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Beyond inherent robustness: strong stability of MPC despite plant-model mismatch*

Steven J. Kuntz and James B. Rawlings
Department of Chemical Engineering
University of California, Santa Barbara

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Abstract

We consider the asymptotic stability of MPC under plant-model mismatch, considering primarily models where the origin remains a steady state despite mismatch. Our results differ from prior results on the inherent robustness of MPC, which guarantee only convergence to a neighborhood of the origin, the size of which scales with the magnitude of the mismatch. For MPC with quadratic costs, continuous differentiability of the system dynamics is sufficient to demonstrate exponential stability of the closed-loop system despite mismatch. For MPC with general costs, a joint comparison function bound and scaling condition guarantee asymptotic stability despite mismatch. The results are illustrated in both algebraic and engineering examples. The tools developed to establish these results can address the stability of offset-free MPC, an open and interesting question in the MPC research literature.

1 Introduction

Plant-model mismatch is an ever-present challenge in model predictive control (MPC) practice. In industrial implementations, the main driver of MPC performance is model quality (Qin and Badgwell, 2003; Darby and Nikolaou, 2012). There has been recent progress

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on improving model quality and MPC performance through disturbance modeling and estimator tuning (Kuntz and Rawlings, 2022, 2024; Simpson et al., 2024), simultaneous state and parameter estimation (Baumgärtner et al., 2022; Muntwiler et al., 2023; Schiller and Müller, 2023), and even direct data-driven MPC design (Berberich et al., 2021, 2022a,b), to name a few methods. However, there is not yet a sharp theoretical understanding of the robustness of MPC to plant-model mismatch.

Before discussing MPC robustness, let us first define *robustness*. In the stability literature, *robust asymptotic stability* has been used to refer to both (i) input-to-state stability (ISS) (Jiang and Wang, 2001) and (ii) asymptotic stability despite disturbances (Kellett and Teel, 2005). To avoid confusion, we reserve the term *robust asymptotic stability* for (i) and use *strong asymptotic stability* to refer to (ii).[1] When such properties are given by a nominal MPC,[2] we call it *inherently robust* or *inherently strongly stabilizing*. Robust and strong exponential stability are defined similarly.

It is well-known that MPC is stabilizing under certain assumptions on the terminal ingredients (Rawlings et al., 2020, Ch. 2). To achieve robust stability in the presence of parameter errors, estimation errors, and exogenous perturbations, a disturbance model can be included.The simplest manner of handling disturbances is with feedback. For MPC this would require future knowledge of the disturbance trajectory, or at least a forecast of it, to implement the controller. While this is a strong requirement, it would confer strong stability rather than robust stability. Alternatively, a disturbance model may be included. Several MPC variants include disturbance models in their design, such as offset-free (Pannocchia et al., 2015), stochastic (McAllister, 2022), tube-based (Rawlings et al., 2020, Ch. 3), and min-max MPC (Limon et al., 2006). For a survey of these methods, see (Rawlings et al., 2020, Ch. 1, 3).

Even in the absence of a disturbance model, a wide range of nominal MPC designs are inherently robust to disturbances. Continuity of the control law was first proven to be a sufficient condition for inherent robustness (De Nicolao et al., 1996; Scokaert et al., 1997). Later, Grimm et al. (2004) proved continuity of the optimal value function is sufficient for inherent robustness, and stated MPC examples with discontinuous optimal value functions that are nominally stable but otherwise not robust to disturbances. A special class of time-varying terminal constraints were proven to confer robust stability to nominal MPC by Grimm et al. (2007), and to suboptimal MPC by Lazar and Heemels (2009). In Pannocchia et al. (2011); Allan et al. (2017), the inherent robustness of optimal and suboptimal MPC, using a class of time-invariant terminal constraints, was proven. With the same terminal constraints, the inherent stochastic robustness (in probability, expectation, and distribution) of nominal MPC was shown by McAllister and Rawlings (2022a,b, 2024). Recently, direct data-driven MPC was shown to be inherently robust to

[1] The latter term is borrowed from the differential inclusion literature (Clarke et al., 1998) (see Jiang and Wang (2002); Kellett and Teel (2005) for discrete-time definitions). Some authors (Jiang and Wang, 2001, 2002) use the term *uniform asymptotic stability* to refer to (ii), but we wish to avoid confusion with the time-varying case.

[2] By *nominal* MPC, we mean any MPC designed without a disturbance model, possibly admitting parameter errors. This includes not only standard nonlinear MPC, but also suboptimal, offset-free, and (some) data-driven MPC.

noisy data by Berberich et al. (2022a).

If the origin remains a steady state under mismatch (e.g., for some kinematic and inventory problems), we might expect strong asymptotic stability. In unconstrained linear optimal control problems (LQR/LQG), the margin of stability (maximum perturbation to the open-loop gain that still gives a closed-loop system) is always nonzero. However, it is important to note that there is no guaranteed relative value of this margin below which the closed loop is stable, save a few exceptional cases such as a single input, or with diagonally-weighted stage costs (Doyle, 1978; Lehtomaki et al., 1981; Zhang and Fu, 1996). Examples are shown by Doyle (1978); Zhang and Fu (1996) in which arbitrarily small perturbations to the gain matrix destabilize the system. These examples use *multiplicative disturbances* that, while persistent in the aforementioned papers, do not need to be time-invariant for the results to hold. The disturbances treated in the MPC literature are typically *additive disturbances* entering the states and measurements (Rawlings et al., 2020, Ch. 3). In the multiplicative case, borrowing from knowledge of linear systems, we should expect strong exponential stability. However, in the additive case, we should expect only robust exponential stability. To the best of our knowledge, the inherent strong stability of nominal MPC to plant-model mismatch has been discussed by only Santos and Biegler (1999); Santos et al. (2008). For unconstrained systems with a sufficiently small bound on the mismatch, nominal MPC is shown to stabilize the plant to the origin. While exact penalty functions are considered for handling constraints, there is no guarantee of recursive feasibility.

In this paper, we extend the work of Santos et al. (2008) to include input constraints and stabilizing terminal constraints. The tools developed in this work can be used to address the open problem of the stability of offset-free MPC. In Section 2, we define the system, state the MPC problem and assumptions, review nominal MPC stability, and present a motivating example exhibiting both robust and strong stability under plant-model mismatch. In Section 3, we formally define robust and strong stability and review the relevant Lyapunov theory. In Section 4, we review inherent robustness of MPC. In Section 5, we present the main results. For MPC with quadratic costs, it is shown in Theorem 10 that the closed loop is strongly exponentially stable under (i) a fixed steady state, (ii) a mild differentiability condition, and (iii) the standard MPC assumptions used by Pannocchia et al. (2011); Allan et al. (2017). For MPC with general, positive definite cost functions, we show a *joint $\mathcal{K}$-function* bound holds on the increase in the optimal value function (Proposition 9), but strong stability is implied only if this bound decays sufficiently quickly near the origin (Theorem 9). To illustrate the main results, we present three examples in Section 6. The first example is a continuous yet nondifferentiable system with a general cost MPC that is not strongly stable, demonstrating inherent strong stability is not a guaranteed property of nonlinear MPC. The second example is a nondifferentiable system for which the quadratic cost MPC is strongly stabilizing. In the third and final example, we use the upright pendulum problem to showcase several types of plant-model mismatch that are covered by the main results, namely, discretization errors, unmodeled dynamics, and errors in estimated parameters. We conclude the paper and discuss future work in Section 7.

Notation, definitions, and basic facts Let $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, and $\mathbb{R}_{>0}$ denote the real, nonnegative real, and positive real numbers, respectively. Let $\mathbb{I}$, $\mathbb{I}_{\geq 0}$, $\mathbb{I}_{>0}$, and $\mathbb{I}_{m:n}$ denote the integers, nonnegative integers, positive integers, and integers from m to n (inclusive), respectively. Let $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote real n-vectors and $n \times m$ matrices, respectively. Let $\overline{\mathbb{R}}_{\geq 0} := \mathbb{R}_{\geq 0} \cup \{\, \infty \,\}$ denote the extended nonnegative reals. For any function $V : \mathbb{R}^n \to \overline{\mathbb{R}}_{\geq 0}$ and finite $\rho \geq 0$, we define the sublevel set $\text{lev}_\rho V := \{\, x \in \mathbb{R}^n \mid V(x) \leq \rho \,\}$. We say $V : \mathbb{R}^n \to \overline{\mathbb{R}}_{\geq 0}$ is lower semicontinuous (l.s.c.) if $\text{lev}_\rho V$ is closed for each $\rho \geq 0$. We say a symmetric matrix $P = P^\top \in \mathbb{R}^{n\times n}$ is positive definite (semidefinite) if $x^\top P x > 0$ ($x^\top P x \geq 0$) for all $x \in \mathbb{R}^n \setminus \{\, 0 \,\}$. We define the Euclidean and Q-weighted norms by $|x| := \sqrt{x^\top x}$ and $|x|_Q := \sqrt{x^\top Q x}$ for each $x \in \mathbb{R}^n$, where Q is positive definite. Moreover, $|\cdot|_Q$ has the property $\underline{\sigma}(Q)|x|^2 \leq |x|_Q^2 \leq \overline{\sigma}(Q)|x|^2$ for all $x \in \mathbb{R}^n$, where $\underline{\sigma}(Q)$ and $\overline{\sigma}(Q)$ denote the smallest and largest singular values of Q. For any signal $a(k)$, we denote both infinite and finite sequences in bold font as $\mathbf{a} := (a(0), \ldots, a(k))$ and $\mathbf{a} := (a(0), a(1), \ldots)$. We define the infinite and length-k signal norm as $\|\mathbf{a}\| := \sup_{k\geq 0} |a(k)|$ and $\|\mathbf{a}\|_{0:k} := \max_{0\leq i\leq k} |a(i)|$. Let $\mathcal{PD}$ be the class of functions $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $\alpha(0) = 0$ and $\alpha(s) > 0$ for all $s > 0$. Let $\mathcal{K}$ be the class of $\mathcal{PD}$-functions that are continuous and strictly increasing. Let $\mathcal{K}_\infty$ be the class of $\mathcal{K}$-functions that are unbounded. Let $\mathcal{KL}$ be the set of functions $\beta : \mathbb{R}_{\geq 0} \times \mathbb{I}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $\beta(\cdot, k) \in \mathcal{K}$, $\beta(r, \cdot)$ is nonincreasing, and $\lim_{i\to\infty} \beta(r, i) = 0$ for all $(r, k) \in \mathbb{R}_{\geq 0} \times \mathbb{I}_{\geq 0}$.

2 Problem statement

2.1 System of interest

Consider the following discrete-time plant:

$$x^+ = f(x, u, \theta) \tag{1}$$

where $x \in \mathbb{R}^n$ is the plant state, $u \in \mathbb{R}^m$ is the plant input, and $\theta \in \mathbb{R}^{n_\theta}$ is an *unknown* parameter vector. We denote the parameter estimate by $\hat{\theta} \in \mathbb{R}^{n_\theta}$ and the modeled system by

$$x^+ = f(x, u, \hat{\theta}). \tag{2}$$

We assume the parameter estimate is time-invariant, while the parameter vector itself may be time-varying. For simplicity, let $\hat{\theta} = 0$ and denote the model as

$$x^+ = \hat{f}(x, u) := f(x, u, 0). \tag{3}$$

Let $\hat{\phi}(k; x, \mathbf{u})$ denote the solution to (3) at time k, given an initial state x and a sufficiently long input sequence $\mathbf{u}$.

In this paper, we study the behavior of an MPC designed with the model (2), but applied to the plant (1). We adopt a user-oriented perspective in this analysis: while the model is fixed (e.g., via system identification or prior knowledge), the plant behavior is unknown and possibly changing over time as equipment or the environment changes. Under the assumption $\hat{\theta} = 0$, θ takes the role of an estimate residual. In the language of inherent robustness, the model (3) is the nominal system, and the plant (1) is the uncertain system.

2.2 Nominal MPC and basic assumptions

We consider an MPC problem with control constraints $u \in \mathbb{U} \subseteq \mathbb{R}^m$, a horizon length of $N \in \mathbb{I}_{>0}$, a stage cost $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\geq 0}$, a terminal constraint $\mathbb{X}_f \subseteq \mathbb{R}^n$, and a terminal cost $V_f : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$. For an initial state $x \in \mathbb{R}^n$, we define the set of admissible $(x, \mathbf{u})$ pairs (4), admissible input sequences (5), and admissible initial states (6) by

$$\mathcal{Z}_N := \{ (x, \mathbf{u}) \in \mathbb{R}^n \times \mathbb{U}^N \mid \hat{\phi}(N; x, \mathbf{u}) \in \mathbb{X}_f \} \tag{4}$$

$$\mathcal{U}_N(x) := \{ \mathbf{u} \in \mathbb{U}^N \mid (x, \mathbf{u}) \in \mathcal{Z}_N \} \tag{5}$$

$$\mathcal{X}_N := \{ x \in \mathbb{R}^n \mid \mathcal{U}_N(x) \text{ is nonempty} \}. \tag{6}$$

For each $(x, \mathbf{u}) \in \mathbb{R}^{n+Nm}$, we define the MPC objective by

$$V_N(x, \mathbf{u}) := \sum_{k=0}^{N-1} \ell(\hat{\phi}(k; x, \mathbf{u}), u(k)) + V_f(\hat{\phi}(N; x, \mathbf{u})) \tag{7}$$

and for each $x \in \mathcal{X}_N$, we define the MPC problem by

$$V_N^0(x) := \min_{\mathbf{u} \in \mathcal{U}_N(x)} V_N(x, \mathbf{u}). \tag{8}$$

Using the convention of Rockafellar and Wets (1998) for infeasible problems, we take $V_N^0(x) := \infty$ for all $x \notin \mathcal{X}_N$.

Throughout, we use the standard assumptions for inherent robustness of MPC from Allan et al. (2017).

