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Maximum likelihood identification of linear models with integrating disturbances for offset-free control* †

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Abstract

This paper addresses the identification of models for offset-free model predictive control (MPC), where LTI models are augmented with (fictitious) uncontrollable integrating modes, called integrating disturbances. The states and disturbances are typically estimated with a Kalman filter. The disturbance estimates effectively provide integral control, so the quality of the disturbance model (and resulting filter) directly influences the control performance. We implement eigenvalue constraints to protect against undesirable filter behavior (unstable or marginally stable modes, high-frequency oscillations). Specifically, we consider the class of linear matrix inequality (LMI) regions for eigenvalue constraints. These LMI regions are open sets by default, so we introduce a barrier function method to create tightened, but closed, eigenvalue constraints. To solve the resulting nonlinear semidefinite program, we approximate it as a nonlinear program using a Cholesky factorization method that exploits known sparsity structures of semidefinite optimization variables and matrix inequalities. The algorithm is applied to real-world data taken from two physical systems: first, a low-cost benchmark temperature microcontroller suitable for classroom laboratories, and second, an industrialscale chemical reactor at Eastman Chemical's plant in Kingsport, TN.

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[†]Version 2: The text and case studies are updated. The main technical results remain unchanged.

1 Introduction

Offset-free model predictive control (MPC) is a widely-used advanced control method that combines regulation, estimation, and steady-state optimization problems to track prescribed setpoints [1, 2]. In linear offset-free MPC, a stochastic linear time-invariant (LTI) model is augmented with *uncontrollable* integrating modes, called *integrating disturbances*, providing integral action through the estimator—typically a Kalman filter—and allowing offset-free tracking even in the presence of plant-model mismatch and persistent disturbances [3, 4]. We call such a model a *linear augmented disturbance model* (LADM).

The LADM or its corresponding Kalman filter can either "tuned" by hand or identified automatically from data. Common tuning methods include pole placement [5–8], covariance matrix selection [9–11], and filter gain selection [12–14]. Disturbance models can be identified with autocovariance least squares estimation [15] or maximum likelihood (ML) identification [16–19].

Tuning of integrating disturbance models can be a time-consuming and ad-hoc procedure, requiring simplified parameterizations (e.g., diagonal covariance matrices). In prior work, we have suggested identification as the preferred strategy for acquiring LADMs [16, 17]. In this work, we further develop ML identification because of its desirable statistical properties (consistency, asymptotic efficiency) and ability to handle general model structures and constraints [20, 21].

Design constraints can be included in tuning procedures to avoid undesirable filter behaviors (slow response time, fictitious high frequencies) that are passed to the control performance through the integrating disturbance estimates. Control-relevant design constraints and prior knowledge have sometimes been incorporated into identification problems [22–24]. However, there are no general approaches to shaping the closed-loop filter behavior in ML identification. To address this gap, we consider ML identification with eigenvalue constraints implemented via the LMI regions commonly used in robust control [25, 26].

LMI region constraints have been used in subspace identification [27]. However, subspace identification cannot be used for LADM identification as it is not possible to impose the required disturbance model structure. Open-loop stability constraints have been included in the expectation maximization (EM) algorithm [28], but this formulation is not obviously generalized to filter stability or general LMI region constraints.

While EM is an algorithm for ML, it does not have strong convergence guarantees. While it can be shown that the EM iterates produce, almost surely, an increasing sequence of likelihood values [29, 30], slow convergence at low noise levels has been reported on a range of problems [28, 31–35]. Interior point, and even gradient methods [35], are therefore preferable to the EM approach.

As originally posed by [25, 26], LMI regions are strict semidefinite matrix inequalities. While [27] used relaxed LMI regions with nonstrict inequalities, as we show in Section 4, the constraint sets are not closed, and thus problematic as optimization constraints. To address this issue, we formulate tightened LMI region constraints that define a closed constraint set. This formulation introduces nonlinear matrix inequalities and semidefinite matrix arguments, making the ML problem a nonlinear semidefinite program (NSDP). To efficiently convert the NSDP to a nonlinear program (NLP), we generalize the Burer-Monteiro-Zhang (BMZ) method [36, 37], which was originally used to convert sparse semidefinite matrix arguments into vector arguments with minimal dimension. An additional advantage of the BMZ method over standard Cholesky factor substitution is that structural knowledge of the plant design (e.g., flowsheet or network structure) can be imposed in the model parameterization in an efficient manner. Finally, while this work is primarily motivated by identification of LADMs and offset-free MPC implementations, we remark that any linear Gaussian state-space model can be identified, with eigenvalue constraints, using this approach.

The remainder of this section is devoted to outlining the paper and establishing notation. In Section 2, the ML identification problem is stated. In Section 3 the algorithm is outlined (Algorithm 1). In Section 4, we introduce tightened LMI region constraints show they define closed sets of system matrices (Theorem 2). In Section 5, we present our substitution and elimination scheme for approximating NSDPs as NLPs (Theorems 4 and 5). In Section 6, we solve the reformulated ML identification problem for two realworld applications of offset-free MPC: first, a benchmark temperature microcontroller used for classroom laboratories and prototyping [38], and second, an industrial-scale chemical reactor at Eastman Chemical's plant in Kingsport, TN [17]. Finally, in Section 7 we discuss broader implications and potential future research.

This report is an extended version of a submitted work, and contains additional review, discussion, and proofs of minor results that were omitted from the journal version due to page limitations. Compared to the journal version, this report contains the following additions:

- a longer discussion of problem formulations in Section 2;
- a characterization of models that can be converted to innovation form (Proposition 1, see Appendix A for proof);
- additional basic LMI regions in Section 3 (see Lemma 1);
- an explicit counterexample of [27, Thm. 1], which (incorrectly) characterized the eigenvalues of relaxed LMI regions, in Conjecture 1;
- a proof of our (correct) characterization of the eigenvalues of relaxed LMI regions (Proposition 2) in Appendix B;
- a proof of the fact that relaxed LMI regions define neither open nor closed sets of system matrices (Proposition 4(b,c)) in Appendix C;
- an additional results on solution uniqueness and minimum/infimum equivalences for the BMZ and generalized BMZ method properties (Lemma 2, Proposition 5, and Theorems 3 and 4) in Section 5;
- and additional remarks throughout.

Notation For any $z \in \mathbb{C}$, let \overline{z} denote its complex conjugate. Denote the set of $n \times n$ symmetric, positive definite, and positive semidefinite matrices by \mathbb{S}^n , \mathbb{S}^n_{++} , and \mathbb{S}^n_+ . Denote the set of lower triangular matrices and lower triangular matrices with positive diagonal entries by \mathbb{L}^n and \mathbb{L}^n_{++} . Recall $M \in \mathbb{R}^{n \times n}$ is positive definite if and only if there exists a unique $L \in \mathbb{L}^n_{++}$, called the *Cholesky factor*, such that $M = LL^{\top}$. A 2 × 2 Hermitian matrix $M = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \in \mathbb{H}^2$ is positive (semi)definite if and only if a, c > 0 ($a, c \ge 0$) and $ac > |b|^2$ ($ac \ge |b|^2$). Denote the matrix direct sum and the Kronecker product by \oplus and \otimes , respectively, defined as in [39]. Define the set of eigenvalues of a matrix $A \in \mathbb{R}^{n \times n}$ by $\lambda(A) \subset \mathbb{C}$. The spectral radius and spectral abscissa are defined as $\rho(A) \coloneqq \max_{\lambda \in \lambda(A)} |\lambda|$ and $\alpha(A) \coloneqq \max_{\lambda \in \lambda(A)} \operatorname{Re}(\lambda)$, respectively. We say a matrix A is Schur (Hurwitz) stable if $\rho(A) < 1$ ($\alpha(A) < 0$). We use \sim as a shorthand for "distributed as" and $\stackrel{\text{id}}{\sim}$ as a shorthand for "independent and identically distributed as." The complement, interior, closure, and boundary of a set S are denoted S^c , $\operatorname{int}(S)$, $\operatorname{cl}(S)$, and ∂S , respectively.

2 Problem statement

We consider stochastic LTI models in innovation form:

$$\hat{x}_{k+1} = A(\theta)\hat{x}_k + B(\theta)u_k + K(\theta)e_k \tag{1a}$$

$$y_k = C(\theta)\hat{x}_k + D(\theta)u_k + e_k \tag{1b}$$

$$e_k \stackrel{\text{iid}}{\sim} \mathcal{N}(0, R_e(\theta))$$
 (1c)

where $\hat{x} \in \mathbb{R}^n$ are the model states, $u \in \mathbb{R}^m$ are the inputs, $y \in \mathbb{R}^p$ are the outputs, $e \in \mathbb{R}^p$ are the innovation errors, and $\theta \in \Theta$ are the model parameters. The model functions $\mathcal{M}(\cdot) \coloneqq (A(\cdot), B(\cdot), C(\cdot), D(\cdot), \hat{x}_0(\cdot), K(\cdot), R_e(\cdot))$ are assumed to be known. While the model \mathcal{M} is kept fairly general throughout, it is advantageous to assume the model is identifiable in Θ . Last, for brevity, we often drop the dependence on the parameters $\theta \in \Theta$ and write the model functions as $\mathcal{M} = (A, B, C, D, \hat{x}_0, K, R_e)$.

While the subsequent developments apply to any model of the form (1), our main motivation is to identify the LADM,

$$\hat{s}_{k+1} = A_s(\theta)\hat{s}_k + B_d(\theta)\hat{d}_k + B_s(\theta)u_k + K_s(\theta)e_k$$
(2a)

$$l_{k+1} = d_k + K_d(\theta)e_k \tag{2b}$$

$$y_k = C_s(\theta)\hat{s}_k + C_d(\theta)\hat{d}_k + D(\theta)u_k + e_k \tag{2c}$$

$$e_k \stackrel{\text{iid}}{\sim} \mathcal{N}(0, R_e(\theta))$$
 (2d)

where $\hat{s} \in \mathbb{R}^{n_s}$ denote plant states and $\hat{d} \in \mathbb{R}^{n_d}$ denote integrating disturbances. The LADM (2) is clearly a special case of (1) and can be put back into the standard form (1)

by consolidating the plant and disturbance states $\hat{x}_k \coloneqq \begin{bmatrix} \hat{s}_k^\top & \hat{d}_k^\top \end{bmatrix}^\top$ and defining

$A \coloneqq \begin{bmatrix} A_s & B_d \\ 0 & I \end{bmatrix},$	$B \coloneqq \begin{bmatrix} B_s \\ 0 \end{bmatrix}$,
$C \coloneqq \begin{bmatrix} C_s & C_d \end{bmatrix},$	$K \coloneqq \begin{bmatrix} K_s \\ K_d \end{bmatrix}$

Typically the LADM (2) is parameterized with (A_s, C_s) in observability canonical form [40], (B_d, C_d) fixed,¹ (B_s, K_s, K_d, R_e) fully parameterized, and $(D, \hat{s}_0, \hat{d}_0) = (0, 0, 0)$. Alternatively, we could choose a physics-based or gray-box plant model for the plant dynamics (A_s, B_s, C_s, D) .

2.1 Constrained maximum likelihood identification

The ML identification problem is defined as follows:

$$\min_{\theta \in \Theta} L_N(\theta) \coloneqq \frac{N}{2} \ln \det R_e(\theta) + \frac{1}{2} \sum_{k=0}^{N-1} |e_k(\theta)|^2_{[R_e(\theta)]^{-1}}$$
(3)

where the $e_k(\theta)$ are given by the recursion (1) [20, p. 557], [21, p. 219]. Often, we may wish to regularize with respect to a previous parameter estimate $\overline{\theta}$, or incorporate an available prior distribution of the parameters $p_0(\theta)$. In either case, we consider the maximum a posteriori (MAP) estimation problem,

$$\min_{\theta \in \Theta} L_N(\theta) + R_0(\theta) \tag{4}$$

where $R_0(\theta) \propto -\ln p_0(\theta)$ is the regularization term, typically chosen as a distance from $\overline{\theta}$ [42, 43].

For a Gaussian prior or generalized ℓ_2 penalty, we use

$$R_0(\theta) \coloneqq \frac{1}{2} |\operatorname{vec}(\theta) - \operatorname{vec}(\overline{\theta})|_{V^{-1}}^2$$
(5)

where $\overline{\theta} \in \Theta$ is the prior estimate, vec is a vectorization operator, ² and $V \succ 0$ is the prior estimate variance. Such penalties are useful for model updating and re-identification. We typically use the penalty (5) with $V = \rho^{-1}I$. Later on, we transform the parameters θ into a more convenient space for optimization and find it more convenient to define the prior directly in the transformed space.

For plants of the form (1), the ML estimates are consistent and asymptotically efficient [20]. In a standard setting, the plant is of the form (1) with A - KC stable, and its

¹With (A_s, B_s, C_s, D) fixed, all (B_d, C_d) such that (2) is observable are equivalent up to a similarity transform [41]. Thus, (B_d, C_d) are chosen by the practitioner to maximize interpretability of the disturbance estimates.

²The vectorization operator may depend on the parameterization, as θ may contain both a vector portion and a sparse (semidefinite) matrix portion. The vectorization should only preserve the uniquely defined nonzero elements of the sparse matrix.

coefficients are asymptotically recovered by (3). With sufficient data, the identified filter is stable. However, the LADM (2) is an intentional misspecification of the plant. Under certain regularity assumptions, we are consistent with respect to the estimates nearest in *relative entropy rate*, taken between the plant and model measurement distributions,

$$\theta^* \coloneqq \min_{\theta \in \Theta} N^{-1} \mathbb{E}[L_N(\theta)]$$

where the expectation is taken over the true distribution of measurements $(y_k)_{k=0}^{N-1}$ [44, 45]. Identified LADM filters do not necessarily inherit stability from the plant, so we must design Θ to guarantee offset-free control.

2.2 Constraints

The constraint set Θ should capture both estimator design specifications and system knowledge. At a bare minimum, we require nondegeneracy of the innovation errors,

$$R_e(\theta) \succ 0 \tag{6}$$

and stability of the estimator,

$$\rho(A(\theta) - K(\theta)C(\theta)) < 1.$$
(7)

Other useful constraints include spectral abscissa bounds,

$$\alpha(-\tilde{A}(\theta)) < 0,\tag{8}$$

and bounds on the argument of the eigenvalues,

$$0 < |\mathrm{Im}(\mu)|/\mathrm{Re}(\mu) < \tan(\omega), \qquad \qquad \forall \mu \in \lambda(\hat{A}(\theta)) \tag{9}$$

for either the open-loop stability $\tilde{A} = A$ or estimator stability $\tilde{A} = A - KC$ matrices, to eliminate artificial high-frequency dynamics that may affect the control performance.

Chemical processes exhibit sparse interactions between units (mass and energy flows), especially for large-scale plants [46, 47]. Sparse parameterizations of (A, B, C, D, K) are easily encoded, but the sparse parameterization of R_e is less obviously accomplished. While the covariance R_e for a centralized Kalman filter is dense even for sparse plants, correlations between distant units are small [48]. Thus, it suffices to consider only nearest-neighbor correlations, e.g.,

$$R_{e} = \begin{bmatrix} R_{1,1} & R_{1,2} & & \\ R_{1,2}^{\top} & R_{2,2} & \ddots & \\ & \ddots & \ddots & R_{N_{u}-1,N_{u}} \\ & & R_{N_{u}-1,N_{u}}^{\top} & R_{N_{u},N_{u}} \end{bmatrix}$$
(10)

where $R_{i,j} \in \mathbb{R}^{p_u \times p_u}$ is the covariance between the innovations of the *i*-th and *j*-th process unit innovations. In (10), the sparse formulation introduces just $O(N_u p_u^2)$ variables compared to $O(N_u^2 p_u^2)$ variables for the dense formulation. Another algorithm goal is to simultaneously and efficiently enforce both (6) and (10). Finally, we remark that such constraints can be applied to the ML identification of any networked system with a timeinvariant topology, as in [49].

2.3 Other parameterizations

The remainder of this section presents some other formulations of the ML identification problem. While we do not consider these formulations explicitly in our algorithm formulation (Section 3) or case studies (Section 6), the methods are readily generalized to these formulations.

2.3.1 Time-varying Kalman filter formulations

More generally, we could consider models of the following form:

$$x_{k+1} = A(\theta)x_k + B(\theta)u_k + w_k \tag{11a}$$

$$y_k = C(\theta)x_k + D(\theta)u_k + v_k \tag{11b}$$

$$x_0 \sim \mathcal{N}(\hat{x}_0(\theta), \dot{P}_0(\theta))$$
 (11c)

$$\begin{bmatrix} w_k \\ v_k \end{bmatrix} \stackrel{\text{iid}}{\sim} \mathcal{N}(0, S(\theta)) \tag{11d}$$

where $w \in \mathbb{R}^n$ and $v \in \mathbb{R}^p$ are the process and measurement noises and $\mathcal{M} \coloneqq (A, B, C, D, \hat{x}_0, \hat{P}_0, S)$ are the model functions. The noise covariance matrix $S(\theta)$ may be partitioned as

$$S(\theta) = \begin{bmatrix} Q_w(\theta) & S_{wv}(\theta) \\ [S_{wv}(\theta)]^\top & R_v(\theta) \end{bmatrix}$$
(12)

where $Q_w(\theta) \in \mathbb{S}^n_+$ is the process noise covariance, $S_{wv}(\theta)$ is the cross-covariance, and $R_v(\theta) \in \mathbb{S}^p_+$ is the measurement noise covariance. Throughout, we impose the stronger requirement $R_v(\theta) \succ 0$ on the measurement noise covariance.

For the model (11), the ML problem is defined as

$$\min_{\theta \in \Theta} L_N(\theta) \coloneqq \frac{1}{2} \sum_{k=0}^{N-1} \ln \det \mathcal{R}_k(\theta) + |e_k(\theta)|^2_{[\mathcal{R}_k(\theta)]^{-1}}$$
(13)

where the e_k and \mathcal{R}_k are defined by the Kalman filtering equations

$$\hat{x}_{k+1} = A\hat{x}_k + Bu_k + \mathcal{K}_k e_k \tag{14a}$$

$$y_k = C\hat{x}_k + Du_k + e_k \tag{14b}$$

$$e_k \sim \mathcal{N}(0, \mathcal{R}_k) \quad \text{(indep.)}$$
 (14c)

where

$$\hat{P}_{k+1} \coloneqq A\hat{P}_k A^\top + Q_w - \mathcal{K}_k \mathcal{R}_k \mathcal{K}_k^\top \tag{14d}$$

$$\mathcal{K}_k \coloneqq (A\hat{P}_k C^\top + S_{wv}) \mathcal{R}_k^{-1} \tag{14e}$$

$$\mathcal{R}_k \coloneqq C\hat{P}_k C^\top + R_v. \tag{14f}$$

We remark that $R_v \succ 0$ suffices to guarantee the innovations are uniformly nondegenerate, i.e., $\mathcal{R}_k \succ 0$. However, stability of the filter is more difficult to guarantee as the early iterates $A - \mathcal{K}_k C$ may not be stable, even though the overall filter is stable, or vice versa. Instead, it is necessary to check that a stabilizing solution to the Riccati equation exists, which we elaborate on in the next formulation.

