TWCCC ⋆ Texas – Wisconsin – California Control Consortium

Technical report number 2024-01

Maximum likelihood identification of uncontrollable linear time-invariant models for offset-free control[*][†]

Steven J. Kuntz and James B. Rawlings[‡]

June 12, 2024

Abstract

Maximum likelihood identification of linear time-invariant models is a difficult problem because it is, in general, a nonlinear semidefinite program, with semidefinite covariance matrix arguments and semidefinite filter stability constraints. To enforce filter stability, we establish a general theory of closed constraints on the system eigenvalues using LMI regions. To solve the identification problem, we employ a Cholesky factorization method that reduces the semidefinite program to a standard nonlinear program. Finally, we apply the identification algorithm to a class of linear plant and disturbance models commonly used in offset-free model predictive control applications. Specifically, we consider models that are structured with uncontrollable, integrating disturbance states. We solve this disturbance modeling problem, and validate the resulting controller and estimator performance, in two real-world case studies: first, a low-cost benchmark temperature control laboratory, and second, an industrial-scale chemical reactor at Eastman Chemical's Kingsport plant.

1 Introduction

Linear system identification is an important problem in control applications and theory, with a longstanding history of applied use and a large body of literature on its theory [1–3].

[*]This report is an extended version of a submitted paper.

[†]This work was supported by the National Science Foundation (NSF) under Grant 2138985.

[‡]The authors are with the Department of Chemical Engineering, University of California, Santa Barbara, CA 93106 USA (e-mail: skuntz@ucsb.edu; jbraw@ucsb.edu). *Corresponding author: Steven J. Kuntz.*

In particular, stochastic linear time-invariant (LTI) state-space models are used in a variety of control contexts to represent dynamics with process and measurement uncertainty. Maximum likelihood (ML) identification is the preferable method in parametric identification for its desirable statistical properties (consistency, asymptotic efficiency) and ability to handle general parameterizations, constraints, and stochastic noise models [4–6].

The main computational challenge to ML identification of LTI models is that, in general, the problem is a nonlinear semidefinite program (SDP), with semidefinite matrix arguments and semidefinite filter stability constraints. In the ML identification literature, nonlinear SDPs are avoided either by using the expectation maximization (EM) algorithm [7–10], by invoking simplified covariance matrix parameterizations (positive diagonal matrices, scaled rotation matrices) [11–14], or by minimizing the determinant of the sample covariance [4–6, 15–17]. Each of these strategies impose a specific structure on the estimates. The chief advantage of the EM algorithm is that, for black-box models, there are closed-form solutions for the iterates. When further structure on the model is imposed, it may invoke an optimization problem within the EM algorithm iterates, significantly slowing down the computation [18]. Diagonal covariances are a highly constrained structure, and the type of simple covariance structures available in the literature do not scale to high-dimensional systems. Finally, the minimum determinant approach requires a fully parameterized covariance matrix for the Kalman filter innovations. Moreover, these last two approaches introduce a filter stability constraint that is not explicitly enforced in the current literature.

EM does not have strong convergence guarantees even in the best case scenario. While it can be shown that the EM iterates produce, almost surely, an increasing sequence of likelihood values [7, 9], slow convergence at low noise levels has been reported on a range of problems [18–23], with hundreds or thousands of iterates being common for small ML problems in even the latest EM works [18]. Interior point, and even gradient methods [23], are therefore preferable to the standard EM approach.

Linear identification of nonlinear systems In a wide variety of control applications, including chemical processes [24–26], aerospace vehicles [17, 27], combustion engines [28], nautical vehicles [11, 29], and speech recognition [8], linear approximations of the nonlinear plant are beneficial for the convenience of linear identification relative to that of nonlinear identification and the ability to meet strict computational constraints, e.g., for online optimal control. Linear black-box models are particularly useful when first-principles knowledge of the plant dynamics is not available.

The main difficulty of linear identification of nonlinear systems is plant-model mismatch. With ML identification, properties of the estimates are dependent on the plant's stochastic behavior [30, 31]. For stationary, input-free models, the solution to the mismatched problem can be interpreted as (asymptotically) minimizing the Kullback-Leibler divergence between the power spectral densities of the model and plant [32]. However, there are still gaps in the treatment of inputs, state-space models, and arbitrary nonlinear plants. Rather than address this theory, we turn to control and estimation applications that specifically address plant-model mismatch.

Application to offset-free control Model predictive control (MPC) is a widely-used advanced control method in which an optimal control problem is solved on-line, based on the current state or state estimate, and the first input in the solution trajectory is injected into the plant [33, 34]. Model quality is the main contributor to the performance of an MPC implementation [35, 36], and therefore high-quality identification algorithms are of relevance to MPC implementations.

Inherent to the use of identified plant models in MPC is the problem of plant-model mismatch. Moreover, many applications require the rejection of a stochastic, possibly nonstationary, disturbance process with unknown or un-modeled components (e.g., environmental or upstream disturbances, demand changes). In linear offset-free MPC, the stochastic LTI state-space model is augmented with *uncontrollable* integrating disturbance modes to achieve offset-free control in the presence of plant-model mismatch and persistent disturbances [37, 38]. We refer to these models as *linear augmented disturbance models*.

Linear augmented disturbance models (or the resulting observer) can either be tuned or identified. Tuning of disturbance models can be roughly divided into three categories: pole placement [39–42], diagonal covariance matrix tuning [25, 43, 44], and direct filter gain tuning [45–47]. On the other hand, disturbance models have been identified only via autocovariance least squares (ALS) estimation [48] and (approximate) ML estimation [49–51]. Only our prior work in [50, 51] integrates the plant and disturbance identification in a single step. However, the method in [50, 51] uses a nested ML estimation approach in which unstructured stochastic LTI models are augmented with integrating disturbance modes. The nested ML method can be improved by consolidation to a single step ML step.

Direct data-driven control The approach discussed so far is an indirect data-driven control design of offset-free MPC. A potential alternative is the direct data-driven control approach, where the control law is designed according to data [52–55]. The drawback of this approach is its reliance on Willem's Fundamental Lemma [56], which does not admit the required linear augmented disturbance model structure.

Contributions and outline The main contributions of this work are (i) a method for directly solving constrained ML identification problems as NLPs, more efficiently and on a wider class of systems than the state-of-the-art, and (ii) real-world case studies of the application of this algorithm to offset-free control. The ML problem is stated in Section 2. In Section 3, stability and other eigenvalue constraints are formulated. Inspired by the smoothed spectral radius and abscissa formulations of [57, 58], and we present a novel barrier function theory of constraints on the system eigenvalues via the linear matrix inequality (LMI) regions of [59]. In Section 4, we present a novel modification of the Burer-Monteiro-Zhang (BMZ) method [60], reformulating a class of nonlinear SDPs as NLPs via Cholesky factorization. In identification problems, Cholesky factor subsitution has only been used for replacing semidefinite covariance matrices [49], not for general matrix inequalities. In Section 5, we used the ML identification algorithm for two real-world applications of offset-free MPC: first, a benchmark temperature microcontroller [61], and second, an industrial-scale chemical reactor at Eastman Chemical's Kingsport plant [51].

The advantage of ML identification over other disturbance modeling techniques is demonstrated. Finally, in Section 6, we conclude with a discussion of future work on applying the identification problem to other parts of the control architecture (e.g., performance monitoring, steady-state optimization, and automated MPC upkeep) and extensions to direct data-driven control.

This report is an extended version of a submitted work, and contains proofs of minor results and details on the industrial-scale reactor case study that were omitted from the journal version due to page limitations. Compared to the journal version, this report additional contains the following additions:

- the proof of Proposition 6 in Appendix A;
- a longer version of Lemma 13;
- an additional discussion of LMI region properties in Section 3;
- an explicit counterexample of [62, Thm. 1] in Conjecture 18;
- the proof of Proposition 20 in Appendix B;
- the proof of Proposition 21(b,c) in Appendix C; and
- additional figures, data, and discussion of the industrial-scale reactor study in Section 5.

Notation Denote the set of $n \times n$ symmetric, positive definite, positive semidefinite matrices, lower triangular, and positive lower triangular by $\mathbb{S}^n$, $\mathbb{S}^n_{++}$, $\mathbb{S}^n_{+}$, $\mathbb{L}^n$, and $\mathbb{L}^n_{++}$, respectively. Recall $M \in \mathbb{R}^{n\times n}$ is positive definite if and only if there exists a unique $L \in \mathbb{L}^n_{++}$, called the *Cholesky factor*, such that $M = LL^\top$. Denote the matrix direct sum and the Kronecker product by $\oplus$ and $\otimes$, respectively, defined as in [63]. Define the set of eigenvalues of a matrix $A \in \mathbb{R}^{n\times n}$ by $\lambda(A) \subset \mathbb{C}$. The spectral radius and spectral abscissa are defined as $\rho(A) := \max_{\lambda\in\lambda(A)} |\lambda|$ and $\alpha(A) := \max_{\lambda\in\lambda(A)} \mathrm{Re}(\lambda)$, respectively. We say a matrix A is Schur (Hurwitz) stable if $\rho(A) < 1$ ($\alpha(A) < 0$). We use $\sim$ as a shorthand for "distributed as" and $\overset{\text{iid}}{\sim}$ as a shorthand for "independent and identically distributed as." The complement, interior, closure, and boundary of a set S are denoted S^c, $\mathrm{int}(S)$, $\mathrm{cl}(S)$, and ∂S, respectively.

2 Maximum likelihood identification

In this section, we formulate a maximum likelihood problem for identifying models of the following form:

$$x_{k+1} = A(\theta)x_k + B(\theta)u_k + w_k \tag{1a}$$
$$y_k = C(\theta)x_k + D(\theta)u_k + v_k \tag{1b}$$
$$x_0 \sim \mathcal{N}(\hat{x}_0(\theta), \hat{P}_0(\theta)) \tag{1c}$$
$$\begin{bmatrix} w_k \\ v_k \end{bmatrix} \overset{\text{iid}}{\sim} \mathcal{N}(0, S(\theta)) \tag{1d}$$

where $x \in \mathbb{R}^n$ are the model states, $u \in \mathbb{R}^m$ are the inputs, $y \in \mathbb{R}^p$ are the outputs, $w \in \mathbb{R}^n$ and $v \in \mathbb{R}^p$ are the process and measurement noises, and $\mathcal{M} := (A, B, C, D, \hat{x}_0, \hat{P}_0, S)$ are functions, to be defined, that map the model parameters $\theta \in \Theta$ to system matrices or vectors of appropriate dimensions. The ML estimate $\hat{\theta}_N$ is defined a maximizer of $p(\mathbf{y}_{N-1}|\mathbf{u}_{N-1}, \theta)$, or equivalently, a solution to

$$\min_{\theta\in\Theta} f_N(\theta) := -\sum_{k=0}^{N-1} \ln p(y_k|\mathbf{u}_{k-1}, \mathbf{y}_{k-1}, \theta). \tag{2}$$

The noise covariance matrix $S(\theta)$ may be partitioned as

$$S(\theta) = \begin{bmatrix} Q_w(\theta) & S_{wv}(\theta) \\ [S_{wv}(\theta)]^\top & R_v(\theta) \end{bmatrix} \tag{3}$$

where $Q_w(\theta) \in \mathbb{S}^n_+$ is the process noise covariance, $S_{wv}(\theta)$ is the cross-covariance, and $R_v(\theta) \in \mathbb{S}^p_+$ is the measurement noise covariance. Throughout, we impose the stronger requirement $R_v(\theta) \succ 0$ on the measurement noise covariance.

2.1 The constraint set

The main difficulty of the ML problem (2) is that the parameter constraint set Θ must necessarily contain matrix inequalities that make (2) a nonlinear SDP. For example, we expect consistency (i.e., positive semidefiniteness) of the covariance matrices and nondegeneracy of the measurement noise distribution.

Assumption 1. For all $\theta \in \Theta$, we have $\hat{P}_0(\theta) \succeq 0$, $S(\theta) \succeq 0$, and $R_v(\theta) \succ 0$.

Other matrix inequalities may arise as stability, eigenvalue, or other system-level constraints. Stability and other eigenvalue constraints are explicitly covered in Section 3.

We structure the constraint set to facilitate the Cholesky factor-based reformulation in Section 4. To define this structure, we first need to define some additional notation. Consider the index sets $\mathcal{L}^n := \{ (i,j) \in \mathbb{N}^2 \mid 1 \le i \le j \le n \}$ and $\mathcal{D}^n := \{ (i,i) \in \mathbb{N}^2 \}$ corresponding to the sparsity patterns of $n \times n$ lower triangular and diagonal matrices, respectively. With a slight abuse of notation, we define the direct sum of index sets $\mathcal{I} \subseteq \mathcal{L}^n$ and $\mathcal{J} \subseteq \mathcal{L}^m$ by

$$\mathcal{I} \oplus \mathcal{J} := \mathcal{I} \cup \{ (i+n, j+n) \mid (i,j) \in \mathcal{L}^m \} \subseteq \mathcal{L}^{n+m}.$$

For each $\mathcal{I} \subseteq \mathcal{L}^n$, define the sets

$$\begin{aligned}
\mathbb{S}^n[\mathcal{I}] &\coloneqq \{\, S \in \mathbb{S}^n \mid S_{ij} = 0 \;\forall\, (i,j) \in \mathcal{L}^n \setminus \mathcal{I} \,\} \\
\mathbb{L}^n[\mathcal{I}] &\coloneqq \{\, L \in \mathbb{L}^n \mid L_{ij} = 0 \;\forall\, (i,j) \notin \mathcal{I} \,\} \\
\mathbb{L}^n_{++}[\mathcal{I}] &\coloneqq \{\, L \in \mathbb{L}^n_{++} \mid L_{ij} = 0 \;\forall\, (i,j) \notin \mathcal{I} \,\}.
\end{aligned}$$

Using this notation, we make the following assumptions about the structure of the parameter constraint set Θ.

Assumption 2. There exist index sets $\mathcal{D}^{n_\Sigma} \subseteq \mathcal{I}_\Sigma \subseteq \mathcal{L}^{n_\Sigma}$ and $\mathcal{D}^{n_\mathcal{A}} \subseteq \mathcal{I}_\mathcal{A} \subseteq \mathcal{L}^{n_\mathcal{A}}$ and differentiable functions $g : \mathbb{R}^{n_\beta} \times \mathbb{S}^{n_\Sigma} \to \mathbb{R}^{n_g}$, $h : \mathbb{R}^{n_\beta} \times \mathbb{S}^{n_\Sigma} \to \mathbb{R}^{n_h}$, $H : \mathbb{R}^{n_\beta} \to \mathbb{S}^{n_\Sigma}$, and $\mathcal{A} : \mathbb{R}^{n_\beta} \times \mathbb{S}^{n_\Sigma} \to \mathbb{S}^{n_\mathcal{A}}[\mathcal{I}_\mathcal{A}]$ such that

$$\Theta = \{\ (\beta, \Sigma) \in \mathbb{R}^{n_\beta} \times \mathbb{S}^{n_\Sigma}[\mathcal{I}_\Sigma] \mid g(\beta, \Sigma) = 0,\ h(\beta, \Sigma) \le 0,\ \Sigma \succeq H(\beta),\ \mathcal{A}(\beta, \Sigma) \succeq 0\ \}.$$

Moreover, $\mathrm{cl}(\Theta_{++}) = \Theta$ where

$$\Theta_{++} \coloneqq \{\ (\beta, \Sigma) \in \mathbb{R}^{n_\beta} \times \mathbb{S}^{n_\Sigma}[\mathcal{I}_\Sigma] \mid g(\beta, \Sigma) = 0,\ h(\beta, \Sigma) \le 0,\ \Sigma \succ H(\beta),\ \mathcal{A}(\beta, \Sigma) \succ 0\ \}.$$

Under Assumption 2, the parameter set Θ contains four constraints: two standard vector equality and inequality constraints and two matrix inequalities. The purpose of the elimination algorithm is to transform the matrix inequalities into vector equalities while introducing as few new variables as possible. The first matrix inequality enforces a lower bound $H(\beta)$ on the sparse symmetric matrix argument Σ. In many cases this lower bound is simply zero or a small diagonal matrix, but we have left the bound general for illustrative purposes. This inequality is in the form used by [60], where variables in the Cholesky factorization

$$\Sigma = L_\Sigma L_\Sigma^\top + H(\beta), \qquad\qquad L_\Sigma \in \mathbb{L}^{n_\Sigma}_{++} \tag{4}$$

are algorithmically eliminated to write Σ in terms of just $H(\beta)$ and a sparse lower triangular matrix $L^{\mathcal{I}_\Sigma} \in \mathbb{L}^{n_\Sigma}_{++}[\mathcal{I}_\Sigma]$. Similarly, in Section 4, we use the sparsity structure in the second matrix inequality $\mathcal{A}(\beta, \Sigma)$ to algorithmically eliminate variables in the squared slack variable transformation

$$\mathcal{A}(\beta, \Sigma) = L_\mathcal{A} L_\mathcal{A}^\top, \qquad\qquad L_\mathcal{A} \in \mathbb{L}^{n_\mathcal{A}}_{++} \tag{5}$$

by writing $L_\mathcal{A}$ in terms of a sparse lower triangular matrix $L^{\mathcal{I}_\mathcal{A}} \in \mathbb{L}^{n_\mathcal{A}}_{++}[\mathcal{I}_\mathcal{A}]$. The second part of Assumption 2 guarantees the existence and uniqueness of these Cholesky factors $(L_\Sigma, L_\mathcal{A})$, helps to avoid divisions by zero during the variable elimination procedure, and allows taking limits.

Remark 3. Assumption 2 rules out direct use of the strict inequality $R_v(\theta) \succ 0$. To satisfy Assumption 1, we use the closed constraint $R_v(\theta) \succeq \delta I_p$ with a small backoff $\delta > 0$.

Remark 4. The index set $\mathcal{I}_\Sigma$ provides enough sparsity in Σ to reduce the problem to $n_\beta + |\mathcal{I}_\Sigma|$ variables (excluding slack terms). This is important when Σ has a particular structure. The most common example is the block diagonal structure $\Sigma = \hat{P}_0 \oplus Q_w \oplus R_v \in \mathbb{S}^{2n+p}[\mathcal{I}_\Sigma]$ where $\mathcal{I}_\Sigma := \mathcal{L}^n \oplus \mathcal{L}^n \oplus \mathcal{L}^p$. We may further restrict Q_w and R_v to take block tridiagonal and diagonal structures, e.g.,

$$Q_w = \begin{bmatrix} Q_{1,1} & Q_{1,2} & & \\ Q_{1,2}^\top & Q_{2,2} & \ddots & \\ & \ddots & \ddots & Q_{\tilde{n}-1,\tilde{n}} \\ & & Q_{\tilde{n}-1,\tilde{n}}^\top & Q_{\tilde{n},\tilde{n}} \end{bmatrix} \qquad R_v = R_1 \oplus \ldots \oplus R_{\tilde{n}}$$

that arise in sequentially interconnected processes such as chemical plants. Adding a $Q_{1,\tilde{n}}$ block can account for an overall recycle loop. Note that if we parameterize the block tridiagonal Q_w via a sparse shaping matrix (i.e., $Q_w = G_w G_w^\top$), then there are more parameters than if the sparsity of Q_w is known *unless* the rank of G_w is known to be low.

Remark 5. The index set $\mathcal{I}_\mathcal{A}$ provides sparsity on the range of $\mathcal{A}$ to eliminate all but $|\mathcal{I}_\mathcal{A}|$ entries of the squared slack term $L_\mathcal{A}$. Typically $\mathcal{A}$ is a block diagonal matrix of eigenvalue constraints, so the dense matrix $L_\mathcal{A}$ is reduced to a sparse matrix $L_\mathcal{A} = L_{\mathcal{A},1} \oplus \ldots \oplus L_{\mathcal{A},\tilde{n}_\mathcal{A}}$.

2.2 Kalman filtering and the log-likelihood

In this subsection, we derive exact and approximate expressions for the negative log-likelihood. For brevity, we drop the dependence on θ where appropriate and write

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + w_k \\ y_k &= Cx_k + Du_k + v_k \\ x_0 &\sim \mathcal{N}(\hat{x}_0, \hat{P}_0) \\ \begin{bmatrix} w_k \\ v_k \end{bmatrix} &\overset{\text{iid}}{\sim} \mathcal{N}(0, S) \end{aligned}$$

and $\mathcal{M} = (A, B, C, D, \hat{x}_0, \hat{P}_0, S)$.

2.2.1 Time-varying Kalman filter formulation

Consider the Kalman filter in innovations form

$$\hat{x}_{k+1} = A\hat{x}_k + Bu_k + \mathcal{K}_k e_k \tag{6a}$$
$$y_k = C\hat{x}_k + Du_k + e_k \tag{6b}$$
$$e_k \sim \mathcal{N}(0, \mathcal{R}_k) \quad \text{(indep.)} \tag{6c}$$

where

$$\hat{P}_{k+1} := A\hat{P}_k A^\top + Q_w - \mathcal{K}_k \mathcal{R}_k \mathcal{K}_k^\top \tag{6d}$$
$$\mathcal{K}_k := (A\hat{P}_k C^\top + S_{wv})\mathcal{R}_k^{-1} \tag{6e}$$
$$\mathcal{R}_k := C\hat{P}_k C^\top + R_v. \tag{6f}$$

Since the e_k in (6) are mutually independent, we have the negative log-likelihood

$$f_N(\theta) \propto \frac{1}{2} \sum_{k=0}^{N-1} \ln\det \mathcal{R}_k(\theta) + |e_k(\theta)|^2_{[\mathcal{R}_k(\theta)]^{-1}}$$

and we can write (2) equivalently as

$$\min_{\theta \in \Theta} \frac{1}{2} \sum_{k=0}^{N-1} \ln\det \mathcal{R}_k(\theta) + |e_k(\theta)|^2_{[\mathcal{R}_k(\theta)]^{-1}} \tag{7}$$

where the $e_k(\theta)$ and $\mathcal{R}_k(\theta)$ are given by the recursion (6). For black-box covariance models

$$\mathcal{M}(\beta, \hat{P}_0 \oplus S) = (A(\beta), B(\beta), C(\beta), D(\beta), \hat{x}_0(\beta), \hat{P}_0, S)$$

the covariance consistency constraint

$$\Theta \coloneqq \{\, (\beta, \Sigma) \in \mathbb{R}^{n_\beta} \times \mathbb{S}^{n+p}[\mathcal{L}^n \oplus \mathcal{L}^{n+p}] \mid \Sigma \succeq \left[\begin{smallmatrix} 0 & \\ & \varepsilon I_p \end{smallmatrix}\right] \,\}$$

suffices, without any vector constraints or general matrix inequalities. These constraints say nothing about stability of the filter (6), so unstable filters may be realized during optimization, producing numerically infinite or undefined values.