Assumption 1 (Continuity). The functions $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n_\theta} \to \mathbb{R}^n$, $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\geq 0}$, and $V_f : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ are continuous and $\hat{f}(0,0) = 0$, $\ell(0,0) = 0$, and $V_f(0) = 0$.

Assumption 2 (Constraint properties). The set $\mathbb{U}$ is compact and contains the origin. The set $\mathbb{X}_f$ is defined by $\mathbb{X}_f := \text{lev}_{c_f} V_f$ for some $c_f > 0$.

Assumption 3 (Terminal control law). There exists a terminal control law $\kappa_f : \mathbb{X}_f \to \mathbb{U}$ such that

$$V_f(\hat{f}(x, \kappa_f(x))) \leq V_f(x) - \ell(x, \kappa_f(x)), \qquad \forall\, x \in \mathbb{X}_f.$$

Assumption 4 (Stage cost bound). There exists a function $\alpha_1 \in \mathcal{K}_\infty$ such that

$$\ell(x, u) \geq \alpha_1(|(x, u)|), \qquad \forall\, (x, u) \in \mathbb{R}^n \times \mathbb{U}. \tag{9}$$

Remark 1. Assumptions 2 and 3 imply $V_f(\hat{f}(x, \kappa_f(x))) \leq V_f(x) \leq c_f$ for all $x \in \mathbb{X}_f$ and therefore $\mathbb{X}_f$ is positive invariant for $x^+ = \hat{f}(x, \kappa_f(x))$.

Quadratic stage and terminal costs are of particular interest in this work. Throughout, we call an MPC satisfying the following assumption a *quadratic cost MPC*.

Assumption 5 (Quadratic cost). We have

$$\ell(x,u) := |x|_Q^2 + |u|_R^2, \qquad V_f(x) := |x|_{P_f}^2 \tag{10}$$

for all $(x,u) \in \mathbb{R}^n \times \mathbb{R}^m$ and positive definite Q, R, and P_f.

Under Assumptions 1 and 2, the existence of solutions to (8) follows from (Rawlings et al., 2020, Prop. 2.4). We denote any such solution by $\mathbf{u}^0(x) = (u^0(0;x), \ldots, u^0(N-1;x))$, denote the optimal state sequence by $\hat{x}^0(k;x) := \hat{\phi}(k;x,\mathbf{u}^0(x))$ for each $k \in \mathbb{I}_{0:N}$, and define the MPC control law $\kappa_N : \mathcal{X}_N \to \mathbb{U}$ by $\kappa_N(x) := u^0(0;x)$. It is also useful to define the following suboptimal input sequence:

$$\tilde{\mathbf{u}}(x) := (u^0(1;x), \ldots, u^0(N-1;x), \kappa_f(\hat{x}^0(N;x))).$$

Consider the *modeled* closed-loop system

$$x^+ = \hat{f}_c(x) := \hat{f}(x, \kappa_N(x)). \tag{11}$$

From Assumptions 1 to 4, it can be shown $x^+ = \hat{f}_c(x)$ is asymptotically stable in $\mathcal{X}_N$ with the Lyapunov function V_N^0 (Rawlings et al., 2020, Thm. 2.19). For completeness, we include a sketch of the proof in Appendix A.

Theorem 1 (Thm. 2.19 of Rawlings et al. (2020)). Suppose Assumptions 1 to 4 hold. Then

(a) $\mathcal{X}_N$ is positive invariant for $x^+ = \hat{f}_c(x)$;

(b) there exists $\alpha_2 \in \mathcal{K}_\infty$ such that, for each $x \in \mathcal{X}_N$,

$$\alpha_1(|x|) \leq V_N^0(x) \leq \alpha_2(|x|) \tag{12a}$$

$$V_N^0(\hat{f}_c(x)) \leq V_N^0(x) - \alpha_1(|x|); \tag{12b}$$

(c) and $x^+ = \hat{f}_c(x)$ is asymptotically stable on $\mathcal{X}_N$.

Similarly, it is shown in (Rawlings et al., 2020, Sec. 2.5.5) that, under Assumptions 1 to 3 and 5, the quadratic cost MPC *exponentially* stabilizes the closed-loop system (11) on any sublevel set of the optimal value function $\mathcal{S} := \text{lev}_\rho V_N^0$. Note that, because V_N^0 is only defined on $\mathcal{X}_N$, we have $\mathcal{S} \subseteq \mathcal{X}_N$ by the definition of the sublevel set. For completeness, we restate the conclusion of (Rawlings et al., 2020, Sec. 2.5.5) in the theorem below and include a sketch of the proof in Appendix A.

Theorem 2. Suppose Assumptions 1 to 3 and 5 hold. Let $\rho > 0$ and $\mathcal{S} := \text{lev}_\rho V_N^0$. Then

(a) $\mathcal{S}$ is positive invariant for $x^+ = \hat{f}_c(x)$;

(b) there exists a constant $c_2 > 0$ such that

$$c_1|x|^2 \leq V_N^0(x) \leq c_2|x|^2 \tag{13a}$$

$$V_N^0(\hat{f}_c(x)) \leq V_N^0(x) - c_1|x|^2 \tag{13b}$$

for each $x \in \mathcal{S}$, where $c_1 := \underline{\sigma}(Q)$; and

(c) $x^+ = \hat{f}_c(x)$ is exponentially stable on $\mathcal{S}$.

To show strong stability of the MPC with mismatch, we eventually require one or both of the following assumptions.

Assumption 6 (Steady state). The origin is a steady state, uniformly in $\theta \in \mathbb{R}^{n_\theta}$, i.e., $f(0, 0, \theta) = 0$ for all $\theta \in \mathbb{R}^{n_\theta}$.

Assumption 7 (Differentiability). The function $f(\cdot, \cdot, \theta)$ is continuously differentiable for each $\theta \in \mathbb{R}^{n_\theta}$.

Remark 2. Assumption 6 limits our results to problems where the steady state is known and fixed (e.g., path-planning and inventory problems). If the steady state depends on θ, i.e., $x_s(\theta) = f(x_s(\theta), u_s(\theta), \theta)$, we can still work with deviation variables $(\delta x, \delta u) := (x - x_s(\theta), u - u_s(\theta))$, but (i) we have to estimate the steady-state pair $(x_s(\theta), u_s(\theta))$ (e.g., via an integrating disturbance model (Rawlings et al., 2020, Ch. 1)), and (ii) we only achieve strong stability in the case where the steady-state map is continuous, the parameters are asymptotically constant, and the estimation errors converge.

2.3 Motivating example

We close this section with a motivating example exhibiting many types of stability under persistent mismatch. Recall from the introduction we define *robust stability* as an ISS property for parameter errors, and *strong stability* as convergence to the origin despite mismatch. While precise definitions are given in Section 3, these informal definitions suffice for the example.

Consider the scalar system

$$x^+ = f(x, u, \theta) := x + (1 + \theta)u. \tag{14}$$

The plant (14) is a prototypical integrating system, such as a storage tank or vehicle on a track, with an uncertain input gain. As usual the system is modeled with $\hat{\theta} = 0$,

$$x^+ = \hat{f}(x, u) := f(x, u, 0) = x + u. \tag{15}$$

We define a nominal MPC with $\mathbb{U} := [-1, 1]$, $\ell(x, u) := (1/2)(x^2 + u^2)$, $V_f(x) := (1/2)x^2$, $\mathbb{X}_f := [-1, 1]$, and $N := 2$. Notice that the terminal set can be reached in $N = 2$ moves if and only if $|x| \leq 3$, so we have the steerable set $\mathcal{X}_2 = [-3, 3]$. *Without* the terminal constraint (i.e., $\mathbb{X}_f = \mathbb{R}$), the optimal control sequence is

$$\mathbf{u}^0(x) = \begin{cases} (-3x/5, -x/5), & |x| \leq 5/3 \\ (-\mathrm{sgn}(x), -x/2 + \mathrm{sgn}(x)/2), & 5/3 < |x| \leq 3 \end{cases}$$

and the control law is $\kappa_2(x) := -\mathrm{sat}(3x/5)$ (Rawlings et al., 2020, p. 104). However, the optimal input sequence gives

$$\hat{x}^0(2; x) = \begin{cases} x/5, & |x| \leq 5/3 \\ x/2 - \mathrm{sgn}(x)/2, & 5/3 < |x| \leq 3 \end{cases}$$

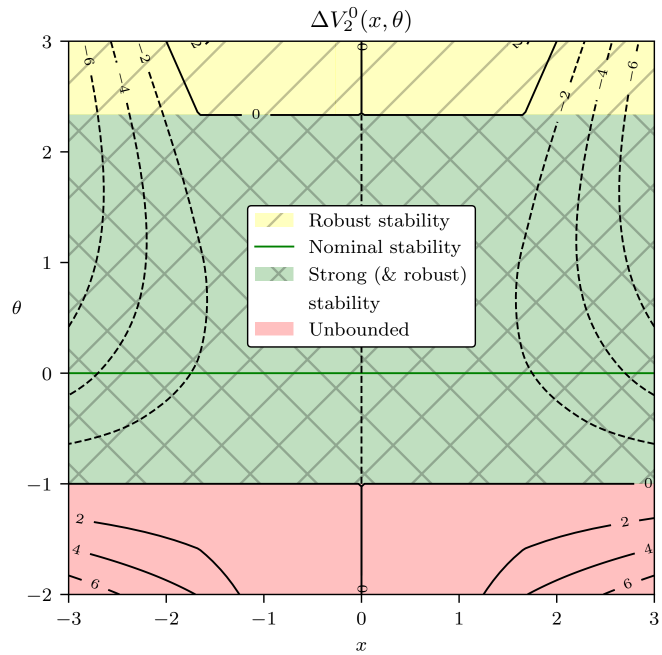


Figure 1: Contours of the cost difference as a function of the initial state x and the parameter θ.

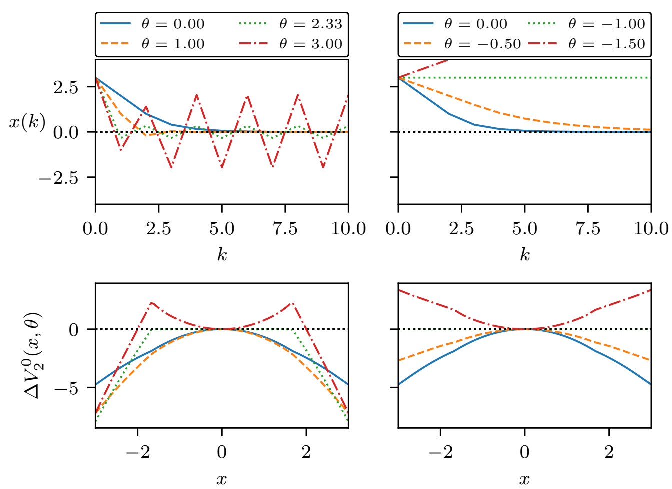


Figure 2: For (left) positive and (right) negative values of θ, the (top) closed-loop trajectories with initial state $x = 3$, and (bottom) cost differences as a function of x, along with the nominal values.

so the terminal constraint $\mathbb{X}_f = [-1, 1]$ is automatically satisfied for all $|x| \leq 3$. Therefore $\kappa_2(x) = -\text{sat}(3x/5)$ is also the control law of the problem *with* the terminal constraint.

In Figure 1 we plot contours of the cost difference $\Delta V_2^0(x, \theta) := V_2^0(f(x, \kappa_2(x), \theta)) - V_2^0(x)$, and in Figure 2, we plot closed-loop trajectories and the cost difference curve $\Delta V_2^0(\cdot, \theta)$ for several values of θ. The system is strongly stable for all $-1 < \theta < 7/3$ as the cost difference curve is negative definite. When $\theta < -1$, the entire cost difference curve is positive definite, so the trajectories become unbounded. This is because the disturbance cancels out the effect of the controller and drives the system in the opposite direction. On the other hand, when $\theta > 7/3$, the cost difference curve is only positive definite near the origin, but negative elsewhere, so the trajectories remain bounded for all time, although they do not converge to the origin. In this case, high parameter values push the system in the same direction as the input, and input saturation moderates the effect of overshoot at high parameter values. We point out the existing literature on inherent robustness is not sufficient to predict strong stability whenever $-1 < \theta < 7/3$.

3 Robust and strong stability

Consider the closed-loop system

$$x^+ = f_c(x, \theta) := f(x, \kappa_N(x), \theta), \qquad \theta \in \Theta \tag{16}$$

where $\Theta \subseteq \mathbb{R}^{n_\theta}$. Let $\phi_c(k; x, \boldsymbol{\theta})$ denote solutions to (16) at time k, given an initial state $x \in \mathcal{X}_N$ and a sufficiently long parameter sequence $\theta \in \Theta$. If $\Theta := \{ \theta \in \mathbb{R}^{n_\theta} \mid |\theta| \leq \delta \}$, it is convenient to write (16) as $x^+ = f_c(x, \theta), |\theta| \leq \delta$.

In this section, we review stability definitions and results for (16). For brevity, asymptotic and exponential definitions and results are consolidated into the same statement. We define robustly positive invariant (RPI) sets as follows.

Definition 1 (Robust positive invariance). A set $X \subseteq \mathbb{R}^n$ is *robustly positive invariant* for the system $x^+ = f_c(x, \theta), \theta \in \Theta$ if $f_c(x, \theta) \in X$ for all $x \in X$ and $\theta \in \Theta$.

3.1 Robust stability

We define robust asymptotic stability (RAS) similarly to input-to-state stability (ISS) from Jiang and Wang (2001). Likewise, we define robust exponential stability (RES) similarly to input-to-state exponential stability (ISES) from Grüne et al. (1999).