2.3.2 Time-invariant Kalman filter formulations

In most situations, the state error covariance matrix converges exponentially fast to a steady-state solution $\hat{P}_k \to \hat{P}$, so it suffices to consider the original steady-state filter model (1). In terms of the model (11), the steady-state filter takes the form $K := (A\hat{P}C^{\top} + S_{wv})R_e^{-1}$ and $R_e := C\hat{P}C^{\top} + R_v$, where \hat{P} is the unique, stabilizing solution to the discrete algebraic Riccati equation (DARE),

$$\hat{P} = A\hat{P}A^{\top} + Q_w - (A\hat{P}C^{\top} + S_{wv}) \times (C\hat{P}C^{\top} + R_v)^{-1} (A\hat{P}C^{\top} + S_{wv})^{\top}.$$
 (15)

Recall a solution to the DARE (15) is stabilizing if the resulting $A_K \coloneqq A - KC$ is stable.

Convergence of \hat{P}_k to \hat{P} is equivalent to the solution to the DARE (15) being unique and stabilizing. We generally assume such a solution exists, but for completeness, we state the following proposition, adapted from [50, Thm. 18(iii)] (see Appendix A for proof).

Proposition 1. Assume $R_v \succ 0$ and consider the full rank factorization

$$\begin{bmatrix} Q_w & S_{wv} \\ S_{wv}^\top & R_v \end{bmatrix} = \begin{bmatrix} \tilde{B} \\ \tilde{D} \end{bmatrix} \begin{bmatrix} \tilde{B}^\top & \tilde{D}^\top \end{bmatrix}$$

Then the following statements are equivalent:

- 1. The DARE (15) has a unique, stabilizing solution $\hat{P} \succeq 0$.
- 2. The error covariance converges exponentially fast $\hat{P}_k \to \hat{P}$ for any $\hat{P}_0 \succeq 0$.
- 3. (A, C) is detectable and $(A FC, \tilde{B} F\tilde{D})$ is stabilizable for all $F \in \mathbb{R}^{n \times p}$.

Remark 1. The hypothesis of Proposition 1 holds if we constrain A to be stable or (A, C) to be observable.

Remark 2. The cross-covariance S_{wv} complicates the filter stability analysis. With $S_{wv} = 0$, it would suffice to assume (A, C) detectable and (A, Q_w) stabilizable. With nonzero S_{wv} , however, a more elaborate stabilizability condition is needed. [50, Thm. 18] considers the regulation problem with a cross-weighting term and semidefinite input weights. Proposition 1 specializes this result to the filter problem with positive definite R_v .

Remark 3. While $R_e(\theta)$ and $K(\theta)$ could be defined via $\hat{P}(\theta)$, taken as the function that returns solutions to the DARE (15) and therefore enforcing filters stability, it is more convenient to directly parameterize these matrices as in (1).

2.3.3 Minimum determinant formulation

Suppose in the model (1), that R_e is parameterized fully, and separately from the other terms, i.e.,

$$\mathcal{M}(\tilde{\theta}, R_e) = \left(A(\tilde{\theta}), B(\tilde{\theta}), C(\tilde{\theta}), D(\tilde{\theta}), \hat{x}_0(\tilde{\theta}), K(\tilde{\theta}), R_e \right).$$

Moreover, assume R_e is constrained separately as well, i.e.,

$$\Theta = \Theta \times \mathbb{S}_{++}^p$$

Then we can always solve (3) stagewise, first in R_e , and then in the remaining variables θ . Solving the inner problem gives the solution

$$\hat{R}_e(\tilde{\theta}) \coloneqq \frac{1}{N} \sum_{k=0}^{N-1} e_k(\tilde{\theta}) [e_k(\tilde{\theta})]^\top$$

where we use the fact that e_k is only dependent on $\tilde{\theta}$, and we assume $\hat{R}_e(\tilde{\theta}) \succ 0$ for all $\tilde{\theta} \in \tilde{\Theta}$. The outer problem can be written

$$\min_{\tilde{\theta} \in \tilde{\Theta}} \det \hat{R}_e(\tilde{\theta}).$$
(16)

The problem (16) is of relevance because it avoids posing (3) as a NSDP. It has been used both in the early ML identification literature [51–53] and in recent works [54–56]. None of these works consider filter stability constraints. To the best of our knowledge, only [28] consider the ML problem (13) with stability constraints, but they consider openloop stability (i.e., $\rho(A) < 1$) and use the EM algorithm.

Remark 4. For real-world data, det $\hat{R}_e(\beta, \tilde{\Sigma}) = 0$ is not attainable because that would imply some direction of y_k were perfectly modeled. Therefore, $\hat{R}_e(\tilde{\theta}) \succ 0$ for all $\tilde{\theta} \in \tilde{\Theta}$ is a reasonable assumption.

3 Algorithm outline

3.1 Constraint set formulation

More generally, we seek to (i) impose eigenvalue constraints on any model function $A(\theta)$ and (ii) impose a sparsity structure on any semidefinite model function $\tilde{Q}(\theta)$.

3.1.1 Eigenvalue constraints

First, we define a LMI region.

Definition 1. We call $\mathcal{D} \subseteq \mathbb{C}$ an *LMI region* if

$$\mathcal{D} = \{ z \in \mathbb{C} \mid f_{\mathcal{D}}(z) \coloneqq M_0 + M_1 z + M_1^{\top} \overline{z} \succ 0 \}$$

for some generating matrices $(M_0, M_1) \in \mathbb{S}^m \times \mathbb{R}^{m \times m}$. We call $f_{\mathcal{D}} : \mathbb{C} \to \mathbb{S}^m$ the characteristic function of \mathcal{D} .

The following lemma defines the four basic LMI regions: shifted half-planes, circles centered on the real axis, conic sections, and horizontal bands. For a general discussion of LMI regions properties, see [25, 57].

Lemma 1. For each $s, x_0 \in \mathbb{R}$, the subsets

$$\mathcal{D}_1(s) \coloneqq \{ z \in \mathbb{C} \mid \operatorname{Re}(z) > s \}$$
$$\mathcal{D}_2(s, x_0) \coloneqq \{ z \in \mathbb{C} \mid |z - x_0| < s \}$$
$$\mathcal{D}_3(s, x_0) \coloneqq \{ z \in \mathbb{C} \mid |\operatorname{Im}(z)| < s(\operatorname{Re}(z) - x_0) \}$$
$$\mathcal{D}_4(s) \coloneqq \{ z \in \mathbb{C} \mid |\operatorname{Im}(z)| < s \}$$

are LMI regions with characteristic functions

$$f_{\mathcal{D}_{1}(x_{0})}(z) \coloneqq -2x_{0} + z + \overline{z}$$

$$f_{\mathcal{D}_{2}(s,x_{0})}(z) \coloneqq \begin{bmatrix} s & -x_{0} \\ -x_{0} & s \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} z + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \overline{z}$$

$$f_{\mathcal{D}_{3}(s,x_{0})}(z) \coloneqq -2sx_{0}I_{2} + \begin{bmatrix} s & 1 \\ -1 & s \end{bmatrix} z + \begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix} \overline{z}$$

$$f_{\mathcal{D}_{4}(s)}(z) \coloneqq -2sI_{2} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} z + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \overline{z}.$$

Proof. The first identity follows from the formula $2\operatorname{Re}(z) = z + \overline{z}$. For the second identity, we have $f_{\mathcal{D}_2(s,x_s)}(z) = \begin{bmatrix} s & z-x_0 \\ \overline{z}-x_0 \end{bmatrix} \succ 0$ if and only if s > 0 and $s^2 > |z-x_0|^2$, or equivalently, $|z-x_0| < s$. For the third identity, we have $f_{\mathcal{D}_3(s,x_0)}(z) = \begin{bmatrix} 2s(\operatorname{Re}(z)-x_0) & 2\iota\operatorname{Im}(z) \\ -2\iota\operatorname{Im}(z) & 2s(\operatorname{Re}(z)-x_0) \end{bmatrix} \succeq 0$ if and only if $2s(\operatorname{Re}(z)-x_0) > 0$ and $4s^2(\operatorname{Re}(z)-x_0)^2 > 4|\operatorname{Im}(z)|^2$, or equivalently, $|\operatorname{Im}(z)| < s(\operatorname{Re}(z)-x_0)$. For the fourth identity, we have $f_{\mathcal{D}_4(s)}(z) = \begin{bmatrix} 2s & 2\iota\operatorname{Im}(z) \\ -2\iota\operatorname{Im}(z) & 2s \end{bmatrix} \succeq 0$ if and only if 2s > 0 and $4s^2 > 4|\operatorname{Im}(z)|^2$, or equivalently, $|\operatorname{Im}(z)| < s$.

Remark 5. For continuous-time systems, $-\mathcal{D}_1(\alpha)$ corresponds to a minimum decay rate of $\alpha > 0$, $\mathcal{D}_3(-\tan(\omega), 0)$ corresponds to a minimum damping ratio $\cos(\omega)$, and $\mathcal{D}_2(r, 0) \cap \mathcal{D}_3(-\tan(\omega), 0)$ implies to a maximum undamped natural frequency $r \sin(\omega)$, where $\alpha, r > 0$ and $\omega \in [0, \pi/2]$ [25]. For discrete-time systems, $\mathcal{D}_2(r, 0)$ corresponds to a minimum decay rate of $-\ln r$, and $\mathcal{D}_2(r, 0) \cap \mathcal{D}_3(\tan(\omega), 0)$ implies a minimum damping ratio $-\cos(\tan^{-1}(\omega/\ln r))$ and maximum natural frequency $(\ln(r)^2 + \omega^2)/\Delta$, where r > 0, $\omega \in [0, \pi/2]$, and Δ is the sample time.

Remark 6. For any LMI region \mathcal{D} (including those in Lemma 1), the set \mathcal{D} is convex, open, and symmetric about the real axis. The intersection of two LMI regions $\mathcal{D} := \mathcal{D}_1 \cap \mathcal{D}_2$ is an LMI region with the characteristic function $f_{\mathcal{D}}(z) = f_{\mathcal{D}_1}(z) \oplus f_{\mathcal{D}_2}(z)$. By this property, we can construct any convex polyhedron that is symmetric about the real axis by intersecting left and right half-planes, horizontal strips, and conic sections. Moreover, since any convex region can be approximated, to any desired accuracy, by a convex polyhedron, the set of LMI regions is dense in the space of convex subsets of \mathbb{C} that are symmetric about the real axis. An LMI region \mathcal{D} with characteristic function $f_{\mathcal{D}}$ also has characteristic function $Mf_{\mathcal{D}}(\cdot)M^{\top}$ for any nonsingular $M \in \mathbb{R}^{m \times m}$. For an in-depth discussion of LMI region geometry and other properties, see [57].

In [25], it is shown a matrix $\tilde{A} \in \mathbb{R}^{\tilde{n} \times \tilde{n}}$ has eigenvalues in a LMI region \mathcal{D} if and only if the following system of matrix inequalities is feasible:

$$M_{\mathcal{D}}(\tilde{A}, P) \succ 0, \qquad P \succ 0$$
 (17)

where the matrix characteristic function $M_{\mathcal{D}}: \mathbb{R}^{\tilde{n} \times \tilde{n}} \times \mathbb{S}^{\tilde{n}} \to \mathbb{S}^{\tilde{n}\tilde{m}}$ of \mathcal{D} is defined by

$$M_{\mathcal{D}}(\tilde{A}, P) \coloneqq M_0 \otimes P + M_1 \otimes (\tilde{A}P) + M_1^\top \otimes (\tilde{A}P)^\top.$$
⁽¹⁸⁾

From this equivalence, we can build tractable eigenvalue constraints. For the constraint (7), Lemma 1 gives the generating matrices $(M_0, M_1) := (\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and we have the matrix inequalities

$$\begin{bmatrix} P & (A - KC)P \\ P(A - KC)^\top & P \end{bmatrix} \succ 0, \qquad \qquad P \succ 0$$

which is a well-known LMI for checking stability [58]. Similarly, to implement (8), we can use the generating matrices $(M_0, M_1) \coloneqq (0, 1)$, and to implement (9), we can use $(M_0, M_1) \coloneqq \left(\begin{bmatrix} -2\tan(\omega) & 0 \\ 0 & -2\tan(\omega) \end{bmatrix}, \begin{bmatrix} \tan(\omega) & 1 \\ -1 & \tan(\omega) \end{bmatrix} \right)$. The system of matrix inequalities (17) contains only strict inequalities, but we can

"tighten" them as follows:

$$M_{\mathcal{D}}(\tilde{A}, P) \succeq M, \qquad P \succeq 0, \qquad \operatorname{tr}(VP) \le \varepsilon^{-1}$$
(19)

where $\varepsilon > 0, V \in \mathbb{S}^{\tilde{n}}_{++}$, and $M \in \mathbb{S}^{\tilde{n}\tilde{m}}_{++}$. The set of $\tilde{A} \in \mathbb{R}^{\tilde{n} \times \tilde{n}}$ for which (19) is feasible defines a closed set for which $\lambda(\tilde{A}) \subseteq \mathcal{D}$. In Section 4, we show this fact and other properties of the constraint (19).

3.1.2Sparsity structure

To encode sparsity information, we adapt the notation of [36]. Define the index sets $\mathcal{L}^n \coloneqq \{ (i,j) \in \mathbb{I}^2_{1:n} \mid i \ge j \}$ and $\mathcal{D}^n \coloneqq \{ (i,i) \in \mathbb{I}^2_{1:n} \}$ corresponding to the sparsity patterns of $n \times n$ lower triangular and diagonal matrices. With a slight abuse of notation, we define the direct sum of index sets $\mathcal{I} \subset \mathcal{L}^n$ and $\mathcal{J} \subset \mathcal{L}^m$ by

$$\mathcal{I} \oplus \mathcal{J} \coloneqq \mathcal{I} \cup \{ (i+n, j+n) \mid (i, j) \in \mathcal{J} \} \subseteq \mathcal{L}^{n+m}.$$

For each $\mathcal{I} \subseteq \mathcal{L}^n$, define the sets

$$\mathbb{S}^{n}[\mathcal{I}] \coloneqq \{ S \in \mathbb{S}^{n} \mid S_{ij} = 0 \ \forall \ (i,j) \in \mathcal{L}^{n} \setminus \mathcal{I} \} \\ \mathbb{L}^{n}[\mathcal{I}] \coloneqq \{ L \in \mathbb{L}^{n} \mid L_{ij} = 0 \ \forall \ (i,j) \notin \mathcal{I} \} \\ \mathbb{L}^{n}_{++}[\mathcal{I}] \coloneqq \{ L \in \mathbb{L}^{n}_{++} \mid L_{ij} = 0 \ \forall \ (i,j) \notin \mathcal{I} \}.$$

Finally, let $\operatorname{vecs}_{\mathcal{I}} : \mathbb{S}^n \to \mathbb{R}^{|\mathcal{I}|}$ denote the operator that vectorizes the $|\mathcal{I}|$ entries of the argument corresponding to the index set \mathcal{I} .

3.1.3Constraint definition

To combine the LMI region and sparsity constraints, we partition the parameter into vector and sparse symmetric matrix parts, i.e., $\theta = (\beta, \Sigma)$, and define the constraint set Θ by

$$\Theta = \{ (\beta, \Sigma) \in \mathbb{R}^{n_{\beta}} \times \mathbb{S}^{n_{\Sigma}}[\mathcal{I}_{\Sigma}] \mid g(\beta, \Sigma) = 0, \ h(\beta, \Sigma) \le 0, \ \Sigma \succeq H(\beta), \ \mathcal{A}(\beta, \Sigma) \succeq 0 \}$$
(20)

where $\mathcal{D}^{n_{\Sigma}} \subseteq \mathcal{I}_{\Sigma} \subseteq \mathcal{L}^{n_{\Sigma}}$, $\mathcal{D}^{n_{\mathcal{A}}} \subseteq \mathcal{I}_{\mathcal{A}} \subseteq \mathcal{L}^{n_{\mathcal{A}}}$, $g : \mathbb{R}^{n_{\beta}} \times \mathbb{S}^{n_{\Sigma}} \to \mathbb{R}^{n_{g}}$, $h : \mathbb{R}^{n_{\beta}} \times \mathbb{S}^{n_{\Sigma}} \to \mathbb{R}^{n_{h}}$, $H : \mathbb{R}^{n_{\beta}} \to \mathbb{S}^{n_{\Sigma}}$, and $\mathcal{A} : \mathbb{R}^{n_{\beta}} \times \mathbb{S}^{n_{\Sigma}} \to \mathbb{S}^{n_{\mathcal{A}}}[\mathcal{I}_{\mathcal{A}}]$. The purpose of the partition $\theta = (\beta, \Sigma)$ is to clearly delineate the sparse semidefinite matrix argument Σ from the remaining parameters β . The index set \mathcal{I}_{Σ} defines the sparsity pattern of Σ and H, and the index set $\mathcal{I}_{\mathcal{A}}$ defines the sparsity pattern of \mathcal{A} .

Remark 7. Assumption 1 rules out direct use strict inequalities, e.g., $R_e(\theta) \succ 0$ or $R_v(\theta) \succ 0$. O. To satisfy nondegeneracy requirements, we use the closed constraint $R_e(\theta) \succeq \delta I_p$ with a small backoff $\delta > 0$.

Remark 8. Typically, the index set \mathcal{I}_{Σ} encodes block diagonal structures, e.g., for the model (11), $\Sigma = \hat{P}_0 \oplus Q_w \oplus R_v \in \mathbb{S}^{2n+p}[\mathcal{I}_{\Sigma}]$ where $\mathcal{I}_{\Sigma} \coloneqq \mathcal{L}^n \oplus \mathcal{L}^n \oplus \mathcal{L}^p$. However, more general structures (e.g., (10)) can be stated. For the time-varying formulation (13), we may further restrict Q_w and R_v to take block tridiagonal and diagonal structures, e.g.,

$$Q_{w} = \begin{bmatrix} Q_{1,1} & Q_{1,2} & & & \\ Q_{1,2}^{\top} & Q_{2,2} & \ddots & & \\ & \ddots & \ddots & Q_{\tilde{n}-1,\tilde{n}} \\ & & Q_{\tilde{n}-1,\tilde{n}}^{\top} & Q_{\tilde{n},\tilde{n}} \end{bmatrix}, \qquad \qquad R_{v} = R_{1} \oplus \ldots \oplus R_{\tilde{n}}$$

that arise in sequentially interconnected processes such as chemical plants. Adding a $Q_{1,\tilde{n}}$ block can account for an overall recycle loop. Note that if we parameterize the block tridiagonal Q_w via a sparse shaping matrix (i.e., $Q_w = G_w G_w^{\top}$), then there are more parameters than if the sparsity of Q_w is known.

Remark 9. As alluded to in Section 2, the Riccati equation solution has a dense solution, but the entries far from the core sparsity pattern decay rapidly. Thus, we can approximate an eigenvalue constraint, e.g., $P - APA^{\top} \succ 0$, as a function that maps to the same sparsity pattern as A [48, 59–61].

3.2 Cholesky factorization and elimination

At this juncture, the ML and MAP problems (3) and (4) with the constraints (20) are in standard NSDP form and can be solved with any dedicated NSDP solver, e.g., [62, 63]. However, such solvers are neither as widely available nor as well-understood as NLP solvers such as IPOPT [64].

