, $\kappa_N(\cdot) = K_N(x)$ is also not Borel measurable.

Stochastic properties of interest for the closed-loop system are defined by the following Lebesgue integrals

$$\begin{aligned}
\mathbb{E}\left[|\phi(k; x, \mathbf{w}_k)|\right] &:= \int_\Omega |\phi(k; x, \mathbf{w}_k(\omega))| dP(\omega) \\
\mathbb{E}\left[V_N^0(\phi(k; x, \mathbf{w}_k))\right] &:= \int_\Omega V_N^0(\phi(k; x, \mathbf{w}_k(\omega))) dP(\omega) \\
\mathbb{E}\left[\ell(\phi(k; x, \mathbf{w}_k), \kappa_N(\phi(k; x, \mathbf{w}_k)))\right] &:= \int_\Omega \ell(\phi(k; x, \mathbf{w}_k(\omega)), \kappa_N(\phi(k; x, \mathbf{w}_k(\omega)))) dP(\omega) \\
\Pr\left(\phi(k; x, \mathbf{w}_k) \in S\right) &:= \int_\Omega I_S\left(\phi(k; x, \mathbf{w}_k(\omega))\right) dP(\omega)
\end{aligned}$$

in which S is a closed set. With the following result we guarantee that $\phi(k; x, \mathbf{w}_k)$ is a Borel measurable function and all of these stochastic properties are well-defined.

Proposition 9. *Let Assumption 1-5 hold. Then the function $\phi(k; x, \mathbf{w}_k(\omega))$ for all $k \in \mathbb{I}_{\geq 0}$ are measurable w.r.t. the measure space $(\Omega, \mathcal{F}, P)$. Furthermore, the integral*

$$\int_{\Omega} g(\phi(k; x, \mathbf{w}_k(\omega)))dP(\omega)$$

is well-defined for all $x \in \mathcal{X}_N$, $k \in \mathbb{I}_{\geq 0}$, and any lower bounded, Borel measurable function $g : \mathcal{X}_N \to \mathbb{R}$. Furthermore, the function $V_N^0 : \mathcal{X}_N \to \mathbb{R}$ and $\ell(\cdot, \kappa_N(\cdot)) : \mathcal{X}_N \to \mathbb{R}$ are lower bounded and Borel measurable.

Proof. Adapted from Proposition 4 in Grammatico et al. (2013). From Proposition 8 and Assumption 5 we have that $\kappa_N : \mathcal{X}_N \to \mathbb{U}$ is Borel measurable. Since $f(\cdot)$ is continuous, $f_c(x, w) = f(x, \kappa_N(x), w)$ is Borel measurable. From Lemma 6, we know that $\phi(k; x, \mathbf{w}_k)$ is well-defined for all $x \in \mathcal{X}_N$, $\mathbf{w}_k \in \mathbb{W}^k$, and $k \in \mathbb{I}_{\geq 0}$.

We proceed by induction. For some $k \in \mathbb{I}_{\geq 0}$ let $\phi(k; x, \mathbf{w}_k)$ be Borel measurable. Then

$$\phi(k+1; x, \mathbf{w}_{k+1}) = f_c(\phi(k; x, \mathbf{w}_k), w_k)$$

is also Borel measurable. Since $\phi(1; x, \mathbf{w}_1) = f_c(x, w_0)$ is Borel measurable, we have that for all $k \in \mathbb{I}_{\geq 0}$, $\phi(k; x, \mathbf{w}_k)$ is Borel measurable. By definition, $\mathbf{w}_k(\omega)$ is measurable w.r.t. $\omega \in \Omega$ and therefore $\phi(k; x, \mathbf{w}_k(\omega))$ is also Borel measurable w.r.t. $\omega \in \Omega$.