Definition 2 (Robust stability). A system $x^+ = f_c(x, \theta), \theta \in \Theta$ is *robustly asymptotically stable* (in a RPI set $X \subseteq \mathbb{R}^n$) if there exists $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that

$$|\phi_c(k; x, \boldsymbol{\theta})| \leq \beta(|x|, k) + \gamma(\|\boldsymbol{\theta}\|_{0:k-1}) \tag{17}$$

for all $k \in \mathbb{I}_{\geq 0}$, $x \in X$, and $\boldsymbol{\theta} \in \Theta^k$. If, additionally, $\beta(s, k) = cs\lambda^k$ for some $c > 0$ and $\lambda \in (0, 1)$, we say $x^+ = f_c(x, \theta), \theta \in \Theta$ is *robustly exponentially stable* (in X).

Definition 3 (ISS/ISES Lyapunov function). A function $V : X \to \mathbb{R}_{\geq 0}$ is an *ISS Lyapunov function* (in an RPI set $X \subseteq \mathbb{R}^n$, for the system $x^+ = f_c(x, \theta), \theta \in \Theta$) if there exists functions $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ and $\sigma \in \mathcal{K}$ such that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \tag{18a}$$

$$V(f_c(x, \theta)) \leq V(x) - \alpha_3(|x|) + \sigma(|\theta|). \tag{18b}$$

for all $x \in X$ and $\theta \in \Theta$. If, additionally, $\alpha_i(\cdot) := a_i(\cdot)^b$ for some $a_i, b > 0$ and each $i \in \{1, 2, 3\}$, we say V is an *ISES Lyapunov function* (in X, for $x^+ = f_c(x, \theta), \theta \in \Theta$).

Next, we state a minor generalization of (Allan et al., 2017, Prop. 19). For completeness, we provide the proof of the exponential case in Appendix B.

Theorem 3 (ISS/ISES Lyapunov theorem). The system $x^+ = f_c(x, \theta), \theta \in \Theta$ is RAS (RES) in an RPI set $X \subseteq \mathbb{R}^n$ if it admits an ISS (ISES) Lyapunov function in X.

Remark 3. Whereas (Allan et al., 2017, Prop. 19) only considers disturbance sets of the form $\Theta := \{ \theta \in \mathbb{R}^{n_\theta} \mid |\theta| \leq \delta \}$ for some $\delta > 0$, it is trivial to modify the proof to use a general constraint set.

3.2 Strong stability

We take strong asymptotic stability (SAS) as a time-invariant version of the conclusion of (Jiang and Wang, 2002, Prop. 2.2). Strong exponential stability (SES) is defined similarly.

Definition 4 (Strong stability). A system $x^+ = f_c(x, \theta), \theta \in \Theta$ is *strongly asymptotically stable* (in a RPI set $X \subseteq \mathbb{R}^n$) if there exists $\beta \in \mathcal{KL}$ such that

$$|\phi_c(k; x, \boldsymbol{\theta})| \leq \beta(|x|, k)$$

for all $k \in \mathbb{I}_{\geq 0}$, $x \in X$, and $\boldsymbol{\theta} \in \Theta^k$. If, additionally, $\beta(s, k) := cs\lambda^k$ for all $s \geq 0$ and $k \in \mathbb{I}_{\geq 0}$, and some $c > 0$ and $\lambda \in (0, 1)$, we say $x^+ = f_c(x, \theta), \theta \in \Theta$ is *strongly exponentially stable* (in X).

Definition 5 (Lyapunov function). A function $V : X \to \mathbb{R}_{\geq 0}$ is a *Lyapunov function* (in a RPI set $X \subseteq \mathbb{R}^n$, for the system $x^+ = f(x, \theta), \theta \in \Theta$), if there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and a continuous function $\sigma \in \mathcal{PD}$ such that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \tag{19a}$$

$$V(f_c(x, \theta)) \leq V(x) - \sigma(|x|) \tag{19b}$$

for all $x \in X$ and $\theta \in \Theta$. If, additionally, $\alpha_i(\cdot) := a_i(\cdot)^b$ for some $a_i, b > 0$ and each $i \in \mathbb{I}_{1:3}$, we say V is an *exponential Lyapunov function* (in X, for $x^+ = f_c(x, \theta), \theta \in \Theta$).

The following Lyapunov theorem combines from (Allan et al., 2017, Prop. 13) and (Pannocchia et al., 2011, Lem. 15).

Theorem 4. The system $x^+ = f_c(x, \theta), \theta \in \Theta$ is SAS (SES) in a RPI set $X \subseteq \mathbb{R}^n$ if it admits a Lyapunov function (an exponential Lyapunov function) in X.

Remark 4. In (Allan et al., 2017, Prop. 13), the Lyapunov function requires a class-$\mathcal{K}_\infty$ bound rather than a continuous class-$\mathcal{PD}$ bound. However, it is shown in (Jiang and Wang, 2002, Lem. 2.8) that a continuous function $\sigma \in \mathcal{PD}$ suffices.

4 Inherent robustness of MPC

Assumptions 1 to 4 are in fact sufficient to show inherent robustness of the nominal MPC. The theorem below is a minor generalization of the results in (Rawlings et al., 2020, Sec. 3.2.4), as well as a special case of (Allan et al., 2017, Thm. 21).

Theorem 5. Suppose Assumptions 1 to 4 hold. Let $\rho > 0$ and $\mathcal{S} := \text{lev}_\rho V_N^0$. Then there exist $\delta > 0$, $\alpha_2 \in \mathcal{K}_\infty$, and $\sigma \in \mathcal{K}$ such that

$$\alpha_1(|x|) \leq V_N^0(x) \leq \alpha_2(|x|) \tag{20a}$$

$$V_N^0(f_c(x, \theta)) \leq V_N^0(x) - \alpha_1(|x|) + \sigma(|\theta|) \tag{20b}$$

for all $x \in \mathcal{S}$ and $|\theta| \leq \delta$, and the system $x^+ = f_c(x, \theta), |\theta| \leq \delta$ is RAS in the RPI set $\mathcal{S}$.

For completeness, we include a proof of Theorem 5 in Appendix C. Before moving on, we note that a key step of the proof of Theorem 5 and the main results is to establish the following robust descent property:

$$V_N^0(f_c(x, \theta)) \leq V_N^0(x) - \ell(x, \kappa_N(x)) + V_N(f_c(x, \theta), \tilde{\mathbf{u}}(x)) - V_N(\hat{f}_c(x), \tilde{\mathbf{u}}(x)). \tag{21}$$

If fact, (21) can be achieved on any sublevel set of V_N^0 and a sufficiently small neighborhood $|\theta| \leq \delta$. We restate this in the following proposition.

Proposition 1. Suppose Assumptions 1 to 4 hold. Let $\rho > 0$ and $\mathcal{S} := \text{lev}_\rho V_N^0$. There exists $\delta > 0$ such that (21) holds for all $x \in \mathcal{S}$ and $|\theta| \leq \delta$ and $\mathcal{S}$ is RPI for $x^+ = f_c(x, \theta), |\theta| \leq \delta$.

With quadratic costs (Assumption 5), Assumptions 1 to 3 also imply inherent *exponential* robustness of MPC. The following theorem is a minor generalization of the results in (Rawlings et al., 2020, Sec. 3.2.4), as well as a special case of (Pannocchia et al., 2011, Thm. 18). A proof of Theorem 6, which follows similarly to that of Theorem 5, is included in Appendix C.

Theorem 6. Suppose Assumptions 1 to 3 and 5 hold. Let $\rho > 0$ and $\mathcal{S} := \text{lev}_\rho V_N^0$. There exist $\delta, c_2 > 0$ and $\sigma \in \mathcal{K}$ such that

$$c_1|x|^2 \leq V_N^0(x) \leq c_2|x|^2 \tag{22a}$$

$$V_N^0(f_c(x, \theta)) \leq V_N^0(x) - c_1|x|^2 + \sigma(|\theta|) \tag{22b}$$

for all $x \in \mathcal{S}$ and $|\theta| \leq \delta$, where $c_1 := \underline{\sigma}(Q)$, and the system $x^+ = f_c(x, \theta), |\theta| \leq \delta$ is RES in the RPI set $\mathcal{S}$.

5 Stability of MPC despite mismatch

In this section, we investigate two approaches to guarantee strong stability of the closed-loop system (16). First, we take a direct approach and assume the existence of an ISS Lyapunov function that achieves a certain maximum increase due to mismatch. In general, an additional scaling condition is required for the mismatch term, although it is automatically satisfied for quadratic cost MPC. Second, we construct error bounds that imply the maximum Lyapunov increase for V_N^0 via the standard MPC assumptions (Assumptions 1 to 5) and one or both of Assumptions 6 and 7.

5.1 Maximum Lyapunov increase

We begin with the direct approach. The goal here is not (necessarily) to provide the means to check if a given MPC is strongly stabilizing, but to (i) identify a set of conditions for which an ISS Lyapunov function also guarantees strong stability and (ii) provide a path towards proving certain classes of nominal MPCs are strongly stabilizing.

5.1.1 Asymptotic case

For inherent robustness, a maximum increase of the form (20b) is proven for the optimal value function V_N^0. However, since the perturbation term $\sigma(|\theta|)$ is uniform in $|x|$, strong stability is not demonstrated for nonzero θ. Under Assumption 6, we might assume the perturbation vanishes in either of the limits $|x| \to 0$ or $|\theta| \to 0$. In this sense, the perturbation should be class-$\mathcal{K}$ in $|x|$ whenever $|\theta|$ is fixed, and vice versa. We call these functions *joint $\mathcal{K}$-functions* or *$\mathcal{K}^2$-functions* and define them as follows.

Definition 6 (Class $\mathcal{K}^2$). The class of *joint $\mathcal{K}$-functions*, denoted $\mathcal{K}^2$ is the class of continuous functions $\gamma : \mathbb{R}^2_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $\gamma(s, \cdot), \gamma(\cdot, s) \in \mathcal{K}$ for all $s > 0$.

To achieve strong stability, we assume the existence of an ISS Lyapunov function with a $\mathcal{K}^2$-function perturbation term, rather than the standard $\mathcal{K}$-function perturbation term. Moreover, we require the perturbation to decay faster than the nominal cost decrease in the limit $|x| \to 0$ so that the descent property of Definition 5 is achieved for sufficiently small θ.

Assumption 8 (Maximum Lyapunov increase). There exists a l.s.c. function $V : \mathbb{R}^n \to \overline{\mathbb{R}}_{\geq 0}$ such that, for each $\rho > 0$, there exist $\delta_0 > 0$, $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$, and $\gamma_V \in \mathcal{K}^2$ such that

(a) $\mathcal{S} := \mathrm{lev}_\rho V \subseteq \mathcal{X}_N$;

(b) for each $x \in \mathcal{S}$ and $|\theta| \leq \delta_0$, we have

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \tag{23a}$$

$$V(f_c(x, \theta)) \leq V(x) - \alpha_3(|x|) + \gamma_V(|x|, |\theta|); \tag{23b}$$

(c) and there exists $\tau > 0$ such that

$$\limsup_{s \to 0^+} \frac{\gamma_V(s, \tau)}{\alpha_3(s)} < 1. \tag{24}$$

With Assumption 8, we have our first main result.

Theorem 7. Suppose Assumption 8 holds with $V : \mathbb{R}^n \to \overline{\mathbb{R}}_{\geq 0}$. For each $\rho > 0$, there exists $\delta > 0$ for which $x^+ = f_c(x, \theta), |\theta| \leq \delta$ is SAS in the RPI set $\mathcal{S} := \text{lev}_\rho V$.

To prove Theorem 7, we require a preliminary result related to the ability of a given $\mathcal{K}^2$-function to lower bound another given $\mathcal{K}$-function (see Appendix D for proof).

Proposition 2. Let $\alpha \in \mathcal{K}_\infty$ and $\gamma \in \mathcal{K}^2$. If there exists $\tau > 0$ such that

$$\limsup_{s \to 0^+} \frac{\gamma(s, \tau)}{\alpha(s)} < 1$$

then, for each $\sigma > 0$, there exists $\delta > 0$ such that $\gamma(s, t) < \alpha(s)$ for all $s \in (0, \sigma]$ and $t \in [0, \delta]$.

Finally, we prove Theorem 7.

Proof of Theorem 7. By Assumption 8(a,b) there exists $\delta_0 > 0$, $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$, and $\gamma_V \in \mathcal{K}^2$ such that $\mathcal{S} \subseteq \mathcal{X}_N$ and (23) holds for each $x \in \mathcal{S}$ and $|\theta| \leq \delta_0$. Let $\varepsilon_0 := \sup_{x \in \mathcal{S}} |x| > 0$.[3] By Assumption 8(c) and Proposition 2, there exists $\delta_1 > 0$ such that $\alpha_3(s) > \gamma_V(s, t)$ for all $s \in (0, \varepsilon_0]$ and $t \in [0, \delta_1]$. With $\delta := \min\{\delta_0, \delta_1\}$, the function

$$\sigma(s) := \begin{cases} \alpha_3(s) - \gamma_V(s, \delta), & 0 \leq s \leq \varepsilon_0 \\ \alpha_3(\varepsilon_0) - \gamma_V(\varepsilon_0, \delta), & s > \varepsilon_0 \end{cases}$$

is both class-$\mathcal{PD}$ and continuous. By (23b), we have

$$V(f_c(x, \theta)) - V(x) \leq -\alpha_3(|x|) + \gamma_V(|x|, \delta) = -\sigma(|x|)$$

for all $x \in \mathcal{S}$ and $|\theta| \leq \delta$. Moreover, $V(x) \leq \rho$ implies

$$V(f_c(x, \theta)) \leq V(x) - \sigma(|x|) \leq \rho$$

so $\mathcal{S} = \text{lev}_\rho V$ must be RPI. Finally, $x^+ = f_c(x, \theta), |\theta| \leq \delta$ is SAS in $\mathcal{S}$ by Theorem 4. □

Remark 5. One might naïvely assume that the closed-loop system (16) is SAS under only Assumption 8(a,b). However, if the scaling condition Assumption 8(c) does not hold, then it may be the case that we cannot shrink t small enough to make $\alpha_3(\cdot) - \gamma_V(\cdot, t)$ positive definite in a sufficiently large neighborhood of the origin, let alone any neighborhood at all. Thus Assumption 8(a,b) alone are insufficient to show V is a Lyapunov function for the closed-loop system (16). This is illustrated in the example of Section 6.1 and in the following examples.