2.2.2 Steady-state Kalman filter formulation

In most situations, the state error covariance matrix converges exponentially fast to a steady-state solution $\hat{P}_k \to \hat{P}$, so it suffices to consider the following steady-state filter:

$$\hat{x}_{k+1} = A\hat{x}_k + Bu_k + Ke_k \tag{8a}$$

$$y_k = C\hat{x}_k + Du_k + e_k \tag{8b}$$

$$e_k \overset{\text{iid}}{\sim} \mathcal{N}(0, R_e) \tag{8c}$$

where $K \coloneqq (A\hat{P}C^\top + S_{wv})R_e^{-1}$, $R_e \coloneqq C\hat{P}C^\top + R_v$, and $\hat{P}$ is the unique, stabilizing solution to the discrete algebraic Riccati equation (DARE),

$$\hat{P} = A\hat{P}A^\top + Q_w - (A\hat{P}C^\top + S_{wv}) \times (C\hat{P}C^\top + R_v)^{-1}(A\hat{P}C^\top + S_{wv})^\top. \tag{9}$$

Recall a solution to the DARE (9) is stabilizing if the resulting $A_K \coloneqq A - KC$ is stable.

Convergence of $\hat{P}_k$ to $\hat{P}$ is equivalent to the solution to the DARE (9) being unique and stabilizing. We generally assume such a solution exists, but for completeness, we state the following proposition, adapted from [64, Thm. 18(iii)] (see Appendix A for proof).

Proposition 6. *Assume $R_v \succ 0$ and consider the full rank factorization*

$$\begin{bmatrix} Q_w & S_{wv} \\ S_{wv}^\top & R_v \end{bmatrix} = \begin{bmatrix} \tilde{B} \\ \tilde{D} \end{bmatrix} \begin{bmatrix} \tilde{B}^\top & \tilde{D}^\top \end{bmatrix}$$

Then the following statements are equivalent:

1. *The DARE (9) has a unique, stabilizing solution $\hat{P} \succeq 0$.*
2. *The error covariance converges exponentially fast $\hat{P}_k \to \hat{P}$ for any $\hat{P}_0 \succeq 0$.*
3. *(A, C) is detectable and $(A - FC, \tilde{B} - F\tilde{D})$ is stabilizable for all $F \in \mathbb{R}^{n\times p}$.*

Under the steady-state approximation, the likelihood is

$$\tilde{f}_N(\theta) := \frac{N}{2} \ln\det R_e(\theta) + \frac{1}{2}\sum_{k=0}^{N-1} |e_k(\theta)|^2_{[R_e(\theta)]^{-1}}$$

and we can approximate solutions to (2) by solving

$$\min_{\theta\in\Theta} \frac{N}{2} \ln\det R_e(\theta) + \frac{1}{2}\sum_{k=0}^{N-1} |e_k(\theta)|^2_{[R_e(\theta)]^{-1}} \tag{10}$$

where the $e_k(\theta)$ are given by the recursion (8) and $R_e(\theta)$ is found by solving the DARE (9).

While $R_e(\theta)$ and $K(\theta)$ could be defined via $\hat{P}(\theta)$, taken as the function that returns solutions to the DARE (9) and therefore enforcing filters stability, it is more convenient to directly parameterize these matrices. In fact, it is equivalent to consider the model structure

$$\mathcal{M} = \left(A, B, C, D, \hat{x}_0, 0, \begin{bmatrix} KR_eK^\top & KR_e \\ R_eK^\top & R_e \end{bmatrix}\right)$$

where $K : \Theta \to \mathbb{R}^{n\times p}$ and $R_e : \Theta \to \mathbb{S}^p_{++}$ are now given functions that are not explicitly related to the DARE (9). We write this model parameterization as

$$\mathcal{M}_{\mathrm{KF}} = (A, B, C, D, \hat{x}_0, K, R_e) \tag{11}$$

using the subscript KF to denote that $\mathcal{M}_{\mathrm{KF}}$ represents a Kalman filter in innovation form (11). Since the model structure (11) no longer enforces filter stability, the following stability assumption is required.

Assumption 7. Given the model structure (11), we have $\rho(A_K(\theta)) < 1$ for all $\theta \in \Theta$.

Since ρ is continuous but not differentiable, Assumption 7 cannot be directly implemented in a form satisfying Assumption 2, even with a backoff term to make the inequality non-strict, i.e., $\rho(A_K(\theta)) \leq 1 - \delta$ where $\delta > 0$. We deal with enforcing the stability constraint in Section 3.

2.2.3 Minimum determinant formulation

Suppose, in the model structure (11), that R_e is parameterized fully, and separately from the other terms, i.e.,

$$\mathcal{M}_{\mathrm{KF}}(\beta, \tilde{\Sigma} \oplus R_e) = \left(A(\beta,\tilde{\Sigma}), B(\beta,\tilde{\Sigma}), C(\beta,\tilde{\Sigma}), D(\beta,\tilde{\Sigma}), \hat{x}_0(\beta,\tilde{\Sigma}), K(\beta,\tilde{\Sigma}), R_e\right).$$

Moreover, assume R_e is constrained separately as well, i.e.,

$$\Theta = \{\ (\beta, \tilde{\Sigma} \oplus R_e) \in \mathbb{R}^{n_\beta} \times \mathbb{S}^{n_\Sigma}[\mathcal{I}_\Sigma] \mid \tilde{g}(\beta, \tilde{\Sigma}) = 0,\ \tilde{h}(\beta, \tilde{\Sigma}) \leq 0,\ \tilde{\Sigma} \succeq \tilde{H}(\beta), \\ R_e \succeq \varepsilon I_p,\ \tilde{\mathcal{A}}(\beta, \tilde{\Sigma}) \succeq 0\ \}$$

for some sufficiently small $\varepsilon > 0$, differentiable functions $(\tilde{g}, \tilde{h}, \tilde{H}, \tilde{\mathcal{A}})$, and index set $\mathcal{I}_\Sigma :=$ $\mathcal{I}_{\tilde{\Sigma}} \oplus \mathcal{L}^p$ where $\mathcal{D}^{\tilde{n}_\Sigma} \subseteq \mathcal{I}_{\tilde{\Sigma}} \subseteq \mathcal{L}^{\tilde{n}_\Sigma}$ and $n_{\tilde{\Sigma}} := n_\Sigma - p$. Then we can always solve (10) stagewise, first in R_e, and then in the remaining variables $(\beta, \tilde{\Sigma})$. Solving the inner problem gives the solution

$$\hat{R}_e(\beta, \tilde{\Sigma}) := \frac{1}{N} \sum_{k=0}^{N-1} e_k(\beta, \tilde{\Sigma})[e_k(\beta, \tilde{\Sigma})]^\top$$

where we use the fact that e_k is only dependent on $(\beta, \tilde{\Sigma})$, and we assume $\hat{R}_e(\beta, \tilde{\Sigma}) \succeq \varepsilon I_p$ for all $(\beta, \tilde{\Sigma}) \in \tilde{\Theta}$ where

$$\tilde{\Theta} := \{\ (\beta, \tilde{\Sigma}) \in \mathbb{R}^{n_\beta} \times \mathbb{S}^{\tilde{n}_\Sigma}[\mathcal{I}_{\tilde{\Sigma}}] \mid \tilde{g}(\beta, \tilde{\Sigma}) = 0,\ \tilde{h}(\beta, \tilde{\Sigma}) \leq 0,\ \tilde{\Sigma} \succeq \tilde{H}(\beta),\ \tilde{\mathcal{A}}(\beta, \tilde{\Sigma}) \succeq 0\ \}.$$

The outer problem can be written

$$\min_{(\beta, \tilde{\Sigma}) \in \tilde{\Theta}} \det \hat{R}_e(\beta, \tilde{\Sigma}). \tag{12}$$

The problem (12) is of relevance for avoiding the nonlinear SDP formulation of (10), both in the early ML identification literature [4–6] and in recent works [15–17]. None of these works consider filter stability constraints. To the best of our knowledge, only [18] consider the ML problem (7) with stability constraints, but they consider open-loop stability (i.e., $\rho(A) < 1$) and use the EM algorithm. To satisfy Assumption 7, we must consider filter stability constraints (i.e., $\rho(A_K) < 1$).

Remark 8. For real-world data, $\det \hat{R}_e(\beta, \tilde{\Sigma}) = 0$ is not attainable because that would imply some direction of y_k were perfectly modeled. Therefore, a constant $\varepsilon > 0$ exists such that the lower bound $\hat{R}_e(\beta, \tilde{\Sigma}) \succeq \varepsilon I_p$ is satisfied for all $(\beta, \tilde{\Sigma}) \in \tilde{\Theta}$. Moreover, we need not explicitly choose $\varepsilon > 0$ to satisfy the constraint $R_e \succ 0$ of Assumption 1.

3 Eigenvalue constraints

In this section, we describe stability and other eigenvalue constraints conforming to Assumption 2. Notably, we consider the smoothed spectral radius and abscissa formulations of [57, 58], the linear matrix inequality (LMI) region approach of [59], and a novel barrier function method that combines the two approaches. While we consider the matrix A in this section, constraints can be added to any square matrix of interest, such as A, A_K, or some submatrix thereof.

3.1 Spectral radius and abscissa bounds

Recall that, for any $Q \in \mathbb{S}^n_{++}$, a matrix A is Schur stable if and only if there exists $P \in \mathbb{S}^n_{++}$ such that

$$P - APA^\top = Q. \tag{13}$$

Likewise, for any $Q \in \mathbb{S}^n_{++}$, a matrix A is Hurwitz stable if and only if there eixsts $P \in \mathbb{S}^n_{++}$ such that

$$AP + PA^\top = -Q. \tag{14}$$

Moreover, solutions to (13) and (14), when they exist, are uniquely given by $P_d(A,Q) := \sum_{k=0}^\infty A^k Q (A^\top)^k$ and $P_c(A,Q) := \int_{t=0}^\infty e^{At} Q e^{A^\top t} dt$, respectively. It is a well-known observation that discrete- and continuous-time stability of A are equivalent to finiteness of the matrices $P_d(A,Q)$ and $P_c(A,Q)$, respectively.

Lemma 9 ([57, Lem. 5.1] and [58, Lem. 2.1])**.** *Let $Q \in \mathbb{S}^n_{++}$ and $\|\cdot\| : \mathbb{R}^{n\times n} \to \mathbb{R}_{\geq 0}$ be a submultiplicative norm. A matrix $A \in \mathbb{R}^{n\times n}$ is Schur stable (Hurwitz stable) if and only if $\|P_d(A,Q)\|$ ($\|P_c(A,Q)\|$) is finite.*

Inspired by this observation, [57, 58] impose an upper bound on the norm of P satisfying either (13) or (14).

3.1.1 Smoothed spectral radius

Let $W, V \in \mathbb{S}^n_{++}$ and consider the function $\phi_d(A,s) := \mathrm{tr}(VP_d(A/s, W))$. In [57], the implicit function theorem is used to show the existence of a *smoothed spectral radius* $\rho_\varepsilon(A)$ satisfying

$$\phi_d(A, \rho_\varepsilon(A)) = \varepsilon^{-1}. \tag{15}$$

Properties of $\rho_\varepsilon(A)$ are reiterated in the following theorem.

Theorem 10 ([57, Thms. 5.4, 5.6])**.** *There exists a function $\rho_{(\cdot)}(\cdot) : \mathbb{R}_{>0} \times \mathbb{R}^{n\times n} \to \mathbb{R}$ such that, for each $A \neq 0$ and $\varepsilon > 0$, $\rho_\varepsilon(A)$ uniquely solves* (15)*, and $\rho_\varepsilon(0) = 0$. Moreover, $\rho_{(\cdot)}(\cdot)$ has the following properties:*

1. *$\rho_{(\cdot)}(\cdot)$ is analytic on $\mathbb{R}_{>0} \times \mathbb{R}^{n\times n} \setminus \{0\}$ and continuous on $\mathbb{R}_{>0} \times \{0\}$;*
2. *$\rho_\varepsilon(A) > \rho(A)$ for all $A \neq 0$ and $\varepsilon > 0$;*
3. *$\rho_\varepsilon(A) \searrow \rho(A)$ as $\varepsilon \searrow 0$ for all $A \in \mathbb{R}^{n\times n}$;*
4. *for each $\varepsilon, s > 0$, $\rho_\varepsilon(A) \leq s$ if and only if there exists $P \succeq 0$ such that $s^2 P - APA^\top = W$ and $\mathrm{tr}(VP) \leq \varepsilon^{-1}$.*

The first property of Theorem 10 establishes the smoothness of ρ_ε. The second and fourth properties of Theorem 10 let us construct constraint sets satisfying Assumptions 2 and 7. Finally, the third property demonstrates that ρ_ε is an approximation of ρ in the sense that ε can be made arbitrarily small to keep $\rho_\varepsilon(A) - \rho(A)$ arbitrarily small.

3.1.2 Smoothed spectral abscissa

Let $W, V \in \mathbb{S}^n_{++}$ and consider the function $\phi_c(A, s) \coloneqq \operatorname{tr}(VP_c(A - sI, W))$. In [58], the implicit function theorem is used to show the existence of a *smoothed spectral abscissa* $\alpha_\varepsilon(A)$ satisfying

$$\phi_c(A, \alpha_\varepsilon(A)) = \varepsilon^{-1}. \tag{16}$$

Properties of $\alpha_\varepsilon(A)$ are reiterated in the following theorem.

Theorem 11 ([58, Thms. 2.5, 2.6])**.** *There exists a function $\alpha_{(\cdot)}(\cdot) : \mathbb{R}_{>0} \times \mathbb{R}^{n\times n} \to \mathbb{R}$ such that, for each $A \in \mathbb{R}^{n\times n}$ and $\varepsilon > 0$, $\alpha_\varepsilon(A)$ uniquely solves* (15)*. Moreover, $\alpha_{(\cdot)}(\cdot)$ has the following properties:*

1. *$\alpha_{(\cdot)}(\cdot)$ is analytic on $\mathbb{R}_{>0} \times \mathbb{R}^{n\times n}$;*
2. *$\alpha_\varepsilon(A) > \alpha(A)$ for all $A \in \mathbb{R}^{n\times n}$ and $\varepsilon > 0$;*
3. *$\alpha_\varepsilon(A) \searrow \alpha(A)$ as $\varepsilon \searrow 0$ for all $A \in \mathbb{R}^{n\times n}$; and*
4. *for each $\varepsilon, s > 0$, $\alpha_\varepsilon(A) \le s$ if and only if there exists $P \succeq 0$ such that $(A - sI)P + P(A - sI)^\top = -W$ and $\operatorname{tr}(VP) \le \varepsilon^{-1}$.*

Similarly to the properties of Theorem 10, the first property of Theorem 11 gives α_ε its smoothness property, the second and fourth properties allow us to construct constraints that satisfy Assumption 2, and the third property demonstrates that α_ε is an approximation of α.

3.2 $\mathcal{D}$-stability constraints

We may wish to place the eigenvalues in a specified region. In [59] the problem of placing eigenvalues in any convex, open region of the complex plane is posed in terms of solving a linear matrix inequality (LMI). These so-called "LMI regions" are defined as follows.

Definition 12. A subset $\mathcal{D} \subseteq \mathbb{C}$ that takes the form

$$\mathcal{D} = \{\, z \in \mathbb{C} \mid f_{\mathcal{D}}(z) \succ 0 \,\}$$

is called an *LMI region* with the *characteristic function* $f_{\mathcal{D}} : \mathbb{C} \to \mathbb{C}^{m\times m}$, defined as

$$f_{\mathcal{D}}(z) \coloneqq M_0 + M_1 z + M_1^\top \overline{z}$$

and *generating matrices* $(M_0, M_1) \in \mathbb{S}^n \times \mathbb{R}^{n\times n}$.

The following lemma defines the four basic LMI regions: shifted half-planes, circles centered on the real axis, conic sections, and horizontal bands.

Lemma 13. *For each $s, x_0 \in \mathbb{R}$, the subsets*

$$\begin{aligned}
\mathcal{D}_1 &\coloneqq \{\, z \in \mathbb{C} \mid \mathrm{Re}(z) < s \,\} \\
\mathcal{D}_2 &\coloneqq \{\, z \in \mathbb{C} \mid |z - x_0| < s \,\} \\
\mathcal{D}_3 &\coloneqq \{\, z \in \mathbb{C} \mid |\mathrm{Im}(z)| < s(\mathrm{Re}(z) - x_0) \,\} \\
\mathcal{D}_4 &\coloneqq \{\, z \in \mathbb{C} \mid |\mathrm{Im}(z)| < s \,\}
\end{aligned}$$

are LMI regions with characteristic functions

$$\begin{aligned}
f_{\mathcal{D}_1}(z) &\coloneqq 2s - z - \overline{z} \\
f_{\mathcal{D}_2}(z) &\coloneqq \begin{bmatrix} s & -x_0 \\ -x_0 & s \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} z + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \overline{z} \\
f_{\mathcal{D}_3}(z) &\coloneqq -2sx_0 I_2 + \begin{bmatrix} s & 1 \\ -1 & s \end{bmatrix} z + \begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix} \overline{z} \\
f_{\mathcal{D}_4}(z) &\coloneqq -2s I_2 + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} z + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \overline{z}.
\end{aligned}$$

Proof. The first identity follows from the formula $2\mathrm{Re}(z) = z + \overline{z}$. For the second identity, we have $f_{\mathcal{D}_2}(z) = \left[\begin{smallmatrix} s & z - x_0 \\ \overline{z} - x_0 & s \end{smallmatrix}\right] \succ 0$ if and only if $s > 0$ and $s^2 > |z - x_0|^2$, or equivalently, $|z - x_0| < s$. For the third identity, we have $f_{\mathcal{D}_3}(z) = \left[\begin{smallmatrix} 2s(\mathrm{Re}(z) - x_0) & 2\imath\mathrm{Im}(z) \\ -2\imath\mathrm{Im}(z) & 2s(\mathrm{Re}(z) - x_0) \end{smallmatrix}\right] \succ 0$ if and only if $2s(\mathrm{Re}(z) - x_0) > 0$ and $4s^2(\mathrm{Re}(z) - x_0)^2 > 4|\mathrm{Im}(z)|^2$, or equivalently, $|\mathrm{Im}(z)| < s(\mathrm{Re}(z) - x_0)$. For the fourth identity, we have $f_{\mathcal{D}_4}(z) = \left[\begin{smallmatrix} 2s & 2\imath\mathrm{Im}(z) \\ -2\imath\mathrm{Im}(z) & 2s \end{smallmatrix}\right] \succ 0$ if and only if $2s > 0$ and $4s^2 > 4|\mathrm{Im}(z)|^2$, or equivalently, $|\mathrm{Im}(z)| < s$. $\square$

Remark 14. For continuous-time systems, $\mathcal{D}_1$ corresponds to a minimum decay rate of $s > 0$, $\mathcal{D}_3$ corresponds to a minimum damping ratio $-\cos(\theta)$, and $\mathcal{D}_2 \cap \mathcal{D}_3$ implies to a maximum natural frequency $r \sin(\theta)$, where $\theta = \tan^{-1}(s)$ and $s < 0$ [59]. For discrete-time systems, $\mathcal{D}_2$ corresponds to a minimum decay rate of $-\ln r$, and $\mathcal{D}_2 \cap \mathcal{D}_3$ implies a minimum damping ratio $-\cos(\tan^{-1}(\theta/\ln r))$ and maximum natural frequency $(\ln(r)^2 + \theta^2)/\Delta$, where $\theta = \tan^{-1}(s)$, $s > 0$, and Δ is the sample time.

Remark 15. An LMI region $\mathcal{D}$ is convex, open, and symmetric about the imaginary axis. The intersection of two LMI regions $\mathcal{D} \coloneqq \mathcal{D}_1 \cap \mathcal{D}_2$ is an LMI region with the characteristic function $f_{\mathcal{D}}(z) = f_{\mathcal{D}_1}(z) \oplus f_{\mathcal{D}_2}(z)$. An LMI region $\mathcal{D}$ with characteristic function $f_{\mathcal{D}}$ also has characteristic function $M f_{\mathcal{D}}(\cdot) M^\top$ for any nonsingular $M \in \mathbb{R}^{m \times m}$. By this property, we can construct any convex polyhedron that is symmetric about the real axis by intersecting left and right half-planes, horizontal strips, and conic sections. Moreover, since any convex region can be approximated, to any desired accuracy, by a convex polyhedron, the set of LMI regions is dense in the space of convex subsets of $\mathbb{C}$ that are symmetric about the real axis. For an in-depth discussion of LMI region geometry and other properties, see [65].

Throughout, assume the LMI region $\mathcal{D}$ is nonempty, not equal to $\mathbb{C}$, and its characteristic function $f_{\mathcal{D}}$ and generating matrices (M_0, M_1) are fixed. We seek LMI conditions under which the eigenvalues of $A \in \mathbb{R}^{n \times n}$ lie in $\mathcal{D}$ or $\mathrm{cl}(\mathcal{D})$.

3.2.1 $\mathcal{D}$-stability

In the following definition, we generalize the notion of asymptotic stability to include pole placement within a given LMI region $\mathcal{D}$.

Definition 16. We say the matrix $A \in \mathbb{R}^{n\times n}$ is *$\mathcal{D}$-stable* if $\lambda(A) \subset \mathcal{D}$.

In [59], $\mathcal{D}$-stability of a matrix $A \in \mathbb{R}^{n\times n}$ is shown to be equivalent to the strict feasibility of the following system of matrix inequalities:

$$M_{\mathcal{D}}(A,P) \succ 0, \qquad P \succ 0 \tag{17}$$

where $M_{\mathcal{D}} : \mathbb{R}^{n\times n} \times \mathbb{S}^n \to \mathbb{S}^{nm}$ is defined by

$$M_{\mathcal{D}}(A,P) := M_0 \otimes P + M_1 \otimes (AP) + M_1^\top \otimes (AP)^\top. \tag{18}$$

This fact is restated in the following theorem.