The Burer-Monteiro-Zhang (BMZ) method is a Cholesky factorization-based substitution and elimination algorithm that can convert certain NSDPs to NLPs [36, 37]. In Section 5, we consider a generalization of this algorithm to (approximately) transform a given NSDP into a NLP while only introducing $|\mathcal{I}_{\mathcal{A}}|$ new variables. This generalization requires the following assumption:

Assumption 1. The model functions \mathcal{M} are twice differentiable and the constraint functions \mathcal{C} are differentiable. Moreover, $cl(\Theta_{++}) = \Theta$ where

$$\Theta_{++} \coloneqq \{ (\beta, \Sigma) \in \mathbb{R}^{n_{\beta}} \times \mathbb{S}^{n_{\Sigma}}[\mathcal{I}_{\Sigma}] \mid g(\beta, \Sigma) = 0, \ h(\beta, \Sigma) \le 0, \ \Sigma \succ H(\beta), \ \mathcal{A}(\beta, \Sigma) \succ 0 \} .$$
(21)

In Section 5, we construct functions

$$\mathcal{T}: \mathbb{R}^{n_{\beta}} \times \mathbb{L}^{n_{\Sigma}}_{++}[\mathcal{I}_{\Sigma}] \times \mathbb{L}^{n_{\mathcal{A}}}_{++}[\mathcal{I}_{\mathcal{A}}] \to \mathbb{R}^{n_{\beta}} \times \mathbb{S}^{n_{\Sigma}}[\mathcal{I}_{\Sigma}]$$
$$\mathcal{A}_{\mathcal{T}}: \mathbb{R}^{n_{\beta}} \times \mathbb{L}^{n_{\Sigma}}_{++}[\mathcal{I}_{\Sigma}] \times \mathbb{L}^{n_{\mathcal{A}}}_{++}[\mathcal{I}_{\mathcal{A}}] \to \mathbb{S}^{n_{\mathcal{A}}}_{++}[\mathcal{I}_{\mathcal{A}}]$$

and define transformed constraint functions

$$g_{\mathcal{T}}(\phi) \coloneqq \begin{bmatrix} g(\mathcal{T}(\phi)) \\ \operatorname{vecs}_{\mathcal{I}_{\mathcal{A}}}(\mathcal{A}(\mathcal{T}(\phi)) - \mathcal{A}_{\mathcal{T}}(\phi)) \end{bmatrix}$$
(22a)

$$h_{\mathcal{T}}(\phi) \coloneqq h(\mathcal{T}(\phi)) \tag{22b}$$

and a transformed constraint set

$$\Phi \coloneqq \{ \phi \in \mathbb{R}^{n_{\beta}} \times \mathbb{L}_{++}^{n_{\Sigma}}[\mathcal{I}_{\Sigma}] \times \mathbb{L}_{++}^{n_{\mathcal{A}}}[\mathcal{I}_{\mathcal{A}}] \mid g_{\mathcal{T}}(\phi) = 0, \ h_{\mathcal{T}}(\phi) \le 0 \}$$
(23)

such that, under Assumption 1, \mathcal{T} is an invertible map from Φ to Θ_{++} . Finally, to eliminate the strict inequalities on the diagonal entries of $(L_{\Sigma}, L_{\mathcal{A}}) \in \mathbb{L}_{++}^{n_{\Sigma}}[\mathcal{I}_{\Sigma}] \times \mathbb{L}_{++}^{n_{\mathcal{A}}}[\mathcal{I}_{\mathcal{A}}]$, we introduce a fixed lower bound $\varepsilon > 0$ on the diagonal entries,

$$\Phi_{\varepsilon} \coloneqq \{ \phi \in \mathbb{R}^{n_{\beta}} \times \mathbb{L}_{\varepsilon}^{n_{\Sigma}}[\mathcal{I}_{\Sigma}] \times \mathbb{L}_{\varepsilon}^{n_{\mathcal{A}}}[\mathcal{I}_{\mathcal{A}}] \mid g_{\mathcal{T}}(\phi) = 0, \ h_{\mathcal{T}}(\phi) \le 0 \}$$
(24)

where we have defined, for any $\varepsilon > 0$ and $\mathcal{I} \subseteq \mathcal{L}^n$,

$$\mathbb{L}_{\varepsilon}^{n}[\mathcal{I}] \coloneqq \{ L \in \mathbb{L}^{n}[\mathcal{I}] \mid L_{ii} \geq \varepsilon \; \forall i \in \mathbb{I}_{1:n} \} \,.$$

We define the *approximate* transformed problem as

$$\min_{\phi \in \Phi_{\varepsilon}} L_N(\mathcal{T}(\phi)) + R_0(\mathcal{T}(\phi)).$$
(25)

If $\hat{\phi}$ solves the problem (25), then $\hat{\theta} \coloneqq \mathcal{T}(\hat{\phi})$ approximately solves the MAP problem (4). We recover the ML problem (3) and its approximate solutions with $R_0 \equiv 0$.

3.3 Algorithm summary

Algorithm 1 provides an example of our approach towards solving the identification problem (4) with eigenvalue constraints and the Cholesky factor-based substitution and elimination scheme.

4 Eigenvalue constraints

In this section, we elaborate on the LMI region constraints previewed in Section 3. Throughout, assume the LMI region \mathcal{D} is nonempty, not equal to \mathbb{C} , and its characteristic function $f_{\mathcal{D}}$ and generating matrices (M_0, M_1) are fixed. Our goal in this section is to define, using only matrix inequalities, a *closed* set of matrices $A \in \mathbb{R}^{n \times n}$ such that $\lambda(A) \subseteq \mathcal{D}$. For this section, the matrix $A \in \mathbb{R}^{n \times n}$ need not have any relation to the model function in (1), and can in fact be any square matrix of any dimension (e.g., the filter stability matrix A - KC, the plant stability matrix A_s from (2), or any submatrix thereof). Throughout this section, we assume the matrix characteristic function $M_{\mathcal{D}}$ is fixed. Algorithm 1 Identification of an innovation form model (1) with eigenvalue constraints and the Cholesky factor-based substitution and elimination scheme.

- **Require:** Model functions $\mathcal{M} = (A, B, C, D, \hat{x}_0, K, R_e)$, regularization term R_0 , initial parameters $\theta_0 = (\beta, \Sigma_0)$ constraint functions $(g, h_0, H_0, \mathcal{A}_0)$ and sparsity patterns $(\mathcal{I}_{\Sigma_0}, \mathcal{I}_{\mathcal{A}_0})$, a series of LMI region constraints $(\mathcal{D}_i, \tilde{A}_i(\cdot))_{i=1}^{n_c}$, and small $\varepsilon, \varepsilon_i > 0$.
- 1: For each $i \in \mathbb{I}_{1:n_c}$, let $M_{\mathcal{D}_i} : \mathbb{R}^{n_i \times n_i} \times \mathbb{S}^{n_i} \to \mathbb{S}^{n_i m_i}$ denote the matrix characteristic function for \mathcal{D}_i .
- 2: Extend the parameters $\Sigma \coloneqq \Sigma_0 \oplus (\bigoplus_{i=1}^{n_c} P_i)$ and $\theta \coloneqq (\beta, \Sigma)$ with $P_i \in \mathbb{S}^{n_i}$.
- 3: Extend the constraint functions $h(\theta) \coloneqq \left[[h_0(\theta_0)]^\top \operatorname{tr}(V_1 P_1) \varepsilon_1^{-1} \dots \operatorname{tr}(V_{n_c} P_{n_c}) \varepsilon_{n_c}^{-1} \right]^\top$ $H(\beta) \coloneqq H_0(\beta) \oplus \left(\bigoplus_{i=1}^{n_c} 0_{n_i \times n_i}\right), \text{ and } \mathcal{A}(\theta) \coloneqq \mathcal{A}_0(\theta_0) \oplus \left(\bigoplus_{i=1}^{n_c} M_{\mathcal{D}_i}(\tilde{\mathcal{A}}_i(\theta_0), P_i) - \varepsilon I\right).$ 4: Extend the index sets $\mathcal{I}_{\Sigma} \coloneqq \mathcal{I}_{\Sigma_0} \oplus \left(\bigoplus_{i=1}^{n_c} \mathcal{L}^{n_i}\right) \text{ and } \mathcal{I}_{\mathcal{A}} \coloneqq \mathcal{I}_{\mathcal{A}_0} \oplus \left(\bigoplus_{i=1}^{n_c} \mathcal{L}^{n_i m_i}\right).$
- 5: Form the functions $\mathcal{T}, \mathcal{T}^{-1}$, and $\tilde{\mathcal{A}}$ as in Section 5.
- 6: Form the transformed constraint functions (22).
- 7: Solve (24) and (25), and let $\hat{\phi}$ denote the solution.
- 8: Let $\hat{\theta} \coloneqq \mathcal{T}(\hat{\phi})$.

4.1LMI region constraints

Originally, [25] proved the following theorem relating the eigenvalues of $A \in \mathbb{R}^{n \times n}$ to feasibility of a system of matrix inequalities.

Theorem 1 ([25, Thm. 2.2]). For any $A \in \mathbb{R}^{n \times n}$, we have $\lambda(A) \subseteq \mathcal{D}$ if and only if

$$M_{\mathcal{D}}(A, P) \succ 0, \qquad P \succ 0.$$
 (26)

holds for some $P \in \mathbb{S}^n$.

Ultimately, we seek matrix inequalities that define a *closed* set of constraints. Due to the strictness of the inequalities (26), it is unlikely that [25, Thm. 2.2] achieves this goal.

4.2**Relaxed** constraints

In [27], the following relaxation of (26) was considered,

$$M_{\mathcal{D}}(A, P) \succeq 0, \qquad P \succ 0.$$
 (27)

Since $M_{\mathcal{D}}(A, P)$ is linear in P, feasibility of (28) is equivalent to feasibility of

$$M_{\mathcal{D}}(A, P) \succeq 0, \qquad P \succeq P_0$$
 (28)

for some fixed $P_0 \in \mathbb{S}^{n}_{++}$.³

An attempt was made in [27, Thm. 1] to characterize the eigenvalues of matrices $A \in$ $\mathbb{R}^{n \times n}$ for which (27) is feasible, but this theorem does not correctly treat eigenvalues on the LMI region's boundary $\partial \mathcal{D}$. We restate [27, Thm. 1] below as a conjecture and disprove it with a simple counterexample.

³For any $P_0 \succ 0$ and P satisfying (27), define the scaling factor $\gamma := \|P_0\|_2 \|P^{-1}\|_2$ and a rescaled solution $P^* \coloneqq \gamma P$. Then $P^* \succeq P_0$ and $M_{\mathcal{D}}(A, P^*) = \gamma M_{\mathcal{D}}(A, P) \succeq 0$.

Conjecture 1 ([27, Thm. 1]). The matrix $A \in \mathbb{R}^{n \times n}$ satisfies $\lambda(A) \subset cl(\mathcal{D})$ if and only if (27) holds for some $P \in \mathbb{S}^n$.

Counterexample. Let \mathcal{D} be the left half-plane, consider the Jordan block $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, and suppose $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \in \mathbb{S}^2$ such that (27) holds. Then $\lambda(A) \subset \operatorname{cl}(\mathcal{D})$ and

$$0 \leq M_{\mathcal{D}}(A, P) = - \begin{bmatrix} 2p_{12} & p_{22} \\ p_{22} & 0 \end{bmatrix}$$

which implies $p_{12} = p_{22} = 0$, a contradiction of (27).

The correction to Conjecture 1 requires a more careful treatment of eigenvalues lying on the the LMI region's boundary $\partial \mathcal{D}$. Specifically, we show in the following proposition that feasibility of (27) for a given $A \in \mathbb{R}^{n \times n}$ is equivalent to the eigenvalues of A being in $cl(\mathcal{D})$, with all non-simple eigenvalues lying in \mathcal{D} (see Appendix B for proof).

Proposition 2. The matrix $A \in \mathbb{R}^{n \times n}$ satisfies $\lambda(A) \subseteq cl(\mathcal{D})$ and $\lambda \in \mathcal{D}$ for all non-simple eigenvalues $\lambda \in \lambda(A)$ if and only if (27) holds for some $P \in \mathbb{S}^n$.

4.3 Tightened constraints

Instead of the "relaxed" constraints (27), we consider "tightened" constraints of the form

$$M_{\mathcal{D}}(A, P) \succeq M, \qquad P \succeq 0, \qquad \operatorname{tr}(VP) \le \varepsilon^{-1}$$
(29)

where $M \in \mathbb{S}^{nm}_+$ and $V \in \mathbb{S}^{n}_{++}$ are fixed and chosen in a way that (29) implies (26). While we allow M to be semidefinite,⁴ in the following proposition, we show $M \succ 0$ always suffices.

Proposition 3. Suppose $M \in \mathbb{S}^{nm}_{++}$ and $V \in \mathbb{S}^{n}_{++}$. Then (29) implies (26) for all $A \in \mathbb{R}^{n \times n}$ and $\varepsilon > 0$.

Proof. With $M \succ 0$ and (29), we automatically have $M_{\mathcal{D}}(A, P) \succ 0$. It remains to show (29) implies $P \succ 0$. For contradiction suppose (29) and $M \succ 0$, but $P \not\succeq 0$. Then there exists a nonzero $v \in \mathbb{R}^n$ such that Pv = 0, and

$$(I_m \otimes v)^\top M_{\mathcal{D}}(A, P)(I_m \otimes v) = M_0 \otimes (v^\top P v) + M_1 \otimes (v^\top A P v) + M_1^\top \otimes (v^\top P A^\top v) = 0$$

a contradiction of the assumption $M_{\mathcal{D}}(A, P) \succeq M \succ 0$.

Remark 10. The tightened constraint (29) was inspired by a similar set of constraints was introduced by Diehl and colleagues [65] to "smooth" the spectral radius. Specifically, feasibility of the nonlinear system

$$s^2 P - A P A^{\top} = W, \qquad P \succeq 0, \qquad \operatorname{tr}(VP) \le \varepsilon^{-1}$$
(30)

*

⁴For some LMI regions, $M \succeq 0$ is advantageous. For example, we can always take $M \coloneqq \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes Q$ with $Q \succ 0$ for circular LMI regions. Then we can reduce the constraint dimension by taking the Schur complement.

implies $\rho(A) < s$ where $W, V \in \mathbb{S}_{++}^n$ and $s, \varepsilon > 0$ are fixed [65, Thms. 5.4, 5.6]. Similarly, the spectral abscissa was "smoothed" in [66, Thms. 2.5, 2.6], and it is straightforward to generalize [65, Thms. 5.4, 5.6] to show feasibility of

$$(A - sI)P + P(A - sI)^{\top} = -W, \qquad P \succeq 0, \qquad \operatorname{tr}(VP) \le \varepsilon^{-1} \qquad (31)$$

implies $\alpha(A) < s$ where $W, V \in \mathbb{S}_{++}^n$, $s \in \mathbb{R}$, and $\varepsilon > 0$ are fixed. The authors do not discuss LMI regions and the results are not obviously generalizable to them.

4.4 Constraint topology

Consider the constraint sets,

$$\mathbb{A}^{n}_{\mathcal{D}} \coloneqq \{ A \in \mathbb{R}^{n \times n} \mid \exists P \in \mathbb{S}^{n} : (26) \text{ holds} \}$$
$$\tilde{\mathbb{A}}^{n}_{\mathcal{D}} \coloneqq \{ A \in \mathbb{R}^{n \times n} \mid \exists P \in \mathbb{S}^{n} : (27) \text{ holds} \}$$
$$\mathbb{A}^{n}_{\mathcal{D}}(\varepsilon) \coloneqq \{ A \in \mathbb{R}^{n \times n} \mid \exists P \in \mathbb{S}^{n} : (29) \text{ holds} \}.$$

The following proposition characterizes the topology of $\mathbb{A}^n_{\mathcal{D}}$ and $\tilde{\mathbb{A}}^n_{\mathcal{D}}$ (see Appendix C for proof).

Proposition 4. (a) $\mathbb{A}^n_{\mathcal{D}}$ is open.

- (b) $\tilde{\mathbb{A}}_{\mathcal{D}}^{n}$ is not open if (i) $n \geq 2$ or (ii) $\partial \mathcal{D} \cap \mathbb{R}$ is nonempty.
- (c) $\tilde{\mathbb{A}}_{\mathcal{D}}^{n}$ is not closed if (i) $n \geq 4$ or (ii) $\partial \mathcal{D} \cap \mathbb{R}$ is nonempty and $n \geq 2$.
- $(d) \operatorname{cl}(\mathbb{A}^{n}_{\mathcal{D}}) = \{ A \in \mathbb{R}^{n \times n} \mid \lambda(A) \subset \operatorname{cl}(\mathcal{D}) \}.$

Proposition 4 reveals a weakness of the relaxed constraints (27) and (28). Since $\tilde{\mathbb{A}}_{\mathcal{D}}^n$ is not closed, any feasible path towards a matrix $A \in \operatorname{cl}(\mathbb{A}_{\mathcal{D}}^n) \setminus \tilde{\mathbb{A}}_{\mathcal{D}}^n$ has no feasible limiting P. In fact, P will grow unbounded along the path of iterates.

To analyze the topology of $\mathbb{A}^n_{\mathcal{D}}(\varepsilon)$, we take a barrier function approach. Consider the parameterized linear SDP,

$$\phi_{\mathcal{D}}(A) \coloneqq \inf_{P \in \mathbb{S}^n_+} \operatorname{tr}(VP) \text{ subject to } M_{\mathcal{D}}(A, P) \succeq M.$$
(32)

The optimal value function $\phi_{\mathcal{D}} : \mathbb{R}^{n \times n} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ is a barrier function for the constraint $A \in \mathbb{A}^n_{\mathcal{D}}$. Theorem 2 establishes properties of $\phi_{\mathcal{D}}$ and its ε^{-1} -sublevel sets (see Appendix D for proof).

Theorem 2. Let $V \in \mathbb{S}_{++}^n$ and $M \in \mathbb{S}_{+}^n$ such that $M_{\mathcal{D}}(A, P) \succeq M$ implies $M_{\mathcal{D}}(A, P) \succ 0$. Then

- (a) $\phi_{\mathcal{D}}$ is continuous on $\mathbb{A}_{\mathcal{D}}$;
- (b) for each $\varepsilon > 0$, $\mathbb{A}^n_{\mathcal{D}}(\varepsilon)$ is equivalent to the ε^{-1} -sublevel set of $\phi_{\mathcal{D}}$, i.e.,

$$\mathbb{A}^{n}_{\mathcal{D}}(\varepsilon) = \{ A \in \mathbb{R}^{n \times n} \mid \phi_{\mathcal{D}}(A) \le \varepsilon^{-1} \}$$
(33)

and both are closed; and

(c) $\mathbb{A}^n_{\mathcal{D}}(\varepsilon) \nearrow \mathbb{A}^n_{\mathcal{D}}$ as $\varepsilon \searrow 0$.

Remark 11. To reconstruct (30) via Theorem 2, we set $M = sW \oplus 0_{n \times n}$ for any $W, V \succ 0$ and s > 0 and apply the Schur complement lemma to $M_{\mathcal{D}_2}(A, P)/s - M/s$, where \mathcal{D}_2 is the circle defined in Lemma 1 with $x_0 = 0$, and $M_{\mathcal{D}_2}$ is defined by the generating matrices used in Lemma 1. Then the ε^{-1} -sublevel set of $\phi_{\mathcal{D}_2}$ equals the set of $A \in \mathbb{R}^{n \times n}$ for which (30) is feasible.