Example 1. Let $\alpha_3(s) := s^2$, $\gamma_V(s, t) := st$, and $L := \limsup_{s \to 0^+} \frac{\gamma_V(s,t)}{\alpha_3(s)}$. Then $\alpha_3 \in \mathcal{K}_\infty$ and $\gamma_V \in \mathcal{K}^2$, but $L = \lim_{s \to 0^+} t/s = \infty$ for each $t > 0$. In fact, since $\sigma_t(s) := \alpha_3(s) - \gamma_V(s, t) = s^2 - st$, σ_t is negative definite near the origin for each $t > 0$.

[3] If $\mathcal{S} = \{0\}$, the conclusion would hold trivially, so we can assume $\mathcal{S} \neq \{0\}$ without loss of generality.

Example 2. Let $\alpha_3(s) := s$, $\gamma_V(s,t) := \frac{2st}{s+t}$, and $L := \limsup_{s\to 0^+} \frac{\gamma_V(s,t)}{\alpha_3(s)}$. Then $\alpha_3 \in \mathcal{K}_\infty$ and $\gamma_V \in \mathcal{K}^2$, but $L = \lim_{s\to 0^+} \frac{2t}{s+t} = 2$ for each $t > 0$. Moreover, since $\sigma_t(s) := \alpha_3(s) - \gamma_V(s,t) = s - \frac{2st}{s+t} = \frac{s^2 - st}{s+t}$, σ_t is negative definite near the origin for each $t > 0$.

Remark 6. While Assumption 4 implies (23) can be satisfied with $\alpha_3 := \alpha_1$, it may be the case that (24) is not satisfied. For example, suppose in some neighborhood of the origin, that $\ell(x,u) := |x|^2 + |u|$, $\kappa_N(x) := -x$, and (f, ℓ, V_f) are Lipschitz on compact sets. Then $\gamma_V(s,t) := Lst$, $\alpha_1(s) := s^2$, and $\alpha_3(s) := s^2 + s$ satisfy (9), (23b), and (32) for some $L > 0$. While $\limsup_{s\to 0^+} \gamma_V(s,t)/\alpha_1(s) = \infty$ for each $t > 0$, we have $\limsup_{s\to 0^+} \gamma_V(s,t)/\alpha_3(s) = Lt$ and therefore (24) holds for any $\tau \in [0, 1/L)$.

Remark 7. To achieve Assumption 8(a), it is necessary to have $V(x) = \infty$ for all $x \notin \mathcal{X}_N$. Under Assumptions 1 to 4, this is automatically achieved by the optimal value function V_N^0, since, according to the convention of Rockafellar and Wets (1998), we have $V_N^0(x) = \infty$ for infeasible problems.

Remark 8. A restricted version of Assumption 6 is automatically satisfied under Assumption 8(b). To see this, we set $x = 0$ in (23) to give $f_c(0,\theta) = f(0, \kappa_N(0), \theta) = 0$ for all $|\theta| \le \delta$ and some $\delta > 0$. If, additionally, Assumptions 1, 2 and 4 are satisfied, we have

$$\tilde{\alpha}_1(|(x, \kappa_N(x))|) \le \tilde{\alpha}_1(|(x, \kappa_N(x))|) \le V_N^0(x) \le \tilde{\alpha}_2(|x|)$$

for some $\tilde{\alpha}_1, \tilde{\alpha}_2 \in \mathcal{K}_\infty$, which implies $\kappa_N(0) = 0$, so $f(0,0,\theta) = 0$ for all $|\theta| \le \delta$.

5.1.2 Exponential case

To achieve strong *exponential* stability, Assumption 8 is strengthened to require power law versions of the bounds in (23). Since identical exponents are required, the scaling condition Assumption 8(c) is automatically satisfied.

Assumption 9 (Max. Lyapunov incr. (exp.)). There exists a l.s.c. function $V : \mathbb{R}^n \to \overline{\mathbb{R}}_{\ge 0}$ such that, for each $\rho > 0$, there exist $\delta_0, a_1, a_2, a_3, b > 0$ and $\sigma_V \in \mathcal{K}_\infty$ satisfying

(a) $\mathcal{S} := \mathrm{lev}_\rho V \subseteq \mathcal{X}_N$; and

(b) for each $x \in \mathcal{S}$ and $|\theta| \le \delta_0$, we have

$$a_1|x|^b \le V(x) \le a_2|x|^b \tag{25a}$$

$$V(f_c(x,\theta)) \le V(x) - a_3|x|^b + \sigma_V(|\theta|)|x|^b. \tag{25b}$$

With Assumption 9, we have our second main result.

Theorem 8. Suppose Assumption 9 holds with $V : \mathbb{R}^n \to \overline{\mathbb{R}}_{\ge 0}$. For each $\rho > 0$, there exists $\delta > 0$ for which $x^+ = f_c(x,\theta), |\theta| \le \delta$ is SES in the RPI set $\mathcal{S} := \mathrm{lev}_\rho V$.

Proof. Assumption 9 gives $\delta_0, a_1, a_2, a_3, b > 0$ such that $\mathcal{S} \subseteq \mathcal{X}_N$ and (25) holds for each $x \in \mathcal{S}$ and $|\theta| \leq \delta_0$. Let $\delta_1 \in (0, \sigma_V^{-1}(a_3))$ and $\delta := \min\{\delta_0, \delta_1\} > 0$. Then, by (25b),

$$V(f_c(x,\theta)) - V(x) \leq -[a_3 - \sigma_V(\delta)]|x|^b = -a_4|x|^b$$

for all $x \in \mathcal{S}$ and $|\theta| \leq \delta$, where $a_4 := a_3 - \sigma_V(\delta) \geq a_3 - \sigma_V(\delta_1) > 0$. But this means that $V(x) \leq \rho$ implies

$$V(f_c(x,\theta)) \leq V(x) - a_4|x|^b \leq \rho$$

so $\mathcal{S} = \text{lev}_\rho V$ must be RPI. Finally, $x^+ = f_c(x,\theta), |\theta| \leq \delta$ is SES in $\mathcal{S}$ by Theorem 4. ☐

Remark 9. Remark 7 also applies to Assumption 9(a): we require $V(x) = \infty$ for all $x \notin \mathcal{X}_N$.

Remark 10. A restricted version of Assumption 6 is automatically satisfied under Assumption 9(b). Setting $x = 0$ in (25) gives $f_c(0,\theta) = f(0, \kappa_N(0), \theta) = 0$ for all $|\theta| \leq \delta$ and some $\delta > 0$. If, additionally, Assumptions 1, 2 and 5 are satisfied, we have

$$c_1|(x, \kappa_N(x))|^2 \leq c_1|(x, \kappa_N(x))|^2 \leq V_N^0(x) \leq c_2|x|^2$$

for some $c_1, c_2 > 0$, which implies $\kappa_N(0) = 0$, so $f(0,0,\theta) = 0$ for all $|\theta| \leq \delta$.

5.2 Error bounds

While the maximum Lyapunov increases (23b) and (25b) are difficult to verify directly, they are in fact satisfied for the optimal value function (i.e., $V := V_N^0$) under fairly general conditions. To show this, however, we require bounds on the error due to mismatch.

5.2.1 Model error bounds

Stability of MPC under mismatch was first investigated by Santos and Biegler (1999); Santos et al. (2008), who considered, for a fixed parameter $\theta \in \mathbb{R}^{n_\theta}$, the following power law bound:

$$|f(x,u,\theta) - \hat{f}(x,u)| \leq c|x| \tag{26}$$

for some $c > 0$ and all $(x,u) \in \mathbb{R}^n \times \mathbb{R}^m$. However, the bound (26) does not account for changing or unknown $\theta \in \mathbb{R}^{n_\theta}$ and is uniform in $u \in \mathbb{R}^m$, thus ruling out the motivating example from Section 2.3. To handle the former issue, we can take $c = \sigma_f(|\theta|)$ for some $\sigma_f \in \mathcal{K}_\infty$. For the latter issue, it suffices to either replace $|x|$ with $|(x,u)|$, i.e.,

$$|f(x,u,\theta) - \hat{f}(x,u)| \leq \sigma_f(|\theta|)|(x,u)| \tag{27}$$

or consider a bound on the closed-loop error, i.e.,

$$|f_c(x,\theta) - \hat{f}_c(x)| \leq \tilde{\sigma}_f(|\theta|)|x| \tag{28}$$

for all $x \in \mathcal{S}$, $u \in \mathbb{U}$, and $\theta \in \mathbb{R}^{n_\theta}$, where $\sigma_f, \tilde{\sigma}_f \in \mathcal{K}_\infty$ and $\mathcal{S} \subseteq \mathbb{R}^n$ is an appropriately chosen compact set.

For illustrative purposes, consider the following examples of the bounds (27). Note that, for a robustly exponentially stabilizing MPC with quadratic costs (satisfying (22)), the control law satisfies $|\kappa_N(x)| \leq \sqrt{c_2/\underline{\sigma}(R)}|x|$, so (27) implies (28).

Example 3. The linear system $x^+ = Ax + Bu$ achieves (27) with θ defined as the vectorization of $\begin{bmatrix} A & B \end{bmatrix}$ and $\sigma_f(\cdot) = (\cdot) \in \mathcal{K}_\infty$. More generally, we could consider arbitrary parameterizations of (A, B) that are continuous at $\theta = 0$, i.e., $x^+ = A(\theta)x + B(\theta)u$ where $\overline{\sigma}(\begin{bmatrix} A(\theta) & B(\theta) \end{bmatrix} - \begin{bmatrix} A(0) & B(0) \end{bmatrix}) \leq \sigma_f(|\theta|)$ and $\sigma_f \in \mathcal{K}_\infty$ is guaranteed by Proposition 11 in Appendix A.

Example 4. Consider the discretized pendulum system

$$x^+ = f(x, u, \theta) := \begin{bmatrix} x_1 + \Delta x_2 \\ x_2 + \Delta(\theta_1 \sin x_1 - \theta_2 x_2 + \theta_3 u) \end{bmatrix}$$

where $\theta \in \mathbb{R}^3_{>0}$ is a vector of lumped parameters and $\Delta > 0$ is the sample time. For a real pendulum system, the discretization will introduce numerical errors, but since the errors are $O(\Delta^2)$, we may assume $\Delta > 0$ is sufficiently small so that they can be safely ignored. For this system we have

$$|f(x, u, \theta) - \hat{f}(x, u, \hat{\theta})| \leq \Delta|\theta - \hat{\theta}||(x, u)|.$$

where $\hat{\theta} \in \mathbb{R}^3_{>0}$. Shifting θ by $-\hat{\theta}$ gives the bound (27).

In the following propositions, we derive the bounds (27) and (28) using Taylor's theorem and Assumptions 1 to 3 and 5 to 7 (see Appendix D for proofs).

Proposition 3. Suppose Assumptions 1, 2, 6 and 7 hold. For each compact set $\mathcal{S} \subseteq \mathbb{R}^n$, there exists $\sigma_f \in \mathcal{K}_\infty$ such that (27) holds for all $x \in \mathcal{S}$, $u \in \mathbb{U}$, and $\theta \in \mathbb{R}^{n_\theta}$.

Proposition 4. Suppose Assumptions 1 to 3 and 5 to 7 hold. For each compact set $\mathcal{S} \subseteq \mathcal{X}_N$, there exists $\tilde{\sigma}_f \in \mathcal{K}_\infty$ such that (28) holds for all $x \in \mathcal{S}$ and $\theta \in \mathbb{R}^{n_\theta}$.

More generally, we could consider $\mathcal{K}^2$-function bounds,

$$|f(x, u, \theta) - \hat{f}(x, u)| \leq \gamma_f(|(x, u)|, |\theta|) \tag{29}$$

$$|f_c(x, \theta) - \hat{f}_c(x)| \leq \tilde{\gamma}_f(|x|, |\theta|) \tag{30}$$

for all $x \in \mathcal{S}$ and $\theta \in \Theta$, where $\gamma_f, \tilde{\gamma}_f \in \mathcal{K}^2$, and $\mathcal{S} \subseteq \mathbb{R}^n$ and $\Theta \subseteq \mathbb{R}^{n_\theta}$ are appropriately chosen compact sets. In the following propositions, we derive the bounds (29) and (30) using Assumptions 1 to 3, 5 and 6 (see Appendix D for proofs).

Proposition 5. Suppose Assumptions 1, 2 and 6 hold. For any compact sets $\mathcal{S} \subseteq \mathbb{R}^n$ and $\Theta \subseteq \mathbb{R}^{n_\theta}$, there exists $\gamma_f \in \mathcal{K}^2$ satisfying (29) for all $x \in \mathcal{S}$, $u \in \mathbb{U}$, and $\theta \in \Theta$.

Proposition 6. Suppose Assumptions 1 to 4 and 6 hold. For any compact sets $\mathcal{S} \subseteq \mathcal{X}_N$ and $\Theta \subseteq \mathbb{R}^{n_\theta}$, there exists $\tilde{\gamma}_f \in \mathcal{K}^2$ satisfying (30) for all $x \in \mathcal{S}$ and $\theta \in \Theta$.

5.2.2 Suboptimal cost error bounds

Ultimately, we require a maximum Lyapunov increase of the form (23b) or (25b). The robust descent property (21) suggests a path through imposing an error bound on the suboptimal cost function $V_N(f_c(x,\theta), \tilde{\mathbf{u}}(x))$, i.e.,

$$|V_N(f_c(x,\theta), \tilde{\mathbf{u}}(x)) - V_N(\hat{f}_c(x), \tilde{\mathbf{u}}(x))| \leq \sigma_V(|\theta|)|x|^2 \tag{31}$$

where $\sigma_V \in \mathcal{K}_\infty$. In Proposition 7, we establish (31) under Assumptions 1 to 3 and 5 to 7 (see Appendix D for proof).