Theorem 17 ([59, Thm. 2.2])**.** *The matrix $A \in \mathbb{R}^{n\times n}$ is $\mathcal{D}$-stable if and only if* (17) *holds for some $P \in \mathbb{S}^n$.*

3.2.2 Marginal $\mathcal{D}$-stability

The drawback of Theorem 17 is strictness of the matrix inequalities (17), making them inadmissible in constraint sets satisfying Assumption 2. Suppose we relax the first inequality,

$$M_{\mathcal{D}}(A,P) \succeq 0, \qquad P \succ 0. \tag{19}$$

Since $M_{\mathcal{D}}(A,P)$ is linear in P, feasibility of (19) is equivalent to feasibility of

$$M_{\mathcal{D}}(A,P) \succeq 0, \qquad P \succeq P_0 \tag{20}$$

for some fixed $P_0 \in \mathbb{S}^n_{++}$.[1] Therefore the system (19) is admissible to constraint sets satisfying Assumption 2 with minor alteration.

An attempt was made in [62, Thm. 1] to characterize the class of matrices $A \in \mathbb{R}^{n\times n}$ for which (19), but this theorem does not correctly treat eigenvalues on the boundary $\partial\mathcal{D}$. We restate [62, Thm. 1] below as a conjecture and disprove it with a simple counterexample.

Conjecture 18 ([62, Thm. 1])**.** *The matrix $A \in \mathbb{R}^{n\times n}$ satisfies $\lambda(A) \subset \mathrm{cl}(\mathcal{D})$ if and only if* (19) *holds for some $P \in \mathbb{S}^n$.*

Counterexample. Let $\mathcal{D}$ be the left half-plane, consider the Jordan block $A = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$, and suppose $P = \left[\begin{smallmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{smallmatrix}\right] \in \mathbb{S}^2$ such that (19) holds. Then $\lambda(A) \subset \mathrm{cl}(\mathcal{D})$ and

$$0 \preceq M_{\mathcal{D}}(A,P) = -\begin{bmatrix} 2p_{12} & p_{22} \\ p_{22} & 0 \end{bmatrix}$$

which implies $p_{12} = p_{22} = 0$, a contradiction of (19). ※

[1] For any $P_0 \succ 0$ and P satisfying (19), define the scaling factor $\gamma := \|P_0\|_2 \|P^{-1}\|_2$ and a rescaled solution $P^* := \gamma P$. Then $P^* \succeq P_0$ and $M_{\mathcal{D}}(A,P^*) = \gamma M_{\mathcal{D}}(A,P) \succeq 0$.

The correction to Conjecture 18 requires a more careful treatment of eigenvalues lying on the the LMI region's boundary $\partial\mathcal{D}$. Recall that a matrix A is marginally stable (in continuous- or discrete-time) if all its eigenvalues lie in the closure of the stability region (left half-plane or unit disc) *and* the non-simple eigenvalues lie strictly in the interior of that region. Replacing the stability region with $\mathcal{D}$, we propose the following definition of *marginal $\mathcal{D}$-stability*.

Definition 19. We say a matrix $A \in \mathbb{R}^{n\times n}$ is *marginally $\mathcal{D}$-stable* if $\lambda(A) \subseteq \mathrm{cl}(\mathcal{D})$ and $\lambda \in \mathcal{D}$ for all non-simple eigenvalues $\lambda \in \lambda(A)$.

In the following proposition, we show $\mathcal{D}$-stability of $A \in \mathbb{R}^{n\times n}$ is equivalent to feasibility of (19) (see Appendix B for proof).

Proposition 20. *The matrix $A \in \mathbb{R}^{n\times n}$ is marginally $\mathcal{D}$-stable if and only if* (19) *holds for some $P \in \mathbb{S}^n$.*

3.2.3 $\mathcal{D}$-stability barrier functions

Thus far, we have proposed LMI equivalences without regard to the topology of the set of $\mathcal{D}$-stable and marginally $\mathcal{D}$-stable matrices

$$\begin{aligned}\mathbb{A}^n_{\mathcal{D}} &\coloneqq \{\, A \in \mathbb{R}^{n\times n} \mid A \text{ is } \mathcal{D}\text{-stable} \,\}\\ \tilde{\mathbb{A}}^n_{\mathcal{D}} &\coloneqq \{\, A \in \mathbb{R}^{n\times n} \mid A \text{ is marginally } \mathcal{D}\text{-stable} \,\}.\end{aligned}$$

The following proposition characterizes the topology of $\mathbb{A}^n_{\mathcal{D}}$ and $\mathbb{A}^n_{\mathcal{D}}$ (see Appendix C for proof).

Proposition 21. *(a) $\mathbb{A}^n_{\mathcal{D}}$ is open.*

(b) $\tilde{\mathbb{A}}^n_{\mathcal{D}}$ is not open if (i) $n \geq 2$ or (ii) $\partial\mathcal{D} \cap \mathbb{R}$ is nonempty.

(c) $\tilde{\mathbb{A}}^n_{\mathcal{D}}$ is not closed if (i) $n \geq 4$ or (ii) $\partial\mathcal{D} \cap \mathbb{R}$ is nonempty and $n \geq 2$.

(d) $\mathrm{cl}(\mathbb{A}^n_{\mathcal{D}}) = \{\, A \in \mathbb{R}^{n\times n} \mid \lambda(A) \subset \mathrm{cl}(\mathcal{D}) \,\}$.

Proposition 21 reveals a weakness of the marginal $\mathcal{D}$-stability constraints (19) and (20). Since $\tilde{\mathbb{A}}^n_{\mathcal{D}}$ is not closed and $P \in \mathbb{S}^n_+$ is neither bounded nor regularized, the optimizer may fail to converge in P as it seeks an A that is not a limit point in $\tilde{\mathbb{A}}^n_{\mathcal{D}}$. While the optimizer can approach A in this case, P grows unbounded along the path of iterates, and the optimizer itself does not converge.

Motivated by the smoothed spectral radius and abscissa, we upper bound $\mathrm{tr}(VP)$ for some $V \in \mathbb{S}^n_{++}$. This still allows rescaling of P, however, so we also lower bound $M_{\mathcal{D}}(A,P)$,

$$M_{\mathcal{D}}(A,P) \succeq M, \qquad P \succeq 0, \qquad \mathrm{tr}(VP) \leq \varepsilon^{-1} \tag{21}$$

and assume $M \in \mathbb{S}^{nm}_+$ is chosen in a way that implies (17). Consider the parameterized linear SDP,

$$\phi_{\mathcal{D}}(A) \coloneqq \inf_{P \in \mathbb{S}^n_+} \mathrm{tr}(VP) \text{ subject to } M_{\mathcal{D}}(A,P) \succeq M. \tag{22}$$

The optimal value function $\phi_{\mathcal{D}} : \mathbb{R}^{n\times n} \to \mathbb{R}_{\geq 0} \cup \{\, \infty \,\}$ is a barrier function for the constraint $A \in \mathbb{A}^n_{\mathcal{D}}$. Proposition 22 establishes properties of $\phi_{\mathcal{D}}$ and its ε^{-1}-sublevel sets (see Appendix D for proof).

Proposition 22. *Let $V \in \mathbb{S}^n_{++}$ and $M \in \mathbb{S}^n_{+}$ such that $M_{\mathcal{D}}(A,P) \succeq M$ implies $M_{\mathcal{D}}(A,P) \succ 0$. Then*

(a) $\phi_{\mathcal{D}}$ is continuous on $\mathbb{A}_{\mathcal{D}}$;

(b) for each $\varepsilon > 0$, the ε^{-1}-sublevel set of $\phi_{\mathcal{D}}$,

$$\mathbb{A}^n_{\mathcal{D}}(\varepsilon) := \{\, A \in \mathbb{R}^{n\times n} \mid \phi_{\mathcal{D}}(A) \leq \varepsilon^{-1} \,\} = \{\, A \in \mathbb{R}^{n\times n} \mid \exists P \succeq 0 : (21) \text{ holds} \,\} \tag{23}$$

is closed; and

(c) $\mathbb{A}^n_{\mathcal{D}}(\varepsilon) \nearrow \mathbb{A}^n_{\mathcal{D}}$ as $\varepsilon \searrow 0$.

Remark 23. It is sufficient, but not necessary, to choose a positive definite lower bound $M \succ 0$.

Remark 24. To reconstruct the s-sublevel sets of the smoothed spectral radius via Proposition 22, we set $M = sW \oplus 0_{n\times n}$ for any $W, V \succ 0$ and $s > 0$ and apply the Schur complement lemma to $M_{\mathcal{D}_2}(A,P)/s - M/s$, where $\mathcal{D}_2$ is the circle defined in Lemma 13 with $x_0 = 0$, and $M_{\mathcal{D}_2}$ is defined by the generating matrices used in Lemma 13. Then the ε^{-1}-sublevel set of $\phi_{\mathcal{D}_2}$ equals the s-sublevel set of ρ_ε.

Remark 25. The ε^{-1}-sublevel sets of $\phi_{\mathcal{D}_1}$ equal the s-sublevel sets of α_ε, where $\mathcal{D}_1$ is the shifted half-plane defined in Lemma 13, and $M = W$ for any $W, V \succ 0$.

3.3 Discussion

We conclude this section with a discussion on implementing eigenvalue constraints. Discrete LTI models with eigenvalues having positive real parts have a one-to-one correspondence with continuous LTI models [66]. As such, it is important to satisfy not only the filter stability constraint $\rho(A_K) < 1$ (and, if desired, open-loop stability constraint $\rho(A) < 1$), but it is also desirable to satisfy differentiability constraints $\alpha(-A) < 0$ and $\alpha(-A_K) < 0$. In practice, we find $\rho_\varepsilon(A_K) \leq 1 - \delta$ and $\alpha_\varepsilon(-A_K) \leq 0$ to be important constraints in ML identification of linear augmented disturbance models. Without the former constraint, the optimizer will frequently pick unstable filters that evaluate to infinite objective values, and without the latter constraint, the optimal filter equations degenerate to aphysical solutions that incorrectly estimate $u \mapsto y$ gain matrices. Last, we refer the reader to [67] for other examples of matrix inequalities (e.g., detectability, minimum phase) that may be useful in system identification.

4 Cholesky reparameterizations

In this section, we seek to transform the ML problems (2), (7), and (10) from nonlinear SDPs to standard NLPs, while introducing as few new variables as possible. The main idea is to combine a squared slack variable substitution (4) and (5) with an elimination scheme that adds only $|\mathcal{I}_\mathcal{A}|$ variables to the optimization problem.

For this section, we define the following notation. For each $\mathcal{I} \subseteq \mathcal{L}^n$, let $\pi^L_\mathcal{I} : \mathbb{R}^{n\times n} \to \mathbb{L}^n[\mathcal{I}]$ and $\pi_\mathcal{I} : \mathbb{R}^{n\times n} \to \mathbb{S}^n[\mathcal{I}]$ denote the orthogonal projections (in the Frobenius norm) from $\mathbb{R}^{n\times n}$ onto the subspaces $\mathbb{L}^n[\mathcal{I}]$ and $\mathbb{S}^n[\mathcal{I}]$, respectively. Let $\mathrm{chol} : \mathbb{S}^n_{++} \to \mathbb{L}^n_{++}$ denote the invertible function that maps a positive definite matrix to its Cholesky factor.

4.1 Burer-Monteiro-Zhang method

In the Burer-Monteiro-Zhang (BMZ) method [60], an invertible map is constructed between the interior of

$$\Theta_{\mathrm{BMZ}} := \{ (\beta, \Sigma) \in \mathbb{R}^{n_\beta} \times \mathbb{S}^{n_\Sigma}[\mathcal{I}_\Sigma] \mid \Sigma \succeq H(\beta) \} \tag{24}$$

and

$$\Theta_{\mathrm{Chol}} := \mathbb{R}^{n_\beta} \times \mathbb{L}^{n_\Sigma}_{++}[\mathcal{I}_\Sigma] \tag{25}$$

where $H : \mathbb{R}^{n_\beta} \to \mathbb{S}^{n_\Sigma}$ and $\mathcal{I}_\Sigma$ is some index set satisfying $\mathcal{D}^{n_\Sigma} \subseteq \mathcal{I}_\Sigma \subseteq \mathcal{L}^{n_\Sigma}$. The set Θ_{BMZ} satisfies Assumption 2 so long as H is differentiable since

$$\mathrm{int}(\Theta_{\mathrm{BMZ}}) = \{ (\beta, \Sigma) \in \mathbb{R}^{n_\beta} \times \mathbb{S}^{n_\Sigma}[\mathcal{I}_\Sigma] \mid \Sigma \succ H(\beta) \} .$$

Recall $\Sigma \succ H(\beta)$ if and only if $\Sigma = L_\Sigma L_\Sigma^\top + H(\beta)$ for some $L_\Sigma \in \mathbb{L}^{n_\Sigma}_{++}$. With $\mathcal{J}_\Sigma := \mathcal{L}^{n_\Sigma} \setminus \mathcal{I}_\Sigma$, we split L_Σ into the sum of $L^{\mathcal{I}_\Sigma} \in \mathbb{L}^{n_\Sigma}_{++}[\mathcal{I}_\Sigma]$ and $L^{\mathcal{J}_\Sigma} \in \mathbb{L}^{n_\Sigma}[\mathcal{J}_\Sigma]$,

$$\Sigma = (L^{\mathcal{I}_\Sigma} + L^{\mathcal{J}_\Sigma})(L^{\mathcal{I}_\Sigma} + L^{\mathcal{J}_\Sigma})^\top + H. \tag{26}$$

But $\Sigma \in \mathbb{S}^{n_\Sigma}[\mathcal{I}_\Sigma]$, so we can take

$$\pi_{\mathcal{J}_\Sigma}[(L^{\mathcal{I}_\Sigma} + L^{\mathcal{J}_\Sigma})(L^{\mathcal{I}_\Sigma} + L^{\mathcal{J}_\Sigma})^\top + H] = 0 \tag{27}$$

to produce $|\mathcal{J}_\Sigma|$ equations to eliminate the $|\mathcal{J}_\Sigma|$ variables of $L^{\mathcal{J}_\Sigma}$. For each $(H, L^{\mathcal{I}_\Sigma}) \in \mathbb{S}^{n_\Sigma} \times \mathbb{L}^{n_\Sigma}[\mathcal{I}_\Sigma]$, define

$$L^{\mathcal{J}_\Sigma}_{ij} := -\frac{1}{L^{\mathcal{I}_\Sigma}_{jj}} \left(H_{ij} + \sum_{k=1}^{j-1} (L^{\mathcal{I}_\Sigma}_{ik} + L^{\mathcal{J}_\Sigma}_{ik})(L^{\mathcal{I}_\Sigma}_{jk} + L^{\mathcal{J}_\Sigma}_{jk}) \right) \tag{28}$$

for each $(i,j) \in \mathcal{J}_\Sigma$, in a top-to-bottom and left-to-right order. So long as we never divide by zero, each $L^{\mathcal{J}_\Sigma}$ is fully defined by H and $L^{\mathcal{I}_\Sigma}$ via (28), and we have the following lemma.

Lemma 26 ([60, Lem. 1])**.** *For each $(\beta, L^{\mathcal{I}_\Sigma}) \in \mathbb{R}^{n_\beta} \times \mathbb{L}^{n_\Sigma}[\mathcal{I}_\Sigma]$ such that $L^{\mathcal{I}_\Sigma}_{ii} \neq 0$ for each $i \in \mathbb{I}_{1:n_\Sigma}$, there is a unique $L^{\mathcal{J}_\Sigma} \in \mathbb{L}^{n_\Sigma}[\mathcal{J}_\Sigma]$ satisfying* (27)*.*

Let $L^{\mathcal{J}_\Sigma} : \mathbb{R}^{n_\beta} \times \mathbb{L}^{n_\Sigma}_{++}[\mathcal{I}_\Sigma] \to \mathbb{L}^{n_\Sigma}[\mathcal{J}_\Sigma]$ be the function that maps each $(\beta, L^{\mathcal{I}_\Sigma}) \in \Theta_{\text{Chol}}$ to the $L^{\mathcal{J}_\Sigma} \in \mathbb{L}^{n_\Sigma}[\mathcal{J}_\Sigma]$ defined by (28) with $H = H(\beta)$. Similarly, let $\Sigma : \mathbb{R}^{n_\beta} \times \mathbb{L}^{n_\Sigma}_{++}[\mathcal{I}_\Sigma] \to \mathbb{S}^{n_\Sigma}[\mathcal{I}_\Sigma]$ be the map defined by (26) for each $(\beta, L^{\mathcal{I}_\Sigma}) \in \Theta_{\text{Chol}}$ with $H = H(\beta)$ and $L^{\mathcal{J}_\Sigma} = L^{\mathcal{J}_\Sigma}(\beta, L^{\mathcal{I}_\Sigma})$. Finally, we let

$$\Psi(\beta, L^{\mathcal{I}_\Sigma}) := \big(\beta, \Sigma(\beta, L^{\mathcal{I}_\Sigma})\big) \tag{29}$$

which has the inverse

$$\Psi^{-1}(\beta, \Sigma) := \big(\beta, \pi^L_{\mathcal{I}_\Sigma}[\text{chol}(\Sigma - H(\beta))]\big) \tag{30}$$

and we have the following lemma.

Lemma 27 ([60, Lem. 2])**.** *The function* Ψ *defined by* (29) *is a bijection between* Θ_{Chol} *and* $\text{int}(\Theta_{\text{BMZ}})$*.*

Differentiability of Ψ and Ψ^{-1} follow from differentiability of H and the algorithm (28). In fact, these functions are as smooth as H is, so if H is analytic, so are Ψ and Ψ^{-1}. More importantly, the bijection Ψ allows us to transform the minimum of a continuous function in Θ_{BMZ} to an infimum of a function in Θ_{Chol}, given by the following theorem.

Theorem 28 ([60, Thm. 1])**.** *For any continuous function* $f : \mathbb{R}^{n_\beta} \times \mathbb{S}^{n_\Sigma} \to \mathbb{R}$ *that attains a minimum in* Θ_{BMZ}*,*

$$\min_{(\beta,\Sigma)\in\Theta_{\text{BMZ}}} f(\beta, \Sigma) = \inf_{(\beta, L^{\mathcal{I}_\Sigma})\in\Theta_{\text{Chol}}} f_\Psi(\beta, L^{\mathcal{I}_\Sigma}) \tag{31}$$

where $f_\Psi := f \circ \Psi$*.*

We reiterate the proof of Theorem 28 for illustrative purposes.

Proof. Continuity of f implies its minimum over Θ_{BMZ} equals its infimum over $\text{int}(\Theta_{\text{BMZ}})$, i.e.,

$$\min_{(\beta,\Sigma)\in\Theta_{\text{BMZ}}} f(\beta, \Sigma) = \inf_{(\beta,\Sigma)\in\text{int}(\Theta_{\text{BMZ}})} f(\beta, \Sigma)$$

Since Ψ is a bijection, we can transform the optimization variables as follows:

$$\inf_{(\beta,\Sigma)\in\text{int}(\Theta_{\text{BMZ}})} f(\beta, \Sigma) = \inf_{(\beta, L^{\mathcal{I}_\Sigma})\in\Psi^{-1}(\text{int}(\Theta_{\text{BMZ}}))} f(\Psi(\beta, L^{\mathcal{I}_\Sigma}))$$

and $\Theta_{\text{Chol}} = \Psi^{-1}(\text{int}(\Theta_{\text{BMZ}}))$ and $f_\Psi = f \circ \Psi$ imply (31). □

4.2 Modified Burer-Monteiro-Zhang method

For constraint sets Θ satisfying Assumption 2, the BMZ method only suffices to eliminate the variables Σ and the matrix inequality $\Sigma \succeq H(\beta)$. The addition of the general matrix inequality $\mathcal{A}(\beta, \Sigma) \succeq 0$ requires a similar procedure to introduce squared slack variables, although the elimination procedure will no longer eliminate as many variables as are introduced. We further complicate the generalization of Lemma 27 and Theorem 28 by requiring general vector equality and inequality constraints $g(\beta, \Sigma) = 0$ and $h(\beta, \Sigma) \leq 0$.

Lemma 27 gives the invertible map $\Psi : \Theta_{\text{Chol}} \to \Theta_{\text{BMZ}}$ with which we can define

$$g_\Psi \coloneqq g \circ \Psi, \qquad h_\Psi \coloneqq h \circ \Psi, \qquad \mathcal{A}_\Psi \coloneqq \mathcal{A} \circ \Psi.$$

In this notation, consider the set

$$\Theta_{\text{NLP}} \coloneqq \{ (\beta, L^{\mathcal{I}_\Sigma}) \in \mathbb{R}^{n_\beta} \times \mathbb{L}^{n_\Sigma}_{++}[\mathcal{I}_\Sigma] \mid g_\Psi(\beta, L^{\mathcal{I}_\Sigma}) = 0, \\ h_\Psi(\beta, L^{\mathcal{I}_\Sigma}) = 0,\ \mathcal{A}_\Psi(\beta, L^{\mathcal{I}_\Sigma}) \succ 0 \}. \tag{32}$$

Recall $\Sigma(\beta, L^{\mathcal{I}_\Sigma}) \succ H(\beta)$ holds automatically for all $(\beta, L^{\mathcal{I}_\Sigma}) \in \Theta_{\text{Chol}} \coloneqq \mathbb{R}^{n_\beta} \times \mathbb{L}^{n_\Sigma}_{++}[\mathcal{I}_\Sigma]$. Then by a simple change of variables we have $\Psi^{-1}(\Theta_{++}) = \Theta_{\text{NLP}}$. Since Θ_{NLP} and Θ_{++} are subsets of Θ_{Chol} and $\text{int}(\Theta_{\text{BMZ}})$, respectively, we have proven the following lemma.