Similarly, we can reconstruct the set of $A \in \mathbb{R}^{n \times n}$ for which (31) is feasible as ε^{-1} sublevel sets of $\phi_{\mathcal{D}_1}$, where \mathcal{D}_1 is the shifted half-plane defined in Lemma 1, and M = Wfor any $W, V \succ 0$.

5 Cholesky substitution and elimination

In this section, we seek to approximate certain NSDPs by NLPs. Specifically, we consider the NSDP

$$\min_{(\beta,\Sigma)\in\Theta} f(\beta,\Sigma) \tag{34}$$

where Θ is defined as in (20). This covers both ML (3) and MAP (4) problems with constraints (20). We combine Cholesky factor-based substitution with an elimination scheme to convert the NSDP to a NLP while adding just $|\mathcal{I}_{\mathcal{A}}|$ variables to the optimization problem.

For this section, we define the following notation. For each $\mathcal{I} \subseteq \mathcal{L}^n$, let $\pi_{\mathcal{I}}^L : \mathbb{R}^{n \times n} \to \mathbb{L}^n[\mathcal{I}]$ and $\pi_{\mathcal{I}} : \mathbb{R}^{n \times n} \to \mathbb{S}^n[\mathcal{I}]$ denote the orthogonal projections (in the Frobenius norm) from $\mathbb{R}^{n \times n}$ onto the subspaces $\mathbb{L}^n[\mathcal{I}]$ and $\mathbb{S}^n[\mathcal{I}]$, respectively. Let chol : $\mathbb{S}^n_{++} \to \mathbb{L}^n_{++}$ denote the invertible function that maps a positive definite matrix to its Cholesky factor.

5.1 Burer-Monteiro-Zhang method

We first consider the simplified constraint set

$$\mathcal{P} \coloneqq \{ (x, Q) \in \mathbb{R}^m \times \mathbb{S}^n[\mathcal{I}] \mid Q \succeq H(x) \}$$
(35)

where $\mathcal{D}^n \subseteq \mathcal{I} \subseteq \mathcal{L}^n$ and $H : \mathbb{R}^m \to \mathbb{S}^n$. As in [36], we parameterize the matrix argument Q in a way that automatically enforces the constraint $Q \succ H(x)$ while introducing just n scalar inequality constraints.

Recall $Q \succ H$ if and only if $Q = H + LL^{\top}$ for the *unique* matrix $L = \operatorname{chol}(Q - H) \in \mathbb{L}_{++}^n$. With $\mathcal{J} \coloneqq \mathcal{L}^n \setminus \mathcal{I}$, we can split L into a sum of $L^{\mathcal{I}} \in \mathbb{L}_{++}^n[\mathcal{I}]$ and $L^{\mathcal{J}} \in \mathbb{L}^n[\mathcal{J}]$, giving

$$Q = H + (L^{\mathcal{I}} + L^{\mathcal{J}})(L^{\mathcal{I}} + L^{\mathcal{J}})^{\top}.$$
(36)

But $Q \in \mathbb{S}^n[\mathcal{I}]$, so we can apply the vectorization operator vecs_{\mathcal{I}} on both sides to give

$$\operatorname{vecs}_{\mathcal{J}}(H + (L^{\mathcal{I}} + L^{\mathcal{J}})(L^{\mathcal{I}} + L^{\mathcal{J}})^{\top}) = 0.$$
(37)

Equation (37) defines $|\mathcal{J}|$ equations to solve for the $|\mathcal{J}|$ variables of $L^{\mathcal{J}}$, where each $L_{ij}^{\mathcal{J}}$ is fully specified by H_{ij} and the $L_{i'j'}$ with (i', j') < (i, j).⁵ In Algorithm 2, we compute the $L_{ij}^{\mathcal{J}}$ in ascending lexicographic order via Cholesky factorization.

⁵The lexicographic order < on \mathbb{I}^2 is defined by (i, j) < (i', j') if i < i' or $(i = i') \land (j < j')$.

Algorithm 2 Cholesky factorization algorithm for solving systems of the form (37) based on [36, Lem. 1].

Require: $\mathcal{D}^n \subseteq \mathcal{I} \subseteq \mathcal{L}^n, L^{\mathcal{I}} \in \mathbb{L}_{++}^n [\mathcal{I}], \text{ and } H \in \mathbb{S}^n$ 1: $(\mathcal{J}, L^{\mathcal{J}}) \leftarrow (\mathcal{L}^n \setminus \mathcal{I}, 0_{n \times n})$ 2: **for** each $(i, j) \in \mathcal{J}$ in ascending lexicographic order **do** 3: $L_{ij}^{\mathcal{J}} \leftarrow -\frac{1}{L_{jj}^{\mathcal{I}}} (H_{ij} + \sum_{k=1}^{j-1} (L_{ik}^{\mathcal{I}} + L_{ik}^{\mathcal{J}}) (L_{jk}^{\mathcal{I}} + L_{jk}^{\mathcal{J}}))$ 4: **end for** 5: **return** $L^{\mathcal{J}}$

Notice that each $L^{\mathcal{J}}$ is fully defined by H and $L^{\mathcal{I}}$ via algorithm 2, so we have proven the following lemma.

Lemma 2 ([36, Lem. 1]). For each $(H, L^{\mathcal{I}}) \in \mathbb{S}^n \times \mathbb{L}^n[\mathcal{I}]$ such that $L_{ii}^{\mathcal{I}} \neq 0$ for each $i \in \mathbb{I}_{1:n}$, there is a unique $L^{\mathcal{J}} \in \mathbb{L}^n[\mathcal{J}]$ satisfying (37).

With a slight abuse of notation, we let $L^{\mathcal{J}} = L^{\mathcal{J}}(H, L^{\mathcal{I}})$ denote the function defined by Algorithm 2, which maps each $(H, L^{\mathcal{I}}) \in \mathbb{S}^n \times \mathbb{L}^n_{++}[\mathcal{I}]$ to the matrix $L^{\mathcal{J}} \in \mathbb{L}^n[\mathcal{J}]$ uniquely satisfying (37). Moreover, we let

$$Q(H, L^{\mathcal{I}}) \coloneqq H + (L^{\mathcal{I}} + L^{\mathcal{J}}(H, L^{\mathcal{I}}))(L^{\mathcal{I}} + L^{\mathcal{J}}(H, L^{\mathcal{I}}))^{\top}$$

as in (36). Clearly $Q(H, L^{\mathcal{I}}) \succ H$ is satisfied by definition. Finally, we define the transformation

$$T(x, L^{\mathcal{I}}) \coloneqq \left(x, Q(H(x), L^{\mathcal{I}})\right) \tag{38}$$

which has the inverse

$$T^{-1}(x,Q) \coloneqq \left(x, \pi_{\mathcal{I}}^{L}[\operatorname{chol}(Q - H(x))]\right)$$
(39)

and we have the following lemma.

Lemma 3 ([36, Lem. 2]). The function T defined by (38) is a bijection between $\mathbb{R}^m \times \mathbb{L}_{++}^n[\mathcal{I}]$ and $\operatorname{int}(\mathcal{P})$.

Differentiability of T and T^{-1} follow from differentiability of H and algorithm 2. In fact, these functions are as smooth as H. More importantly, the bijection T allows us to transform the minimum of a continuous function over \mathcal{P} to an infimum over $\mathbb{R}^m \times \mathbb{L}^n_{++}[\mathcal{I}]$, given by the following theorem.

Theorem 3 ([36, Thm. 1]). If $f : \mathcal{P} \to \mathbb{R}$ is continuous and attains a minimum in \mathcal{P} , then

$$\min_{(x,Q)\in\mathcal{P}} f(x,Q) = \inf_{(x,L^{\mathcal{I}})\in\mathbb{R}^m\times\mathbb{L}^n_{++}[\mathcal{I}]} f(T(x,L^{\mathcal{I}})).$$
(40)

We reiterate the proof of Theorem 3 for illustrative purposes.

Proof. Continuity of f implies its minimum over \mathcal{P} equals its infimum over $int(\mathcal{P})$, i.e.,

$$\min_{(x,Q)\in\mathcal{P}} f(x,Q) = \inf_{(x,Q)\in \operatorname{int}(\mathcal{P})} f(x,Q)$$

Since T is a bijection, we can transform the optimization variables as follows:

$$\inf_{(x,Q)\in \operatorname{int}(\mathcal{P})} f(x,Q) = \inf_{(x,L^{\mathcal{I}})\in T^{-1}(\operatorname{int}(\mathcal{P}))} f(T(x,L^{\mathcal{I}})).$$

Finally, since $\mathbb{R}^m \times \mathbb{L}^n_{++}[\mathcal{I}] = T^{-1}(\operatorname{int}(\mathcal{P}))$, we have (40).

5.2 Generalized Burer-Monteiro-Zhang method

We return to constraints of the form (20). Recall Assumption 1 requires the matrix inequalities are strictly feasible in the constraint set. We use a similar procedure to Section 5.1, but Algorithm 2 must be applied to *each* strict inequality $\Sigma \succ H$ and $\mathcal{A}(\beta, \Sigma) \succ 0$.

For the sparse symmetric matrix Σ and matrix inequality $\Sigma \succ H(\beta)$, the procedure is the same as in Section 5.1. Let $L^{\mathcal{J}_{\Sigma}} = L^{\mathcal{J}_{\Sigma}}(H, L^{\mathcal{I}_{\Sigma}})$ denote the function defined by Algorithm 2 with $L^{\mathcal{I}} = L^{\mathcal{I}_{\Sigma}}$, $\mathcal{I} = \mathcal{I}_{\Sigma}$, and $n = n_{\Sigma}$. Then

$$\Sigma(\beta, L^{\mathcal{I}_{\Sigma}}) \coloneqq H + (L^{\mathcal{I}_{\Sigma}} + L^{\mathcal{J}_{\Sigma}}(H, L^{\mathcal{I}_{\Sigma}}))(L^{\mathcal{I}_{\Sigma}} + L^{\mathcal{J}_{\Sigma}}(H, L^{\mathcal{I}_{\Sigma}}))^{\top}$$

guarantees $\Sigma(H, L^{\mathcal{I}_{\Sigma}}) \succ H$ and $\Sigma(H, L^{\mathcal{I}_{\Sigma}}) \in \mathbb{S}^{n_{\Sigma}}[\mathcal{I}_{\Sigma}]$ for all $(H, L^{\mathcal{I}_{\Sigma}}) \in \mathbb{S}^{n_{\Sigma}} \times \mathbb{L}_{++}^{n_{\Sigma}}[\mathcal{I}_{\Sigma}]$. In other words, Σ is fully defined and the constraint $\Sigma \succ H$ automatically satisfied by $(H, L^{\mathcal{I}_{\Sigma}}) \in \mathbb{S}^{n_{\Sigma}} \times \mathbb{L}_{++}^{n_{\Sigma}}[\mathcal{I}_{\Sigma}]$.

For the general matrix inequality $\mathcal{A}(\beta, \Sigma) \succeq 0$, the procedure is slightly different. Let $L^{\mathcal{I}_{\mathcal{A}}} = L^{\mathcal{I}_{\mathcal{A}}}(L^{\mathcal{I}_{\mathcal{A}}})$ denote function defined by Algorithm 2 with $L^{\mathcal{I}} = L^{\mathcal{I}_{\mathcal{A}}}, \mathcal{I} = \mathcal{I}_{\mathcal{A}}, n = n_{\mathcal{A}}$, and H = 0. Define the functions

$$\mathcal{A}(L^{\mathcal{I}_{\mathcal{A}}}) \coloneqq (L^{\mathcal{I}_{\mathcal{A}}} + L^{\mathcal{J}_{\mathcal{A}}}(L^{\mathcal{I}_{\mathcal{A}}}))(L^{\mathcal{I}_{\mathcal{A}}} + L^{\mathcal{J}_{\mathcal{A}}}(L^{\mathcal{I}_{\mathcal{A}}}))^{\top}$$

which guarantees $\mathcal{A}(L^{\mathcal{I}_{\mathcal{A}}}) \in \mathbb{S}_{++}^{n_{\mathcal{A}}}[\mathcal{I}_{\mathcal{A}}]$ for all $L^{\mathcal{I}_{\mathcal{A}}} \in \mathbb{L}_{++}^{n_{\mathcal{A}}}[\mathcal{I}_{\mathcal{A}}]$. However, the constraint is not fully eliminated; we are left with $|\mathcal{I}_{\mathcal{A}}|$ equality constraints in the transform space,

$$\operatorname{vecs}_{\mathcal{I}_{\mathcal{A}}}(\mathcal{A}(\beta, \Sigma(H(\beta), L^{\mathcal{I}_{\Sigma}})) - \mathcal{A}(L^{\mathcal{I}_{\mathcal{A}}})) = 0$$

with the other $|\mathcal{L}^{n_{\mathcal{A}}} \setminus \mathcal{I}_{\mathcal{A}}|$ constraints automatically guaranteed by Algorithm 2.

To define the new constraints, we require the variable transformations

$$\mathcal{T}(\beta, L^{\mathcal{I}_{\Sigma}}, L^{\mathcal{I}_{\mathcal{A}}}) \coloneqq \left(\beta, \Sigma(H(\beta), L^{\mathcal{I}_{\Sigma}})\right) \tag{41a}$$

$$\mathcal{A}_{\mathcal{T}}(\beta, L^{\mathcal{I}_{\Sigma}}, L^{\mathcal{I}_{\mathcal{A}}}) \coloneqq \mathcal{A}(L^{\mathcal{I}_{\mathcal{A}}})$$
(41b)

which are well-defined for all $(\beta, L^{\mathcal{I}_{\Sigma}}, L^{\mathcal{I}_{\mathcal{A}}}) \in \mathbb{R}^{n_{\beta}} \times \mathbb{L}^{n_{\Sigma}}[\mathcal{I}_{\Sigma}] \times \mathbb{L}^{n_{\mathcal{A}}}[\mathcal{I}_{\mathcal{A}}]$. With the functions (41), we define the transformed constraint functions $(g_{\mathcal{T}}, h_{\mathcal{T}})$ and the transformed constraint set $\Phi \subseteq \mathbb{R}^{n_{\beta}} \times \mathbb{L}^{n_{\Sigma}}[\mathcal{I}_{\Sigma}] \times \mathbb{L}^{n_{\mathcal{A}}}[\mathcal{I}_{\mathcal{A}}]$ according to (22) and (23). The inverse transform is

$$\mathcal{T}^{-1}(\beta, \Sigma) \coloneqq \left(\beta, \pi_{\mathcal{I}_{\Sigma}}^{L}[\operatorname{chol}(\Sigma - H(\beta))], \pi_{\mathcal{I}_{\mathcal{A}}}^{L}[\operatorname{chol}(\mathcal{A}(\beta, \Sigma))]\right)$$
(42)

for all $(\beta, \Sigma) \in \Theta_{++}$, and we have the following lemma.

Lemma 4. The function \mathcal{T} defined by (41) is a bijection between Φ and Θ_{++} .

Proof. First, we have $\mathcal{T}(\Phi) \subseteq \Theta_{++}$ since the transformed constraints guarantee the constraints $g(\beta, \Sigma) = 0$, $h(\beta, \Sigma) \leq 0$, $\Sigma \succ H(\beta)$, and $\mathcal{A}(\beta, \Sigma) \succ 0$ for any $(\beta, \Sigma) \coloneqq \mathcal{T}(\phi)$ and $\phi \in \Phi$. Next, it is clear by construction that $\mathcal{T}^{-1} \circ \mathcal{T}$ is the identity map on Φ . Therefore \mathcal{T} is injective. Similarly, we have $\mathcal{T}^{-1}(\Theta_{++}) \subseteq \Phi$ by construction, and $\mathcal{T} \circ \mathcal{T}^{-1}$ is the identity map on Θ_{++} , so $\mathcal{T} : \Phi$ is surjective.

Under Assumption 1, the functions $\mathcal{T}, \mathcal{T}^{-1}$, and $\mathcal{A}_{\mathcal{T}}$ are as smooth as H, and moreover, the bijection \mathcal{T} transforms a minimum over Θ into an infimum over Φ .

Proposition 5. If Assumption 1 holds and $f: \Theta \to \mathbb{R}$ is continuous and attains a minimum in Θ , then

$$\min_{\theta \in \Theta} f(\theta) = \inf_{\phi \in \Phi} f(\mathcal{T}(\phi))$$

Proof. The proof follows that of Theorem 3, noting that Assumption 1 gives $cl(\Theta_{++}) = \Theta$ and therefore the minimum of f over Θ equals the infimum of f over Θ_{++} .

5.3 Approximate solutions

As mentioned in Section 3, we consider a lower bound $\varepsilon > 0$ on the diagonal elements of $(L^{\mathcal{I}_{\Sigma}}, L^{\mathcal{I}_{\mathcal{A}}})$. We define the tightened constraint set Φ_{ε} by (24). In the following theorem we show, under Assumption 1 and continuity of f, the infimum of $f \circ \mathcal{T}$ over Φ_{ε} converges to the minimum of f over Θ (see Appendix E for proof).

Theorem 4. Suppose f is continuous and attains a minimum in Θ . Define $\mu_0 := \min_{\theta \in \Theta} f(\theta)$ and

$$\mu_{\varepsilon} \coloneqq \inf_{\phi \in \Phi_{\varepsilon}} f(\mathcal{T}(\phi)).$$
(43)

If Assumption 1 holds, then $\mu_{\varepsilon} \searrow \mu$ as $\varepsilon \searrow 0$.

In fact, with a few additional requirements on the objective f, convergence of approximate problem solutions to the solution of the original problem is guaranteed by the following theorem (see Appendix E for proof).

Theorem 5. Suppose f is continuous and Assumption 1 holds. Consider the set-valued function $\hat{\theta} : \mathbb{R}_{\geq 0} \to \mathcal{P}(\Theta)$, defined as $\hat{\theta}_{\varepsilon} \coloneqq \operatorname{argmin}_{\theta \in \mathcal{T}(\Phi_{\varepsilon})} f(\theta)$ for all $\varepsilon > 0$, and $\hat{\theta}_0 \coloneqq \operatorname{argmin}_{\theta \in \Theta} f(\theta)$. If there exists $\alpha \in \mathbb{R}$ and compact $C \subseteq \Theta$ such that

$$\Theta_{f < \alpha} \coloneqq \{ \theta \in \Theta \mid f(\theta) \le \alpha \}$$

is contained in C and $\Theta_{f \leq \alpha} \cap \Theta_{++}$ is nonempty, then there exists $\overline{\varepsilon} > 0$ such that, for all $\varepsilon_0 \in [0, \overline{\varepsilon})$,

- (a) f achieves a minimum in Θ and $\hat{\theta}_0$ is nonempty;
- (b) if $\varepsilon_0 > 0$, then f achieves a minimum in $\mathcal{T}(\Theta_{\varepsilon_0})$ and $\hat{\theta}_{\varepsilon_0}$ is nonempty;
- (c) μ_{ε} is continuous and $\hat{\theta}_{\varepsilon}$ is outer semicontinuous at $\varepsilon = \varepsilon_0$; and



Figure 1: Benchmark temperature Control Laboratory (TCLab) [38].