Proposition 7. Suppose Assumptions 1 to 3 and 5 to 7 hold and let $\mathcal{S} \subseteq \mathcal{X}_N$ be compact. Then there exists $\sigma_V \in \mathcal{K}_\infty$ such that (31) holds for all $x \in \mathcal{S}$ and $\theta \in \mathbb{R}^{n_\theta}$.

Similarly, we can derive a $\mathcal{K}^2$-function version of (31) under Assumptions 1 to 4 and 6 (see Appendix D for proof).

Proposition 8. Suppose Assumptions 1 to 4 and 6 hold. Let $\mathcal{S} \subseteq \mathcal{X}_N$ and $\Theta \subseteq \mathbb{R}^{n_\theta}$ be compact. Then there exists $\gamma_V \in \mathcal{K}^2$ such that, for each $x \in \mathcal{S}$ and $\theta \in \Theta$,

$$|V_N(f_c(x,\theta), \tilde{\mathbf{u}}(x)) - V_N(\hat{f}_c(x), \tilde{\mathbf{u}}(x))| \leq \gamma_V(|x|, |\theta|). \tag{32}$$

5.3 Stability despite mismatch

5.3.1 General costs

Finally, we are in a position to construct a maximum Lyapunov increase (23b) or (25b). For general costs, this is accomplished in the following proposition.

Proposition 9. Suppose Assumptions 1 to 4 and 6 hold. Then Assumption 8(a,b) hold with $V := V_N^0$.

Proof. Let $\rho > 0$, $\mathcal{S} := \mathrm{lev}_\rho V_N^0$, and $V := V_N^0$. Then $\mathcal{S} \subseteq \mathcal{X}_N$ trivially. Since V_N^0 is l.s.c. (Bertsekas and Shreve, 1978, Lem. 7.18), $\mathcal{S}$ is closed. By Theorem 5, there exists $\alpha_2 \in \mathcal{K}_\infty$ satisfying (23a) for all $x \in \mathcal{S}$. Then $|x| \leq \alpha_1^{-1}(V(x)) \leq \alpha_1^{-1}(\rho)$ for all $x \in \mathcal{S}$, so $\mathcal{S}$ is compact.

By Proposition 1, there exists $\delta_0 > 0$ such that $\mathcal{S}$ is RPI for $x^+ = f_c(x,\theta), |\theta| \leq \delta_0$ and (21) holds for all $x \in \mathcal{S}$ and $|\theta| \leq \delta_0$. Moreover, for each $x \in \mathcal{S}$ and $|\theta| \leq \delta_0$, (32) holds for some $\gamma_V \in \mathcal{K}^2$ by Proposition 8. Finally, combining (9), (21), and (32) gives (23b) with $\alpha_3 := \alpha_1$. □

Assumption 8(a,b) alone do not guarantee strong stability. However, we can strengthen the hypothesis of Proposition 9 with a scaling requirement to guarantee strong stability.

Theorem 9. Suppose Assumptions 1 to 4 and 6 hold. Let $\rho > 0$ and $\mathcal{S} := \mathrm{lev}_\rho V_N^0$. Then (23) holds for all $x \in \mathcal{S}$ and $|\theta| \leq \delta_0$ with $V := V_N^0$ and some $\delta_0 > 0$, $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$, and $\gamma_V \in \mathcal{K}^2$. If, additionally, there exists $\tau > 0$ satisfying (24), then there exists $\delta > 0$ such that $x^+ = f_c(x,\theta), |\theta| \leq \delta$ is SES in the RPI set $\mathcal{S}$.

Proof. The first part follows from Proposition 9, and the second part follows from Theorem 7. □

5.3.2 Quadratic costs

For quadratic costs, we construct (25b) in the following proposition.

Proposition 10. Suppose Assumptions 1 to 3 and 5 to 7 hold. Then Assumption 9 holds with $b := 2$ and $V := V_N^0$.

Proof. Let $\rho > 0$, $V := V_N^0$, and $\mathcal{S} := \text{lev}_\rho V$. Since Assumption 5 implies Assumption 4, we have from the first paragraph of the proof of Proposition 9 that $\mathcal{S}$ is compact.

Theorem 6 also implies (25a) holds for all $x \in \mathcal{S}$, with $a_1, a_2 > 0$ and $b := 2$. By Proposition 1, there exists $\delta_0 > 0$ such that $\mathcal{S}$ is RPI for $x^+ = f_c(x, \theta), |\theta| \leq \delta_0$ and (21) holds for all $x \in \mathcal{S}$ and $|\theta| \leq \delta_0$. Moreover, for each $x \in \mathcal{S}$ and $|\theta| \leq \delta_0$, (31) holds for some $\sigma_V \in \mathcal{K}_\infty$ by Proposition 8, and combining (21) and (32) gives (25b). □

Our third and final main result follows immediately from Theorem 7 and Proposition 10.

Theorem 10. Suppose Assumptions 1 to 4, 6 and 7 holds. For each $\rho > 0$, there exists $\delta > 0$ for which $x^+ = f_c(x, \theta), |\theta| \leq \delta$ is SES in the RPI set $\mathcal{S} := \text{lev}_\rho V_N^0$.

Proof. By Proposition 10, Assumption 9 holds with $V := V_N^0$, and by Theorem 8, there exists $\delta > 0$ for which $\mathcal{S}$ is RPI and $x^+ = f_c(x, \theta), |\theta| \leq \delta$ is SES in $\mathcal{S}$. □

6 Examples

In this section, we illustrate the nuances of Assumptions 8 and 9 through several examples. First, we consider a non-differentiable system that satisfies Assumption 8(a,b) but not Assumption 8(c), and is not SAS. Second, we consider a non-differentiable example that nonetheless satisfies Assumption 9 and is therefore SES. Finally, we consider the inverted pendulum system to showcase how the nominal MPC handles different types of mismatch. Notably, we consider (i) discretization errors, (ii) unmodeled dynamics, and (iii) incorrectly estimated input gains.

6.1 Strong asymptotic stability counterexample

Consider the scalar system

$$x^+ = f(x, u, \theta) := \sigma(x + (1 + \theta)u) \tag{33}$$

where σ is the *signed square root* defined as $\sigma(y) := \text{sgn}(y)\sqrt{|y|}$ for each $y \in \mathbb{R}$. We define a nominal MPC with $\mathbb{U} := [-1, 1]$, $\ell(x, u) := x^2 + u^2$, $V_f(x) := 4x^2$, $\mathbb{X}_f := [-1, 1]$, and $N := 1$.

In Appendix E, it is shown the closed-loop system $x^+ = f(x, \kappa_1(x), \theta), |\theta| \leq 3$ is RES on $\mathcal{X}_1 = [-2, 2]$ with the nominal control law $\kappa_1(x) := -\text{sat}(x)$. Additionally, it is shown Assumption 8(a,b) is satisfied with $V := V_1^0$, and (23b) holds for all $x \in \mathcal{S} := \text{lev}_2 V_1^0 = [-1, 1]$ and $|\theta| \leq \delta_0 := 3$ with $\alpha_3(s) := 2s^2$, and $\gamma_V(s, t) := st + 4\sqrt{st}$. But this implies $\lim_{s \to 0^+} \gamma_V(s, t)/\alpha_3(s) = \infty$ for each $t > 0$, so Assumption 8(c) is not satisfied.

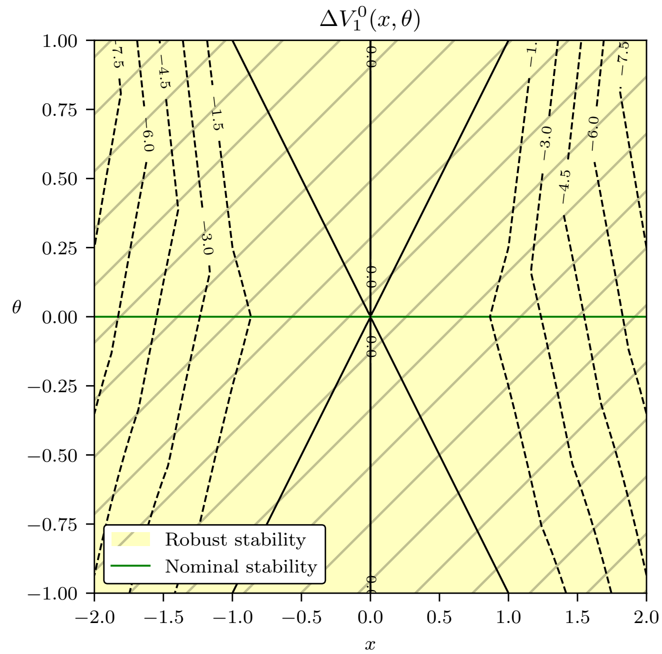


Figure 3: Contours of the cost difference for the MPC of (33).

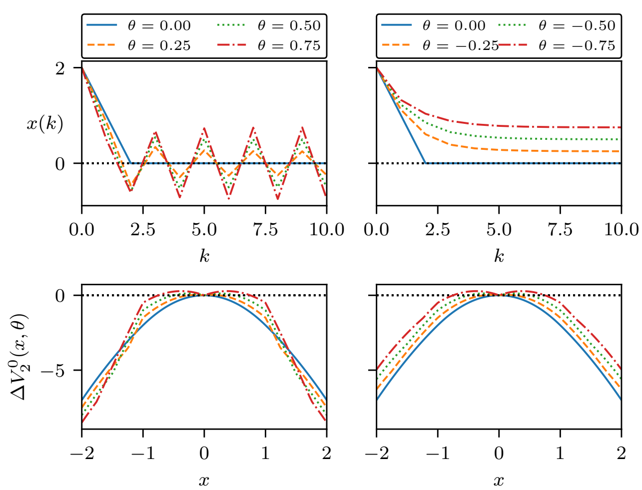


Figure 4: For (left) nonnegative and (right) nonpositive values of θ, the (top) closed-loop trajectories for the MPC of (33) with initial state $x = 2$, and (bottom) cost differences of the same MPC as a function of x.

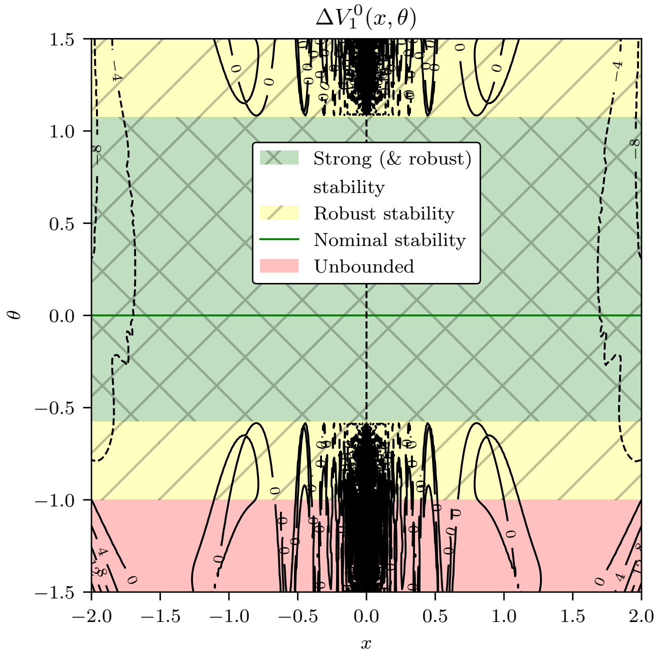


Figure 5: Contours of the cost difference for the MPC of (34).

However, Assumption 8 is only sufficient, not necessary, for establishing strong stability. But we have $V_1^0(x) = 2x^2$ and

$$\begin{aligned}\Delta V_1^0(x,\theta) &:= V_1^0(f(x,\kappa_1(x),\theta)) - V_1^0(x) \\ &= 2[\sigma(\theta x)]^2 - 2x^2 = 2(|\theta| - |x|)|x| > 0.\end{aligned}$$

for each $0 < |x| < |\theta| \leq 1$, so the state always gets pushed out of $(-|\theta|, |\theta|)$ unless it starts at the origin or $\theta = 0$. In other words, the MPC only provides inherent robustness, not strong stability, even though Assumption 8(a,b) is satisfied.

In Figure 3, we plot contours of the cost difference $\Delta V_1^0(x,\theta)$, and in Figure 4 we plot closed-loop trajectories and the cost difference curve $\Delta V_1^0(\cdot,\theta)$ for several values of θ. Only with $\theta = 0$ does the trajectory converge to the origin and the cost difference curve remain negative definite. For each $\theta \neq 0$, the cost difference is positive definite near the origin, and the trajectory does not converge to the origin.

6.2 Non-differentiable yet strongly exponential stable

Consider the scalar system

$$x^+ = f(x,u,\theta) := x + (1/2)\gamma(x) + (1+\theta)u \tag{34}$$

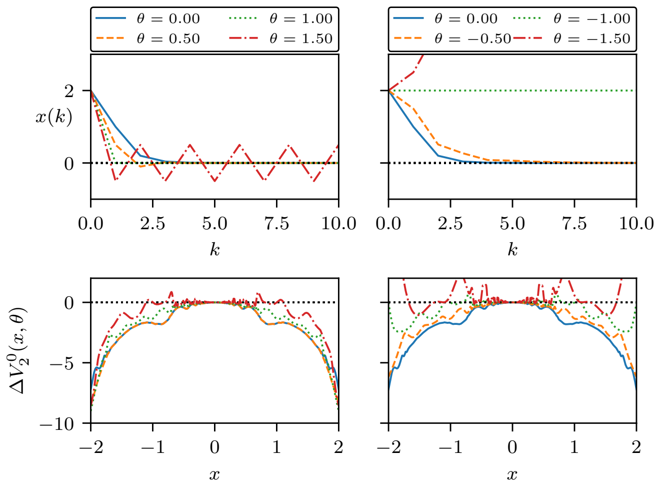


Figure 6: For (left) nonnegative and (right) nonpositive values of θ, the (top) closed-loop trajectories for the MPC of (34) with initial state $x = 2$, and (bottom) cost differences of the same MPC as a function of x.

where $\gamma : \mathbb{R} \to \mathbb{R}$ is defined as

$$\gamma(x) := \begin{cases} 0, & x = 0, \\ |x| \sin(2\pi/x), & x \neq 0. \end{cases}$$

While the function γ is continuous, it is not differentiable at the origin. We define a nominal MPC with $\mathbb{U} := [-1, 1]$, $\ell(x, u) := x^2 + u^2$, $V_f(x) := 4x^2$, $\mathbb{X}_f := [-1, 1]$, and $N := 1$.