Lemma 29. *The function Ψ defined by* (29) *is a bijection between Θ_{NLP} and Θ_{++}.*

Next, we remove the matrix inequality $\mathcal{A}_\Psi(\beta, L^{\mathcal{I}_\Sigma}) \succ 0$ from our representation of the constraint set Θ_{NLP}. Recall $\mathcal{A}_\Psi(\beta, L^{\mathcal{I}_\Sigma}) \succ 0$ if and only if $\mathcal{A}_\Psi(\beta, L^{\mathcal{I}_\Sigma}) = L_\mathcal{A} L_\mathcal{A}^\top$ for some $L_\mathcal{A} \in \mathbb{L}^{n_\mathcal{A}}_{++}$. With $\mathcal{J}_\mathcal{A} \coloneqq \mathcal{L}^{n_\mathcal{A}} \setminus \mathcal{I}_\mathcal{A}$, we split $L_\mathcal{A}$ into the sum of $L^{\mathcal{I}_\mathcal{A}} \in \mathbb{L}^{n_\mathcal{A}}_{++}[\mathcal{I}_\mathcal{A}]$ and $L^{\mathcal{J}_\mathcal{A}} \in \mathbb{L}^{n_\mathcal{A}}[\mathcal{J}_\mathcal{A}]$,

$$\mathcal{A}_\Psi(\beta, L^{\mathcal{I}_\Sigma}) = (L^{\mathcal{I}_\mathcal{A}} + L^{\mathcal{J}_\mathcal{A}})(L^{\mathcal{I}_\mathcal{A}} + L^{\mathcal{J}_\mathcal{A}})^\top. \tag{33}$$

But $\mathcal{A}_\Psi(\beta, L^{\mathcal{I}_\Sigma}) \in \mathbb{S}^{n_\mathcal{A}}[\mathcal{I}_\mathcal{A}]$, so we can take

$$\pi_{\mathcal{J}_\mathcal{A}}[(L^{\mathcal{I}_\mathcal{A}} + L^{\mathcal{J}_\mathcal{A}})(L^{\mathcal{I}_\mathcal{A}} + L^{\mathcal{J}_\mathcal{A}})^\top] = 0$$

to produce $|\mathcal{J}_\mathcal{A}|$ equalities with which to eliminate the $|\mathcal{J}_\mathcal{A}|$ free variables in $L^{\mathcal{J}_\mathcal{A}}$. For each $L^{\mathcal{I}_\mathcal{A}} \in \mathbb{L}^{n_\mathcal{A}}_{++}[\mathcal{I}_\mathcal{A}]$, we define

$$L^{\mathcal{J}_\mathcal{A}}_{ij} \coloneqq -\frac{1}{L^{\mathcal{I}_\mathcal{A}}_{jj}} \sum_{k=1}^{j-1} (L^{\mathcal{I}_\mathcal{A}}_{ik} + L^{\mathcal{J}_\mathcal{A}}_{ik})(L^{\mathcal{I}_\mathcal{A}}_{jk} + L^{\mathcal{J}_\mathcal{A}}_{jk}) \tag{34}$$

for each $(i, j) \in \mathcal{J}_\mathcal{A}$, in a top-to-bottom and left-to-right order. Each $L^{\mathcal{J}_\mathcal{A}}$ satisfying (33) is thus fully defined by $L^{\mathcal{I}_\mathcal{A}}$ via (34), and we have the following corollary of Lemma 26.

Corollary 30. *For each $L^{\mathcal{I}_\mathcal{A}} \in \mathbb{L}^{n_\mathcal{A}}[\mathcal{I}_\mathcal{A}]$ such that $L^{\mathcal{I}_\mathcal{A}}_{ii} \neq 0$ for each $i \in \mathbb{I}_{1:n_\mathcal{A}}$, there exists a unique $L^{\mathcal{J}_\mathcal{A}} \in \mathbb{L}^{n_\mathcal{A}}[\mathcal{J}_\mathcal{A}]$ satisfying* (33).

Let $L^{\mathcal{J}_\mathcal{A}} : \mathbb{L}^{n_\mathcal{A}}_{++} \to \mathbb{L}^{n_\mathcal{A}}[\mathcal{I}_\mathcal{A}]$ denote the map described by Corollary 30, and let $\tilde{\mathcal{A}}(L^{\mathcal{I}_\mathcal{A}}) \coloneqq (L^{\mathcal{I}_\mathcal{A}} + L^{\mathcal{J}_\mathcal{A}}(L^{\mathcal{I}_\mathcal{A}}))(L^{\mathcal{I}_\mathcal{A}} + L^{\mathcal{J}_\mathcal{A}}(L^{\mathcal{I}_\mathcal{A}}))^\top$. Then $\mathcal{A}_\Psi(\beta, L^{\mathcal{I}_\Sigma}) \succ 0$ if and only if $\mathcal{A}_\Psi(\beta, L^{\mathcal{I}_\Sigma}) = \tilde{\mathcal{A}}(L^{\mathcal{I}_\mathcal{A}})$ for some $L^{\mathcal{I}_\mathcal{A}} \in \mathbb{L}^{n_\mathcal{A}}_{++}[\mathcal{I}_\mathcal{A}]$. Subsuming this constraint g_Ψ, we have

$$\tilde{g}(\beta, L^{\mathcal{I}_\Sigma}, L^{\mathcal{I}_\mathcal{A}}) \coloneqq \begin{bmatrix} g_\Psi(\beta, L^{\mathcal{I}_\Sigma}) \\ \text{vecs}_{\mathcal{I}_\mathcal{A}} \left(\mathcal{A}_\Psi(\beta, L^{\mathcal{I}_\Sigma}) - \tilde{\mathcal{A}}(L^{\mathcal{I}_\mathcal{A}}) \right) \end{bmatrix}$$

for all $(\beta, L^{\mathcal{I}_\Sigma}, L^{\mathcal{I}_\mathcal{A}}) \in \mathbb{R}^{n_\beta} \times \mathbb{L}^{n_\Sigma}_{++}[\mathcal{I}_\Sigma] \times \mathbb{L}^{n_\mathcal{A}}_{++}[\mathcal{I}_\mathcal{A}]$, where $\text{vecs}_{\mathcal{I}_\mathcal{A}} : \mathbb{S}^{n_\mathcal{A}} \to \mathbb{R}^{|\mathcal{I}_\mathcal{A}|}$ vectorizes the $|\mathcal{I}_\mathcal{A}|$ entries of the argument corresponding to the index set $\mathcal{I}_\mathcal{A}$. We have proven the following lemma.

Lemma 31. *The set* Θ_{NLP}*, defined by* (32)*, equals*

$$\{ (\beta, L^{\mathcal{I}_\Sigma}) \in \mathbb{R}^{n_\beta} \times \mathbb{L}_{++}^{n_\Sigma}[\mathcal{I}_\Sigma] \mid \exists L^{\mathcal{I}_\mathcal{A}} \in \mathbb{L}_{++}^{n_\mathcal{A}}[\mathcal{I}_\mathcal{A}] : \tilde{g}(\beta, L^{\mathcal{I}_\Sigma}, L^{\mathcal{I}_\mathcal{A}}) = 0,\ h_\Psi(\beta, L^{\mathcal{I}_\Sigma})) \leq 0 \} .$$

Finally, we have the following equivalence between minimization problems over the parameter sets Θ and Θ_{NLP}.

Proposition 32. *For any continuous function* $f : \mathbb{R}^{n_\beta} \times \mathbb{S}^{n_\Sigma} \to \mathbb{R}$ *that attains a minimum in* Θ*,*

$$\min_{(\beta,\Sigma)\in\Theta} f(\beta, \Sigma) = \inf_{(\beta, L^{\mathcal{I}_\Sigma})\in\Theta_{\mathrm{NLP}}} f_\Psi(\beta, L^{\mathcal{I}_\Sigma})$$

where $f_\Psi \coloneqq f \circ \Psi$*.*

Proof. The proof follows that of Theorem 28, noting that Assumption 2 gives $\mathrm{cl}(\Theta_{++}) = \Theta$ and therefore the minimum of f over Θ equals the infimum of f over Θ_{++}. $\square$

4.3 ε-approximate solutions

The strict inequalities implied by constraints $\mathbb{L}_\varepsilon^{n_\Sigma}[\mathcal{I}_\Sigma]$ and $\mathbb{L}_\varepsilon^{n_\mathcal{A}}[\mathcal{I}_\mathcal{A}]$ are not amenable to implementation in standard NLP software. In [60], a log-barrier approach is used to achieve global convergence for a class of linear SDPs. Instead, we consider a constant backoff on the inequalities, providing a small but nonzero lower bound on the diagonal elements. For each $\mathcal{D}^n \subseteq \mathcal{I} \subseteq \mathcal{L}^n$, we define

$$\mathbb{L}_\varepsilon^n[\mathcal{I}] \coloneqq \{ L \in \mathbb{L}_{++}^n[\mathcal{I}] \mid L_{ii} \geq \varepsilon\ \forall i \in \mathbb{I}_{1:n} \} .$$

These sets imply an additional n inequality constraints, but do not have any ill-posed strictness requirements. Therefore we can optimize over the restricted set

$$\Theta_\varepsilon \coloneqq \{ (\beta, L^{\mathcal{I}_\Sigma}) \in \mathbb{R}^{n_\beta} \times \mathbb{L}_\varepsilon^{n_\Sigma}[\mathcal{I}_\Sigma] \mid \exists L^{\mathcal{I}_\mathcal{A}} \in \mathbb{L}_\varepsilon^{n_\mathcal{A}}[\mathcal{I}_\mathcal{A}] : \tilde{g}(\beta, L^{\mathcal{I}_\Sigma}, L^{\mathcal{I}_\mathcal{A}}) = 0,\ h_\Psi(\beta, L^{\mathcal{I}_\Sigma}) \leq 0 \}. \quad (35)$$

In the following proposition we show, for any continuous function f, the infimum of f over Θ_ε converges to the minimum of f over Θ, so long as Assumption 2 is satisfied (see Appendix E for proof).

Proposition 33. *For any continuous* $f : \mathbb{R}^{n_\beta} \times \mathbb{S}^{n_\Sigma} \to \mathbb{R}$ *that attains a minimum in* Θ*, let* $\mu_0 \coloneqq \min_{(\beta,\Sigma)\in\Theta} f(\beta, \Sigma)$ *and*

$$\mu_\varepsilon \coloneqq \inf_{(\beta, L^{\mathcal{I}_\Sigma})\in\Theta_\varepsilon} f(\beta, L^{\mathcal{I}_\Sigma}) \quad (36)$$

where $f(\beta, L^{\mathcal{I}_\Sigma}) \coloneqq f(\Psi(\beta, L^{\mathcal{I}_\Sigma}))$*. If Assumption 2 is satisfied, then* $\mu_\varepsilon \searrow \mu$ *as* $\varepsilon \searrow 0$*.*

With requirements on the objective f, convergence of ε-approximate solutions to the exact solution is guaranteed by the following proposition (see Appendix E for proof).

Proposition 34. *For any continuous $f : \mathbb{R}^{n_\beta} \times \mathbb{S}^{n_\Sigma} \to \mathbb{R}$, consider the set-valued function $\hat{\theta} : \mathbb{R}_{\geq 0} \to \mathcal{P}(\Theta)$, defined as*

$$\hat{\theta}_\varepsilon \coloneqq \operatorname*{argmin}_{(\beta,\Sigma)\in\Psi(\Theta_\varepsilon)} f(\beta, \Sigma)$$

for all $\varepsilon > 0$, and

$$\hat{\theta}_0 \coloneqq \operatorname*{argmin}_{(\beta,\Sigma)\in\Theta} f(\beta, \Sigma).$$

If there exists $\alpha \in \mathbb{R}$ and compact $C \subseteq \Theta$ such that

$$\Theta_{f\leq\alpha} \coloneqq \{\, (\beta, \Sigma) \in \Theta \mid f(\beta, \Sigma) \leq \alpha \,\}$$

is contained in C and $\Theta_{f\leq\alpha} \cap \Theta_{++}$ is nonempty, then there exists $\overline{\varepsilon} > 0$ such that, for all $\varepsilon_0 \in [0, \overline{\varepsilon})$,

(a) f achieves a minimum in Θ and $\hat{\theta}_0$ is nonempty;

(b) if $\varepsilon_0 > 0$, then f achieves a minimum in $\Psi(\Theta_{\varepsilon_0})$ and $\hat{\theta}_{\varepsilon_0}$ is nonempty;

(c) μ_ε is continuous and $\hat{\theta}_\varepsilon$ is outer semicontinuous at $\varepsilon = \varepsilon_0$; and

(d) if $\hat{\theta}_0$ is a singleton, then $\limsup_{\varepsilon\searrow 0} \hat{\theta}_\varepsilon = \hat{\theta}_0$.

5 Case Studies

In this section, we apply the identification methods outlined in the previous sections to design Kalman filters for the linear augmented disturbance models used in offset-free MPC:

$$\begin{bmatrix} \hat{x}_{k+1} \\ \hat{d}_{k+1} \end{bmatrix} = \begin{bmatrix} A & B_d \\ 0 & I_p \end{bmatrix} \begin{bmatrix} \hat{x}_k \\ \hat{d}_k \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_k + \begin{bmatrix} K_x \\ K_d \end{bmatrix} e_k \tag{37a}$$

$$y_k = \begin{bmatrix} C & C_d \end{bmatrix} \begin{bmatrix} \hat{x}_k \\ \hat{d}_k \end{bmatrix} + e_k \tag{37b}$$

$$e_k \overset{\text{iid}}{\sim} \mathcal{N}(0, R_e) \tag{37c}$$

where $\hat{x} \in \mathbb{R}^n$ denote plant state estimates, $\hat{d} \in \mathbb{R}^{n_d}$ denote disturbance state estimates, $(B_d, C_d) \in \mathbb{R}^{n\times n_d} \times \mathbb{R}^{p\times n_d}$ denote disturbance shaping matrices, and $(K_x, K_d) \in \mathbb{R}^{n\times p} \times \mathbb{R}^{n_d\times p}$ are plant and disturbance state filter gains.

In the first case study, we consider the TCLab (Figure 1), an Arduino-based temperature control laboratory that serves as a low-cost[2] benchmark for linear MIMO control [61]. We identify the TCLab from open-loop data and use the resulting model to design an offset-free MPC. We compare closed-loop control and estimation performance of these models to that of offset-free MPCs designed with the identification methods from [50, 51]. In

[2]The TCLab is available for under $40 from https://apmonitor.com/heat.htm and https://www.amazon.com/gp/product/B07GMFWMRY.

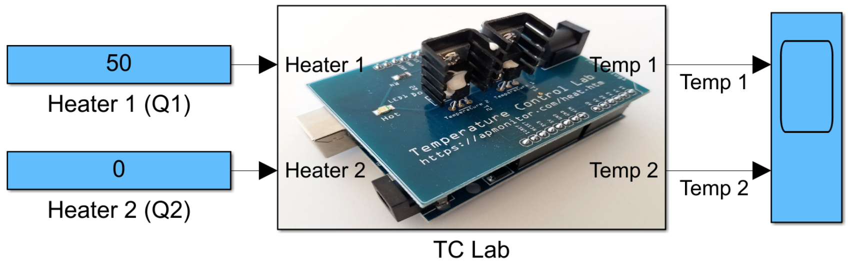

Figure 1: Benchmark temperature Control Laboratory (TCLab) [61].

the second case study, data from an industrial-scale chemical reactor is used to design Kalman filters for the linear augmented disturbance model, and the closed-loop estimation performance is compared to that of the designs proposed in [51].

To aid in optimizer convergence and guarantee uniqueness of the solution, to any likelihood function $f_N(\theta)$, we add a regularization term or prior distribution,

$$\min_{\theta \in \Theta} f_N(\theta) - \ln p(\theta)$$

where $p(\theta)$ is the prior density. Throughout, we use the following prior,

$$-\ln p(\beta, \Sigma) \propto \frac{\rho}{2} \left(\|\beta - \beta_0\|_2^2 + \operatorname{tr}(\Sigma - \Sigma_0) \right) \tag{38}$$

where $\rho > 0$ and $(\beta_0, \Sigma_0) \in \Theta_{++}$ is the initial guess given to the optimizer. The prior (38) is equivalent to a Frobenius norm regularizer in the Cholesky factor space,

$$-\ln p(\Psi(\beta, L^{\mathcal{I}_\Sigma})) \propto \frac{\rho}{2} \left(\|\beta - \beta_0\|_2^2 + \|L^{\mathcal{I}_\Sigma} - L_0^{\mathcal{I}_\Sigma}\|_{\mathrm{F}}^2 + \|L^{\mathcal{J}_\Sigma}(\beta, L^{\mathcal{I}_\Sigma}) - L_0^{\mathcal{J}_\Sigma}\|_{\mathrm{F}}^2 \right) \tag{39}$$

where $L_0^{\mathcal{I}_\Sigma} := \pi_{\mathcal{I}_\Sigma}^L[\operatorname{chol}(\Sigma_0 - H(\beta_0))]$ and $L_0^{\mathcal{J}_\Sigma} := L^{\mathcal{J}_\Sigma}(\beta_0, L_0^{\mathcal{I}_\Sigma})$. When Σ is block diagonal, $L^{\mathcal{J}_\Sigma}(\beta, L^{\mathcal{I}_\Sigma}) = 0$ and the last term of (39) vanishes. The squared slack variables $L_{\mathcal{A}} = L^{\mathcal{I}_{\mathcal{A}}} + L^{\mathcal{J}_{\mathcal{A}}}$ are not regularized.

Throughout these experiments, we use the steady-state filter likelihood (10) and a Cholesky factor diagonal backoff ε of 10^{-6}. When constraints are considered, they either take the form of smoothed spectral radius and abscissa constraints,

$$P_d - \frac{A_K P_d A_K^\top}{(1-\delta)^2} = \varepsilon_d I_{n+p}, \qquad P_d \succeq 0, \qquad \operatorname{tr}(P_d) \le \varepsilon_d^{-1} \tag{40a}$$

$$A_K P_c + P_c A_K^\top = \varepsilon_c I_{n+p}, \qquad P_c \succeq 0, \qquad \operatorname{tr}(P_c) \le \varepsilon_c^{-1} \tag{40b}$$

or the LMI region barrier function constraints,

$$P_d - \frac{A_K P_d A_K^\top}{(1-\delta)^2} \succeq \varepsilon_d I, \qquad P_d \succeq 0, \qquad \operatorname{tr}(P_d) \le \varepsilon_d^{-1} \tag{41a}$$

$$A_K P_c + P_c A_K^\top \succeq \varepsilon_c I, \qquad P_c \succeq 0, \qquad \operatorname{tr}(P_c) \le \varepsilon_c^{-1}. \tag{41b}$$

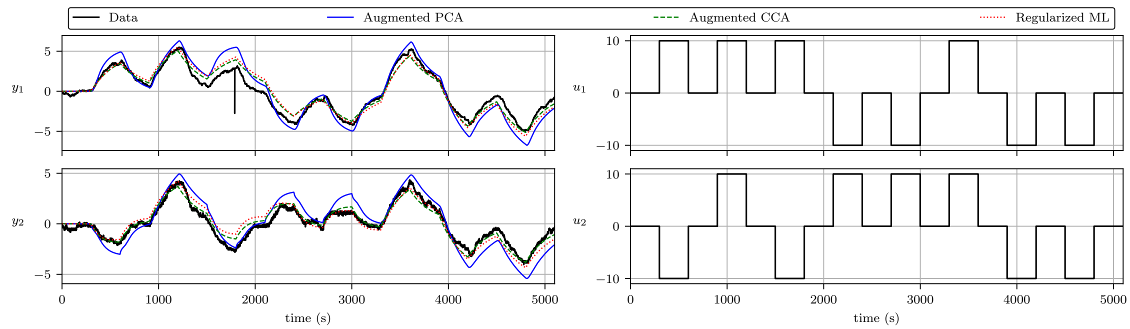

Figure 2: TCLab identification data and noise-free responses $\hat{y}_k = \sum_{j=1}^{k} \hat{C}\hat{A}^{j-1}\hat{B}u_{k-j}$ of a few selected models.

Table 1: TCLab model fitting results. * The augmented PCA/CCA identification methods are not iterative. ** The maximum number of iterations was set at 500.

Model	**Time (s)**	**Iterations**	**Log-likelihood**
Augmented PCA	0.01	N/A*	3823.4
Augmented CCA	0.04	N/A*	2415.8
Unregularized ML	121.9**	500**	-9431.4
Regularized ML	7.5	17	-9411.7
Constrained ML 1	36.3	61	-9407.5
Constrained ML 2	45.1	78	-9412.5
Constrained ML 3	14.2	23	-9407.5
Constrained ML 4	25.1	54	-9412.5

Each optimization problem is formulated in CasADi as a ε-approximate problem in Cholesky factor form (35) and (36), and solved with IPOPT. Wall times for a single-thread run on an Intel Core i9-10850K processor are reported. The initial guesses for these models are based on a nested ML identification approach described in [50, 51]. This approach effectively augments a standard identification method (e.g., PCA, Ho-Kalman, canonical correlation analysis algorithms), so we refer to the initial guess models as "augmented" versions of the standard method being used.

5.1 TCLab

Unless otherwise specified, the TCLab is modeled as a two-state system of the form (1), with internal temperatures as plant states $x = \begin{bmatrix} T_1 & T_2 \end{bmatrix}^\top$, heater voltages as inputs $u = \begin{bmatrix} V_1 & V_2 \end{bmatrix}^\top$, and measured temperatures $y = \begin{bmatrix} T_{m,1} & T_{m_2} \end{bmatrix}^\top$ as outputs. Throughout, we choose $n_d = p$ to satisfy the offset-free necessary conditions in [37, 38], and we con-

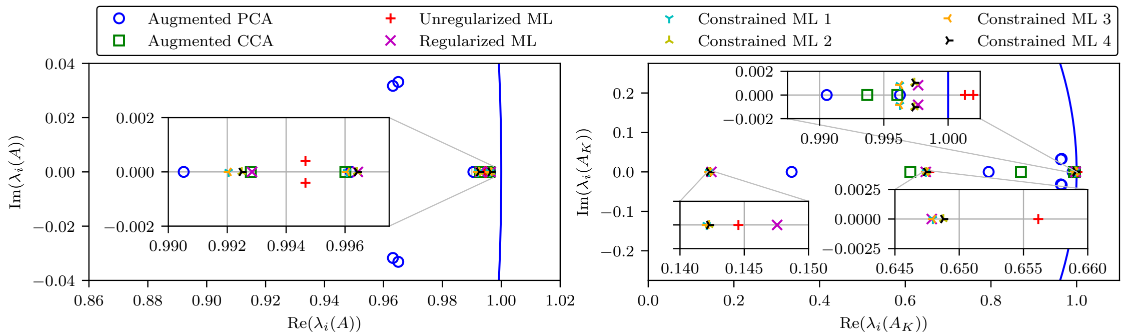

Figure 3: TCLab models open-loop and closed-loop (filter) eigenvalues.

sider output disturbance models $(B_d, C_d) = (0_{2\times 2}, I_2)$. We use an observability canonical form [68] with (A, B) fully parameterized and $C = I_2$ to guarantee model identifiability and make the states interpretable as internal temperatures. The remaining model terms (K_x, K_d, R_e) are fully parameterized. As in (37), the models do not include a passthrough term, i.e., $D = 0$.

Eight TCLab models were considered:

1. **Augmented PCA**: the 6-state TCLab model used in [50], where principle component analysis on a 400×5100 data Hankel matrix is used to determine the states in the disturbance-free model.

2. **Augmented CCA**: a 2-state augmented canonical correlation analysis (CCA) model, based on the canonical method of [69].

3. **Unregularized ML**: a 2-state model, fit directly to (10) without regularization or constraints, using Augmented CCA as the initial guess.