For nonnegative, real-valued, Borel-measurable functions, Lebesgue integrals are well-defined (although not necessarily finite). Because $g(\cdot)$ is lower bounded, there exists finite $c \in \mathbb{R}$ such that $g(x) \geq c \; \forall x \in \mathcal{X}_N$. We define $h : \mathcal{X}_N \to \mathbb{R}_{\geq 0}$ as $h(x) := g(x) - c$. Therefore, $h(\cdot)$ is a nonnegative, real-valued, Borel-measurable function and

$$\begin{aligned} \int_{\Omega} g(\phi(k; x, \mathbf{w}(\omega)))dP(\omega) &= \int_{\Omega} (c + h(\phi(k; x, \mathbf{w}(\omega))))dP(\omega) \\ &= c + \int_{\Omega} (h(\phi(k; x, \mathbf{w}(\omega)))dP(\omega) \end{aligned}$$

is well-defined.[7]

From Assumption 2 we have that $\ell : \mathbb{X} \times \mathbb{U} \to \mathbb{R}$ and $V_N : \mathbb{X} \times \mathbb{V}^N \to \mathbb{R}$ are lower bounded. Thus, there exists $M \in \mathbb{R}$ such that $V_N(z) \geq M$ for all $z \in \mathcal{Z}_N$ and by properties of constrained minimization $V_N^0(x) \geq M$ for all $x \in \mathcal{X}_N$ as well. From Proposition 8 and Standing Assumption 5, we know that $V_N^0(\cdot)$ and $\kappa_N(\cdot)$ are Borel measurable. Therefore, $\ell(x, \kappa_N(x))$ and $V_N^0(x)$ are lower bounded and Borel measurable functions for $x \in \mathcal{X}_N$. □

Thus, Proposition 9 ensures that all stochastic properties of interest for the closed-loop system are well-defined. In addition, we note that the control laws admitted by Proposition 9 subsume the control law defined by nominal MPC since we can define $\hat{\mathbb{W}} = 0$ and $\mu(\{0\}) = 1$ in the SMPC problem to recover the nominal MPC problem. Therefore, nominal MPC applied to a stochastic system also produces measurable closed-loop systems. See Appendix B for a further discussion of nominal MPC and measurability.

[7] Since we are considering a probability space, we know that $\int_{\Omega} c dP(\omega) = c$.

4 Discussion and Conclusions

In this report we established a collection of critical results that are required to properly analyze the closed-loop stochastic properties of SMPC. First, we established under suitable regularity assumptions that the stochastic optimization problem for SMPC is well-defined, i.e., the minimum exists. We achieve this result by leveraging Lebesgue integration theory (specifically, Fatou's Lemma). Furthermore, we are able to guarantee that the stochastic cost function $V_N(\cdot)$ is lower semicontinuous and that the set of admissible initial conditions and parameter trajectories $\mathcal{Z}_N$ is in fact closed despite including probabilistic constraints. We then establish that these same regularity conditions are sufficient to guarantee that the optimal control law mapping is Borel measurable. We conclude with Proposition 9, in which we establish that all stochastic properties of interest for the closed-loop system are well-defined. These results constitute a critical foundation for any and all analysis of closed-loop stochastic properties for SMPC.

Although many of these results are often tacitly and harmlessly assumed in SMPC literature, there are a few important considerations raised by conducting the formal analysis in this report. We note discuss a few specific insights here.

First, we note that Assumption 2 requires that $\pi(\cdot)$ is a continuous function. This requirement is not always discussed explicitly in SMPC literature and sometimes this requirement is relaxed to allow any Borel measurable function $\pi(\cdot)$. However, if we allow $\pi(\cdot)$ to be a discontinuous (albeit measurable) function, the functions $J_N(\cdot)$ and therefore $V_N(\cdot)$ are not necessarily continuous or even lower semicontinuous. Consequently, the optimization problem $\mathbb{P}_N(x)$ is not necessarily well-defined and the minimum may not exist.

Similarly, we require $\tilde{\mathbb{X}}$ to be a closed set and the probabilistic constraint to be defined by an inequality. Although this requirement is often enforced in formulating the SMPC problem, the analysis conducted in this report indicates that such a formulation is important to guarantee that $\mathbb{Z}$ is closed (and by extension ensure that $\mathbb{P}_N(x)$ is well-defined). If we modify this constraint (require a strict inequality or use an open set $\tilde{\mathbb{X}}$), we may not be able to guarantee that $\mathbb{Z}$ is closed.

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A Measurable Selection Rules

In this section, we elaborate on the nuances of measurable selection rules. We begin by presenting an example to illustrate that, in theory, we can select a non-measurable single-valued control law $\kappa_N(x)$ from a measurable set-valued control law $K_N(x)$.