(d) if $\hat{\theta}_0$ is a singleton, then $\limsup_{\varepsilon \geq 0} \hat{\theta}_{\varepsilon} = \hat{\theta}_0$.

Remark 12. Originally, [36] used a log-barrier approach to handle the strict inequalities implied by $L \in \mathbb{L}_{++}^{n}[\mathcal{I}]$ and achieve global convergence for a class of linear SDPs. For problems of the form (40), the log-barrier term eliminates all remaining constraints. However, for problems of the form (34) many constraints remain in addition to the strict inequalities on the diagonal elements of $(L^{\mathcal{I}_{\Sigma}}, L^{\mathcal{I}_{\mathcal{A}}}) \in \mathbb{L}_{++}^{n_{\Sigma}}[\mathcal{I}_{\Sigma}] \times \mathbb{L}_{++}^{n_{\mathcal{A}}}[\mathcal{I}_{\mathcal{A}}].$

6 **Case Studies**

In this section, we present two real-world case studies in which Algorithm 1 is used to identify the LADM (2) and implement offset-free MPC. In the first case study, we consider the TCLab (Figure 1), an Arduino-based temperature control laboratory that serves as a low-cost⁶ benchmark for linear MIMO control [38]. We identify the TCLab from open-loop data and use the resulting model to design an offset-free MPC. We compare closed-loop control and estimation performance of these models to that of offset-free MPCs designed with the identification methods from [16, 17]. In the second case study, data from an industrial-scale chemical reactor is used to design Kalman filters for the linear augmented disturbance model, and the closed-loop estimation performance is compared to that of the designs proposed in [17].

Throughout these experiments, we use an ℓ_2 regularization term in the transformed space, 7,8

$$-\ln p_0(\beta, L^{\mathcal{I}_{\Sigma}}) \propto R_0(\beta, L^{\mathcal{I}_{\Sigma}}) \coloneqq \frac{\rho}{2} \left(|\beta - \overline{\beta}|^2 + \|L_{\Sigma}(\beta, L^{\mathcal{I}_{\Sigma}}) - L_{\Sigma}(\overline{\beta}, \overline{L}^{\mathcal{I}_{\Sigma}})\|_{\mathrm{F}}^2 \right).$$
(44)

where $\rho \geq 0$ is the regularization weight and $(\overline{\beta}, \overline{L}^{\mathcal{I}_{\Sigma}}, \overline{L}^{\mathcal{I}_{\mathcal{A}}})$ denote the initial guess for the

⁶The TCLab is available for under \$40 from https://apmonitor.com/heat.htm and https://www. amazon.com/gp/product/B07GMFWMRY.

⁷With $L_{\Sigma}(\bar{\beta}, \overline{L}^{\mathcal{I}_{\Sigma}}) = 0$, the last term of (44) becomes proportional to $\operatorname{tr}(L_{\Sigma}L_{\Sigma}^{\top}) = \operatorname{tr}(\Sigma)$ where $L_{\Sigma} = L_{\Sigma}(\beta, L^{\mathcal{I}_{\Sigma}})$ and $\Sigma = \Sigma(\beta, L^{\mathcal{I}_{\Sigma}})$. ⁸With $L^{\mathcal{I}_{\Sigma}}(\beta, L^{\mathcal{I}_{\Sigma}}) \equiv 0$ (e.g., Σ is block diagonal and $H(\beta) \equiv 0$) the last term of (44) is proportional to $\|L^{\mathcal{I}_{\Sigma}} - \overline{L}^{\mathcal{I}_{\Sigma}}\|_{\mathrm{F}}^{2} = |\operatorname{vec}_{\mathcal{I}_{\Sigma}}(L^{\mathcal{I}_{\Sigma}} - \overline{L}^{\mathcal{I}_{\Sigma}})|^{2}$.



Figure 2: TCLab identification data and noise-free responses $\hat{y}_k = \sum_{j=1}^k \hat{C} \hat{A}^{j-1} \hat{B} u_{k-j}$ of a few selected models.

optimizer. The variable $L_{\mathcal{A}}$ is not regularized. With $\rho = 0$, the MAP problem (4) with the regularizer (44) simplifies to the standard ML identification problem (3).

The initial guess for the ML and MAP problems is based on a nested ML estimation approach described in [16, 17]. The initial guess methods effectively augment standard identification methods (e.g., principal component analysis (PCA), Ho-Kalman (HK), canonical correlation analysis (CCA) algorithms), so we refer to the initial guess models as "augmented" versions of the standard method being used (e.g., augmented PCA, augmented HK, augmented CCA). Each optimization problem is formulated in CasADi via Algorithm 1 and solved with IPOPT. Information about each model fit and configuration is presented in Table 1. Wall times for a single-thread of an Intel Core i9-10850K processor are reported.

6.1 Benchmark temperature controller

Unless otherwise specified, the TCLab is modeled as a two-state, two-disturbance system of the form (2), with internal temperatures as plant states $s = \begin{bmatrix} T_1 & T_2 \end{bmatrix}^{\top}$, heater voltages as inputs $u = \begin{bmatrix} V_1 & V_2 \end{bmatrix}^{\top}$, and measured temperatures $y = \begin{bmatrix} T_{m,1} & T_{m_2} \end{bmatrix}^{\top}$ as outputs. Throughout, we choose $n_d = p$ to satisfy the offset-free necessary conditions in [3, 4], and we consider output disturbance models $(B_d, C_d) = (0_{2\times 2}, I_2)$. We use (A_s, B_s) fully parameterized and $C = I_2$ to guarantee model identifiability and make the states interpretable as internal temperatures. For the remaining model terms, we have (K_x, K_d, R_e) fully parameterized and $(D, \hat{s}_0, \hat{d}_0) = (0, 0, 0)$.

Eight TCLab models are presented.

- 1. Augmented PCA: the 6-state TCLab model used in [16], where principle component analysis on a 400×5100 data Hankel matrix is used to determine the states in the disturbance-free model.
- 2. Augmented ARX: a VARX(1,1) model, equivalent to a stochastic LTI model with process noise but zero measurement noise.

Table 1: TCLab model fitting results. * The augmented PCA/ARX identification methods are not iterative. ** The maximum number of iterations was set at 500.

Model	Results			Configuration				
	Time (s)	Iterations	$L_N(\hat{\theta})$	Method	ρ	\mathcal{D}	ε	ε_i
Augmented PCA	0.01	N/A^*	3823.4	[16]	N/A	N/A	N/A	N/A
Augmented ARX	0.04	N/A^*	3807.3	see text	N/A	N/A	N/A	N/A
Unregularized ML	119.46	500**	-9430.9	Algo. 1	0	C	10^{-6}	N/A
Regularized ML 1	120.06	500**	-9431.7	Algo. 1	0.002	C	10^{-6}	N/A
Regularized ML 2	9.06	21	-9416.6	Algo. 1	0.005	C	10^{-6}	N/A
Constrained ML 1	71.92	97	-9347.2	Algo. 1	0	$\mathcal{D}_1(0.3) \cap \mathcal{D}_2(0.998, 0)$	10^{-6}	0.03
Constrained ML 2	50.69	62	-9358.2	Algo. 1	0	$\mathcal{D}_1(0.3) \cap \mathcal{D}_2(0.999, 0)$	10^{-6}	0.03
Reg. & Cons. ML	36.59	40	-9338.4	Algo. 1	0.001	$\mathcal{D}_1(0.3) \cap \mathcal{D}_2(0.998, 0)$	10^{-6}	0.03



Figure 3: TCLab models open-loop and closed-loop (filter) eigenvalues.



Figure 4: TCLab setpoint tracking tests.

- 3–5) Unregularized ML, Regularized ML 1 and 2: classic ML and MAP models.
- 6–8) Constrained ML 1 and 2, Reg. & Cons. ML: eigenvalue-constrained ML and MAP models. LMI region constraints enforce filter stability and impose a lower bound on the real part of the filter eigenvalues.

Each ML model uses Augmented ARX as the initial guess as it has the smallest number of states. The augmented PCA model is, in effect, an unsupervised learner of the state estimates, and therefore does not produce a parsimonious state description.

In Figure 2, the identification data is presented along with the noise-free responses $\hat{y}_k = \sum_{j=1}^k \hat{C} \hat{A}^{j-1} \hat{B} u_{k-j}$ of a few selected models. Computation times, numbers of IPOPT iterations, and *unregularized* log-likelihood $L_N(\hat{\theta})$ values are reported in Table 1. The open-loop A and closed-loop $A_K \coloneqq A - KC$ eigenvalues of each model are plotted in Figure 3.

Except for the augmented PCA model, all of the open-loop eigenvalues cluster around the same region of the complex plane (figure 3). The closed-loop filter eigenvalues are also placed similarly, although the classic ML models (Unregularized ML, Regularized ML 1 and 2) suffer from slow or even unstable filter eigenvalues, despite achieving lower $L_N(\hat{\theta})$ values than their eigenvalue-constrained counterparts. The models with unstable eigenvalues fail to converge (Table 1) as the unstable filter modes make the problem extremely sensitive to changes in the parameter values. While sufficiently high ρ is sufficient to achieve filter stability, there are no clear minimum value of ρ to achieve this. On the other hand, the constrained ML models have stable filter eigenvalues without regularization, and have well-defined estimator performance guarantees based on the applied constraints.

To test offset-free control performance, we performed two sets of closed-loop experiments on offset-free MPCs designed with the models. In Figure 4, identical setpoint changes were applied to a TCLab running at a steady-state power output of 50%. The setpoint changes were tracked with the offset-free MPC design described in [16]. In Figure 5, step disturbances in the output p_i and the input m_i are injected into a plant trying to maintain a given steady-state temperature. The setpoints are tracked with the offset-free MPC design described in [16].



Figure 5: TCLab disturbance rejection tests.



Figure 6: TCLab setpoint tracking test performance.



Figure 7: TCLab disturbance rejection test performance.



Figure 8: TCLab identification index data for (left) setpoint tracking and (right) disturbance rejection tests.

Control performance is quantified by the squared distance from the setpoint $\ell_k := |y_k - y_{sp,k}|^2$. Estimation performance is quantified by the squared filter errors $e_k^{\top} e_k$. For any signal a_k , we define a *T*-sample moving average by $\langle a_k \rangle_T := T^{-1} \sum_{j=0}^{T-1} a_{k-j}$. Setpoint tracking performance is reported in Figure 6, and disturbance rejection performance is reported in Figure 7. The worst performing models are those with unstable filters (Unregularized ML and Regularized ML 1). These models shut off over the course of the experiment as the integrating disturbance estimates grow unbounded. The remaining classic ML model (Regularized ML 2) has slow filter eigenvalues that contribute to poor control performance on the disturbance rejection test (Figure 7, left). The augmented models (Augmented PCA/ARX) perform poorly in either control or estimation aspect on both test. The best performance is achieved by the remaining ML models, which all perform approximately the same across the tests.

To investigate the distributional accuracy of the models, we quantify performance with the identification index $q := e^{\top} R_e^{-1} e$. Recall the signal $e_k \in \mathbb{R}^2$ is an i.i.d., zero-mean Gaussian process, i.e., $e_k \stackrel{\text{iid}}{\sim} \mathcal{N}(0, R_e)$, and therefore the index q_k is i.i.d. with a χ_2^2 distribution. Moreover, the moving average $\langle q_k \rangle_T$ is distributed as χ_{2T}^2/T , although it is no longer independent in time. In Figure 8, histograms of $\langle q \rangle_T, T \in \{1, 10, 100\}$ are plotted against their expected distribution for a few selected models (Augmented PCA/ARX, Un-



Figure 9: Schematic of the DMT reactor and MPC control strategy.

regularized ML, and Reg. & Cons. ML). The extreme discrepancies between the augmented models' performance index $\langle q \rangle_T$ and the reference distribution χ^2_{2T}/T are primarily due to the augmented models significantly overestimating R_e compared to the ML models,

$$\hat{R}_{e}^{\text{Aug. PCA}} = \begin{bmatrix} 0.5871 & 0.3365 \\ 0.3365 & 0.2878 \end{bmatrix}, \qquad \qquad \hat{R}_{e}^{\text{Aug. ARX}} = \begin{bmatrix} 0.5084 & 0.2198 \\ 0.2198 & 0.2980 \end{bmatrix}, \\ \hat{R}_{e}^{\text{Unreg. ML}} = \begin{bmatrix} 0.0106 & 0.0007 \\ 0.0007 & 0.008 \end{bmatrix}, \qquad \qquad \hat{R}_{e}^{\text{Reg. Cons. ML}} = \begin{bmatrix} 0.0107 & 0.0007 \\ 0.0007 & 0.008 \end{bmatrix}.$$

The reference distribution and the ML models' $\langle q \rangle_T$ distribution diverge at large T since, due to plant-model mismatch, the filter's innovation errors are slightly autocorrelated. Frequent right-tail errors from the unregularized ML model are due to filter instability.

6.2 Eastman reactor

A schematic of the chemical reactor considered in the next case study is presented in Figure 9. The control objective of the chemical reactor is to track three setpoints (the output, a specified reactor temperature y = T, and the flowrates $\begin{bmatrix} u_1 & u_3 \end{bmatrix}^{\top} = \begin{bmatrix} F_1 & F_2 \end{bmatrix}^{\top}$), without offset, by controlling the three inputs (the reactant flow rates and utility temperatures $u = \begin{bmatrix} F_1 & T_H & F_2 \end{bmatrix}^{\top}$).⁹ See [17] for more details about the reactor operation. As in Section 6.1, we choose $n_d = p$ and consider output disturbance models $(B_d, C_d) = (0_{2\times 1}, 1)$. This time, we use an observability canonical form [40] with $A_s = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix}$ and

⁹The flowrates are both manipulated variables and controlled variables. At steady-state, we should reach the setpoints in y = T and $\begin{bmatrix} u_1 & u_3 \end{bmatrix}^{\top} = \begin{bmatrix} F_1 & F_2 \end{bmatrix}^{\top}$, but $u_2 = T_H$ will not reach a predefined setpoint.



Figure 10: Training data and noise-free responses for the Eastman reactor models (Augmented HK and ML models using Augmented HK as the initial guess).

Table 2: Eastman reactor model fitting results. * The augmented identification methods are not iterative. ** The maximum number of iterations was set at 500.

Model		Configuration						
	Time (s)	Iterations	$L_N(\hat{\theta})$	Method	ρ	\mathcal{D}	ε	ε_i
Augmented CCA	0.09	N/A*	-11399.3	[17]	N/A	N/A	N/A	N/A
Unregularized ML	5.59	19	-14383.1	Algo. 1	0	C	10^{-6}	N/A
Regularized ML 1	5.46	17	-14362.5	Algo. 1	0.0	C	10^{-6}	N/A
Regularized ML 2	5.75	20	-14346.7	Algo. 1	0.1	C	10^{-6}	N/A
Regularized ML 3	4.89	13	-14108.0	Algo. 1	1.0	\mathbb{C}	10^{-6}	N/A
Constrained ML 1 $$	19.89	92	-13944.9	Algo. 1	0	$D_1(0.3)$	10^{-6}	0.01
Constrained ML 2	16.58	73	-13941.1	Algo. 1	0	$D_1(0.3)$	10^{-6}	0.02
Constrained ML 3	14.01	58	-13928.5	Algo. 1	0	$\mathcal{D}_1(0.3)$	10^{-6}	0.04

 $C_s = \begin{bmatrix} 1 & 0 \end{bmatrix}$. For the remaining model terms, we have (B_s, K_x, K_d, R_e) fully parameterized and $(D, \hat{s}_0, \hat{d}_0) = (0, 0, 0)$.

Eight reactor models were fit to the closed-loop data from [17]:

- 1. Augmented CCA: a CCA model [67] augmented with a disturbance model, as detailed in [17].¹⁰
- 3–5) Unregularized ML, Regularized ML 1 to 3: classic ML and MAP models.
- 6–8) **Constrained ML 1** to **3**: eigenvalue-constrained ML and MAP models. LMI region constraints impose a lower bound on the real part of the filter eigenvalues.

Each ML model uses the augmented CCA model as the initial guess. In Figure 10, the closed-loop identification data and noise-free responses are presented. Computational details, the *unregularized* log-likelihood value, and model configuration details are reported in Table 2. The open-loop A_s and closed-loop A_K eigenvalues are plotted in Figure 11.

The main difference between eigenvalues of the unconstrained ML models (Unregularized ML and Regularized ML 1–3) and the constrained ML models (Constrained ML 1–3)

¹⁰This is not the same model used in [17], as a different input-output model is considered, although the same data is used. Specifically, [17] considered a model with both regulatory-layer setpoints and measured values.



Figure 11: Eastman reactor models open-loop and closed-loop (filter) eigenvalues.



Figure 12: Test performance for the Eastman reactor models on the test data sets from [17].

are faster open-loop eigenvalues and closed-loop eigenvalues with possibly negative real part (Figure 11). For the constrained ML models, the real part of this fast filter eigenvalue is bounded from below using the LMI region constraint $\mathcal{D}_1(0.3)$. As in the TCLab case study, sufficiently high ρ is sufficient to avoid the negative eigenvalue, but there is no clear cutoff to achieve this.

The estimation performance for these filters are compared on two test data sets (from [17]) in Figure 12. While the unconstrained models appear to have the best test performance, it is at a cost of undesirable estimate dynamics. In Figure 13, we plot the filter response to an initial guess equal to the eigenvector corresponding to the smallest eigenvalue of A_K . Those filters with eigenvalues having negative real parts exhibit overshoot in the estimate. The best performing filters without this behavior are the constrained ML models.

Control performance could not be compared on the real plant due to cost and safety considerations. However, the closed-loop responses can be compared in simulation. In Figure 14, we plot simulated responses to a setpoint change. Each simulation considers the nominal closed-loop response (i.e., plant as the model, no noise) using the offset-free



Figure 13: Eastman reactor models' closed-loop (filter) response to the eigenvector corresponding to the fastest eigenvalue.



Figure 14: Eastman reactor models simulated closed-loop test performance.

MPC design in [17] with $Q_s = 1$ and $R_s = \text{diag}(0.01, 1, 0.01)$. The regularized ML models exhibit significant overshoot in the response, whereas the unregularized ML model and constrained ML models do not.

6.3 Discussion

The main limitation of eigenvalued-constrained ML is computational cost. While constrained ML retains linear scaling in sample size N, each LMI region constraint on an arbitrary system matrix $\tilde{A} \in \mathbb{R}^{\tilde{n} \times \tilde{n}}$ requires an additional $O(\tilde{n}^2(m^2 + 1))$ variables and $O(\tilde{n}^2m^2)$ equality constraints. These requirements can be significantly reduced for spectral abscissa bounds $\mathcal{D}_1(s)$ and stability constraints $\mathcal{D}_2(s,0)$. As mentioned in remark 10, these constraints are quite similar to the "smooth" spectral radii and abscissa constraints of [65, 66], which only add $O(\tilde{n}^2)$ variables and $O(\tilde{n}^2)$ equality constraints. For eigenvalues constrained to the LMI regions $\mathcal{D}_1(s)$ or $\mathcal{D}_2(s, x_0)$, implementing these constraints as a special case can reduce the computational cost significantly.