4. **Regularized ML**: the same as Unregularized ML but with an added regularizer (38) with $\rho = 10^{-2}N$.

5. **Constrained ML 1**: the same as Unregularized ML but with the smoothed LMI region constraints (41) with $\delta = \varepsilon_d = 10^{-3}$ and $\varepsilon_c = 10$.

6. **Constrained ML 2**: the same as Constrained ML 1 but with $\delta = 2 \times 10^{-3}$.

7. **Constrained ML 3**: the same as Constrained ML 1, but using the smoothed spectral radius and abscissa constraints (40) (and the same constants).

8. **Constrained ML 4**: the same as Constrained ML 3 (i.e., using (40)) but with $\delta = 2 \times 10^{-3}$.

Each ML model uses Augmented CCA as the initial guess as it has the smallest number of states. The augmented PCA model is, in effect, an unsupervised learner of the state estimates, and therefore does not produce a parsimonious state description.

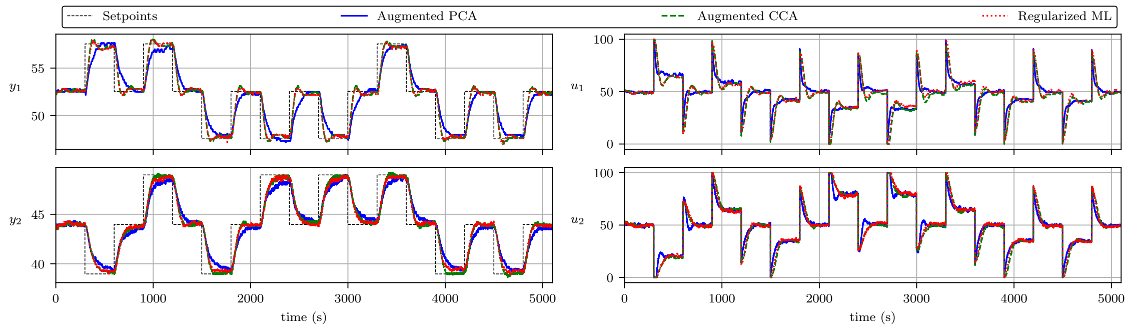

Figure 4: TCLab setpoint tracking tests.

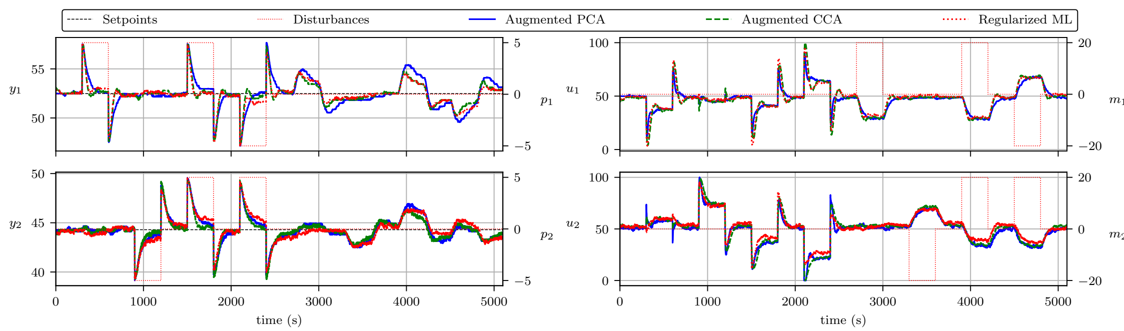

Figure 5: TCLab disturbance rejection tests.

In Figure 2, the identification data is presented along with the noise-free responses $\hat{y}_k = \sum_{j=1}^{k} \hat{C}\hat{A}^{j-1}\hat{B}u_{k-j}$ of a few selected models. Computation time, number of IPOPT iterations, and the *unregularized* log-likelihood value are reported in Table 1. The open-loop A and closed-loop A_K eigenvalues of each model are plotted in Figure 3.

Except for the augmented PCA model, all of the open-loop eigenvalues cluster around the same region of the complex plane (figure 3). The closed-loop filter eigenvalues are placed similarly, with the exception of the unregularized ML model, which has a slightly unstable filter. Despite these differences, the ML models all have about the same unregularized log-likelihood value (Table 1) and appear to have identical noise-free responses (Figure 2). Here, the constraints (40) and (41) appear to enforce filter stability and aid in convergence with only a small computational penalty.

As reported in Table 1, the unregularized ML model fitting did not converge. It is our experience that encountering iterates with unstable filters and filters with eigenvalues having negative real parts can cause convergence issues. This is because the likelihood function becomes sensitive to small changes in the parameter values due to filter instability or rapid oscillations in the filter predictions.

To test offset-free control performance, we performed two sets of closed-loop experiments on offset-free MPCs designed with the models. In Figure 4, identical setpoint

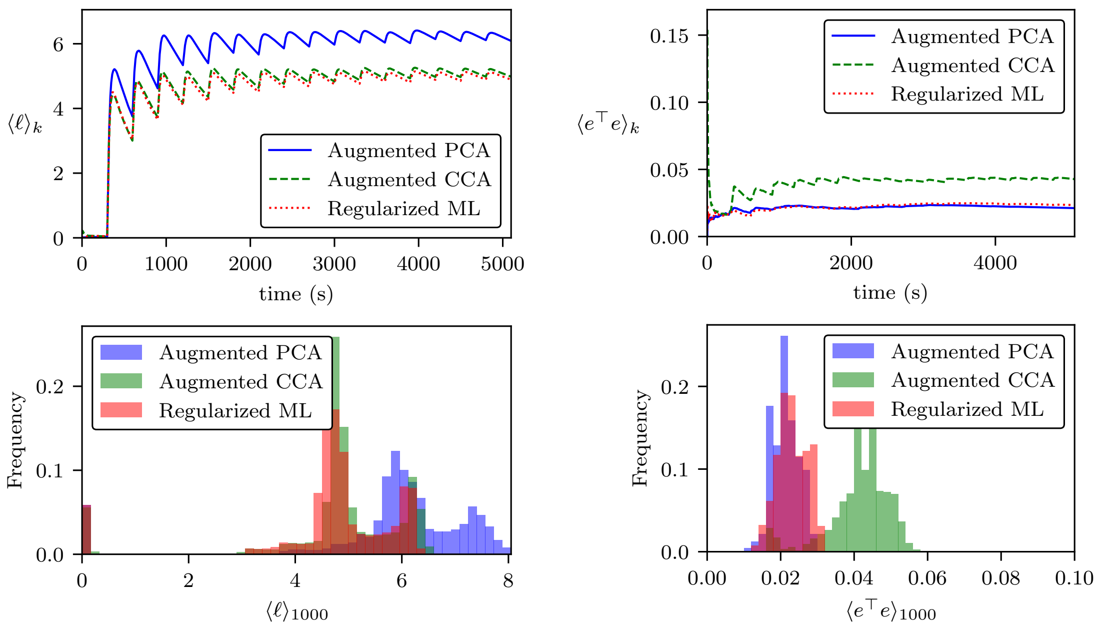

Figure 6: TCLab setpoint tracking test performance.

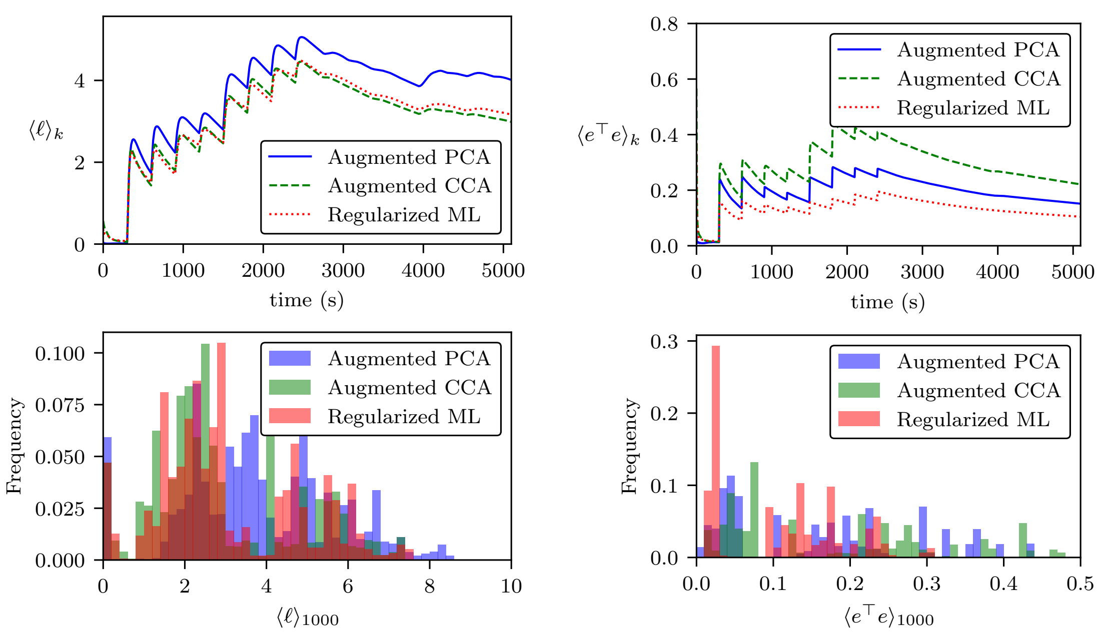

Figure 7: TCLab disturbance rejection test performance.

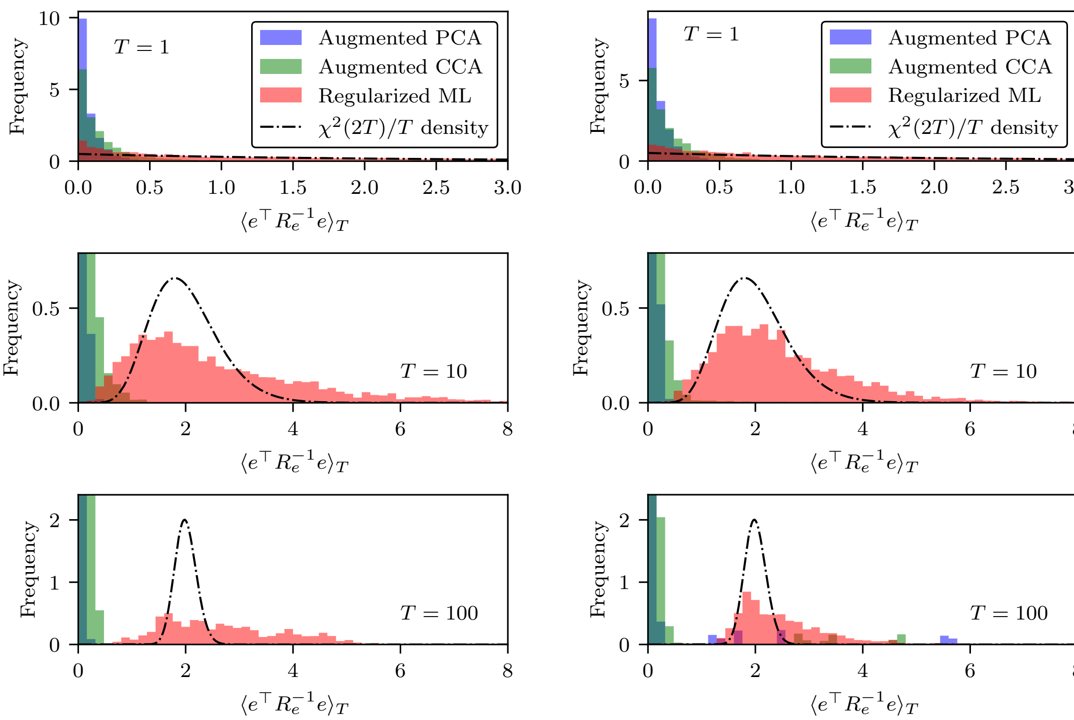

Figure 8: TCLab identification index data for (left) setpoint tracking and (right) disturbance rejection tests.

changes were applied to a TCLab running at a steady-state power output of 50%. The setpoint changes were tracked with the offset-free MPC design described in [50]. In Figure 5, step disturbances in the output p_i and the input m_i are injected into a plant trying to maintain a given steady-state temperature. The setpoints are tracked with the offset-free MPC design described in [50].

Control performance indexed by squared distance from the setpoint $\ell_k := \|y_k - y_{\mathrm{sp},k}\|_2^2$. Estimation performance is indexed by squared filter errors $e_k^\top e_k$. For any signal a_k, we define a T-sample moving average by $\langle a_k\rangle_T := T^{-1}\sum_{j=0}^{T-1} a_{k-j}$. Setpoint tracking performance is reported in Figure 6, and disturbance rejection performance is reported in Figure 7. From Figures 6 and 7, it is clear that the regularized ML model delivers the best overall performance.

To investigate the accuracy of the stochastic models, we consider the inverse-covariance weighted squared norm of the filter errors $q := e^\top R_e^{-1} e$ as an identification index. Recall the signal e_k is an i.i.d., zero-mean Gaussian process, i.e., $e_k \overset{\text{iid}}{\sim} \mathcal{N}(0, R_e)$, and therefore the index q_k is i.i.d. with a χ_p^2 distribution. Moreover, the moving average $\langle q_k\rangle_T$ is distributed as χ_{pT}^2/T, although it is no longer independent in time. In Figure 8, histograms of $\langle q\rangle_T$ are plotted against their expected distribution for $T \in \{1, 10, 100\}$ and the augmented PCA, augmented CCA, and regularized ML models. The extreme discrepancies between the

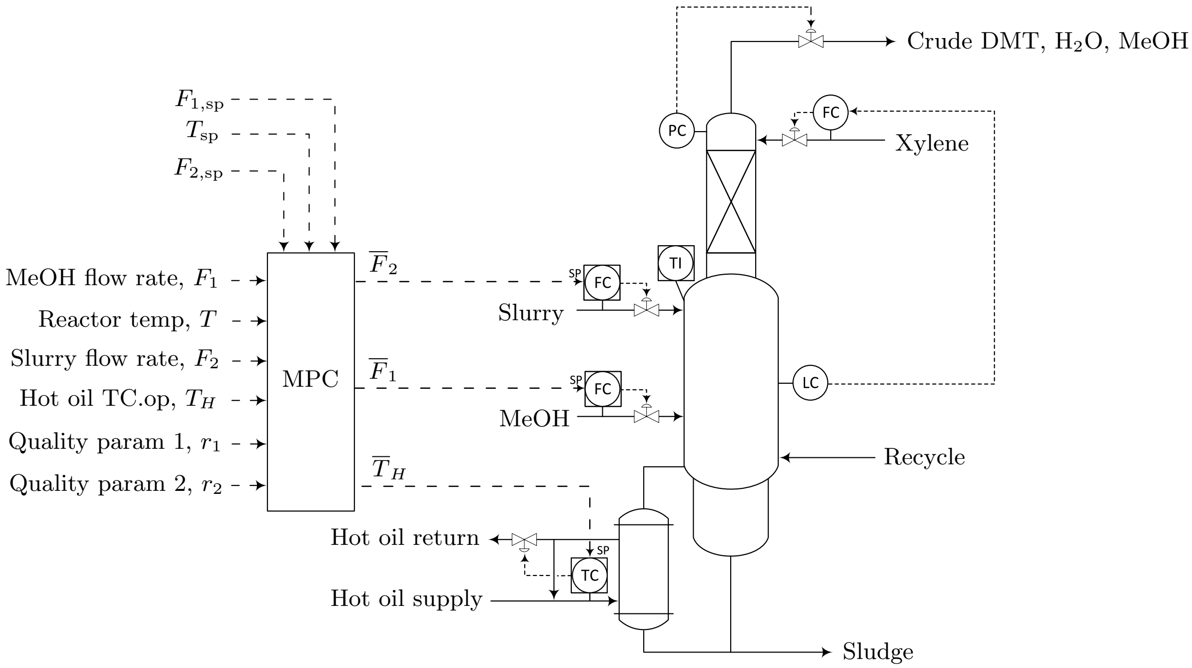

Figure 9: Schematic of the DMT reactor and MPC control strategy.

augmented PCA and CCA models' performance index $\langle q \rangle_T$ and the reference distribution χ^2_{pT}/T are primarily due to the augmented PCA/CCA models significantly overestimating R_e,

$$\hat{R}_e^{\text{aug-PCA}} = \left[\begin{smallmatrix} 0.5871 & 0.3365 \\ 0.3365 & 0.2878 \end{smallmatrix}\right], \qquad \hat{R}_e^{\text{aug-CCA}} = \left[\begin{smallmatrix} 0.5889 & 0.0918 \\ 0.1791 & 0.3152 \end{smallmatrix}\right], \qquad \hat{R}_e^{\text{ML}} = \left[\begin{smallmatrix} 0.0107 & 0.0006 \\ 0.0006 & 0.008 \end{smallmatrix}\right].$$

The reference distribution and the ML model's $\langle q \rangle_T$ distribution diverge at large T since, due to plant-model mismatch, the filter's innovation errors are slightly autocorrelated.

5.2 Eastman reactor

A schematic of the chemical reactor considered in the next case study is presented in Figure 9.The control objective of the chemical reactor considered in the next case study is to pick three inputs (the reactant flow rates and utility temperatures $u = \begin{bmatrix} F_1 & T_H & F_2 \end{bmatrix}^\top$) that steer the system to three setpoints (the output, a specified reactor temperature $y = T$, and the flowrates $\begin{bmatrix} u_1 & u_2 \end{bmatrix}^\top = \begin{bmatrix} F_1 & F_2 \end{bmatrix}^\top$) without offset. See [51] for more details about the reactor operation. As in Subsection 5.1, we choose $n_d = p$, consider output disturbance models $(B_d, C_d) = (0_{3\times 1}, 1)$, and use an observability canonical form [68] this time with the parameterization $A = \left[\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ a_1 & a_2 & a_3 \end{smallmatrix}\right]$ and $C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$. Again, the remaining model terms (B, K_x, K_d, R_e) are fully parameterized. As in (37), the models do not include a passthrough term, i.e., $D = 0$. We fit eight models (two augmented, six ML) of this form to closed-loop data of this core 3-input, 1-output system:

Table 2: Eastman reactor model fitting results. * The augmented HK/CCA identification methods are not iterative.

Model	**Time (s)**	**Iterations**	**Log-likelihood**
Augmented HK	0.08	N/A*	-7143.1
Augmented CCA	0.07	N/A*	-11288.4
Unregularized ML 1	23.7	93	-14345.5
Unregularized ML 2	19.4	73	-14345.5
Regularized ML 1	10.0	28	-13053.6
Regularized ML 2	8.8	22	-13587.7
Constrained ML 1	44.6	115	-13034.1
Constrained ML 2	37.1	91	-13587.7

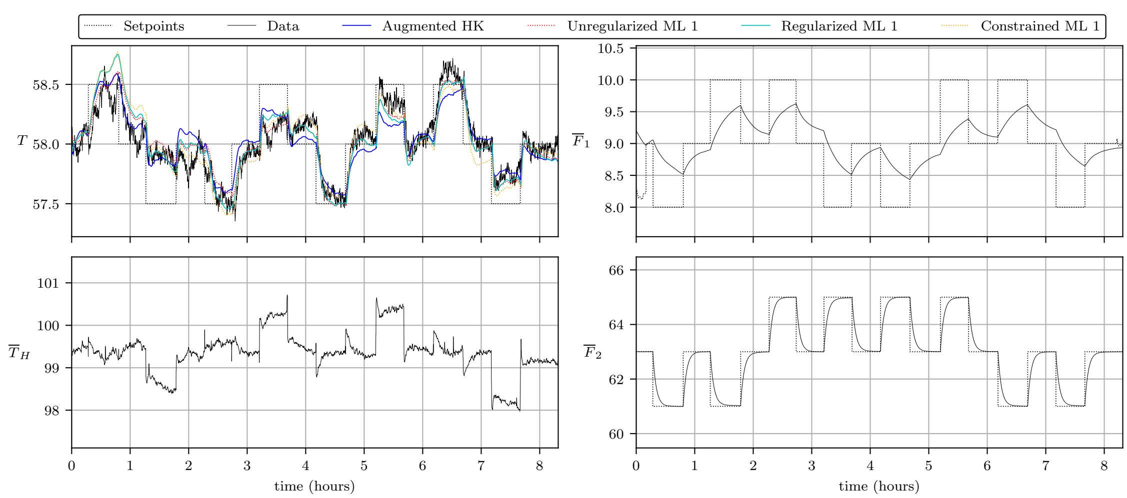

Figure 10: Training data and noise-free responses for the Eastman reactor models (Augmented HK and ML models using Augmented HK as the initial guess).

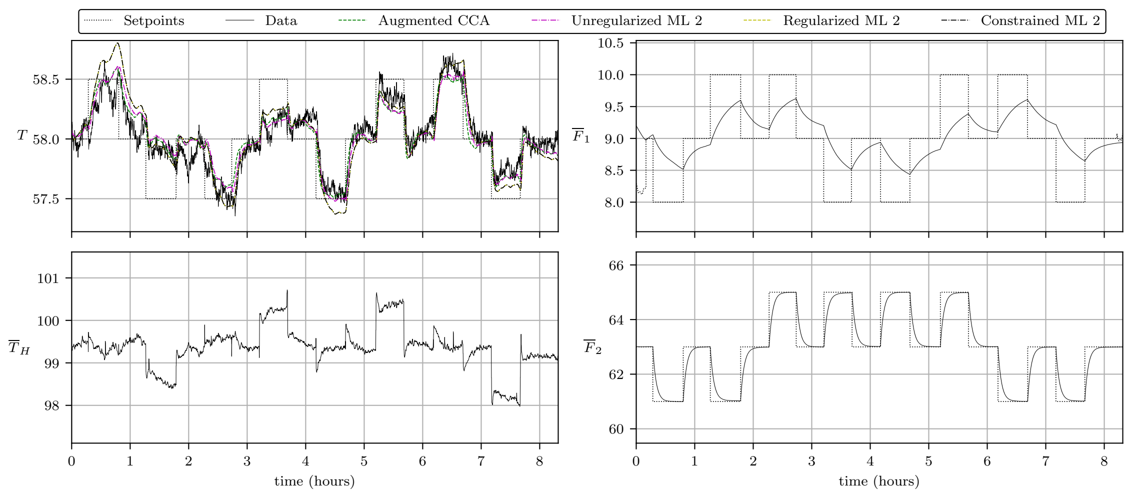

Figure 11: Training data and noise-free responses for the Eastman reactor models (Augmented CCA and ML models using Augmented CCA as the initial guess).