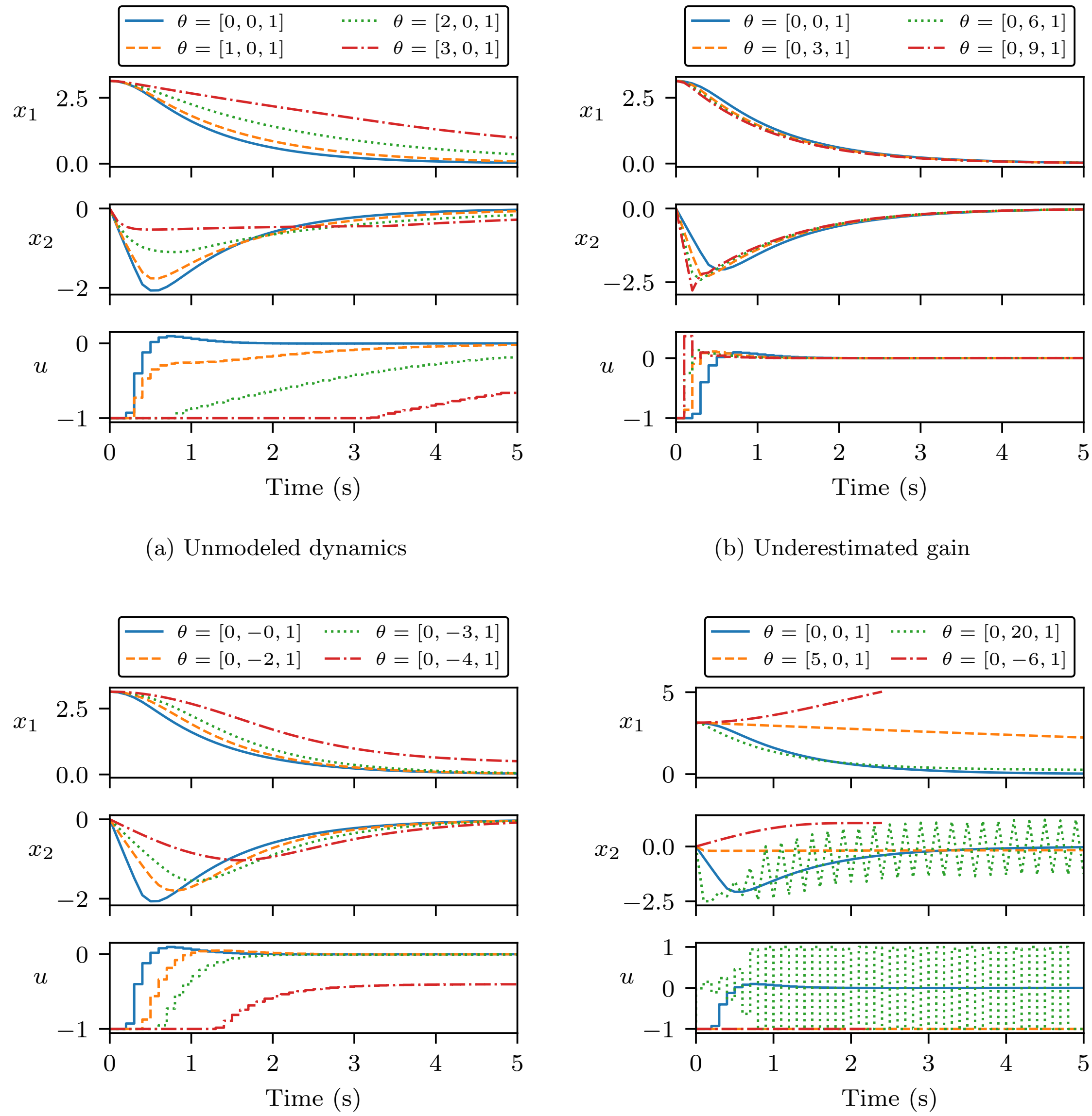
In Appendix E.2, we show the closed-loop system $x^+ = f(x, \kappa_1(x), \theta), |\theta| \leq 1$ is RES on $\mathcal{X}_1 = [-2, 2]$ with the nominal control law $\kappa_1(x) := -\text{sat}((4/5)x + (2/5)\gamma(x))$. Moreover, it is shown that Assumption 9 is satisfied, and by Theorem 8 (and its proof), the closed-loop system $x^+ = f(x, \kappa_1(x), \theta), |\theta| \leq \delta := 0.5$ is SES on $\mathcal{X}_1 = [-2, 2]$.

To establish a clearer picture of robust and strong stability for the closed-loop, we plot in Figure 5 contours of the cost difference $\Delta V_1^0(x, \theta) := V_1^0(f(x, \kappa_1(x), \theta)) - V_1^0(x)$, and in Figure 6 closed-loop trajectories and the cost difference curve $\Delta V_1^0(\cdot, \theta)$ for several values of θ. For θ between $\theta_0 \approx 0.57$ and $\theta_1 \approx 1.08$, the closed-loop system is strongly stable, with trajectories converging to the origin, and a negative definite cost difference curve. Outside of this range but with $\theta \in [-1, 1.5]$, the closed-loop system is still robustly stable, with a cost difference curve of ambiguous sign but trajectories converging to a neighborhood of the origin. Finally, for $\theta < -1$, trajectories are unbounded because $\mathcal{X}_1$ is not RPI.

6.3 Upright pendulum

Consider the nondimensionalized pendulum system

$$\dot{x} = F(x, u, \theta) := \begin{bmatrix} x_2 \\ \sin x_1 - \theta_1^2 x_2 + (\hat{k} + \theta_2)u \end{bmatrix} \tag{35}$$



(a) Unmodeled dynamics

(b) Underestimated gain

(c) Overestimated gain

(d) Breakdown cases

Figure 7: Simulated closed-loop trajectories for the MPC of (35) from the resting position $x(0) = (\pi, 0)$ to the upright position $x_s = (0, 0)$ for various values of $(\theta_1, \theta_2) \in \mathbb{R}^2$.

where $x_1, x_2 \in \mathbb{R}$ are the angle and angular velocity, $u \in [-1, 1]$ is the (signed and normalized) motor voltage, $\theta_1 \in \mathbb{R}$ is an air resistance factor, $\hat{k} > 0$ is the estimated gain of the motor, and $\theta_2 \in \mathbb{R}$ is the error in the motor gain estimate. Let $\psi(t; x, u, \theta)$ denote the solution to the differential equation (35) at time $t \geq 0$ given an initial condition $x(0) = x$, constant input signal $u(t) = u$, and parameters θ. We model the continuous-time system (35) as

$$x^+ = f(x, u, \theta) := x + \Delta F(x, u, \theta) + \theta_3 r(x, u, \theta) \tag{36}$$

where r is a residual function given by

$$r(x, u, \theta) := \int_0^\Delta [F(\psi(t; x, u, \theta), u, \theta) - F(x, u, \theta)]dt.$$

Assuming a zero-order hold on the input u, the system (35) is discretized (exactly) as (36) with $\theta_3 = 1$. Since we model the system with $\theta = 0$ as

$$x^+ = \hat{f}(x, u) := f(x, u, 0) = x + \Delta \begin{bmatrix} x_2 \\ \sin x_1 + \hat{k}u \end{bmatrix} \tag{37}$$

we do not need access to r to design the nominal MPC.

For the following simulations, let the model gain be $\hat{k} = 5$ rad/s^2, the sample time be $\Delta = 0.1$ s, and define a nominal MPC with $N := 20$, $\mathbb{U} := [-1, 1]$, $\ell(x, u) := |x|^2 + u^2$, $V_f(x) := |x|^2_{P_f}$, $\mathbb{X}_f := \text{lev}_{c_f} V_f$, and $c_f := \underline{\sigma}(P_f)/8$, where $P_f = \left[\begin{smallmatrix} 31.133... & 10.196... \\ 10.196... & 10.311... \end{smallmatrix}\right]$ is shown, in Appendix E.3, to satisfy Assumption 3 with the terminal law $\kappa_f(x) := -2x_1 - 2x_2$. Assumptions 1, 2, 5 and 6 are satisfied trivially, and Assumption 7 is satisfied since continuous differentiability of F implies continuous differentiability of ψ (and therefore also r and f) (Hale, 1980, Thm. 3.3). Thus, the conclusion of Theorem 10 holds for some $\delta > 0$, and if we can take $\delta > 1$, the nominal MPC is inherently strongly stabilizing with $\begin{bmatrix} \theta_1 & \theta_2 \end{bmatrix}$ sufficiently small.

In Figure 7, we simulate the closed-loop system $x^+ = f(x, \kappa_{20}(x), \theta)$ for some fixed $\begin{bmatrix} \theta_1 & \theta_2 & 1 \end{bmatrix}^\top \in \mathbb{R}^3$. Note that all of these simulations include discretization errors. Figure 7a showcases the ability of MPC to handle unmodeled dynamics (i.e., a missing air resistance term). In Figure 7b, the gain of the motor is increased until the nominal controller is severely underdamped. In Figure 7c, the gain of the motor is decreased until the motor cannot overcome the force of gravity and strong stability is not achieved. In Figure 7d, we plot cases where the errors as made so extreme as to prevent stability.

7 Conclusion

We establish conditions under which MPC is strongly stabilizing despite plant-model mismatch in the form of parameter errors. Namely, it suffices to assume the existence of a Lyapunov function with a maximum increase, suitably bounded level sets, and a scaling condition (Assumptions 8 and 9). While we are not able to show the assumptions hold in general, when the MPC has quadratic costs it is possible to show that continuous differentiability of the dynamics implies strong stability (Theorem 8). When the $\mathcal{K}^2$-function bound is not properly scaled, the MPC may not be stabilizing, as illustrated in the examples. In this sense, while MPC is not inherently stabilizing under mismatch *in general*, there is a common class of cost functions (quadratic costs) for which nominal MPC is inherently stabilizing under mismatch.

Several questions about the strong stability of MPC remain unanswered. While quadratic costs are used in many control problems, it may be possible to generalize Theorem 10 to other useful classes of stage costs, such as q-norm costs, or costs with exact penalty functions for soft state constraints. We propose the direct approach to strong exponential stability (Assumption 9 and Theorem 8) provides a path to generalizing Theorem 10 to other classes of stage costs, output feedback, or semidefinite costs. We note that the Assumptions 8 and 9 are dependent on the horizon length. This leaves the possibility that some MPC problems are strongly stabilizing at smaller horizon lengths but only inherently robust at longer horizon lengths, or vice versa. However, this remains to be seen.

There are several areas in which future work of this type would be valuable. Nonlinear MPC is computationally difficult to implement online. Therefore it would be worth extending this work to include the suboptimal MPC algorithm from Allan et al. (2017) using the approach therein. While systems with fixed and known setpoints are a useful and interesting class of problems, many systems have setpoints that must be tracked that may change based on the value of the parameters. To accommodate the effect of mismatch on the setpoints, offset-free MPC is sometimes used. Theory on nonlinear offset-free MPC is fairly limited, typically relying on stability of the closed-loop system to guarantee offset-free performance (Pannocchia et al., 2015). Using the tools developed in this paper, we plan to extend the offset-free MPC theory by establishing closed-loop stability and guaranteed offset-free performance for tracking random, asymptotically constant setpoints subject to plant-model mismatch.

A Nominal MPC stability

In this appendix, we provide sketches of the MPC stability results Theorems 1 and 2. First, the lower bound $V_N^0(x) \geq \alpha_1(|x|)$ follows immediately from Assumption 4. Next, consider the following proposition from Allan et al. (2017).

Proposition 11 (Prop. 20 of Allan et al. (2017)). Let $C \subseteq D \subseteq \mathbb{R}^n$, with C compact, D closed, and $f : D \to \mathbb{R}^m$ continuous. Then there exists $\alpha \in \mathcal{K}_\infty$ such that $|f(x) - f(y)| \leq \alpha(|x - y|)$ for all $x \in C$ and $y \in D$.

Under Assumptions 1 to 4, we can establish the following bounds via Proposition 11,[4]

$$V_f(x) \leq \alpha_f(|x|), \qquad \forall x \in \mathbb{X}_f \tag{38}$$

$$V_N^0(x) \leq \alpha_2(|x|), \qquad \forall x \in \mathcal{X}_N \tag{39}$$

[4]Equation (38) follows immediately from Proposition 11 and Assumptions 1 and 2. For (39), see (Rawlings et al., 2020, Prop. 2.16).

for some $\alpha_f, \alpha_2 \in \mathcal{K}_\infty$. To establish the cost difference bound, first note that, under Assumptions 2 and 3, we have

$$V_f(\hat{f}(x, \kappa_f(x))) \leq V_f(x) - \ell(x, \kappa_f(x)) \leq c_f$$

for all $x \in \mathbb{X}_f$. Therefore $\mathbb{X}_f$ is positive invariant for $x^+ = \hat{f}(x, \kappa_f(x))$. But this means $\mathcal{X}_N$ is positively invariant because, for each $x \in \mathcal{X}_N$, $\tilde{\mathbf{u}}(x)$ steers the system into $\mathbb{X}_f$ in $N-1$ moves and keeps it there, meaning $\hat{f}_c(x) \in \mathcal{X}_N$. Finally, Assumption 3 implies

$$V_N^0(\hat{f}_c(x)) \leq V_N(\hat{f}_c(x), \tilde{\mathbf{u}}(x)) \leq V_N^0(x) - \ell(x, \kappa_N(x)) \tag{40}$$

for all $x \in \mathcal{X}_N$ (Rawlings et al., 2020, pp. 116–117). Therefore $V_N^0(\hat{f}_c(x)) \leq V_N^0(x) - \alpha_1(|x|)$ by Assumption 4.