For a standard, black-box LADM (2) with $n_d = p$, a canonical form for (A_s, B_s, C_s) , and $(D, \hat{s}_0, \hat{d}_0) = (0, 0, 0)$, there are $O(n_s(p + m) + p^2)$ variables before constraints are added, and $O(n_s^2)$ variables after. Thus, fitting black-box models of large-scale systems is computationally prohibitive. However, as discussed in Section 2, large-scale chemical plants and networked systems may be represented by significantly fewer variables: $O(N_u n_u (p_u + m_u) + N_u p_u^2)$ without constraints, or $O(N_u n_u^2)$ with constraints, where N_u is the number of units or nodes, and n_u, m_u, p_u are the number of states, inputs, and outputs per unit or node.

7 Conclusion

We propose an algorithm for identifying offset-free MPC-relevant models with ML identification, which includes a method of generating *closed* constraints on the eigenvalues of system matrices, and a Cholesky factor reparameterization of nonlinear SDPs as standard NLPs. The algorithm is validated on real-world data in two case studies: a low-cost benchmark temperature controller, and an industrial-scale reactor.

The code, including sample TCLab datasets and scripts for model fitting, is made available at github.com/rawlings-group/mlid_2024. The Eastman reactor data is proprietary and is not made available. We will develop this code further for reliable applications on large-scale systems.

We conclude with some suggestions of future research. Since ML identified models are more distributionally accurate, they are more suitable to the performance monitoring technique of [68]. Integrated identification and offset-free controller validation may be possible by combining this method with ours. There are limitations to the performance of Kalman filter-based disturbance observers, as shown by [69]. Specifically, the largest real filter eigenvalue is often bounded from below by the largest real open-loop eigenvalue. We find these limitations present in all our case studies. Filter designs with eigenvalue constraints may overcome these limitations and deliver superior offset-free MPC performance.

A Proof of Proposition 1

Silverman [50] contains a more complete characterization of the DARE solutions for regulation problems with cross terms. However, this admits additional nullspace terms into the gain matrix which the Kalman filtering problem does not allow. We avoid nullspace terms through the assumption $R_v \succ 0$ and therefore streamline the proof of Proposition 1.

For the following definitions and lemmas, consider the system matrices $\mathcal{W} \coloneqq (A, B, C, D)$ corresponding to a noise-free system.

Definition 2. The system \mathcal{W} is *left invertible on* $\mathbb{I}_{0:k-1}$ if

$$0 = \begin{bmatrix} D & & & \\ CB & D & & \\ \vdots & \ddots & \ddots & \\ CA^{k-2}B & \dots & CB & D \end{bmatrix} \begin{bmatrix} u_0 \\ \vdots \\ u_{k-1} \end{bmatrix}$$

implies $u_0 = 0$. The system \mathcal{W} is *left invertible* if there is some $j \in \mathbb{I}_{>0}$ such that \mathcal{W} is left invertible on $\mathbb{I}_{0:k-1}$ for all $k \geq j$.

Definition 3. The system \mathcal{W} is strongly detectable if $y_k \to 0$ implies $x_k \to 0$.

The following lemmas are taken directly from [50, Thms. 8, 18(iii)], but the proofs are omitted for the sake of brevity.

Lemma 5 ([50, Thm. 8]). If W is left invertible, then W is strongly detectable if and only if (A - BF, C - DF) is detectable for all F of appropriate dimension.

Lemma 6 ([50, Thm. 18(iii)]). If W is left invertible, then the DARE

$$P = A^{\top}PA - (A^{\top}PB + C^{\top}D)(B^{\top}PB + D^{\top}D)^{-1}(B^{\top}PA + D^{\top}C)$$

has a unique, stabilizing solution¹¹ if and only if W is stabilizable and semistrongly detectable.

For the remainder of this section, we consider the full rank factorization

$$\begin{bmatrix} Q_w & S_{wv} \\ S_{wv}^\top & R_v \end{bmatrix} = \begin{bmatrix} \tilde{B} \\ \tilde{D} \end{bmatrix} \begin{bmatrix} \tilde{B}^\top & \tilde{D}^\top \end{bmatrix}$$

and the dual system $\tilde{\mathcal{W}} \coloneqq (A^{\top}, C^{\top}, \tilde{B}^{\top}, \tilde{D}^{\top})$ to analyze the properties of the original system (11). The following lemma relates the properties $R_v \succ 0$ and left invertability of $\tilde{\mathcal{W}}$.

Lemma 7. If $R_v \succ 0$ then $\tilde{\mathcal{W}}$ is left invertible.

¹¹Contrary to in Section 2, here we mean the solution P is stabilizing when A - BK(P) is stable, where $K(P) \coloneqq (B^{\top}PB + D^{\top}D)^{-1}B^{\top}P$.

Proof. Left invertability on $\mathbb{I}_{0:k-1}$ is equivalent to

$$0 = \begin{bmatrix} \tilde{D}^{\top} & & \\ \tilde{B}^{\top}C^{\top} & \tilde{D}^{\top} & \\ \vdots & \ddots & \ddots & \\ \tilde{B}^{\top}(A^{\top})^{k-2}C^{\top} & \dots & \tilde{B}^{\top}C^{\top} & \tilde{D}^{\top} \end{bmatrix} \begin{bmatrix} u_0 \\ \vdots \\ u_{k-1} \end{bmatrix}$$
(45)

implying $u_0 = 0$. But $R_v = \tilde{D}\tilde{D}^\top \succ 0$, so \tilde{D}^\top has a zero nullspace. For each $k \in \mathbb{I}_{>0}$, the coefficient matrix of (45) has a zero nullspace. Thus, $u_0 = 0$ and $\tilde{\mathcal{W}}$ is left invertible. \Box

Finally, we can prove Proposition 1.

Proof of Proposition 1. By Lemma 7, we have that $\tilde{\mathcal{W}}$ is left invertible. Therefore, by Lemma 6, the DARE (15) has a unique, stabilizing solution if and only if $\tilde{\mathcal{W}}$ is stabilizable and strongly detectable. But by Lemma 5 and duality, the latter statement is true if and only if (A, C) is detectable and $(A - FC, \tilde{B} - F\tilde{D})$ is stabilizable for all $F \in \mathbb{R}^{n \times p}$. \Box

B Proof of Proposition 2

Throughout this appendix, we define the set of $n \times n$ Hermitian, Hermitian positive definite, and Hermitian positive semidefinite matrices as \mathbb{H}^n , \mathbb{H}^n_{++} , and \mathbb{H}^n_+ . Notice that $f_{\mathcal{D}}$ maps to Hermitian matrices so we can write it as $f : \mathbb{C} \to \mathbb{H}^m$. We define the extension of $M_{\mathcal{D}}$ to complex arguments $M_{\mathcal{D}} : \mathbb{C}^{n \times n} \times \mathbb{H}^n_+ \to \mathbb{H}^{nm}$ as

$$M_{\mathcal{D}}(A,P) \coloneqq M_0 \otimes P + M_1 \otimes (AP) + M_1^{\top} \otimes (AP)^{\mathsf{H}}.$$

To show Proposition 2, we need a preliminary result about Hermitian positive semidefinite matrices, generalized from Lemma A.1 in [25].

Lemma 8. For any $M \in \mathbb{H}^n$, if $M \succeq 0$ $(M \succ 0)$ then $\operatorname{Re}(M) \succeq 0$ $(\operatorname{Re}(M) \succ 0)$.

Proof. With $M = \operatorname{Re}(M) + \iota \operatorname{Im}(M)$, it is clear M Hermitian implies $\operatorname{Re}(M)$ is symmetric and $\operatorname{Im}(M)$ is skew-symmetric. Thus $v^{\top}Mv = v^{\top}\operatorname{Re}(M)v$ for all $v \in \mathbb{R}^n$, and positive (semi)definiteness of M implies positive (semi)definiteness of $\operatorname{Re}(M)$.

In proving Proposition 2, we take the approach of [25] but are careful to distinguish eigenvalues on the interior \mathcal{D} from those on the boundary $\partial \mathcal{D}$.

Proof of Proposition 2. (\Leftarrow) Suppose that $M_{\mathcal{D}}(A, P) \succeq 0$ for some $P \succ 0$ and let $\lambda \in \lambda(A)$. Then there exists a nonzero $v \in \mathbb{C}^n$ for which $v^{\mathrm{H}}A = \lambda v^{\mathrm{H}}$. Consider the identity

$$(I_m \otimes v)^{\mathrm{H}} M_{\mathcal{D}}(A, P)(I_m \otimes v) = M_0 \otimes v^{\mathrm{H}} P v + M_1 \otimes (v^{\mathrm{H}} A P v) + M_1^{\top} \otimes (v^{\mathrm{H}} P A^{\top} v)$$

$$= M_0 \otimes v^{\mathrm{H}} P v + M_1 \otimes (\overline{\lambda} v^{\mathrm{H}} P v) + M_1^{\top} \otimes (\lambda v^{\mathrm{H}} P v)$$

$$= v^{\mathrm{H}} P v (M_0 + M_1 \lambda + M_1^{\top} \overline{\lambda})$$

$$= v^{\mathrm{H}} P v f_{\mathcal{D}}(\lambda).$$

The assumption $P \succ 0$ implies $v^{\mathrm{H}} P v > 0$, and $M_{\mathcal{D}}(A, P) \succeq 0$ further implies $f_{\mathcal{D}}(\lambda) \succeq 0$. Therefore $\lambda \in \mathrm{cl}(\mathcal{D})$.

Next suppose $\lambda \in \lambda(A)$ is non-simple and $\lambda \in \partial \mathcal{D}$. Then there exists nonzero $v_1, v_2 \in \mathbb{C}^n$ (linearly independent) such that $v^{\mathrm{H}} f_{\mathcal{D}}(\lambda) v = 0$, $v_1^{\mathrm{H}} A = \lambda v_1^{\mathrm{H}}$, and $v_2^{\mathrm{H}} A = \lambda v_2^{\mathrm{H}} + v_1$. Because \mathcal{D} is open, $\lambda \in \partial \mathcal{D} = \mathrm{cl}(\mathcal{D}) \setminus \mathcal{D}$ must satisfy both $f_{\mathcal{D}}(\lambda) \succeq 0$ and $f_{\mathcal{D}}(\lambda) \neq 0$. Therefore $f_{\mathcal{D}}(\lambda)$ is singular, and there exists a nonzero vector $v \in \mathbb{C}^m$ such that $v^{\mathrm{H}} f_{\mathcal{D}}(\lambda) v = 0$. With the 2×2 matrices

$$\tilde{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \coloneqq \begin{bmatrix} v_1^H \\ v_2^H \end{bmatrix} P \begin{bmatrix} v_1 & v_2 \end{bmatrix} \succ 0$$
$$\tilde{J} \coloneqq \lambda I_2 + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

we have $\begin{bmatrix} v_1 & v_2 \end{bmatrix}^{H} A = \tilde{J} \begin{bmatrix} v_1 & v_2 \end{bmatrix}^{H}$ and therefore

$$(I_m \otimes \begin{bmatrix} v_1 & v_2 \end{bmatrix})^{\mathrm{H}} M_{\mathcal{D}}(A, P) (I_m \otimes \begin{bmatrix} v_1 & v_2 \end{bmatrix}) = M_0 \otimes \tilde{P} + M_1 \otimes \tilde{J}\tilde{P} + M_1^{\mathrm{T}} \otimes (\overline{\tilde{J}}\tilde{P})^{\mathrm{T}} = M_{\mathcal{D}}(\tilde{J}, \tilde{P}) \succeq 0.$$

Next, we have

$$\begin{split} \tilde{M} &\coloneqq K_{2,m} M_{\mathcal{D}}(\tilde{J}, \tilde{P}) K_{2,m}^{\top} \\ &= \tilde{P} \otimes M_0 + \tilde{J} \tilde{P} \otimes M_1 + (\overline{\tilde{J}} \tilde{P})^{\top} \otimes M_1^{\top} \\ &= \tilde{P} \otimes f_{\mathcal{D}}(\lambda) + \begin{bmatrix} p_{12}(M_1 + M_1^{\top}) & p_{22}M_1 \\ p_{22}M_1^{\top} & 0 \end{bmatrix} \succeq 0. \end{split}$$

Finally,

$$(I_2 \otimes v)^{\mathrm{H}} \tilde{M}(I_2 \otimes v) = \begin{bmatrix} p_{12}v^{\mathrm{H}}(M_1 + M_1^{\mathrm{T}})v & p_{22}v^{\mathrm{H}}M_1v \\ p_{22}v^{\mathrm{H}}M_1^{\mathrm{T}}v & 0 \end{bmatrix} \succeq 0$$

But $\tilde{P} \succ 0$ implies $p_{22} > 0$, so the above matrix inequality implies $v^{\mathrm{H}} M_1 v = 0$. Moreover, with $v^{\mathrm{H}} f_{\mathcal{D}}(\lambda) v = 0$, we also have $v^{\mathrm{H}} M_0 v = 0$ and therefore $f(z) \equiv 0$ and \mathcal{D} is empty, a contradiction. Therefore each $\lambda \in \lambda(A)$ non-simple implies $\lambda \in \mathcal{D}$.

 (\Rightarrow) Suppose $\lambda(A) \subset cl(\mathcal{D})$ and $\lambda \in \lambda(A)$ non-simple implies $\lambda \in \mathcal{D}$.

If $A = \lambda$ is a (possibly complex) scalar, then it lies in $cl(\mathcal{D})$ by assumption, with $M_{\mathcal{D}}(\lambda, p) = pf_{\mathcal{D}}(\lambda) \succeq 0$ for all p > 0.

If $A = \lambda I_n + N$ is a (possibly complex) Jordan block, where $N \in \mathbb{R}^{n \times n}$ is a shift matrix and n > 1, then $\lambda \in \mathcal{D}$ and $f_{\mathcal{D}}(\lambda) \succ 0$. Let $T_k := \text{diag}(k^{n-1}, \ldots, k, 1)$ for each $k \in \mathbb{I}_{>0}$. Then $T_k^{-1}AT_k = \lambda I_n + k^{-1}N \to \lambda I_n$ as $k \to \infty$. Moreover, because $M_{\mathcal{D}}$ is continuous, we have

$$M_{\mathcal{D}}(T_k^{-1}AT_k, I_n) \to M_{\mathcal{D}}(\lambda I_n, I_n) = f_{\mathcal{D}}(\lambda) \otimes I_n \succ 0.$$

Therefore there exists some $k_0 \in \mathbb{I}_{>0}$ such that $M_{\mathcal{D}}(T_k^{-1}AT_k, I_n) \succ 0$ for all $k \ge k_0$. With $P := T_k T_k^{\top}$, we have

$$M_{\mathcal{D}}(A, P) = M_0 \otimes T_k T_k^\top + M_1 \otimes (AT_k T_k^\top) + M_1^\top \otimes (AT_k T_k^\top)^\top$$

= $(I_m \otimes T_k)(M_0 \otimes I_n + M_1 \otimes T_k^{-1}AT_k + M_1^\top \otimes (T_k^{-1}AT_k)^\top)(I_m \otimes T_k)^\top$
= $(I_m \otimes T_k)M_{\mathcal{D}}(T_k^{-1}AT_k, I_n)(I_m \otimes T_k)^\top \succ 0.$

Finally, for any $A \in \mathbb{R}^{n \times n}$, let $A = V(\bigoplus_{i=1}^{p} J_i)V^{-1}$ denote the Jordan decomposition of A, where $J_i = \lambda_i I_{n_i} + N_i$, $\lambda_i \in \lambda(A)$, N_i are shift matrices, and $n = \sum_{i=1}^{p} n_i$. We have already shown that for each $i \in \mathbb{I}_{1:p}$, there exists $P_i \succ 0$ such that $M_{\mathcal{D}}(J_i, P_i) \succeq 0$. Then with $\tilde{P} \coloneqq V(\bigoplus_{i=1}^{p} P_i)V^{-1}$, we have

$$(I_m \otimes V^{-1}) M_{\mathcal{D}}(A, \dot{P}) (I_m \otimes V^{-1})^{\mathrm{H}}$$

= $M_0 \otimes \left(\bigoplus_{i=1}^p P_i\right) + M_1 \otimes \left(\bigoplus_{i=1}^p J_i P_i\right) + M_1 \otimes \left(\bigoplus_{i=1}^p J_i P_i\right)^{\mathrm{T}}$
= $K_{n,m} \left(\bigoplus_{i=1}^p K_{m,n_i} M_{\mathcal{D}}(J_i, P_i) K_{m,n_i}^{\mathrm{T}}\right) K_{n,m}^{\mathrm{T}} \succeq 0$

and therefore $M_{\mathcal{D}}(A, \tilde{P}) \succeq 0$. Last, Lemma 8 gives $M_{\mathcal{D}}(A, P) \succeq 0$ with $P \coloneqq \operatorname{Re}(\tilde{P})$ since

$$M_{\mathcal{D}}(A, P) = M_{\mathcal{D}}(A, \operatorname{Re}(\tilde{P})) = \operatorname{Re}(M_{\mathcal{D}}(A, \tilde{P})).$$

C Proof of Proposition 4

To show Proposition 4(a), we first require the following eigenvalue sensitivity result due to [39, Thm. 7.2.3].

Theorem 6 ([39, Thm. 7.2.3]). For any $A \in \mathbb{C}^{n \times n}$, denote its Schur decomposition by $A = Q(D+N)Q^{\mathrm{H}}$, where $Q \in \mathbb{C}^{n \times n}$ is unitary, $D \in \mathbb{C}^{n \times n}$ is diagonal, and $N \in \mathbb{C}^{n \times n}$ is strictly upper triangular.¹² Let p be the smallest positive integer for which $M^p = 0$ where $M_{ij} := |N_{ij}|$. Then, for any $E \in \mathbb{R}^{n \times n}$ and $\mu \in \lambda(A + E)$,

$$\min_{\lambda \in \lambda(A)} |\mu - \lambda| \le \max\left\{ c \|E\|, (c\|E\|)^{1/p} \right\}$$

where $c \coloneqq \sum_{k=0}^{p-1} \|N\|^k$.

Proof of Proposition 4. Throughout this proof, we show a set S is not open (or not closed) by demonstrating that S^c (or S) does not contain all its limit points.

(a)—For any $A \in \mathbb{A}^n_{\mathcal{D}}$, continuity of $f_{\mathcal{D}}$ gives the existence of a function $\delta(\lambda) > 0$ such that $f_{\mathcal{D}}(z) \succ 0$ for all $|z - \lambda| < \delta(\lambda)$ and $\lambda \in \lambda(A)$. Let $\delta := \min_{\lambda \in \lambda(A)} \delta(\lambda)$. By Theorem 6 and norm equivalence, there exist c > 0 and $p \in \mathbb{I}_{1:n}$ such that

$$\max_{\mu \in \lambda(A+E)} \min_{\lambda \in \lambda(A)} |\lambda - \mu| \le \max\left\{ c \|E\|_{\mathcal{F}}, (c\|E\|_{\mathcal{F}})^{1/p} \right\}$$

for all $E \in \mathbb{R}^{n \times n}$. Therefore there exists a $\varepsilon > 0$ such that

$$\max_{\mu \in \lambda(A+E)} \min_{\lambda \in \lambda(A)} |\lambda - \mu| < \delta$$

¹²A matrix U is strictly upper triangular if $U_{ij} = 0$ for all $i \ge j$.

for all $E \in \mathcal{B} := \{ E' \in \mathbb{R}^{n \times n} \mid ||E'||_{\mathrm{F}} < \varepsilon \}$. Finally, $A + \mathcal{B}$ is a neighborhood of A contained in $\mathbb{A}^n_{\mathcal{D}}$, and, since $A \in \mathbb{A}^n_{\mathcal{D}}$ was chosen arbitrarily, $\mathbb{A}^n_{\mathcal{D}}$ is open.