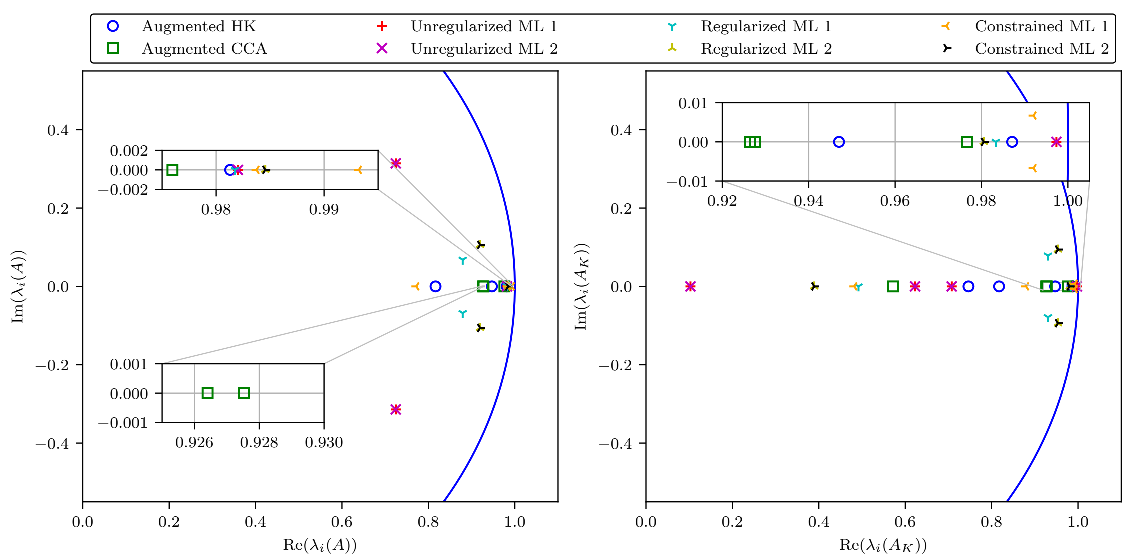

Figure 12: Eastman reactor models open-loop and closed-loop (filter) eigenvalues.

1. **Augmented HK**: a Ho-Kalman-based subspace model is augmented with a disturbance model, as detailed in [50].

2. **Augmented CCA**: a CCA subspace model, based on the method of [69], is augmented with a disturbance model, as detailed in [51].

3,4) **Unregularized ML 1** and **2**: a model is fit directly to (10) without regularization or constraints, using Augmented HK and CCA (resp.) as the initial guesses.

5,6) **Regularized ML 1** and **2**: the same as Unregularized ML 1 and 2, but with an added regularizer (38) with $\rho = 4N$.

7,8) **Constrained ML 1** and **2**: the same as Regularized ML 1 and 2, but with the smoothed spectral radius and abscissa constraints (40) with $\delta = 0$, $\varepsilon_d = 10^{-3}$, and $\varepsilon_c = 10$.

Computation time, number of IPOPT iterations, and the *unregularized* log-likelihood value are reported in Table 1.In Figures 10 and 11, the identification data is presented along with the noise-free responses $\hat{y}_k = \sum_{j=1}^{k} \hat{C}\hat{A}^{j-1}\hat{B}u_{k-j}$ of the Ho-Kalman-based and CCA-based models, respectively. The open-loop A and closed-loop A_K eigenvalues of each model are plotted in Figure 12.

The eigenvalue clustering of the reactor models is more variable than that of the TCLab models. Unregularized models converge to the same solution despite different initial guesses, and regularization pulls the eigenvalues towards the subspace model clustering around $z = 0.98$. A notable feature of the unregularized models is their strong oscillating modes at $\lambda = 0.7246 \pm 0.3141\iota$. While these oscillations are substantially dampened by regularization and constraints, they persist for most models. These oscillations are likely true plant dynamics caused by oscillations in the lower-level PID loops (i.e., level or flow controls). This effect could be seen as an advantage or disadvantage, with the model either correctly fitting desired oscillations so they can be corrected in a supervisory MPC layer, or the model incorrectly overfitting to unwanted process dynamics. Removing unwanted process dynamics may be desirable when experimentation is costly, as it allows the user to "fix" the data rather than run another experiment.

The models' estimation performances are compared against each other in Figures 13 and 14 using one of the test datasets from in [51]. Control performance could not be compared without a costly redesign of the existing MPC strategy. The unregularized models perform the best here, providing over 70% reduction in average filter error for the HK-based models. While the regularized models still capture much of this improvement, they do not have the same level of filter accuracy. This is again evidence of the oscillating modes being a real part of the reactor dynamics rather than a feature of model overfitting.

6 Conclusion

We conclude with a discussion of possible future directions of research for applying our ML identification scheme. An advantage of ML identification is the large body of literature on

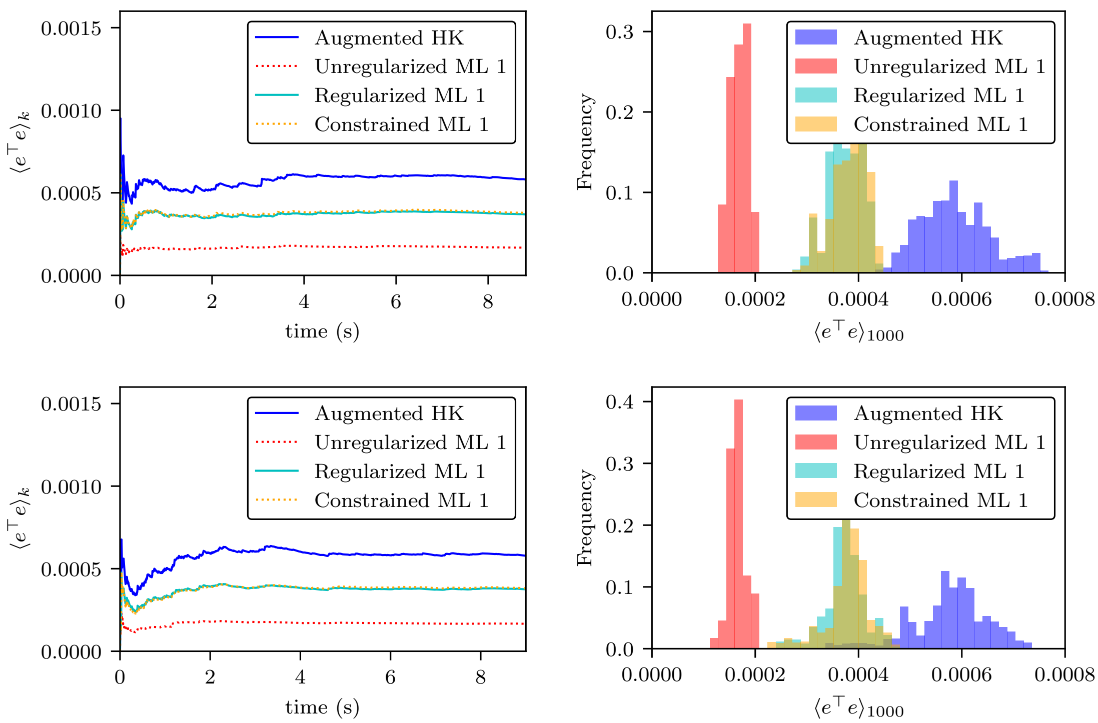

Figure 13: Test performance for the Eastman reactor models (Augmented HK and ML models using Augmented HK as the initial guess).

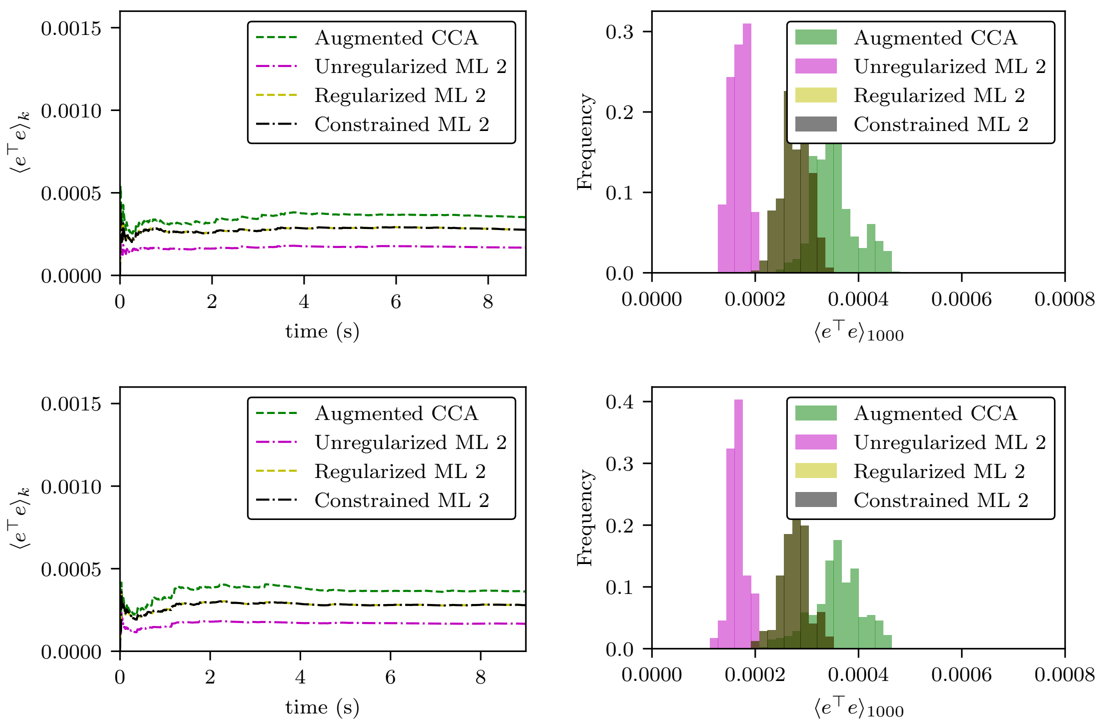

Figure 14: Test performance for the Eastman reactor models (Augmented CCA and ML models using Augmented CCA as the initial guess).

the asymptotic distribution and statistical efficiency of the parameter estimates. From the asymptotic distribution, decision functions can be constructed to map the on-line MPC performance to a re-identification signal. These decision functions can be constructed so as to not alarm unless sufficiently exciting data is available. Decision-theoretic re-identification therefore has lower cost and risk compared to classic adaptive control or online reinforcement learning methods that require a persistently exciting identification signal. This approach could bring statistical data efficiency to the fields of adaptive control and online reinforcement learning. Recent work on direct data-driven control has incorporated likelihood functions with measurement noise models into the control design [70]. To the best of our knowledge, no current work has considered process noise, Kalman filter forms, or structuring the model with uncontrollable integrators for offset-free MPC. There is a future possibility of direct data-driven offset-free MPC design with both optimal control and estimation performance.

A Proof of Proposition 6

Silverman [64] contains a more complete characterization of the DARE solutions for regulation problems with cross terms. However, this admits additional nullspace terms into the gain matrix which the Kalman filtering problem does not allow. We avoid nullspace terms through the assumption $R_v \succ 0$ and therefore streamline the proof of Proposition 6.

For the following definitions and lemmas, consider the system matrices $\mathcal{W} := (A, B, C, D)$ corresponding to a noise-free system.

Definition 35. The system $\mathcal{W}$ is *left invertible on* $\mathbb{I}_{0:k-1}$ if

$$0 = \begin{bmatrix} D & & & \\ CB & D & & \\ \vdots & \ddots & \ddots & \\ CA^{k-2}B & \dots & CB & D \end{bmatrix} \begin{bmatrix} u_0 \\ \vdots \\ u_{k-1} \end{bmatrix}$$

implies $u_0 = 0$. The system $\mathcal{W}$ is *left invertible* if there is some $j \in \mathbb{N}$ such that $\mathcal{W}$ is left invertible on $\mathbb{I}_{0:k-1}$ for all $k \geq j$.

Definition 36. The system $\mathcal{W}$ is *strongly detectable* if $y_k \to 0$ implies $x_k \to 0$.

The following lemmas are taken directly from [64, Thms. 8, 18(iii)], but the proofs are omitted for the sake of brevity.

Lemma 37 ([64, Thm. 8])**.** *If $\mathcal{W}$ is left invertible, then $\mathcal{W}$ is strongly detectable if and only if $(A - BF, C - DF)$ is detectable for all F of appropriate dimension.*

Lemma 38 ([64, Thm. 18(iii)])**.** *If $\mathcal{W}$ is left invertible, then the DARE*

$$P = A^\top PA - (A^\top PB + C^\top D)(B^\top PB + D^\top D)^{-1}(B^\top PA + D^\top C)$$

has a unique, stabilizing solution[3] *if and only if* $\mathcal{W}$ *is stabilizable and semistrongly detectable.*

For the remainder of this section, we consider the full rank factorization

$$\begin{bmatrix} Q_w & S_{wv} \\ S_{wv}^\top & R_v \end{bmatrix} = \begin{bmatrix} \tilde{B} \\ \tilde{D} \end{bmatrix} \begin{bmatrix} \tilde{B}^\top & \tilde{D}^\top \end{bmatrix}$$

and the dual system $\tilde{\mathcal{W}} := (A^\top, C^\top, \tilde{B}^\top, \tilde{D}^\top)$ to analyze the properties of the original system (1). The following lemma relates the properties $R_v \succ 0$ and left invertability of $\tilde{\mathcal{W}}$.

Lemma 39. *If* $R_v \succ 0$ *then* $\tilde{\mathcal{W}}$ *is left invertible.*

Proof. Left invertability on $\mathbb{I}_{0:k-1}$ is equivalent to

$$0 = \begin{bmatrix} \tilde{D}^\top & & & \\ \tilde{B}^\top C^\top & \tilde{D}^\top & & \\ \vdots & \ddots & \ddots & \\ \tilde{B}^\top (A^\top)^{k-2} C^\top & \dots & \tilde{B}^\top C^\top & \tilde{D}^\top \end{bmatrix} \begin{bmatrix} u_0 \\ \vdots \\ u_{k-1} \end{bmatrix} \tag{42}$$

implying $u_0 = 0$. But $R_v = \tilde{D}\tilde{D}^\top \succ 0$, so $\tilde{D}^\top$ has a zero nullspace. For each $k \in \mathbb{N}$, the coefficient matrix of (42) has a zero nullspace. Thus, $u_0 = 0$ and $\tilde{\mathcal{W}}$ is left invertible. □

Finally, we can prove Proposition 6.

Proof of Proposition 6. By Lemma 39, we have that $\tilde{\mathcal{W}}$ is left invertible. Therefore, by Lemma 38, the DARE (9) has a unique, stabilizing solution if and only if $\tilde{\mathcal{W}}$ is stabilizable and strongly detectable. But by Lemma 37 and duality, the latter statement is true if and only if (A, C) is detectable and $(A - FC, \tilde{B} - F\tilde{D})$ is stabilizable for all $F \in \mathbb{R}^{n\times p}$. □

B Proof of Proposition 20

Throughout this appendix, we define the set of $n \times n$ Hermitian, Hermitian positive definite, and Hermitian positive semidefinite matrices as $\mathbb{H}^n$, $\mathbb{H}^n_{++}$, and $\mathbb{H}^n_{+}$. Notice that $f_{\mathcal{D}}$ maps to Hermitian matrices so we can write it as $f : \mathbb{C} \to \mathbb{H}^m$. We define the extension of $M_{\mathcal{D}}$ to complex arguments $M_{\mathcal{D}} : \mathbb{C}^{n\times n} \times \mathbb{H}^n_+ \to \mathbb{H}^{nm}$ as

$$M_{\mathcal{D}}(A, P) := M_0 \otimes P + M_1 \otimes (AP) + M_1^\top \otimes (AP)^{\mathrm{H}}.$$

To show Proposition 20, we need a preliminary result about Hermitian positive semidefinite matrices, generalized from Lemma A.1 in [59].

Lemma 40. *For any* $M \in \mathbb{H}^n$*, if* $M \succeq 0$ *(*$M \succ 0$*) then* $\mathrm{Re}(M) \succeq 0$ *(*$\mathrm{Re}(M) \succ 0$*).*

[3] Contrary to in Section 2, here we mean the solution P is stabilizing when $A - BK(P)$ is stable, where $K(P) := (B^\top PB + D^\top D)^{-1}B^\top P$.

Proof. With $M = \mathrm{Re}(M) + \imath\mathrm{Im}(M)$, it is clear M Hermitian implies $\mathrm{Re}(M)$ is symmetric and $\mathrm{Im}(M)$ is skew-symmetric. Thus $v^\top M v = v^\top \mathrm{Re}(M) v$ for all $v \in \mathbb{R}^n$, and positive (semi)definiteness of M implies positive (semi)definiteness of $\mathrm{Re}(M)$. □

In proving Proposition 20, we take the approach of [59] but are careful to distinguish eigenvalues on the interior $\mathcal{D}$ from those on the boundary $\partial\mathcal{D}$.

Proof of Proposition 20. ($\Leftarrow$) Suppose that $M_\mathcal{D}(A, P) \succeq 0$ for some $P \succ 0$ and let $\lambda \in \lambda(A)$. Then there exists a nonzero $v \in \mathbb{C}^n$ for which $v^\mathrm{H} A = \lambda v^\mathrm{H}$. Consider the identity

$$\begin{aligned}
(I_m \otimes v)^\mathrm{H} M_\mathcal{D}(A, P)(I_m \otimes v) &= M_0 \otimes v^\mathrm{H} P v + M_1 \otimes (v^\mathrm{H} A P v) + M_1^\top \otimes (v^\mathrm{H} P A^\top v) \\
&= M_0 \otimes v^\mathrm{H} P v + M_1 \otimes (\overline{\lambda} v^\mathrm{H} P v) + M_1^\top \otimes (\lambda v^\mathrm{H} P v) \\
&= v^\mathrm{H} P v (M_0 + M_1 \lambda + M_1^\top \overline{\lambda}) \\
&= v^\mathrm{H} P v f_\mathcal{D}(\lambda).
\end{aligned}$$

The assumption $P \succ 0$ implies $v^\mathrm{H} P v > 0$, and $M_\mathcal{D}(A, P) \succeq 0$ further implies $f_\mathcal{D}(\lambda) \succeq 0$. Therefore $\lambda \in \mathrm{cl}(\mathcal{D})$.

Next suppose $\lambda \in \lambda(A)$ is non-simple and $\lambda \in \partial\mathcal{D}$. Then there exists nonzero $v_1, v_2 \in \mathbb{C}^n$ (linearly independent) such that $v^\mathrm{H} f_\mathcal{D}(\lambda) v = 0$, $v_1^\mathrm{H} A = \lambda v_1^\mathrm{H}$, and $v_2^\mathrm{H} A = \lambda v_2^\mathrm{H} + v_1$. Because $\mathcal{D}$ is open, $\lambda \in \partial\mathcal{D} = \mathrm{cl}(\mathcal{D}) \setminus \mathcal{D}$ must satisfy both $f_\mathcal{D}(\lambda) \succeq 0$ and $f_\mathcal{D}(\lambda) \not\succ 0$. Therefore $f_\mathcal{D}(\lambda)$ is singular, and there exists a nonzero vector $v \in \mathbb{C}^m$ such that $v^\mathrm{H} f_\mathcal{D}(\lambda) v = 0$. With the 2×2 matrices

$$\begin{aligned}
\tilde{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} &:= \begin{bmatrix} v_1^\mathrm{H} \\ v_2^\mathrm{H} \end{bmatrix} P \begin{bmatrix} v_1 & v_2 \end{bmatrix} \succ 0 \\
\tilde{J} &:= \lambda I_2 + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}
\end{aligned}$$

we have $\begin{bmatrix} v_1 & v_2 \end{bmatrix}^\mathrm{H} A = \tilde{J} \begin{bmatrix} v_1 & v_2 \end{bmatrix}^\mathrm{H}$ and therefore

$$\begin{aligned}
(I_m \otimes \begin{bmatrix} v_1 & v_2 \end{bmatrix})^\mathrm{H} M_\mathcal{D}(A, P)(I_m \otimes \begin{bmatrix} v_1 & v_2 \end{bmatrix}) &= M_0 \otimes \tilde{P} + M_1 \otimes \tilde{J}\tilde{P} + M_1^\top \otimes (\overline{\tilde{J}}\tilde{P})^\top \\
&= M_\mathcal{D}(\tilde{J}, \tilde{P}) \succeq 0.
\end{aligned}$$

Next, we have

$$\begin{aligned}
\tilde{M} &:= K_{2,m} M_\mathcal{D}(\tilde{J}, \tilde{P}) K_{2,m}^\top \\
&= \tilde{P} \otimes M_0 + \tilde{J}\tilde{P} \otimes M_1 + (\overline{\tilde{J}}\tilde{P})^\top \otimes M_1^\top \\
&= \tilde{P} \otimes f_\mathcal{D}(\lambda) + \begin{bmatrix} p_{12}(M_1 + M_1^\top) & p_{22} M_1 \\ p_{22} M_1^\top & 0 \end{bmatrix} \succeq 0.
\end{aligned}$$

Finally,

$$(I_2 \otimes v)^\mathrm{H} \tilde{M} (I_2 \otimes v) = \begin{bmatrix} p_{12} v^\mathrm{H} (M_1 + M_1^\top) v & p_{22} v^\mathrm{H} M_1 v \\ p_{22} v^\mathrm{H} M_1^\top v & 0 \end{bmatrix} \succeq 0.$$

But $\tilde{P} \succ 0$ implies $p_{22} > 0$, so the above matrix inequality implies $v^{\mathrm{H}} M_1 v = 0$. Moreover, with $v^{\mathrm{H}} f_{\mathcal{D}}(\lambda) v = 0$, we also have $v^{\mathrm{H}} M_0 v = 0$ and therefore $f(z) \equiv 0$ and $\mathcal{D}$ is empty, a contradiction. Therefore each $\lambda \in \lambda(A)$ non-simple implies $\lambda \in \mathcal{D}$.

($\Rightarrow$) Suppose $\lambda(A) \subset \mathrm{cl}(\mathcal{D})$ and $\lambda \in \lambda(A)$ non-simple implies $\lambda \in \mathcal{D}$.

If $A = \lambda$ is a (possibly complex) scalar, then it lies in $\mathrm{cl}(\mathcal{D})$ by assumption, with $M_{\mathcal{D}}(\lambda, p) = p f_{\mathcal{D}}(\lambda) \succeq 0$ for all $p > 0$.