Let $\rho > 0$ and $\mathcal{S} := \mathrm{lev}_\rho V_N^0$. As noted in the main text, we have $\mathcal{S} \subseteq \mathcal{X}_N$ by definition of the sublevel set. Assumptions 2 and 5 implies $\underline{\sigma}(P_f)|x|^2 \leq V_f(x) \leq c_f$ for all $x \in \mathbb{X}_f$, so we have $|x| \leq \varepsilon := \sqrt{c_f/\underline{\sigma}(P_f)}$ for all $x \in \mathbb{X}_f$. Then with $c_2 := \max\{\overline{\sigma}(P_f), \rho/\varepsilon^2\}$, we can write

$$V_N^0(x) \leq \begin{cases} V_f(x) \leq \overline{\sigma}(P_f)|x|^2 \leq c_2|x|^2, & |x| \leq \varepsilon, \\ \rho \leq c_2\varepsilon^2 \leq c_2|x|^2, & |x| \geq \varepsilon. \end{cases}$$

for each $x \in \mathcal{S}$. Finally, V_N^0 is an exponential Lyapunov function in $\mathcal{S}$ for $x^+ = \hat{f}_c(x)$.

B Lyapunov proofs

In this appendix, we prove some of the Lyapunov results of Section 3.

Proof of Theorem 3 *(exponential case).* The case where an ISS Lyapunov function implies RAS for a system is covered by (Allan et al., 2017, Prop. 19), so we only consider the ISES/RES case.

Let $X \subseteq \mathbb{R}^n$ be RPI and suppose $V : X \to \mathbb{R}_{\geq 0}$ is an ISES Lyapunov function, both for the system $x^+ = f_c(x, \theta), \theta \in \Theta$. Then there exist $a_1, a_2, a_3, b > 0$ satisfying (18) for all $x \in X$, where $\alpha_i(\cdot) := a_i(\cdot)^b$ for each $i \in \{1, 2, 3\}$. Suppose, without loss of generality, that $a_3 < a_2$. Then (18) can be rewritten

$$\begin{aligned} V(f_c(x, \theta)) &\leq V(x) - a_3|x|^b + \sigma(|\theta|) \\ &\leq V(x) - \frac{a_3}{a_2}V(x) + \sigma(|\theta|) \\ &= \lambda_0 V(x) + \sigma(|\theta|) \end{aligned}$$

for all $x \in X$ and $\theta \in \Theta$, where $\lambda_0 := 1 - \frac{a_3}{a_2} \in (0,1)$. Since X is RPI, this implies

$$\begin{aligned} V(\phi_c(k;x,\boldsymbol{\theta})) &\leq \lambda_0^k V(x) + \sum_{i=1}^{k} \lambda_0^{i-1}\sigma(|\theta(k-i)|) \\ &\leq \lambda_0^k V(x) + \left(\sum_{i=1}^{k} \lambda_0^{i-1}\right) \max_{i\in\mathbb{I}_{0:k-1}} \sigma(|\theta(i)|) \\ &\leq \lambda_0^k V(x) + \frac{\max_{i\in\mathbb{I}_{0:k-1}} \sigma(|\theta(i)|)}{1-\lambda_0} \\ &= \lambda_0^k V(x) + \frac{\sigma(\|\boldsymbol{\theta}\|_{0:k-1})}{1-\lambda_0} \\ &= a_2|x|^b\lambda_0^k + \frac{\sigma(\|\boldsymbol{\theta}\|_{0:k-1})}{1-\lambda_0} \end{aligned}$$

for all $k \in \mathbb{I}_{\geq 0}$, $x \in X$, and $\boldsymbol{\theta} \in \Theta^k$. If $b \geq 1$, then, by the triangle inequality for the b-norm, we have

$$\begin{aligned} |\phi_c(k;x,\boldsymbol{\theta})| &\leq \left(\frac{V(\phi_c(k;x,\boldsymbol{\theta})}{a_1}\right)^{1/b} \\ &\leq \frac{1}{a_1^{1/b}}\left(a_2|x|^b\lambda_0^k + \frac{\sigma(\|\boldsymbol{\theta}\|_{0:k-1})}{1-\lambda_0}\right)^{1/b} \\ &\leq \left(\frac{a_2}{a_1}\right)^{1/b}|x|(\lambda_0^b)^k + \left(\frac{\sigma(\|\boldsymbol{\theta}\|_{0:k-1})}{a_1(1-\lambda_0)}\right)^{1/b} \\ &\leq c|x|\lambda^k + \gamma(\|\boldsymbol{\theta}\|_{0:k-1}) \end{aligned}$$

for all $k \in \mathbb{I}_{\geq 0}$, $x \in X$, and $\boldsymbol{\theta} \in \Theta^k$, where $\lambda := \lambda_0^{1/b} \in (0,1)$, $c := \left(\frac{a_2}{a_1}\right)^{1/b} > 0$, and $\gamma(\cdot) := \left(\frac{\sigma(\cdot)}{a_1(1-\lambda_0)}\right)^{1/b} \in \mathcal{K}$. On the other hand, if $b \in (0,1)$, then $1/b \geq 1$, so by convexity of $(\cdot)^{1/b}$, we have

$$\begin{aligned} |\phi_c(k;x,\boldsymbol{\theta})| &\leq \left(\frac{2}{a_1}\right)^{1/b}\left(\frac{1}{2}a_2|x|^b\lambda_0^k + \frac{1}{2}\frac{\sigma(\|\boldsymbol{\theta}\|_{0:k-1})}{1-\lambda_0}\right)^{1/b} \\ &\leq \frac{1}{2}\left(\frac{2a_2}{a_1}\right)^{1/b}|x|(\lambda_0^b)^k + \frac{1}{2}\left(\frac{2\sigma(\|\boldsymbol{\theta}\|_{0:k-1})}{a_1(1-\lambda_0)}\right)^{1/b} \\ &\leq c|x|\lambda^k + \gamma(\|\boldsymbol{\theta}\|_{0:k-1}) \end{aligned}$$

for all $k \in \mathbb{I}_{\geq 0}$, $x \in X$, and $\boldsymbol{\theta} \in \Theta^k$, where $\lambda := \lambda_0^{1/b} \in (0,1)$, $c := \frac{1}{2}\left(\frac{2a_2}{a_1}\right)^{1/b} > 0$, and $\gamma(\cdot) := \frac{1}{2}\left(\frac{2\sigma(\cdot)}{a_1(1-\lambda_0)}\right)^{1/b} \in \mathcal{K}$. In either case, (17) is satisfied with $\beta(s,k) := cs\lambda^k$ and $\gamma \in \mathcal{K}$ for some $c > 0$ and $\lambda \in (0,1)$. □

C Proofs of inherent robustness results

This appendix contains proofs of the inherent robustness results from Section 4. From Proposition 11, we have the following proposition.

Proposition 12. Suppose Assumptions 1 and 2 holds and let $\tilde{V}_f(\cdot,\cdot) := V_f(\hat{\phi}(N;\cdot,\cdot))$. Then, for any compact set $\mathcal{S} \subseteq \mathcal{X}_N$, there exist $\alpha_a, \alpha_b, \alpha_\theta \in \mathcal{K}_\infty$ such that, for each $x \in \mathcal{S}$ and $\theta \in \mathbb{R}^{n_\theta}$,

$$|\tilde{V}_f(x^+, \tilde{\mathbf{u}}(x)) - \tilde{V}_f(\hat{x}^+, \tilde{\mathbf{u}}(x))| \leq \alpha_a(|x^+ - \hat{x}^+|) \tag{41}$$

$$|V_N(x^+, \tilde{\mathbf{u}}(x)) - V_N(\hat{x}^+, \tilde{\mathbf{u}}(x))| \leq \alpha_b(|x^+ - \hat{x}^+|) \tag{42}$$

$$|f_c(x, \theta) - \hat{f}_c(x)| \leq \alpha_\theta(|\theta|) \tag{43}$$

where $x^+ := f_c(x,\theta)$ and $\hat{x}^+ := \hat{f}_c(x)$.

Proof. Assumptions 1 and 2 guarantee $\tilde{\mathbf{u}}(x)$ is well-defined for all $x \in \mathcal{X}_N$ (Rawlings et al., 2020, Prop. 2.4). Define $C_0 := \mathcal{S} \times \mathbb{U} \times \{0\}$ and $C_1 := \mathcal{S} \times \mathbb{U}^N$. Then C_0 and C_1 are compact, f is continuous, and $\tilde{V}_f$ and V_N are continuous as they are finite compositions of continuous functions. By Proposition 11, there exist $\alpha_a, \alpha_b, \alpha_\theta \in \mathcal{K}_\infty$ such that

$$|\tilde{V}_f(x^+, \mathbf{u}) - \tilde{V}_f(\hat{x}^+, \hat{\mathbf{u}})| \leq \alpha_a(|(x^+ - \hat{x}^+, \mathbf{u} - \hat{\mathbf{u}})|)$$
$$|V_N(x^+, \mathbf{u}) - V_N(\hat{x}^+, \hat{\mathbf{u}})| \leq \alpha_b(|(x^+ - \hat{x}^+, \mathbf{u} - \hat{\mathbf{u}})|)$$
$$|f(x, u, \theta) - \hat{f}(\hat{x}, \hat{u})| \leq \alpha_\theta(|(x - \hat{x}, u - \hat{u}, \theta)|)$$

for all $(\hat{x}, \hat{u}, 0) \in C_0$, $(\hat{x}^+, \hat{\mathbf{u}}) \in C_1$, $(x, u, \theta) \in \mathbb{R}^{n+m+n_\theta}$, and $(x^+, \mathbf{u}) \in \mathbb{R}^{n+Nm}$. Specializing the above inequalities to $x = \hat{x}$, $\hat{x}^+ = \hat{f}_c(x)$, $x^+ = f_c(x,\theta)$, $u = \hat{u} = \kappa_N(x)$, and $\mathbf{u} = \hat{\mathbf{u}} = \tilde{\mathbf{u}}(x)$ gives (41)–(43) for all $x \in \mathcal{S}$ and $\theta \in \mathbb{R}^{n_\theta}$. □

Next, we can prove Proposition 1.

Proof of Proposition 1. First, we have $\alpha_1, \alpha_2, \alpha_f \in \mathcal{K}_\infty$ satisfying the bounds (9), (12), and (38)–(40) from the assumptions and Theorem 1 (and its proof). Next, we let $x \in \mathcal{S}$ and $\theta \in \mathbb{R}^{n_\theta}$, and define $\tilde{V}_f(\cdot,\cdot) := V_f(\hat{\phi}(N;\cdot,\cdot))$, $x^+ := f_c(x,\theta)$, and $\hat{x}^+ := \hat{f}_c(x)$, throughout. By Proposition 12, there exist $\alpha_a, \alpha_b, \alpha_\theta \in \mathcal{K}_\infty$ satisfying the bounds (41)–(43).

(a)—*Robust feasibility*: By nominal feasibility, we have $\hat{x}^0(N;\hat{x}^+) \in \mathbb{X}_f$ and therefore $V_f(\hat{x}^0(N;x)) \leq c_f$. By construction of the warm start, we have $\hat{\phi}(N;x^+,\tilde{\mathbf{u}}(x)) = \hat{f}(\hat{x}^0(N;x), \kappa_f(\hat{x}^0(N;x)))$ and therefore

$$\begin{aligned}
\tilde{V}_f(\hat{x}^+, \tilde{\mathbf{u}}(x)) &= V_f(\hat{\phi}(N;\hat{x}^+, \tilde{\mathbf{u}}(x))) \\
&= V_f(\hat{f}(\hat{x}^0(N;x), \kappa_f(\hat{x}^0(N;x)))) \\
&\leq V_f(\hat{x}^0(N;x)) - \alpha_1(|\hat{x}^0(N;x)|)
\end{aligned}$$

where the inequality follows from Assumptions 3 and 4. If $V_f(\hat{x}^0(N;x)) \geq c_f/2$, then $|\hat{x}^0(N;x)| \geq \alpha_f^{-1}(c_f/2)$ and $\tilde{V}_f(\hat{x}^+, \tilde{\mathbf{u}}(x)) \leq c_f - \alpha_1(\alpha_f^{-1}(c_f/2))$. On the other hand, if $V_f(\hat{x}^0(N;x)) < c_f/2$, then $\tilde{V}_f(\hat{x}^+, \tilde{\mathbf{u}}(x)) < c_f/2$. In summary,

$$\tilde{V}_f(\hat{x}^+, \tilde{\mathbf{u}}(x)) \leq c_f - \gamma_1$$

where $\gamma_1 := \min\{\, c_f/2, \alpha_1(\alpha_f^{-1}(c_f/2)\,\} > 0$. Combining the above inequality with (41) and (43) gives

$$\tilde{V}_f(x^+, \tilde{\mathbf{u}}(x)) \leq c_f - \gamma_1 + \alpha_a(\alpha_\theta(|\theta|)).$$

Therefore, so long as $|\theta| \leq \delta_1 := \alpha_\theta^{-1}(\alpha_a^{-1}(\gamma_1))$, we have $V_f(\hat{\phi}(N; x^+, \tilde{\mathbf{u}}(x))) = \tilde{V}_f(x^+, \tilde{\mathbf{u}}(x)) \leq c_f$, which implies $\hat{\phi}(N; x^+, \tilde{\mathbf{u}}(x)) \in \mathbb{X}_f$, and therefore $(x^+, \tilde{\mathbf{u}}(x)) \in \mathcal{Z}_N$.