Consider the Borel measurable control law mapping $K_N : [0,1] \rightrightarrows [0,1]$ defined as $K_N(x) = [0,x]$, i.e., the closed set from zero to x. We note that the graph of $K_N(x)$, i.e., the set

$$\{(x,u) \in [0,1]^2 : u \in K_N(x)\} = \{(x,u) \in [0,1]^2 : x - u \leq 0\},$$

is closed and therefore $K_N(\cdot)$ is outer semicontinuous and Borel measurable (Rockafellar and Wets, 1998, Theorem 5.7(a), Exercise 14.9). Define the function

$$\sigma(U) := \begin{cases} \arg\max_{u\in U} |u| & ; \arg\max_{u\in U} |u| \in V \\ \arg\min_{u\in U} |u| & ; \arg\max_{u\in U} |u| \notin V \end{cases}$$

in which $V \subseteq [0,1]$ is the Vitali set (or some other non-measurable set).[8] Thus, the resulting single-valued control law is

$$\kappa_N(x) := \sigma(K_N(x)) = xI_V(x)$$

because $\arg\max_{u\in K_N(x)} |u| = x$ and $\arg\min_{u\in K_N(x)} |u| = 0$ for all $x \in [0,1]$. Also, note that $\kappa_N(x) \in K_N(x)$ for all $x \in [0,1]$, i.e., we have taken a selection of $K_N(x)$ to generate $\kappa_N(x)$. However, $\kappa_N(\cdot)$ is not a (Borel) measurable function because

$$\kappa_N^{-1}((0,1)) = \{x \in [0,1] : xI_V(x) \in (0,1)\} = V \notin \mathcal{B}([0,1])$$

[8] The function $\sigma(\cdot)$ is not technically a selection rule, since $\max_{u\in U} |u|$ is undefined for a open (not closed) sets U. However, since $K_N(x)$ always produces a closed set, we can still apply this function as a pseudo selection rule.

in which V is the Vitali set (or some other non-measurable set). This example illustrates how we can, at least in theory, construct a non-measurable single-valued control law $\kappa_N(x)$ as a selection from a measurable set-valued control law $K_N(x)$.

We argue, however, that accidentally constructing a non-measurable selection rule is very unlikely. Robert Solovay established in 1970 that the axiom of choice is *necessary* to construct a non-measurable set and by extension a non-measurable function (Solovay, 1970). Therefore, to construct a non-measurable selection rule, we must generate such a function through uncountably many arbitrary choices (no algorithm), e.g., by defining a set such as the Vitali set. Presumably, any selection rule (intentionally or unintentionally) defined for the controller is still based on an underlying algorithm and therefore does not require the axiom of choice. Hence, the assumption that $\sigma(\cdot)$ is a Borel measurable function is very reasonable.

B Measurable Closed-Loop Trajectory for Nominal MPC

In this section, we consider the case of nominal MPC applied to the stochastic system in (1). For nominal MPC, we assume that $\hat{\mathbb{W}} := \{0\}$ and $\mathbf{w} = \mathbf{0}$ in the optimization problem and define $\pi(x, v) := v$ and $\mathbb{V} = \mathbb{U}$. We note that Assumptions 1, 2, and 3 admit the nominal MPC formulation and therefore Proposition 8 holds for the nominal MPC control law mapping as well. Indeed, we can strengthen Lemma 5 to establish that $V_N : \mathcal{Z}_N \to \mathbb{R}$ is continuous and lower bounded since $V_N(x, \mathbf{v}) := J_N(x, \mathbf{v}, \mathbf{0})$. Furthermore, we can similarly apply Assumption 5 to construct a Borel measurable, single-valued control law $\kappa_N(x)$ (such that $\kappa_N(x) \in K_N(x)$ for all $x \in \mathcal{X}_N$) for the nominal MPC problem.

Although we assume $\hat{\mathbb{W}} = \{0\}$ in the optimization problem, the underlying closed-loop system is still stochastic with nonzero disturbances, i.e., we determine $\kappa_N(\cdot)$ based on the nominal system, but the underlying closed-loop system is still $x^+ = f(x, \kappa_N(x), w)$ with the disturbance $\mathbf{w}_\infty$ defined on the probability space $(\Omega, \mathcal{F}, P)$ and support $\mathbb{W} \neq \{0\}$. If we are able to establish that Assumption 4 holds for the closed-loop system (despite not addressing the disturbances in the optimization problem directly), we can still apply Proposition 9 to the closed-loop trajectory. Therefore, all stochastic properties of interest for the closed-loop system are still well-defined.