If $A = \lambda I_n + N$ is a (possibly complex) Jordan block, where $N \in \mathbb{R}^{n \times n}$ is a shift matrix and $n > 1$, then $\lambda \in \mathcal{D}$ and $f_{\mathcal{D}}(\lambda) \succ 0$. Let $T_k \coloneqq \mathrm{diag}(k^{n-1}, \dots, k, 1)$ for each $k \in \mathbb{N}$. Then $T_k^{-1} A T_k = \lambda I_n + k^{-1} N \to \lambda I_n$ as $k \to \infty$. Moreover, because $M_{\mathcal{D}}$ is continuous, we have

$$M_{\mathcal{D}}(T_k^{-1} A T_k, I_n) \to M_{\mathcal{D}}(\lambda I_n, I_n) = f_{\mathcal{D}}(\lambda) \otimes I_n \succ 0.$$

Therefore there exists some $k_0 \in \mathbb{N}$ such that $M_{\mathcal{D}}(T_k^{-1} A T_k, I_n) \succ 0$ for all $k \geq k_0$. With $P \coloneqq T_k T_k^\top$, we have

$$\begin{aligned} M_{\mathcal{D}}(A, P) &= M_0 \otimes T_k T_k^\top + M_1 \otimes (A T_k T_k^\top) + M_1^\top \otimes (A T_k T_k^\top)^\top \\ &= (I_m \otimes T_k)(M_0 \otimes I_n + M_1 \otimes T_k^{-1} A T_k + M_1^\top \otimes (T_k^{-1} A T_k)^\top)(I_m \otimes T_k)^\top \\ &= (I_m \otimes T_k) M_{\mathcal{D}}(T_k^{-1} A T_k, I_n)(I_m \otimes T_k)^\top \succ 0. \end{aligned}$$

Finally, for any $A \in \mathbb{R}^{n \times n}$, let $A = V(\bigoplus_{i=1}^p J_i) V^{-1}$ denote the Jordan decomposition of A, where $J_i = \lambda_i I_{n_i} + N_i$, $\lambda_i \in \lambda(A)$, N_i are shift matrices, and $n = \sum_{i=1}^p n_i$. We have already shown that for each $i \in \mathbb{I}_{1:p}$, there exists $P_i \succ 0$ such that $M_{\mathcal{D}}(J_i, P_i) \succeq 0$. Then with $\tilde{P} \coloneqq V(\bigoplus_{i=1}^p P_i) V^{-1}$, we have

$$\begin{aligned} &(I_m \otimes V^{-1}) M_{\mathcal{D}}(A, \tilde{P})(I_m \otimes V^{-1})^{\mathrm{H}} \\ &\quad = M_0 \otimes \left(\bigoplus_{i=1}^p P_i \right) + M_1 \otimes \left(\bigoplus_{i=1}^p J_i P_i \right) + M_1 \otimes \left(\bigoplus_{i=1}^p J_i P_i \right)^\top \\ &\quad = K_{n,m} \left(\bigoplus_{i=1}^p K_{m,n_i} M_{\mathcal{D}}(J_i, P_i) K_{m,n_i}^\top \right) K_{n,m}^\top \succeq 0 \end{aligned}$$

and therefore $M_{\mathcal{D}}(A, \tilde{P}) \succeq 0$. Last, Lemma 40 gives $M_{\mathcal{D}}(A, P) \succeq 0$ with $P \coloneqq \mathrm{Re}(\tilde{P})$ since

$$M_{\mathcal{D}}(A, P) = M_{\mathcal{D}}(A, \mathrm{Re}(\tilde{P})) = \mathrm{Re}(M_{\mathcal{D}}(A, \tilde{P})). \qquad \square$$

C Proof of Proposition 21

To show Proposition 21(a), we first require the following eigenvalue sensitivity result due to [63, Thm. 7.2.3].

Theorem 41 ([63, Thm. 7.2.3])**.** *For any $A \in \mathbb{C}^{n \times n}$, denote its Schur decomposition by $A = Q(D + N)Q^{\mathrm{H}}$, where $Q \in \mathbb{C}^{n \times n}$ is unitary, $D \in \mathbb{C}^{n \times n}$ is diagonal, and $N \in \mathbb{C}^{n \times n}$ is* *strictly upper triangular.*[4] *Let p be the smallest integer for which $M^p = 0$ where $M_{ij} := |N_{ij}|$. Then*

$$\min_{\lambda\in\lambda(A)} |\mu - \lambda| \le \max\{\, c\|E\|, (c\|E\|)^{1/p} \,\}$$

where $c := \sum_{k=0}^{p-1} \|N\|^k$.

Proof of Proposition 21. Throughout this proof, we show a set S is not open (or not closed) by demonstrating that S^c (or S) does not contain all its limit points.

(a)—For any $A \in \mathbb{A}^n_{\mathcal{D}}$, continuity of $f_{\mathcal{D}}$ gives the existence of a function $\delta(\lambda) > 0$ such that $f_{\mathcal{D}}(z) \succ 0$ for all $|z-\lambda| < \delta(\lambda)$ and $\lambda \in \lambda(A)$. Let $\delta := \min_{\lambda\in\lambda(A)} \delta(\lambda)$. By Theorem 41 and norm equivalence, there exist $c > 0$ and $p \in \mathbb{I}_{1:n}$ such that

$$\max_{\mu\in\lambda(A+E)} \min_{\lambda\in\lambda(A)} |\lambda - \mu| \le \max\{\, c\|E\|_{\mathrm{F}}, (c\|E\|_{\mathrm{F}})^{1/p} \,\}$$

for all $E \in \mathbb{R}^{n\times n}$. Therefore there exists a $\varepsilon > 0$ such that

$$\max_{\mu\in\lambda(A+E)} \min_{\lambda\in\lambda(A)} |\lambda - \mu| < \delta$$

for all $E \in \mathcal{B} := \{\, E' \in \mathbb{R}^{n\times n} \mid \|E'\|_{\mathrm{F}} < \varepsilon \,\}$. Finally, $A+\mathcal{B}$ is a neighborhood of A contained in $\mathbb{A}^n_{\mathcal{D}}$, and, since $A \in \mathbb{A}^n_{\mathcal{D}}$ was chosen arbitrarily, $\mathbb{A}^n_{\mathcal{D}}$ is open.

(b)(i)—Because $\mathcal{D}$ is open, nonempty, and not equal to $\mathcal{D}$, $\partial\mathcal{D}$ is nonempty. Let $\lambda \in \partial\mathcal{D}$ and $\lambda_k \in \mathcal{D}^c$ be a sequence for which $\lambda_k \to \lambda$. By symmetry, we also have $\overline{\lambda} \in \mathcal{D}$ and $\overline{\lambda}_k \in \mathcal{D}^c$.

For $n = 2$, we have $A := \begin{bmatrix} \mathrm{Re}(\lambda) & -\mathrm{Im}(\lambda) \\ \mathrm{Im}(\lambda) & \mathrm{Re}(\lambda) \end{bmatrix} \in \mathbb{R}^{2\times 2}$ has eigenvalues $\lambda, \overline{\lambda} \in \mathcal{D}$, and $A_k := \begin{bmatrix} \mathrm{Re}(\lambda_k) & -\mathrm{Im}(\lambda_k) \\ \mathrm{Im}(\lambda_k) & \mathrm{Re}(\lambda_k) \end{bmatrix} \in \mathbb{R}^{2\times 2}$ has eigenvalues $\lambda_k, \overline{\lambda_k} \in \mathcal{D}^c$ for each $k \in \mathbb{N}$. The corresponding eigenvectors are $\begin{bmatrix} \pm\iota \\ 1 \end{bmatrix} \in \mathbb{C}^2$. Therefore $A \in \tilde{\mathbb{A}}^2_{\mathcal{D}}$ but $A_k \in (\tilde{\mathbb{A}}^2_{\mathcal{D}})^c$ for each $k \in \mathbb{N}$, and the limit $A_k \to A$ gives us that $(\tilde{\mathbb{A}}^2_{\mathcal{D}})^c$ does not contain all its limit points.

For $n > 2$, let $A_0 \in \tilde{\mathbb{A}}^{n-2}_{\mathcal{D}}$, and we can extend the prior argument with the sequence $B_k := A_k \oplus A_0 \in (\tilde{\mathbb{A}}^n_{\mathcal{D}})^c, k \in \mathbb{N}$ that converges to $B := A \oplus A_0 \in \tilde{\mathbb{A}}^n_{\mathcal{D}}$.

(b)(ii)—By part (b)(i), it suffices to consider the case $n = 1$. By closure and convexity of $\mathcal{D}$, $\mathcal{D}\cap\mathbb{R}$ is either a closed line segment, a closed ray, or $\mathbb{R}$ itself. In other words, $\mathcal{D}\cap\mathbb{R}$ is open if and only if it has no endpoints. Moreover, since $\partial\mathcal{D}\cap\mathbb{R}$ is the set of the endpoints of $\mathcal{D}\cap\mathbb{R}$, $\mathcal{D}\cap\mathbb{R}$ is open if and only if $\partial\mathcal{D}\cap\mathbb{R}$ is empty. Finally, since $\tilde{\mathbb{A}}^1_{\mathcal{D}} = \mathcal{D}\cap\mathbb{R}$, $\tilde{\mathbb{A}}^1_{\mathcal{D}}$ is open if and only if $\partial\mathcal{D}\cap\mathbb{R}$ is empty.

(c)(i)—Let $\lambda \in \partial\mathcal{D}$. Suppose $n = 4$. Then $\overline{\lambda} \in \partial\mathcal{D}$ by symmetry. Because $\mathcal{D}$ is open, there exists a sequence $\lambda_k \in \mathcal{D}$ such that $\lambda_k \to \lambda$, and by symmetry, we also have $\overline{\lambda}_k \in \mathcal{D}$ and $\overline{\lambda}_k \to \overline{\lambda}$. Consider again the 2×2 matrices A and A_k from part (b)(i), which have eigenvalues $\lambda, \overline{\lambda} \in \mathcal{D}$ and $\lambda_k, \overline{\lambda_k} \in \mathcal{D}^c$, respectively. Then the block matrices $B := \begin{bmatrix} A & I_2 \\ 0 & A \end{bmatrix} \in \mathbb{R}^{4\times 4}$ and $B_k := \begin{bmatrix} A_k & I_2 \\ 0 & A_k \end{bmatrix} \in \mathbb{R}^{4\times 4}$ have the same eigenvalues, but this time the eigenvectors are $\begin{bmatrix} \pm\iota \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \pm\iota \\ 1 \end{bmatrix} \in \mathbb{C}^4$ and the eigenvalues are non-simple. Since λ is a

[4] A matrix U is strictly upper triangular if $U_{ij} = 0$ for all $i \ge j$.

non-simple eigenvalue on the boundary of $\mathcal{D}$, we have $B \notin \tilde{\mathbb{A}}^4_{\mathcal{D}}$. However, λ_k are all in the interior of $\mathcal{D}$, so $B_k \in \tilde{\mathbb{A}}^4_{\mathcal{D}}$. Since $B_k \to B$, the set $\tilde{\mathbb{A}}^4_{\mathcal{D}}$ does not contain all its limit points.

On the other hand, let $\lambda \in \partial\mathcal{D}$ and suppose $n > 4$. Similarly to part (b)(i), with any $\tilde{A}_0 \in \tilde{\mathbb{A}}^{n-4}_{\mathcal{D}}$, we can extend the argument for the $n = 4$ case with the sequence $\tilde{A}_k := B_k \oplus \tilde{A}_0 \in \tilde{\mathbb{A}}^n_{\mathcal{D}}, k \in \mathbb{N}$ that converges to $\tilde{A} := B \oplus \tilde{A}_0 \in (\tilde{\mathbb{A}}^n_{\mathcal{D}})^c$.

(c)(ii)—Let $\lambda \in \partial\mathcal{D} \cap \mathbb{R}$ and $n \geq 2$. Because $\mathcal{D}$ is convex, open, and nonempty, there exists $\varepsilon > 0$ such that exactly one of the real intervals $(\lambda, \lambda+\varepsilon)$ or $(\lambda-\varepsilon, \lambda)$ is contained in $\mathcal{D}$, whereas the other is contained in $\mathrm{int}(\mathcal{D}^c)$. Without loss of generality, assume $(\lambda-\varepsilon, \lambda) \subseteq \mathcal{D}$.[5] Then $A_k := (\lambda-\varepsilon/k)I_n+N_n \in \tilde{\mathbb{A}}^n_{\mathcal{D}}$ for each $k \in \mathbb{N}$, but $A_k \to \lambda I_n+N_n \in (\tilde{\mathbb{A}}^n_{\mathcal{D}})^c$ and therefore $\tilde{\mathbb{A}}^n_{\mathcal{D}}$ does not contain all its limit points. (d)—Since $\overline{\mathbb{A}}^n_{\mathcal{D}} := \{ A \in \mathbb{R}^{n\times n} \mid \lambda(A) \subset \mathrm{cl}(\mathcal{D}) \}$ contains $\mathbb{A}^n_{\mathcal{D}}$, it suffices to show any $A \in \overline{\mathbb{A}}^n_{\mathcal{D}}$ is a limit point of $\mathbb{A}^n_{\mathcal{D}}$. Denote the Jordan form by $A = V \left(\bigoplus_{i=1}^p \mu_i I_{n_i} + N_{n_i}\right) V^{-1}$, where $V \in \mathbb{R}^{n\times n}$ is invertible, $\mu_i \in \lambda(A)$, $n = \sum_{i=1}^p n_i$, and $N_i \in \mathbb{R}^{n_i\times n_i}$ is a shift matrix. Because $\mu_i \in \mathrm{cl}(\mathcal{D})$, there exists a sequence $\mu_{i,k} \in \mathcal{D}$ such that $\mu_{i,k} \to \mu_i$. Then $A_k := V \left(\bigoplus_{i=1}^p \mu_{i,k} I_{n_i} + N_i\right) V^{-1} \in \mathbb{A}^n_{\mathcal{D}}$ and $A_k \to A$. □

D Proof of Proposition 22

To prove Proposition 22(a,b), we use sensitivity results on the value functions of parameterized nonlinear SDPs,

$$V(y) := \inf_{x\in\mathbb{X}(y)} F(x, y) \tag{43}$$

where the set-valued function $\mathbb{X} : \mathbb{R}^m \to \mathcal{P}(\mathbb{R}^n)$ is defined by

$$\mathbb{X}(y) := \{ x \in \mathbb{R}^n \mid G(x, y) \succeq 0 \} .$$

Consider also the graph of the set-valued function $\mathbb{X}$,

$$\mathbb{Z} := \{ (x, y) \in \mathbb{R}^{n+m} \mid G(x, y) \succeq 0 \} .$$

Notice that $\mathbb{Z}$ is closed if G is continuous. We say Slater's condition holds at $y \in \mathbb{R}^m$ if there exists $x \in \mathbb{R}^n$ such that $x \in \mathrm{int}(\mathbb{X}(y))$, or equivalently, $G(x, y) \succ 0$.

In [71, Prop. 4.4], continuity of a general class of optimization problems is considered. In the following proposition, we state the specialization to nonlinear SDPs.

Proposition 42 ([71, Prop. 4.4])**.** *Let $y_0 \in \mathbb{R}^m$ and suppose*

(i) F and G are continuous on $\mathbb{R}^{n+m}$;

(ii) there exist $\alpha \in \mathbb{R}$ and compact $C \subset \mathbb{R}^n$ such that, for each y in a neighborhood of y_0, the level set

$$\mathrm{lev}_{\leq\alpha} F(\cdot, y) := \{ x \in \mathbb{X}(y) \mid F(x, y) \leq \alpha \}$$

is nonempty and contained in C; and

(iii) Slater's condition holds at y_0.

[5] Otherwise, take the reflection about the imaginary axis $-\mathcal{D}$ and $-\tilde{\mathbb{A}}^n_{\mathcal{D}}$.

Then $F(\cdot, y)$ attains a minimum on $\mathbb{X}(y)$ for all $y \in N_y$, and $V(y)$ is continuous at $y = y_0$.

Proof. See [71, Prop. 4.4] and the discussions in [71, pp. 264, 483–484, 491–492]. □

Finally, we prove Proposition 22.

Proof of Proposition 22. Let $\mathrm{vec} : \mathbb{R}^{n\times n} \to \mathbb{R}^{n^2}$ and $\mathrm{vecs} : \mathbb{R}^{n\times n} \to \mathbb{R}^{(1/2)(n+1)n}$ denote the vectorization and symmetric vectorization operators, respectively.

(a)—With $x := \mathrm{vecs}(P)$, $y := \mathrm{vec}(A)$, $F(x, y) := \mathrm{tr}(VP)$, and $G(x, y) := P \oplus (M_{\mathcal{D}}(A, P) - M)$, we can use Proposition 42 to show the continuity of $\phi_{\mathcal{D}}$ on $\mathbb{A}^n_{\mathcal{D}}$. Let $A_0 \in \mathbb{A}^n_{\mathcal{D}}$. Condition (i) of Proposition 42 holds by assumption. Slater's condition (iii) holds because for any $P \succ 0$ such that $M_{\mathcal{D}}(A_0, P) \succ 0$, we can define $P_0 := \gamma P \succ 0$ for some $\gamma > \gamma_0 := \|M\| \times \|[M_{\mathcal{D}}(A_0, P)]^{-1}\|$ to give

$$M_{\mathcal{D}}(A_0, P_0) = \gamma M_{\mathcal{D}}(A_0, P) \succ \gamma_0 M_{\mathcal{D}}(A_0, P) \succeq M.$$

Moreover, by continuity of $M_{\mathcal{D}}$, there exists a neighborhood N_A of A_0 such that $M_{\mathcal{D}}(A, P_0) \succ M$ for all $A \in N_A$. Letting $\alpha := \mathrm{tr}(VP_0) > 0$, we have that the set

$$\{\, P \in \mathbb{S}^n_+ \mid \mathrm{tr}(VP) \le \alpha \,\}$$

is compact and contains the nonempty level set

$$\{\, P \in \mathbb{P}(A) \mid \mathrm{tr}(VP) \le \alpha \,\}$$

for all $A \in N_A$. Taking the image of each of the above sets under the vecs operation gives condition (ii) of Proposition 42. All the conditions of Proposition 42 are thus satisfied for each $A_0 \in \mathbb{A}^n_{\mathcal{D}}$, and we have $\phi_{\mathcal{D}}$ is continuous on $\mathbb{A}^n_{\mathcal{D}}$.

(b)—Continuity of $\phi_{\mathcal{D}}$ on $\mathbb{A}^n_{\mathcal{D}}$ implies closure of the sublevel sets of $\phi_{\mathcal{D}}$, and (23) follows by definition of $\mathbb{A}^n_{\mathcal{D}}(\varepsilon)$.

(c)—First, $M_{\mathcal{D}}(A, P) \succ 0$ implies $P \succ 0$ since, if $M_{\mathcal{D}}(A, P) \succ 0$ and $P \succeq 0$ but $P \not\succ 0$, there exists a nonzero $v \in \mathbb{R}^n$ such that $Pv = 0$ and

$$\begin{aligned}(I_m \otimes v)^\top (M_{\mathcal{D}}(A, P))(I_m \otimes v) = M \otimes (v^\top P v) \\ + M_1 \otimes (v^\top A P v) + M_1^\top \otimes (v^\top P A^\top v) = 0\end{aligned}$$

a contradiction of the assumption $M_{\mathcal{D}}(A, P) \succ 0$. Moreover, for any $P \succ 0$ such that $M_{\mathcal{D}}(A, P) \succ 0$, we have $M_{\mathcal{D}}(A, P) \succeq \gamma M_{\mathcal{D}}(A, P) \succeq M$ with $P := \gamma P$ and $\gamma := \|M\| \times \|[M_{\mathcal{D}}(A, P)]^{-1}\|$, so feasibility of (17) is equivalent to feasibility of

$$M_{\mathcal{D}}(A, P) \succ M, \qquad P \succeq 0$$

and therefore $\bigcup_{\varepsilon>0} \mathbb{A}^n_{\mathcal{D}}(\varepsilon) = \mathbb{A}^n_{\mathcal{D}}$. But $\mathbb{A}^n_{\mathcal{D}}(\varepsilon)$ is monotonically decreasing,[6] so $\mathbb{A}^n_{\mathcal{D}}(\varepsilon) \nearrow \bigcup_{\varepsilon>0} \mathbb{A}^n_{\mathcal{D}}(\varepsilon) = \mathbb{A}^n_{\mathcal{D}}$ as $\varepsilon \searrow 0$. □

[6] By "monotonically decreasing" we mean $\mathbb{A}^n_{\mathcal{D}}(\varepsilon) \supseteq \mathbb{A}^n_{\mathcal{D}}(\varepsilon')$ for all $\varepsilon \le \varepsilon'$.

E Proof of Propositions 33 and 34

Starting with Proposition 33:

Proof of Proposition 33. Since μ_ε is nondecreasing and bounded from below by μ, it suffices to show that for each $\delta > 0$, there exists a $\overline{\varepsilon} > 0$ such that $\mu_{\overline{\varepsilon}} - \mu < \delta$.

Let $(\beta^*, \Sigma^*) \in \Theta$ denote a point for which $\mu = f(\beta^*, \Sigma^*)$. If $(\beta^*, \Sigma^*) \in \Theta_{++}$, we could simply choose $\overline{\varepsilon} > 0$ small enough to put (β^*, Σ^*) in $\Theta_{\overline{\varepsilon}}$ and achieve $\mu_{\overline{\varepsilon}} - \mu = 0 < \delta$.

Instead, we assume $(\beta^*, \Sigma^*) \notin \Theta_{++}$. By Assumption 2, there exists a sequence $(\beta_k, \Sigma_k) \in \Theta_{++}, k \in \mathbb{N}$ such that $\beta_k \to \beta$ and $\Sigma_k \to \Sigma$ as $k \to \infty$. Defining $\nu_k := f(\beta_k, \Sigma_k)$, we have $\nu_k \to \mu$ by continuity of f. Therefore, there exists some $k_0 \in \mathbb{N}$ such that $\nu_k - \mu < \delta$ for all $k \geq k_0$.