(b)(i)—Because \mathcal{D} is open, nonempty, and not equal to \mathcal{D} , $\partial \mathcal{D}$ is nonempty. Let $\lambda \in \partial \mathcal{D}$ and $\lambda_k \in \mathcal{D}^c$ be a sequence for which $\lambda_k \to \lambda$. By symmetry, we also have $\overline{\lambda} \in \mathcal{D}$ and $\overline{\lambda}_k \in \mathcal{D}^c$.

For n = 2, we have $A \coloneqq \begin{bmatrix} \operatorname{Re}(\lambda) & -\operatorname{Im}(\lambda) \\ \operatorname{Im}(\lambda) & \operatorname{Re}(\lambda) \end{bmatrix} \in \mathbb{R}^{2 \times 2}$ has eigenvalues $\lambda, \overline{\lambda} \in \mathcal{D}$, and $A_k \coloneqq \begin{bmatrix} \operatorname{Re}(\lambda_k) & -\operatorname{Im}(\lambda_k) \\ \operatorname{Im}(\lambda_k) & \operatorname{Re}(\lambda_k) \end{bmatrix} \in \mathbb{R}^{2 \times 2}$ has eigenvalues $\lambda_k, \overline{\lambda_k} \in \mathcal{D}^c$ for each $k \in \mathbb{I}_{>0}$. The corresponding eigenvectors are $\begin{bmatrix} \pm \iota \\ 1 \end{bmatrix} \in \mathbb{C}^2$. Therefore $A \in \widetilde{\mathbb{A}}^2_{\mathcal{D}}$ but $A_k \in (\widetilde{\mathbb{A}}^2_{\mathcal{D}})^c$ for each $k \in \mathbb{I}_{>0}$, and the limit $A_k \to A$ gives us that $(\widetilde{\mathbb{A}}^2_{\mathcal{D}})^c$ does not contain all its limit points.

For n > 2, let $A_0 \in \tilde{\mathbb{A}}_{\mathcal{D}}^{n-2}$, and we can extend the prior argument with the sequence $B_k \coloneqq A_k \oplus A_0 \in (\tilde{\mathbb{A}}_{\mathcal{D}}^n)^c, k \in \mathbb{I}_{>0}$ that converges to $B \coloneqq A \oplus A_0 \in \tilde{\mathbb{A}}_{\mathcal{D}}^n$.

(b)(ii)—By part (b)(i), it suffices to consider the case n = 1. By closure and convexity of $\mathcal{D}, \mathcal{D} \cap \mathbb{R}$ is either a closed line segment, a closed ray, or \mathbb{R} itself. In other words, $\mathcal{D} \cap \mathbb{R}$ is open if and only if it has no endpoints. Moreover, since $\partial \mathcal{D} \cap \mathbb{R}$ is the set of the endpoints of $\mathcal{D} \cap \mathbb{R}, \mathcal{D} \cap \mathbb{R}$ is open if and only if $\partial \mathcal{D} \cap \mathbb{R}$ is empty. Finally, since $\tilde{\mathbb{A}}^1_{\mathcal{D}} = \mathcal{D} \cap \mathbb{R}, \tilde{\mathbb{A}}^1_{\mathcal{D}}$ is open if and only if $\partial \mathcal{D} \cap \mathbb{R}$ is empty.

(c)(i)—Let $\lambda \in \partial \mathcal{D}$. Suppose n = 4. Then $\overline{\lambda} \in \partial \mathcal{D}$ by symmetry. Because \mathcal{D} is open, there exists a sequence $\lambda_k \in \mathcal{D}$ such that $\lambda_k \to \lambda$, and by symmetry, we also have $\overline{\lambda}_k \in \mathcal{D}$ and $\overline{\lambda}_k \to \overline{\lambda}$. Consider again the 2 × 2 matrices A and A_k from part (b)(i), which have eigenvalues $\lambda, \overline{\lambda} \in \mathcal{D}$ and $\lambda_k, \overline{\lambda_k} \in \mathcal{D}^c$, respectively. Then the block matrices $B \coloneqq \begin{bmatrix} A & I_2 \\ 0 & A \end{bmatrix} \in \mathbb{R}^{4 \times 4}$ and $B_k \coloneqq \begin{bmatrix} A_k & I_2 \\ 0 & A_k \end{bmatrix} \in \mathbb{R}^{4 \times 4}$ have the same eigenvalues, but this time $\begin{bmatrix} \pm i \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

the eigenvectors are $\begin{bmatrix} \pm \iota \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \pm \iota \end{bmatrix} \in \mathbb{C}^4$ and the eigenvalues are non-simple. Since λ is a non-simple eigenvalue on the boundary of \mathcal{D} , we have $B \notin \tilde{\mathbb{A}}^4_{\mathcal{D}}$. However, λ_k are all in the

interior of \mathcal{D} , so $B_k \in \tilde{\mathbb{A}}_{\mathcal{D}}^4$. Since $B_k \to B$, the set $\tilde{\mathbb{A}}_{\mathcal{D}}^4$ does not contain all its limit points. On the other hand, let $\lambda \in \partial \mathcal{D}$ and suppose n > 4. Similarly to part (b)(i), with

any $\tilde{A}_0 \in \tilde{\mathbb{A}}_{\mathcal{D}}^{n-4}$, we can extend the argument for the n = 4 case with the sequence $\tilde{A}_k := B_k \oplus \tilde{A}_0 \in \tilde{\mathbb{A}}_{\mathcal{D}}^n, k \in \mathbb{I}_{>0}$ that converges to $\tilde{A} := B \oplus \tilde{A}_0 \in (\tilde{\mathbb{A}}_{\mathcal{D}}^n)^c$.

(c)(ii)—Let $\lambda \in \partial \mathcal{D} \cap \mathbb{R}$ and $n \geq 2$. Because \mathcal{D} is convex, open, and nonempty, there exists $\varepsilon > 0$ such that exactly one of the real intervals $(\lambda, \lambda + \varepsilon)$ or $(\lambda - \varepsilon, \lambda)$ is contained in \mathcal{D} , whereas the other is contained in $\operatorname{int}(\mathcal{D}^c)$. Without loss of generality, assume $(\lambda - \varepsilon, \lambda) \subseteq \mathcal{D}$.¹³ Then $A_k := (\lambda - \varepsilon/k)I_n + N_n \in \tilde{\mathbb{A}}_{\mathcal{D}}^n$ for each $k \in \mathbb{I}_{>0}$, but $A_k \to \lambda I_n + N_n \in (\tilde{\mathbb{A}}_{\mathcal{D}}^n)^c$ and therefore $\tilde{\mathbb{A}}_{\mathcal{D}}^n$ does not contain all its limit points. (d)—Since $\overline{\mathbb{A}}_{\mathcal{D}}^n := \{A \in \mathbb{R}^{n \times n} \mid \lambda(A) \subset \operatorname{cl}(\mathcal{D})\}$ contains $\mathbb{A}_{\mathcal{D}}^n$, it suffices to show any

(d)—Since $\mathbb{A}_{\mathcal{D}}^{n} := \{A \in \mathbb{R}^{n \times n} \mid \lambda(A) \subset \operatorname{cl}(\mathcal{D})\}$ contains $\mathbb{A}_{\mathcal{D}}^{n}$, it suffices to show any $A \in \overline{\mathbb{A}}_{\mathcal{D}}^{n}$ is a limit point of $\mathbb{A}_{\mathcal{D}}^{n}$. Denote the Jordan form by $A = V \left(\bigoplus_{i=1}^{p} \mu_{i} I_{n_{i}} + N_{n_{i}}\right) V^{-1}$, where $V \in \mathbb{R}^{n \times n}$ is invertible, $\mu_{i} \in \lambda(A)$, $n = \sum_{i=1}^{p} n_{i}$, and $N_{i} \in \mathbb{R}^{n_{i} \times n_{i}}$ is a shift matrix. Because $\mu_{i} \in \operatorname{cl}(\mathcal{D})$, there exists a sequence $\mu_{i,k} \in \mathcal{D}$ such that $\mu_{i,k} \to \mu_{i}$. Then $A_{k} := V \left(\bigoplus_{i=1}^{p} \mu_{i,k} I_{n_{i}} + N_{i}\right) V^{-1} \in \mathbb{A}_{\mathcal{D}}^{n}$ and $A_{k} \to A$.

¹³Otherwise, take the reflection about the imaginary axis $-\mathcal{D}$ and $-\tilde{\mathbb{A}}_{\mathcal{D}}^{n}$.

D Proof of Theorem 2

To prove Theorem 2(a,b), we use sensitivity results on the value functions of parameterized nonlinear SDPs,

$$V(y) \coloneqq \inf_{x \in \mathbb{X}(y)} F(x, y) \tag{46}$$

where the set-valued function $\mathbb{X} : \mathbb{R}^m \to \mathcal{P}(\mathbb{R}^n)$ is defined by

$$\mathbb{X}(y) \coloneqq \{ x \in \mathbb{R}^n \mid G(x, y) \succeq 0 \}.$$

Consider also the graph of the set-valued function X,

$$\mathbb{Z} \coloneqq \{ (x, y) \in \mathbb{R}^{n+m} \mid G(x, y) \succeq 0 \}.$$

Notice that \mathbb{Z} is closed if G is continuous. We say Slater's condition holds at $y \in \mathbb{R}^m$ if there exists $x \in \mathbb{R}^n$ such that $x \in int(\mathbb{X}(y))$, or equivalently, $G(x, y) \succ 0$. In the following proposition, we specialize [70, Prop. 4.4] to nonlinear SDPs.

Proposition 6 ([70, Prop. 4.4]). Let $y_0 \in \mathbb{R}^m$ and suppose

- (i) F and G are continuous on \mathbb{R}^{n+m} ;
- (ii) there exist $\alpha \in \mathbb{R}$ and compact $C \subset \mathbb{R}^n$ such that, for each y in a neighborhood of y_0 , the level set

 $\operatorname{lev}_{\leq \alpha} F(\cdot, y) \coloneqq \{ x \in \mathbb{X}(y) \mid F(x, y) \le \alpha \}$

is nonempty and contained in C; and

(iii) Slater's condition holds at y_0 .

Then $F(\cdot, y)$ attains a minimum on $\mathbb{X}(y)$ for all $y \in N_y$, and V(y) is continuous at $y = y_0$.

Proof. See [70, Prop. 4.4] and the discussions in [70, pp. 264, 483–484, 491–492]. \Box

Proof of Theorem 2. Let vec : $\mathbb{R}^{n \times n} \to \mathbb{R}^{n^2}$ and vecs : $\mathbb{R}^{n \times n} \to \mathbb{R}^{(1/2)(n+1)n}$ denote the vectorization and symmetric vectorization operators, respectively.

(a)—With $x \coloneqq \operatorname{vecs}(P)$, $y \coloneqq \operatorname{vec}(A)$, $F(x, y) \coloneqq \operatorname{tr}(VP)$, and $G(x, y) \coloneqq P \oplus (M_{\mathcal{D}}(A, P) - M)$, we can use Proposition 6 to show the continuity of $\phi_{\mathcal{D}}$ on $\mathbb{A}^n_{\mathcal{D}}$. Let $A_0 \in \mathbb{A}^n_{\mathcal{D}}$. Condition (i) of Proposition 6 holds by assumption. Slater's condition (iii) holds because for any $P \succ 0$ such that $M_{\mathcal{D}}(A_0, P) \succ 0$, we can define $P_0 \coloneqq \gamma P \succ 0$ for some $\gamma > \gamma_0 \coloneqq \|M\| \times \|[M_{\mathcal{D}}(A_0, P)]^{-1}\|$ to give

$$M_{\mathcal{D}}(A_0, P_0) = \gamma M_{\mathcal{D}}(A_0, P) \succ \gamma_0 M_{\mathcal{D}}(A_0, P) \succeq M.$$

Moreover, by continuity of $M_{\mathcal{D}}$, there exists a neighborhood N_A of A_0 such that $M_{\mathcal{D}}(A, P_0) \succ M$ for all $A \in N_A$. Letting $\alpha := \operatorname{tr}(VP_0) > 0$, we have that the set

$$\{P \in \mathbb{S}^n_+ \mid \operatorname{tr}(VP) \le \alpha\}$$

is compact and contains the nonempty level set

$$\{P \in \mathbb{P}(A) \mid \operatorname{tr}(VP) \leq \alpha\}$$

for all $A \in N_A$. Taking the image of each of the above sets under the vecs operation gives condition (ii) of Proposition 6. All the conditions of Proposition 6 are thus satisfied for each $A_0 \in \mathbb{A}^n_{\mathcal{D}}$, and we have $\phi_{\mathcal{D}}$ is continuous on $\mathbb{A}^n_{\mathcal{D}}$.

(b)—Continuity of $\phi_{\mathcal{D}}$ on $\mathbb{A}^n_{\mathcal{D}}$ implies closure of the sublevel sets of $\phi_{\mathcal{D}}$, and (33) follows by definition of $\mathbb{A}^n_{\mathcal{D}}(\varepsilon)$.

(c)—First, $M_{\mathcal{D}}(A, P) \succ 0$ implies $P \succ 0$ by Proposition 3. Moreover, for any $P \succ 0$ such that $M_{\mathcal{D}}(A, P) \succ 0$, we have $M_{\mathcal{D}}(A, P) \succeq \gamma M_{\mathcal{D}}(A, P) \succeq M$ with $P \coloneqq \gamma P$ and $\gamma \coloneqq \|M\| \times \|[M_{\mathcal{D}}(A, P)]^{-1}\|$, so feasibility of (17) is equivalent to feasibility of

$$M_{\mathcal{D}}(A, P) \succ M, \qquad P \succeq 0$$

and therefore $\bigcup_{\varepsilon>0} \mathbb{A}^n_{\mathcal{D}}(\varepsilon) = \mathbb{A}^n_{\mathcal{D}}$. But $\mathbb{A}^n_{\mathcal{D}}(\varepsilon)$ is monotonically decreasing,¹⁴ so $\mathbb{A}^n_{\mathcal{D}}(\varepsilon) \nearrow \bigcup_{\varepsilon>0} \mathbb{A}^n_{\mathcal{D}}(\varepsilon) = \mathbb{A}^n_{\mathcal{D}}$ as $\varepsilon \searrow 0$.

E Proof of Theorems 4 and 5

Starting with Theorem 4:

Proof of Theorem 4. Since μ_{ε} is nondecreasing and bounded from below by μ , it suffices to show that for each $\delta > 0$, there exists a $\overline{\varepsilon} > 0$ such that $\mu_{\overline{\varepsilon}} - \mu < \delta$.

Let $\theta^* \in \Theta$ denote a point for which $\mu = f(\theta^*)$. If $\theta^* \in \Theta_{++}$, we could simply choose $\overline{\varepsilon} > 0$ large enough to put θ^* in $\mathcal{T}(\Phi_{\overline{\varepsilon}})$ and achieve $\mu_{\overline{\varepsilon}} - \mu = 0 < \delta$.

Instead, we assume $\theta^* \notin \Theta_{++}$. By Assumption 1, there exists a sequence $\theta_k \in \Theta_{++}, k \in \mathbb{I}_{>0}$ such that $\theta_k \to \theta$ as $k \to \infty$. Defining $\nu_k \coloneqq f(\theta_k)$, we have $\nu_k \to \mu$ by continuity of f. Therefore, there exists some $k_0 \in \mathbb{I}_{>0}$ such that $\nu_k - \mu < \delta$ for all $k \ge k_0$. For each $\theta_k \in \Theta_{++}$, there exists a unique $\phi_k = (\beta_k, L_k^{\mathcal{I}_{\Sigma}}, L_k^{\mathcal{I}_A}) \in \Phi$ such that $\theta_k = \mathcal{T}(\phi_k)$ (by Lemma 4). Let $\overline{\varepsilon}$ be the minimum over all the diagonal elements of $L_{k_0}^{\mathcal{I}_{\Sigma}}$ and $L_{k_0}^{\mathcal{I}_A}$. Then $(\beta_{k_0}, L_{k_0}^{\mathcal{I}_{\Sigma}}, L_{k_0}^{\mathcal{I}_A}) \in \Phi_{\overline{\varepsilon}}$ by construction, $\nu_{k_0} \ge \mu_{\overline{\varepsilon}}$ by optimality, and $\mu_{\overline{\varepsilon}} - \mu \le \nu_{k_0} - \mu < \delta$. \Box

As in Appendix D, we use sensitivity results of [70] on optimization problems to prove Theorem 5. This time, however, we consider the continuity of the value function for parameterized NLPs on Banach spaces. Let \mathcal{X} , \mathcal{Y} , and \mathcal{K} be Banach spaces and consider the parameterized NLP,

$$V(y) \coloneqq \inf_{x \in \mathbb{X}(y)} F(x, y) \tag{47}$$

where the set-valued function $\mathbb{X}: \mathcal{Y} \to \mathcal{P}(\mathcal{X})$ is defined by

 $\mathbb{X}(y) \coloneqq \{ x \in \mathcal{X} \mid G(x, y) \in K \}$

¹⁴By "monotonically decreasing" we mean $\varepsilon \leq \varepsilon' \Rightarrow \mathbb{A}^n_{\mathcal{D}}(\varepsilon) \supseteq \mathbb{A}^n_{\mathcal{D}}(\varepsilon')$.

for some $G : \mathcal{X} \times \mathcal{Y} \to \mathcal{K}$ and $K \subseteq \mathcal{K}$ is closed. Let $X^0(y)$ denote the (possibly empty) set of solutions to (47). Define the graph of the set-valued function $\mathbb{X}(\cdot)$ by

$$\mathbb{Z} \coloneqq \{ (x, y) \in \mathcal{X} \times \mathcal{Y} \mid G(x, y) \in K \}.$$

Notice that \mathbb{Z} is closed if G is continuous and K is closed.

Proposition 7 ([70, Prop. 4.4]). Let $y_0 \in \mathcal{Y}$ and assume:

- (i) F and G are continuous on $\mathcal{X} \times \mathcal{Y}$ and K is closed;
- (ii) there exist $\alpha \in \mathbb{R}$ and a compact set $C \subseteq \mathcal{X}$ such that, for every y in a neighborhood of y_0 , the level set

$$\{x \in \mathbb{X}(y) \mid f(x, y) \le \alpha\}$$

is nonempty and contained in C; and

(iii) for any neighborhood N_x of the solution set $X^0(y_0)$, there exists a neighborhood N_y of y_0 such that $N_x \cap \mathbb{X}(y)$ is nonempty for all $y \in N_y$;

then V(y) is continuous and $X^{0}(y)$ is outer semicontinuous at $y = y_{0}$.