For each $(\beta_k, \Sigma_k) \in \Theta_{++}$, there exist unique $L_k^{\mathcal{I}_\Sigma} \in \mathbb{L}_{++}^{n_\Sigma}[\mathcal{I}_\Sigma]$ and $L_k^{\mathcal{I}_\mathcal{A}} \in \mathbb{L}_{++}^{n_\mathcal{A}}[\mathcal{I}_\mathcal{A}]$ such that the constraints

$$\tilde{g}(\beta_k, L_k^{\mathcal{I}_\Sigma}, L_k^{\mathcal{I}_\mathcal{A}}) = 0 \qquad\qquad h(\beta_k, L_k^{\mathcal{I}_\Sigma}) \leq 0$$

are satisfied (by Lemmas 29 and 31). Let $\overline{\varepsilon}$ be the minimum value over all the diagonal elements of $L_{k_0}^{\mathcal{I}_\Sigma}$ and $L_{k_0}^{\mathcal{I}_\mathcal{A}}$. Then $(\beta_{k_0}, L_{k_0}^{\mathcal{I}_\Sigma}) \in \Theta_{\overline{\varepsilon}}$ by construction, so $\nu_{k_0} \geq \mu_{\overline{\varepsilon}}$ by optimality, and therefore $\mu_{\overline{\varepsilon}} - \mu \leq \nu_{k_0} - \mu < \delta$. □

As in Appendix D, we use sensitivity results of [71] on optimization problems to to prove Proposition 34. This time, however, we consider the continuity of the value function for parameterized NLPs on Banach spaces. Let $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{K}$ be Banach spaces and consider the parameterized NLP,

$$V(y) := \inf_{x \in \mathbb{X}(y)} F(x, y) \tag{44}$$

where the set-valued function $\mathbb{X} : \mathcal{Y} \to \mathcal{P}(\mathcal{X})$ is defined by

$$\mathbb{X}(y) := \{\, x \in \mathcal{X} \mid G(x, y) \in K \,\}$$

for some $G : \mathcal{X} \times \mathcal{Y} \to \mathcal{K}$ and $K \subseteq \mathcal{K}$ is closed. Let $X^0(y)$ denote the (possibly empty) set of solutions to (44). Define the graph of the set-valued function $\mathbb{X}(\cdot)$ by

$$\mathbb{Z} := \{\, (x, y) \in \mathcal{X} \times \mathcal{Y} \mid G(x, y) \in K \,\}.$$

Notice that $\mathbb{Z}$ is closed if G is continuous and K is closed.

Proposition 43 ([71, Prop. 4.4])**.** *Let $y_0 \in \mathcal{Y}$ and assume:*

(i) *F and G are continuous on $\mathcal{X} \times \mathcal{Y}$ and K is closed;*

(ii) *there exist $\alpha \in \mathbb{R}$ and a compact set $C \subseteq \mathcal{X}$ such that, for every y in a neighborhood of y_0, the level set*

$$\{\, x \in \mathbb{X}(y) \mid f(x, y) \leq \alpha \,\}$$

is nonempty and contained in C; and

(iii) for any neighborhood N_x of the solution set $X^0(y_0)$, there exists a neighborhood N_y of y_0 such that $N_x \cap \mathbb{X}(y)$ is nonempty for all $y \in N_y$;

then $V(y)$ is continuous and $X^0(y)$ is outer semicontinuous at $y = y_0$.

Proof of Proposition 34. First, we must specify $\overline{\varepsilon}$. For each $\theta \in \Theta_{++}$, let

$$\varepsilon(\theta) := \max \{ \varepsilon > 0 \mid \theta \in \Psi(\Theta_\varepsilon) \}$$

where the maximum is achieved since there is a finite number of diagonal elements of the Cholesky factors that must be lower bounded. Now we specify $\overline{\varepsilon}$ as the supremum of $\varepsilon(\theta)$ over all $\theta \in \Theta_{f\leq\alpha} \cap \Theta_{++}$,

$$\overline{\varepsilon} := \sup \{ \varepsilon(\theta) \mid \theta \in \Theta_{f\leq\alpha} \cap \Theta_{++} \}$$

so that, for any $\varepsilon \in (0, \overline{\varepsilon})$, $\Theta_{f\leq\alpha} \cap \Psi(\Theta_\varepsilon)$ is nonempty and is contained in the compact set C.

(a)—Following the proof of [71, Prop. 4.4], we have (i) F is continuous and (ii) the level set $\Theta_{f\leq\alpha}$ is nonempty and contained in the compact set C, which implies $\Theta_{f\leq\alpha}$ is a compact level set and therefore the minimum of f over $\Theta_{f\leq\alpha}$ is achieved and equals the minimum over Θ. Moreover, $\hat{\theta}_0$ must be nonempty.

(b)—Similarly to part (a), we have, for each $\varepsilon \in (0, \overline{\varepsilon})$, that the level set $\Theta_{f\leq\alpha} \cap \Psi(\Theta_\varepsilon)$ is nonempty and contained in the compact set C, so f achieves its minimum over $\Psi(\Theta_\varepsilon)$ and $\hat{\theta}_\varepsilon$ is nonempty.

(c)—Consider the graph of the constraint function,

$$\mathbb{Z} := \{ (\theta, \varepsilon) \in \Theta \times \mathbb{R}_{\geq 0} \mid \theta \in \Psi(\Theta_\varepsilon) \text{ if } \varepsilon > 0 \}.$$

Consider a sequence $(\theta_k, \varepsilon_k) \in \mathbb{Z}, k \in \mathbb{N}$ that is convergent $(\theta_k, \varepsilon_k) \to (\theta, \varepsilon)$. Then $\varepsilon \geq 0$, otherwise the sequence would not converge. Moreover, $\theta \in \Theta$ since $\theta_k \in \Psi(\Theta_{\varepsilon_k}) \subseteq \Theta$ for all $k \in \mathbb{N}$ and Θ contains all its limit points. If $\varepsilon = 0$, then $(\theta, \varepsilon) \in \mathbb{Z}$ trivially. On the other hand, if $\varepsilon > 0$, then $\varepsilon(\theta_k)$ converges to $\varepsilon(\theta)$ because Ψ is continuous and the max can be taken over a finite number of elements of $\Psi^{-1}(\theta_k)$. Moreover, $\varepsilon(\theta_k)$ and upper bounds ε_k because $\theta_k \in \Psi(\Theta_{\varepsilon_k})$, so $\varepsilon(\theta) \geq \varepsilon$. Finally, we have $\theta \in \Psi(\Theta_\varepsilon)$, $(\theta, \varepsilon) \in \mathbb{Z}$, and $\mathbb{Z}$ is closed.

Let $\varepsilon_0 \geq 0$ and N_θ be a neighborhood of $\hat{\theta}_{\varepsilon_0}$. With

$$\delta := \sup \{ \varepsilon(\theta) \mid \theta \in N_\theta \} > 0$$

we have $N_\theta \cap \Theta$ and $N_\theta \cap \Psi(\Theta_\varepsilon)$ are nonempty for all $\varepsilon \in (0, \varepsilon_0 + \delta)$.

Finally, the requirements of [71, Prop. 4.4] are satisfied for all $\varepsilon_0 \in [0, \overline{\varepsilon})$, so μ_ε is continuous and $\hat{\theta}_\varepsilon$ is outer semicontinuous at $\varepsilon = \varepsilon_0$.

(d)—The last statement follows by the definition of outer semicontinuity and the fact that the $\limsup$ is nonempty. $\square$

References

[1] L. Ljung, *System Identification: Theory for the User*, 2nd ed. New Jersey: Prentice Hall, 1999.

[2] E. J. Hannan and M. Deistler, *The Statistical Theory of Linear Systems*, ser. Classics in Applied Mathematics. Society for Industrial and Applied Mathematics, 2012.

[3] R. H. Shumway and D. S. Stoffer, *Time Series Analysis and its Applications*, 4th ed. Cham, Switzerland: Springer International Publishing, 2017.

[4] K. Åström and P. Eykhoff, "System identification—A survey," *Automatica*, vol. 7, no. 2, pp. 123–162, 1971.

[5] L. Ljung, "Convergence analysis of parametric identification methods," *IEEE Trans. Auto. Cont.*, vol. 23, no. 5, pp. 770–783, 1978.

[6] K. J. Åström, "Special section system identification tutorial: Maximum likelihood and prediction error methods," *Automatica*, vol. 16, no. 5, pp. 551–574, 1980.

[7] R. H. Shumway and D. S. Stoffer, "An approach to time series smoothing and forecasting using the EM algorithm," *J. Time Series Anal.*, vol. 3, pp. 253–264, 1982.

[8] V. Digalakis, J. Rohlicek, and M. Ostendorf, "ML estimation of a stochastic linear system with the EM algorithm and its application to speech recognition," *IEEE Trans. Speech and Audio Proc.*, vol. 1, no. 4, pp. 431–442, 1993.

[9] S. Gibson and B. Ninness, "Robust maximum-likelihood estimation of multivariable dynamic systems," *Automatica*, vol. 41, no. 10, pp. 1667–1682, 2005.

[10] M. Dewar and V. Kadirkamanathan, "A Canonical Space-Time State Space Model: State and Parameter Estimation," *IEEE Trans. Signal Process.*, vol. 55, no. 10, pp. 4862–4870, 2007.

[11] C. G. Källström and K. J. Åström, "Experiences of System Identification Applied to Ship Steering Dynamics," *IFAC Proceedings Volumes*, vol. 12, no. 8, pp. 173–184, 1979.

[12] N. R. Kristensen, H. Madsen, and S. B. Jørgensen, "A method for systematic improvement of stochastic grey-box models," *Comput. Chem. Eng.*, vol. 28, no. 8, pp. 1431–1449, 2004.

[13] E. Shahinfar, M. Bozorg, and M. Bidoky, "Parameter Estimation of an AUV Using the Maximum Likelihood method and a Kalman Filter with Fading Memory," *IFAC Proceedings Volumes*, vol. 43, no. 16, pp. 1–6, 2010.

[14] L. Simpson, A. Ghezzi, J. Asprion, and M. Diehl, "An Efficient Method for the Joint Estimation of System Parameters and Noise Covariances for Linear Time-Variant Systems," in *2023 62nd IEEE Conference on Decision and Control (CDC)*, Dec 2023, pp. 4524–4529.

[15] T. McKelvey, A. Helmersson, and T. Ribarits, "Data driven local coordinates for multivariable linear systems and their application to system identification," *Automatica*, vol. 40, no. 9, pp. 1629–1635, 2004.

[16] T. Ribarits, M. Deistler, and B. Hanzon, "An analysis of separable least squares data driven local coordinates for maximum likelihood estimation of linear systems," *Automatica*, vol. 41, no. 3, pp. 531–544, 2005.

[17] P. Li, I. Postlethwaite, and M. Turner, "Parameter Estimation Techniques for Helicopter Dynamic Modelling," in *2007 American Control Conference*, 2007, pp. 2938–2943.

[18] J. Umenberger, J. Wågberg, I. R. Manchester, and T. B. Schön, "Maximum likelihood identification of stable linear dynamical systems," *Automatica*, vol. 96, pp. 280–292, 2018.

[19] R. A. Redner and H. F. Walker, "Mixture Densities, Maximum Likelihood and the EM Algorithm," *SIAM Rev.*, vol. 26, no. 2, pp. 195–239, 1984.

[20] O. Bermond and J.-F. Cardoso, "Approximate likelihood for noisy mixtures," in *International Workshop on Independent Component Analysis (ICA '99)*, vol. 99, Aussois, France, 1999, pp. 325–330.

[21] K. B. Petersen, O. Winther, and L. K. Hansen, "On the Slow Convergence of EM and VBEM in Low-Noise Linear Models," *Neural Comput.*, vol. 17, no. 9, pp. 1921–1926, 2005.

[22] K. B. Petersen and O. Winther, "Explaining slow convergence of EM in low noise linear mixtures," Informatics and Mathematical Modelling, Technical University of Denmark, Technical Report 2005-2, 2005. [Online]. Available: https://core.ac.uk/download/pdf/13755227.pdf

[23] R. K. Olsson, K. B. Petersen, and T. Lehn-Schiøler, "State-Space Models: From the EM Algorithm to a Gradient Approach," *Neural Comput.*, vol. 19, no. 4, pp. 1097–1111, 2007.

[24] T. Westerlund, "A digital quality control system for an industrial dry process rotary cement kiln," *IEEE Trans. Auto. Cont.*, vol. 26, no. 4, pp. 885–890, 1981.

[25] M. H. Caveness and J. J. Downs, "Reactor control using infinite horizon model predictive control," AIChE Spring Meeting, Atlanta, GA, April 2005.

[26] H. Raghavan, A. K. Tangirala, B. R. Gopaluni, and S. L. Shah, "Identification of chemical processes with irregular output sampling," *Control Eng. Pract.*, vol. 14, no. 5, pp. 467–480, 2006.

[27] L. W. Taylor, "On-Orbit Systems Identification of Flexible Spacecraft," *IFAC Proceedings Volumes*, vol. 18, no. 5, pp. 511–516, 1985.

[28] H. Melgaard, E. Hendricks, and H. Madsen, "Continuous Identification of a Four-Stroke SI Engine," in *1990 American Control Conference*, 1990, pp. 1876–1881.

[29] K. Åström and C. Källström, "Identification of ship steering dynamics," *Automatica*, vol. 12, no. 1, pp. 9–22, 1976.

[30] A. B. W. Jr., "On the numerical solution of initial/boundary-value problems in one space dimension," *SIAM J. Numer. Anal.*, vol. 19, no. 4, pp. 683–697, 1982.

[31] H. White, "Maximum Likelihood Estimation of Misspecified Dynamic Models," in *Misspecification Analysis*, ser. Lecture Notes in Economics and Mathematical Systems, T. K. Dijkstra, Ed. Berlin, Heidelberg: Springer, 1984, pp. 1–19.

[32] B. D. O. Anderson, J. B. Moore, and R. M. Hawkes, "Model approximations via prediction error identification," *Automatica*, vol. 14, no. 6, pp. 615–622, 1978.

[33] S. J. Qin and T. A. Badgwell, "A survey of industrial model predictive control technology," *Control Eng. Pract.*, vol. 11, no. 7, pp. 733–764, 2003.

[34] J. B. Rawlings, D. Q. Mayne, and M. M. Diehl, *Model Predictive Control: Theory, Design, and Computation*, 2nd ed. Santa Barbara, CA: Nob Hill Publishing, 2020, 770 pages, ISBN 978-0-9759377-5-4.

[35] W. M. Canney, "The future of advanced process control promises more benefits and sustained value," *Oil & Gas J.*, vol. 101, no. 16, pp. 48–54, Apr 2003.

[36] M. L. Darby and M. Nikolaou, "MPC: Current practice and challenges," *Control Eng. Pract.*, vol. 20, no. 4, pp. 328 – 342, 2012.

[37] K. R. Muske and T. A. Badgwell, "Disturbance modeling for offset-free linear model predictive control," *J. Proc. Cont.*, vol. 12, no. 5, pp. 617–632, 2002.

[38] G. Pannocchia and J. B. Rawlings, "Disturbance models for offset-free MPC control," *AIChE J.*, vol. 49, no. 2, pp. 426–437, 2003.

[39] M. Wallace, B. Das, P. Mhaskar, J. House, and T. Salsbury, "Offset-free model predictive control of a vapor compression cycle," *J. Proc. Cont.*, vol. 22, no. 7, pp. 1374–1386, 2012.

[40] M. Wallace, P. Mhaskar, J. House, and T. I. Salsbury, "Offset-Free Model Predictive Control of a Heat Pump," *Ind. Eng. Chem. Res.*, vol. 54, no. 3, pp. 994–1005, 2015.

[41] P. Schmid and P. Eberhard, "Offset-free Nonlinear Model Predictive Control by the Example of Maglev Vehicles," *IFAC–P. Online*, vol. 54, no. 6, pp. 83–90, 2021.

[42] F. Xu, Y. Shi, K. Zhang, and X. Xu, "Real-Time Application of Robust Offset-Free MPC in Maglev Planar Machine," *IEEE Trans. Indus. Elec.*, vol. 70, no. 6, pp. 6121–6130, 2023.

[43] R. Huang, S. C. Patwardhan, and L. T. Biegler, "Offset-free Nonlinear Model Predictive Control Based on Moving Horizon Estimation for an Air Separation Unit," *IFAC Proceedings Volumes*, vol. 43, no. 5, pp. 631–636, 2010.

[44] L. N. Petersen, N. K. Poulsen, H. H. Niemann, C. Utzen, and J. B. Jørgensen, "Comparison of three control strategies for optimization of spray dryer operation," *J. Proc. Cont.*, vol. 57, 2017.

[45] S.-K. Kim, D.-K. Choi, K.-B. Lee, and Y. I. Lee, "Offset-Free Model Predictive Control for the Power Control of Three-Phase AC/DC Converters," *IEEE Trans. Indus. Elec.*, vol. 62, no. 11, pp. 7114–7126, 2015.

[46] F. A. Bender, S. Goltz, T. Braunl, and O. Sawodny, "Modeling and Offset-Free Model Predictive Control of a Hydraulic Mini Excavator," *IEEE Trans. Auto. Sci. Eng.*, vol. 14, no. 4, pp. 1682–1694, 2017.

[47] D. Deenen, E. Maljaars, L. Sebeke, B. De Jager, E. Heijman, H. Grüll, and W. P. M. H. Heemels, "Offset-free model predictive control for enhancing MR-HIFU hyperthermia in cancer treatment," *IFAC–P. Online*, vol. 51, no. 20, pp. 191–196, 2018.

[48] B. J. Odelson, A. Lutz, and J. B. Rawlings, "Application of autocovariance least-squares methods to laboratory data," TWMCC, Department of Chemical Engineering, University of Wisconsin-Madison, Tech. Rep. 2003–03, September 2003. [Online]. Available: https://engineering.ucsb.edu/~jbraw/jbrweb-archives/tech-reports/twmcc-2003-03.pdf

[49] M. A. Zagrobelny and J. B. Rawlings, "Identifying the uncertainty structure using maximum likelihood estimation," in *American Control Conference*, Chicago, IL, July 1–3, 2015, pp. 422–427.

[50] S. J. Kuntz and J. B. Rawlings, "Maximum likelihood estimation of linear disturbance models for offset-free model predictive control," in *American Control Conference*, Atlanta, GA, June 8–10, 2022, pp. 3961–3966.

[51] S. J. Kuntz, J. J. Downs, S. M. Miller, and J. B. Rawlings, "An industrial case study on the combined identification and offset-free control of a chemical process," *Comput. Chem. Eng.*, vol. 179, 2023.

[52] F. Dorfler, J. Coulson, and I. Markovsky, "Bridging Direct & Indirect Data-Driven Control Formulations via Regularizations and Relaxations," *IEEE Trans. Auto. Cont.*, vol. 68, no. 2, pp. 883–897, 2022.

[53] J. Berberich, J. Köhler, M. A. Müller, and F. Allgöwer, "Linear Tracking MPC for Nonlinear Systems—Part II: The Data-Driven Case," *IEEE Trans. Auto. Cont.*, vol. 67, no. 9, pp. 4406–4421, 2022.

[54] Z. Yuan and J. Cortés, "Data-Driven Optimal Control of Bilinear Systems," *IEEE Ctl. Sys. Let.*, vol. 6, pp. 2479–2484, 2022.

[55] G. Bianchin, M. Vaquero, J. Cortés, and E. Dall'Anese, "Online Stochastic Optimization for Unknown Linear Systems: Data-Driven Controller Synthesis and Analysis," *IEEE Trans. Auto. Cont.*, pp. 1–15, 2023.

[56] J. C. Willems, P. Rapisarda, I. Markovsky, and B. L. M. De Moor, "A note on persistency of excitation," *Sys. Cont. Let.*, vol. 54, no. 4, pp. 325–329, 2005.

[57] M. Diehl, K. Mombaur, and D. Noll, "Stability Optimization of Hybrid Periodic Systems via a Smooth Criterion," *IEEE Trans. Auto. Cont.*, vol. 54, no. 8, pp. 1875–1880, 2009.

[58] J. Vanbiervliet, B. Vandereycken, W. Michiels, S. Vandewalle, and M. Diehl, "The Smoothed Spectral Abscissa for Robust Stability Optimization," *SIAM J. Optim.*, vol. 20, no. 1, pp. 156–171, 2009.

[59] M. Chilali and P. Gahinet, "H_∞ design with pole placement constraints: an LMI approach," *IEEE Trans. Auto. Cont.*, vol. 41, no. 3, pp. 358–367, 1996.

[60] S. Burer, R. D. Monteiro, and Y. Zhang, "Solving a class of semidefinite programs via nonlinear programming," *Math. Prog.*, vol. 93, no. 1, pp. 97–122, Jun 2002.

[61] J. Park, R. A. Martin, J. D. Kelly, and J. D. Hedengren, "Benchmark temperature microcontroller for process dynamics and control," *Comput. Chem. Eng.*, vol. 135, Apr 2020.

[62] D. N. Miller and R. A. De Callafon, "Subspace identification with eigenvalue constraints," *Automatica*, vol. 49, no. 8, pp. 2468–2473, 2013.

[63] G. H. Golub and C. F. Van Loan, *Matrix Computations*, 4th ed. Baltimore, MD: The Johns Hopkins University Press, 2013.

[64] L. M. Silverman, "Discrete Riccati equations: Alternative algorithms, asymptotic properties, and system theory interpretations," in *Control and Dynamic Systems*, ser. Control and Dynamic Systems, C. T. Leondes, Ed. Academic Press, 1976, vol. 12, pp. 313–386.

[65] O. Y. Kushel, "Geometric properties of LMI regions," 2019, arXiv:1910.10372 [math]. [Online]. Available: http://arxiv.org/abs/1910.10372

[66] I. Kollar, G. Franklin, and R. Pintelon, "On the equivalence of z-domain and s-domain models in system identification," in *Quality Measurement: The Indispensable Bridge between Theory and Reality (No Measurements? No Science! Joint Conference - 1996: IEEE Instrumentation and Measurement Technology Conference and IMEKO Tec*, vol. 1, 1996, pp. 14–19.

[67] R. J. Caverly and J. R. Forbes, "LMI Properties and Applications in Systems, Stability, and Control Theory," 2021, arXiv:1903.08599 [cs, math]. [Online]. Available: http://arxiv.org/abs/1903.08599

[68] M. Denham, "Canonical forms for the identification of multivariable linear systems," *IEEE Trans. Auto. Cont.*, vol. 19, no. 6, pp. 646–656, 1974.

[69] W. E. Larimore, "Canonical variate analysis in identification, filtering, and adaptive control," in *Proceedings of the 29th Conference on Decision and Control*, 1990, pp. 596–604.

[70] M. Yin, A. Iannelli, and R. S. Smith, "Maximum Likelihood Estimation in Data-Driven Modeling and Control," *IEEE Trans. Auto. Cont.*, vol. 68, no. 1, pp. 317–328, 2023.

[71] J. F. Bonnans and A. Shapiro, *Perturbation Analysis of Optimization Problems*, ser. Springer Series in Operations Research. New York, NY: Springer Science & Business Media, 2000.