Proof of Theorem 5. First, we must specify $\overline{\varepsilon}$. For each $\theta \in \Theta_{++}$, let

$$\varepsilon(\theta) \coloneqq \max \{ \varepsilon > 0 \mid \theta \in \mathcal{T}(\Phi_{\varepsilon}) \}$$

where the maximum is achieved since there is a finite number of diagonal elements of the Cholesky factors that must be lower bounded. Now we specify $\overline{\varepsilon}$ as the supremum of $\varepsilon(\theta)$ over all $\theta \in \Theta_{f \leq \alpha} \cap \Theta_{++}$,

$$\overline{\varepsilon} \coloneqq \sup \left\{ \varepsilon(\theta) \mid \theta \in \Theta_{f < \alpha} \cap \Theta_{++} \right\}$$

so that, for any $\varepsilon \in (0,\overline{\varepsilon})$, $\Theta_{f \leq \alpha} \cap \mathcal{T}(\Phi_{\varepsilon})$ is nonempty and is contained in the compact set C.

(a)—Following the proof of [70, Prop. 4.4], we have (i) F is continuous and (ii) the level set $\Theta_{f \leq \alpha}$ is nonempty and contained in the compact set C, which implies $\Theta_{f \leq \alpha}$ is a compact level set and therefore the minimum of f over $\Theta_{f \leq \alpha}$ is achieved and equals the minimum over Θ . Moreover, $\hat{\theta}_0$ must be nonempty.

(b)—Similarly to part (a), we have, for each $\varepsilon \in (0, \overline{\varepsilon})$, that the level set $\Theta_{f \leq \alpha} \cap \mathcal{T}(\Phi_{\varepsilon})$ is nonempty and contained in the compact set C, so f achieves its minimum over $\mathcal{T}(\Phi_{\varepsilon})$ and $\hat{\theta}_{\varepsilon}$ is nonempty.

(c)—Consider the graph of the constraint function,

$$\mathbb{Z} \coloneqq \{ (\theta, \varepsilon) \in \Theta \times \mathbb{R}_{\geq 0} \mid \theta \in \mathcal{T}(\Phi_{\varepsilon}) \text{ if } \varepsilon > 0 \}.$$

Consider a sequence $(\theta_k, \varepsilon_k) \in \mathbb{Z}, k \in \mathbb{I}_{>0}$ that is convergent $(\theta_k, \varepsilon_k) \to (\theta, \varepsilon)$. Then $\varepsilon \ge 0$, otherwise the sequence would not converge. Moreover, $\theta \in \Theta$ since $\theta_k \in \mathcal{T}(\Phi_{\varepsilon_k}) \subseteq \Theta$ for all $k \in \mathbb{I}_{>0}$ and Θ contains all its limit points. If $\varepsilon = 0$, then $(\theta, \varepsilon) \in \mathbb{Z}$ trivially. On the other

hand, if $\varepsilon > 0$, then $\varepsilon(\theta_k)$ converges to $\varepsilon(\theta)$ because \mathcal{T} is continuous and the max can be taken over a finite number of elements of $\mathcal{T}^{-1}(\theta_k)$. Moreover, $\varepsilon(\theta_k)$ and upper bounds ε_k because $\theta_k \in \mathcal{T}(\Phi_{\varepsilon_k})$, so $\varepsilon(\theta) \ge \varepsilon$. Finally, we have $\theta \in \mathcal{T}(\Phi_{\varepsilon})$, $(\theta, \varepsilon) \in \mathbb{Z}$, and \mathbb{Z} is closed.

Let $\varepsilon_0 \geq 0$ and N_{θ} be a neighborhood of θ_{ε_0} . With

$$\delta \coloneqq \sup \left\{ \varepsilon(\theta) \mid \theta \in N_{\theta} \right\} > 0$$

we have $N_{\theta} \cap \Theta$ and $N_{\theta} \cap \mathcal{T}(\Phi_{\varepsilon})$ are nonempty for all $\varepsilon \in (0, \varepsilon_0 + \delta)$.

Finally, the requirements of Proposition 7 are satisfied for all $\varepsilon_0 \in [0, \overline{\varepsilon})$, so μ_{ε} is continuous and $\hat{\theta}_{\varepsilon}$ is outer semicontinuous at $\varepsilon = \varepsilon_0$.

(d)—The last statement follows by the definition of outer semicontinuity and the fact that the lim sup is nonempty. $\hfill \Box$

References

- S. J. Qin and T. A. Badgwell, "A survey of industrial model predictive control technology," *Control Eng. Pract.*, vol. 11, no. 7, pp. 733–764, 2003.
- [2] J. B. Rawlings, D. Q. Mayne, and M. M. Diehl, Model Predictive Control: Theory, Design, and Computation, 2nd ed. Santa Barbara, CA: Nob Hill Publishing, 2020, 770 pages, ISBN 978-0-9759377-5-4.
- [3] K. R. Muske and T. A. Badgwell, "Disturbance modeling for offset-free linear model predictive control," J. Proc. Cont., vol. 12, no. 5, pp. 617–632, 2002.
- [4] G. Pannocchia and J. B. Rawlings, "Disturbance models for offset-free MPC control," AIChE J., vol. 49, no. 2, pp. 426–437, 2003.
- [5] M. Wallace, B. Das, P. Mhaskar, J. House, and T. Salsbury, "Offset-free model predictive control of a vapor compression cycle," *J. Proc. Cont.*, vol. 22, no. 7, pp. 1374–1386, 2012.
- [6] M. Wallace, P. Mhaskar, J. House, and T. I. Salsbury, "Offset-Free Model Predictive Control of a Heat Pump," *Ind. Eng. Chem. Res.*, vol. 54, no. 3, pp. 994–1005, 2015.
- [7] P. Schmid and P. Eberhard, "Offset-free Nonlinear Model Predictive Control by the Example of Maglev Vehicles," *IFAC-P. Online*, vol. 54, no. 6, pp. 83–90, 2021.
- [8] F. Xu, Y. Shi, K. Zhang, and X. Xu, "Real-Time Application of Robust Offset-Free MPC in Maglev Planar Machine," *IEEE Trans. Indus. Elec.*, vol. 70, no. 6, pp. 6121– 6130, 2023.
- [9] M. H. Caveness and J. J. Downs, "Reactor control using infinite horizon model predictive control," AIChE Spring Meeting, Atlanta, GA, April 2005.
- [10] R. Huang, S. C. Patwardhan, and L. T. Biegler, "Offset-free Nonlinear Model Predictive Control Based on Moving Horizon Estimation for an Air Separation Unit," *IFAC Proceedings Volumes*, vol. 43, no. 5, pp. 631–636, 2010.

- [11] L. N. Petersen, N. K. Poulsen, H. H. Niemann, C. Utzen, and J. B. Jørgensen, "Comparison of three control strategies for optimization of spray dryer operation," J. Proc. Cont., vol. 57, 2017.
- [12] S.-K. Kim, D.-K. Choi, K.-B. Lee, and Y. I. Lee, "Offset-Free Model Predictive Control for the Power Control of Three-Phase AC/DC Converters," *IEEE Trans. Indus. Elec.*, vol. 62, no. 11, pp. 7114–7126, 2015.
- [13] F. A. Bender, S. Goltz, T. Braunl, and O. Sawodny, "Modeling and Offset-Free Model Predictive Control of a Hydraulic Mini Excavator," *IEEE Trans. Auto. Sci. Eng.*, vol. 14, no. 4, pp. 1682–1694, 2017.
- [14] D. Deenen, E. Maljaars, L. Sebeke, B. De Jager, E. Heijman, H. Grüll, and W. P. M. H. Heemels, "Offset-free model predictive control for enhancing MR-HIFU hyperthermia in cancer treatment," *IFAC-P. Online*, vol. 51, no. 20, pp. 191–196, 2018.
- [15] B. J. Odelson, A. Lutz, and J. B. Rawlings, "Application of autocovariance least-squares methods to laboratory data," TWMCC, Department of Chemical Engineering, University of Wisconsin-Madison, Tech. Rep. 2003–03, September 2003. [Online]. Available: https://engineering.ucsb.edu/~jbraw/jbrweb-archives/ tech-reports/twmcc-2003-03.pdf
- [16] S. J. Kuntz and J. B. Rawlings, "Maximum likelihood estimation of linear disturbance models for offset-free model predictive control," in *American Control Conference*, Atlanta, GA, June 8–10, 2022, pp. 3961–3966.
- [17] S. J. Kuntz, J. J. Downs, S. M. Miller, and J. B. Rawlings, "An industrial case study on the combined identification and offset-free control of a chemical process," *Comput. Chem. Eng.*, vol. 179, 2023.
- [18] M. A. Zagrobelny and J. B. Rawlings, "Identifying the uncertainty structure using maximum likelihood estimation," in *American Control Conference*, Chicago, IL, July 1–3, 2015, pp. 422–427.
- [19] L. Simpson, A. Ghezzi, J. Asprion, and M. Diehl, "An Efficient Method for the Joint Estimation of System Parameters and Noise Covariances for Linear Time-Variant Systems," in 2023 62nd IEEE Conference on Decision and Control (CDC), Dec 2023, pp. 4524–4529.
- [20] K. J. Åström, "Maximum Likelihood and Prediction Error Methods," IFAC Proceedings Volumes, vol. 12, no. 8, pp. 551–574, Sep 1979.
- [21] L. Ljung, System Identification: Theory for the User, 2nd ed. New Jersey: Prentice Hall, 1999.
- [22] D. Piga, M. Forgione, S. Formentin, and A. Bemporad, "Performance-Oriented Model Learning for Data-Driven MPC Design," *IEEE Ctl. Sys. Let.*, vol. 3, no. 3, pp. 577– 582, 2019.

- [23] S. Formentin and A. Chiuso, "Control-oriented regularization for linear system identification," Automatica, vol. 127, p. 109539, 2021.
- [24] J. Berberich, C. W. Scherer, and F. Allgöwer, "Combining Prior Knowledge and Data for Robust Controller Design," *IEEE Trans. Auto. Cont.*, vol. 68, no. 8, pp. 4618–4633, 2023.
- [25] M. Chilali and P. Gahinet, " H_{∞} design with pole placement constraints: an LMI approach," *IEEE Trans. Auto. Cont.*, vol. 41, no. 3, pp. 358–367, 1996.
- [26] M. Chilali, P. Gahinet, and P. Apkarian, "Robust pole placement in LMI regions," *IEEE Trans. Auto. Cont.*, vol. 44, no. 12, pp. 2257–2269, 1999.
- [27] D. N. Miller and R. A. De Callafon, "Subspace identification with eigenvalue constraints," Automatica, vol. 49, no. 8, pp. 2468–2473, 2013.
- [28] J. Umenberger, J. Wågberg, I. R. Manchester, and T. B. Schön, "Maximum likelihood identification of stable linear dynamical systems," *Automatica*, vol. 96, pp. 280–292, 2018.
- [29] R. H. Shumway and D. S. Stoffer, "An approach to time series smoothing and forecasting using the EM algorithm," J. Time Series Anal., vol. 3, pp. 253–264, 1982.
- [30] S. Gibson and B. Ninness, "Robust maximum-likelihood estimation of multivariable dynamic systems," *Automatica*, vol. 41, no. 10, pp. 1667–1682, 2005.
- [31] R. A. Redner and H. F. Walker, "Mixture Densities, Maximum Likelihood and the EM Algorithm," SIAM Rev., vol. 26, no. 2, pp. 195–239, 1984.
- [32] O. Bermond and J.-F. Cardoso, "Approximate likelihood for noisy mixtures," in International Workshop on Independent Component Analysis (ICA '99), vol. 99, Aussois, France, 1999, pp. 325–330.
- [33] K. B. Petersen, O. Winther, and L. K. Hansen, "On the Slow Convergence of EM and VBEM in Low-Noise Linear Models," *Neural Comput.*, vol. 17, no. 9, pp. 1921–1926, 2005.
- [34] K. B. Petersen and O. Winther, "Explaining slow convergence of EM in low noise linear mixtures," Informatics and Mathematical Modelling, Technical University of Denmark, Technical Report 2005-2, 2005. [Online]. Available: https://core.ac.uk/download/pdf/13755227.pdf
- [35] R. K. Olsson, K. B. Petersen, and T. Lehn-Schiøler, "State-Space Models: From the EM Algorithm to a Gradient Approach," *Neural Comput.*, vol. 19, no. 4, pp. 1097– 1111, 2007.
- [36] S. Burer, R. D. Monteiro, and Y. Zhang, "Solving a class of semidefinite programs via nonlinear programming," *Math. Prog.*, vol. 93, no. 1, pp. 97–122, 2002.

- [37] —, "Interior-Point Algorithms for Semidefinite Programming Based on a Nonlinear Formulation," *Comp. Optim. Appl.*, vol. 22, no. 1, pp. 49–79, 2002.
- [38] J. Park, R. A. Martin, J. D. Kelly, and J. D. Hedengren, "Benchmark temperature microcontroller for process dynamics and control," *Comput. Chem. Eng.*, vol. 135, Apr 2020.
- [39] G. H. Golub and C. F. Van Loan, *Matrix Computations*, 4th ed. Baltimore, MD: The Johns Hopkins University Press, 2013.
- [40] M. Denham, "Canonical forms for the identification of multivariable linear systems," *IEEE Trans. Auto. Cont.*, vol. 19, no. 6, pp. 646–656, 1974.
- [41] M. R. Rajamani, J. B. Rawlings, and S. J. Qin, "Achieving state estimation equivalence for misassigned disturbances in offset-free model predictive control," *AIChE J.*, vol. 55, no. 2, pp. 396–407, Feb 2009.
- [42] J. Sjöberg, T. McKelvey, and L. Ljung, "On the Use of Regularization in System Identification," *IFAC Proceedings Volumes*, vol. 26, no. 2, pp. 75–80, 1993.
- [43] T. A. Johansen, "On Tikhonov regularization, bias and variance in nonlinear system identification," *Automatica*, vol. 33, no. 3, pp. 441–446, 1997.
- [44] H. White, "Maximum Likelihood Estimation of Misspecified Dynamic Models," in *Misspecification Analysis*, ser. Lecture Notes in Economics and Mathematical Systems, T. K. Dijkstra, Ed. Berlin, Heidelberg: Springer, 1984, pp. 1–19.
- [45] R. Douc and E. Moulines, "Asymptotic properties of the maximum likelihood estimation in misspecified hidden Markov models," Ann. Stat., vol. 40, no. 5, pp. 2697–2732, 2012.
- [46] P. Daoutidis, W. Tang, and A. Allman, "Decomposition of control and optimization problems by network structure: Concepts, methods, and inspirations from biology," *AIChE J.*, vol. 65, no. 10, p. e16708, 2019.
- [47] W. Tang, P. Carrette, Y. Cai, J. M. Williamson, and P. Daoutidis, "Automatic decomposition of large-scale industrial processes for distributed MPC on the Shell–Yokogawa Platform for Advanced Control and Estimation (PACE)," *Comput. Chem. Eng.*, vol. 178, p. 108382, 2023.
- [48] N. Motee and A. Jadbabaie, "Distributed Multi-Parametric Quadratic Programming," *IEEE Trans. Auto. Cont.*, vol. 54, no. 10, pp. 2279–2289, 2009.
- [49] M. Zamani, B. Ninness, and J. C. Agüero, "On Identification of Networked Systems with Time-invariant Topology," *IFAC-P. Online*, vol. 48, no. 28, pp. 1184–1189, 2015.
- [50] L. M. Silverman, "Discrete Riccati equations: Alternative algorithms, asymptotic properties, and system theory interpretations," in *Control and Dynamic Systems*, ser.

Control and Dynamic Systems, C. T. Leondes, Ed. Academic Press, 1976, vol. 12, pp. 313–386.

- [51] K. Åström and P. Eykhoff, "System identification—A survey," Automatica, vol. 7, no. 2, pp. 123–162, 1971.
- [52] L. Ljung, "Convergence analysis of parametric identification methods," *IEEE Trans. Auto. Cont.*, vol. 23, no. 5, pp. 770–783, 1978.
- [53] K. J. Åström, "Special section system identification tutorial: Maximum likelihood and prediction error methods," *Automatica*, vol. 16, no. 5, pp. 551–574, 1980.
- [54] T. McKelvey, A. Helmersson, and T. Ribarits, "Data driven local coordinates for multivariable linear systems and their application to system identification," *Automatica*, vol. 40, no. 9, pp. 1629–1635, 2004.
- [55] T. Ribarits, M. Deistler, and B. Hanzon, "An analysis of separable least squares data driven local coordinates for maximum likelihood estimation of linear systems," *Automatica*, vol. 41, no. 3, pp. 531–544, 2005.
- [56] P. Li, I. Postlethwaite, and M. Turner, "Parameter Estimation Techniques for Helicopter Dynamic Modelling," in 2007 American Control Conference, 2007, pp. 2938– 2943.
- [57] O. Y. Kushel, "Geometric properties of LMI regions," 2019, arXiv:1910.10372 [math].
 [Online]. Available: http://arxiv.org/abs/1910.10372
- [58] S. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in Control Theory*. Philadelphia: SIAM, 1994.
- [59] A. Haber and M. Verhaegen, "Sparse solution of the Lyapunov equation for large-scale interconnected systems," *Automatica*, vol. 73, pp. 256–268, 2016.
- [60] N. Motee and Q. Sun, "Sparsity and Spatial Localization Measures for Spatially Distributed Systems," SIAM J. Cont. Opt., vol. 55, no. 1, pp. 200–235, Jan 2017.
- [61] S. Massei and L. Saluzzi, "On the data-sparsity of the solution of Riccati equations with applications to feedback control," 2024, arXiv:2408.16569 [cs, math]. [Online]. Available: http://arxiv.org/abs/2408.16569
- [62] J. Fiala, M. Kočvara, and M. Stingl, "PENLAB: A MATLAB solver for nonlinear semidefinite optimization," 2013.
- [63] M. Kočvara and M. Stingl, "PENNON: Software for linear and nonlinear matrix inequalities," 2015.
- [64] A. Wächter and L. T. Biegler, "On the implementation of a primal-dual interior-point filter line-search algorithm for large-scale nonlinear programming," *Math. Prog.*, vol. 106, no. 1, pp. 25–57, 2006.

- [65] M. Diehl, K. Mombaur, and D. Noll, "Stability Optimization of Hybrid Periodic Systems via a Smooth Criterion," *IEEE Trans. Auto. Cont.*, vol. 54, no. 8, pp. 1875–1880, 2009.
- [66] J. Vanbiervliet, B. Vandereycken, W. Michiels, S. Vandewalle, and M. Diehl, "The Smoothed Spectral Abscissa for Robust Stability Optimization," *SIAM J. Optim.*, vol. 20, no. 1, pp. 156–171, 2009.
- [67] W. E. Larimore, "Canonical variate analysis in identification, filtering, and adaptive control," in *Proceedings of the 29th Conference on Decision and Control*, 1990, pp. 596–604.
- [68] M. A. Zagrobelny, L. Ji, and J. B. Rawlings, "Quis custodiet ipsos custodes?" Annual Rev. Control, vol. 37, pp. 260–270, 2013.
- [69] V. Bageshwar and F. Borrelli, "On a Property of a Class of Offset-Free Model Predictive Controllers," *IEEE Trans. Auto. Cont.*, vol. 54, no. 3, pp. 663–669, 2009.
- [70] J. F. Bonnans and A. Shapiro, *Perturbation Analysis of Optimization Problems*, ser. Springer Series in Operations Research. New York, NY: Springer Science & Business Media, 2000.