(b)—*Descent property*: Suppose $|\theta| \leq \delta_1$. Then $(x^+, \tilde{\mathbf{u}}(x)) \in \mathcal{Z}_N$ by part (a), so the inequality $V_N^0(x^+) \leq V_N(x^+, \tilde{\mathbf{u}}(x))$ follows by optimality. Combining this inequality with the nominal descent property (40) gives the robust descent property (21).

(c)—*Positive invariance of* $\mathcal{S}$: Suppose again that $|\theta| \leq \delta_1$. Then the inequality (43) holds from part (b), and combining it with (21) and (42) gives

$$V_N^0(x^+) \leq V_N^0(x) - \alpha_1(|x|) + \alpha_b(\alpha_\theta(|\theta|)). \tag{44}$$

If $V_N^0(x) \geq \rho/2$, then $|x| \geq \alpha_2^{-1}(\rho/2)$ and $V_N^0(x^+) \leq \rho - \alpha_1(\alpha_2^{-1}(\rho/2)) + \alpha_b(\alpha_\theta(|\theta|))$. On the other hand, if $V_N^0(x) < \rho/2$, then $V_N^0(x^+) < \rho/2 + \alpha_b(\alpha_\theta(|\theta|))$. Then

$$V_N^0(x^+) \leq \rho - \gamma_2 + \alpha_b(\alpha_\theta(|\theta|))$$

where $\gamma_2 := \min\{\, \rho/2, \alpha_1(\alpha_2^{-1}(\rho/2)\,\} > 0$. Therefore $V_N^0(x^+) \leq \rho$ and $x^+ \in \mathcal{S}$ so long as $|\theta| \leq \delta := \min\{\, \delta_1, \delta_2 \,\}$ where $\delta_2 := \alpha_\theta^{-1}(\alpha_b^{-1}(\gamma_2))$. □

Finally, Theorem 5 follows from Propositions 1 and 12 by combining the inequalities (21), (42), and (43).

Proof of Theorem 5. From Theorem 1, there exists $\alpha_2 \in \mathcal{K}_\infty$ such that (12a) holds for all $x \in \mathcal{S} \subseteq \mathcal{X}_N$, where $\alpha_1 \in \mathcal{K}_\infty$ is from Assumption 4. By Proposition 12, there exist $\alpha_b, \alpha_\theta \in \mathcal{K}$ such that (42) and (43) hold for all $x \in \mathcal{S}$ and $\theta \in \mathbb{R}^{n_\theta}$. By Proposition 1, there exists $\delta > 0$ such that (21) holds for all $x \in \mathcal{S}$ and $|\theta| \leq \delta$, and $\mathcal{S}$ is RPI for $x^+ = f_c(x, \theta), |\theta| \leq \delta$. As in the proof of Proposition 1, we can combine (21), (42), and (43) to give (44) for all $x \in \mathcal{S}$ and $|\theta| \leq \delta$, which is the desired cost decrease bound with $\sigma := \alpha_b \circ \alpha_\theta \in \mathcal{K}$. Thus, part (a) is established, and part (b) follows by Theorem 3. □

Proof of Theorem 6. All the conditions of Theorems 2 and 5 are satisfied. Thus, there exists $c_2 > 0$ such that (13) holds for all $x \in \mathcal{S}$ with $c_1 := \underline{\sigma}(Q) > 0$. Moreover, we can substitute $\alpha_1(\cdot) := c_1|\cdot|^2$ and $\alpha_2(\cdot) := c_2|\cdot|^2$ into the proof of Theorem 5 to construct $\delta > 0$ and $\sigma \in \mathcal{K}$ such that (22) holds for all $x \in \mathcal{S}$ and $|\theta| \leq \delta$. Therefore, by Theorem 3, $x^+ = f_c(x, \theta), |\theta| \leq \delta$ is ISES in $\mathcal{S}$. □

D Proofs of strong stability results

In this appendix we prove strong stability results from Section 4.

D.1 Quadratic cost MPC

We first consider results pertaining to strong stability of the quadratic cost MPC (Propositions 3, 4 and 7). Note that several preliminary results are required.

Proposition 13. Suppose Assumptions 1 to 3 and 5 hold. Let $\rho > 0$ and $\mathcal{S} := \text{lev}_\rho V_N^0$. There exist $c_x, c_u > 0$ such that

$$|\hat{x}^0(k;x)| \leq c_x|x|, \qquad \forall\, x \in \mathcal{S},\ k \in \mathbb{I}_{0:N}. \tag{45}$$

$$|u^0(k;x)| \leq c_u|x|, \qquad \forall\, x \in \mathcal{S},\ k \in \mathbb{I}_{0:N-1}. \tag{46}$$

Proof. By Theorem 6, we have the upper bound (22a) for all $x \in \mathcal{S}$ and some $c_2 > 0$. Moreover, since Q, R, P_f are positive definite, we can write, for each $x \in \mathcal{S}$ and $k \in \mathbb{I}_{0:N-1}$,

$$\begin{aligned}
\underline{\sigma}(Q)|\hat{x}^0(k;x)|^2 \leq |\hat{x}^0(k;x)|_Q^2 \leq V_N^0(x) \leq c_2|x|^2 \\
\underline{\sigma}(P_f)|\hat{x}^0(N;x)|^2 \leq |\hat{x}^0(N;x)|_{P_f}^2 \leq V_N^0(x) \leq c_2|x|^2 \\
\underline{\sigma}(R)|u^0(k;x)|^2 \leq |u^0(k;x)|_R^2 \leq V_N^0(x) \leq c_2|x|^2.
\end{aligned}$$

Thus, with $c_x := \max\{\sqrt{c_2/\underline{\sigma}(Q)}, \sqrt{c_2/\underline{\sigma}(P_f)}\}$ and $c_u := \sqrt{c_2/\underline{\sigma}(R)}$, we have (45) and (46). $\square$

Proof of Proposition 3. Let $z := (x, u)$. By Proposition 11, for each $i \in \mathbb{I}_{1:n}$, there exists $\sigma_i \in \mathcal{K}_\infty$ such that

$$\left|\frac{\partial f_i}{\partial z}(z,\theta) - \frac{\partial \hat{f}_i}{\partial z}(\tilde{z})\right| \leq \sigma_i(|(z - \tilde{z}, \theta)|) \tag{47}$$

for all $z, \tilde{z} \in \mathcal{S} \times \mathbb{U}$ and $\theta \in \mathbb{R}^{n_\theta}$. Next, let $\mathcal{Z}$ denote the convex hull of $\mathcal{S} \times \mathbb{U}$. Then $tz \in \mathcal{Z}$ for all $t \in [0,1]$ and $z \in \mathcal{Z}$. By Taylor's theorem (Apostol, 1974, Thm. 12.14), for each $i \in \mathbb{I}_{1:n}$ and $(z,\theta) \in \mathcal{Z} \times \Theta$, there exists $t_i(z,\theta) \in (0,1)$ such that

$$f_i(z,\theta) - \hat{f}_i(z) = \left(\frac{\partial f_i}{\partial z}(t_i(z,\theta)z,\theta) - \frac{\partial \hat{f}_i}{\partial z}(t_i(z,\theta)z)\right) z. \tag{48}$$

Combining (47) and (48) gives, for each $(z,\theta) \in \mathcal{S} \times \mathbb{U} \times \mathbb{R}^{n_\theta}$,

$$|f(z,\theta) - \hat{f}(z)| \leq \sum_{i=1}^n |f_i(z,\theta) - \hat{f}_i(z)| \leq \sum_{i=1}^n \sigma_i(|\theta|)|z|$$

and therefore (27) holds with $\sigma_f := \sum_{i=1}^n \sigma_i$. $\square$

Proof of Proposition 4. By Proposition 13, there exists $c_u > 0$ such that $|\kappa_N(x)| = |u^0(0;x)| \leq c_u|x|$ for all $x \in \mathcal{S}$. Moreover, by Proposition 3, there exists $\sigma_f \in \mathcal{K}_\infty$ such that

$$\begin{aligned}
|f_c(x,\theta) - \hat{f}_c(x)| &\leq \sigma_f(|\theta|)|(x,\kappa_N(x))| \leq \sigma_f(|\theta|)(|x| + |\kappa_N(x)|) \\
&\leq \sigma_f(|\theta|)(|x| + c_u|x|) = \tilde{\sigma}_f(|\theta|)|x|
\end{aligned}$$

for all $x \in \mathcal{S}$ and $\theta \in \mathbb{R}^{n_\theta}$, where $\tilde{\sigma}_f := \sigma_f(1 + c_u) \in \mathcal{K}_\infty$. $\square$

Proposition 14. Suppose Assumptions 1 to 3, 5 and 6 hold and assume $\hat{f}$ is Lipschitz continuous on bounded sets. Let $\rho > 0$, $\mathcal{S} := \mathrm{lev}_\rho V_N^0$, and $\Theta \subseteq \mathbb{R}^{n_\theta}$ be compact. There exist $c_{b,1}, c_{b,2} > 0$ such that, for each $x \in \mathcal{S}$ and $\theta \in \Theta$,

$$|V_N(x^+, \tilde{\mathbf{u}}(x)) - V_N(\hat{x}^+, \tilde{\mathbf{u}}(x))| \leq 2c_{b,1}|x||x^+ - \hat{x}^+| + c_{b,2}|x^+ - \hat{x}^+|^2 \tag{49}$$

where $\hat{x}^+ := \hat{f}_c(x)$ and $x^+ := f_c(x, \theta)$.

Proof. First, we seek to prove the following bound on the incurred terminal penalty $\tilde{V}_f(\cdot,\cdot) := V_f(\hat{\phi}(N;\cdot,\cdot))$: for each $x \in \mathcal{S}$ and $\theta \in \Theta$,

$$|\tilde{V}_f(x^+, \tilde{\mathbf{u}}(x)) - \tilde{V}_f(\hat{x}^+, \tilde{\mathbf{u}}(x))| \leq c_{a,1}|x||x^+ - \hat{x}^+| + c_{a,2}|x^+ - \hat{x}^+|^2 \tag{50}$$

where $x^+ := f_c(x, \theta)$ and $\hat{x}^+ := \hat{f}_c(x)$.

Using the identity $|y|_M^2 - |\hat{y}|_M^2 = |y - \hat{y}|_M^2 + 2(y - \hat{y})^\top M\hat{y}$ for any positive definite M and $y, \hat{y}$ of appropriate dimensions, we have, for each $x \in \mathcal{S}$ and $\theta \in \Theta$,

$$\begin{aligned}\tilde{V}_f(x^+, \tilde{\mathbf{u}}(x)) - \tilde{V}_f(\hat{x}^+, \tilde{\mathbf{u}}(x)) &= |\hat{\phi}(N; x^+, \tilde{\mathbf{u}}(x)) - \hat{\phi}(N; \hat{x}^+, \tilde{\mathbf{u}}(x))|_{P_f}^2 \\ &\quad + 2(\hat{\phi}(N; x^+, \tilde{\mathbf{u}}(x)) - \hat{\phi}(N; \hat{x}^+, \tilde{\mathbf{u}}(x)))^\top P_f \hat{\phi}(N; \hat{x}^+, \tilde{\mathbf{u}}(x)).\end{aligned} \tag{51}$$

where $x^+ := f_c(x, \theta)$ and $\hat{x}^+ := \hat{f}_c(x)$. By Proposition 13, there exists $c_x > 0$ such that $|\hat{x}^0(k; x)| \leq c_x|x|$ and therefore

$$|\hat{\phi}(k; \hat{f}_c(x), \tilde{\mathbf{u}}(x))| = |\hat{x}^0(k+1; x)| \leq c_x|x| \tag{52}$$

for each $k \in \mathbb{I}_{0:N-1}$ and $x \in \mathcal{S}$. By Assumptions 3 and 5, we have, for each $x \in \mathbb{X}_f$,

$$\underline{\sigma}(P_f)|\hat{f}(x, \kappa_f(x))|^2 \leq V_f(\hat{f}(x, \kappa_f(x))) \leq V_f(x) - \underline{\sigma}(Q)|x|^2 \leq [\overline{\sigma}(P_f) - \underline{\sigma}(Q)]|x|^2$$

and therefore

$$|\hat{f}(x, \kappa_f(x))| \leq \gamma_f|x|$$

where $\gamma_f := \sqrt{[\overline{\sigma}(P_f) - \underline{\sigma}(Q)]/\underline{\sigma}(P_f)}$. Then, since $\hat{x}^0(N; x) \in \mathbb{X}_f$ and $\mathbb{X}_f$ is positively invariant for $x^+ = \hat{f}(x, \kappa_f(x))$, we have

$$|\hat{\phi}(N; \hat{f}_c(x), \tilde{\mathbf{u}}(x))| = |\hat{f}(\hat{x}^0(N; x), u^0(N; x))| \leq \gamma_f|\hat{x}^0(N; x)| \leq \gamma_f c_x|x| \tag{53}$$

for each $x \in \mathcal{S}$. Since $(\mathcal{S}, \mathbb{U}, \Theta)$ are each bounded and f is continuous, $\mathcal{S}_0 := f(\mathcal{S}, \mathbb{U}, \Theta)$ is bounded. But this means $\mathcal{S}_{k+1} := \hat{f}(\mathcal{S}_k, \mathbb{U})$ is bounded for each $k \in \mathbb{I}_{\geq 0}$ (by induction), so $\overline{\mathcal{S}} := \bigcup_{k=0}^N \mathcal{S}_k$ is also bounded. Since $\hat{f}$ is Lipschitz continuous on bounded sets, there exists $L_f > 0$ such that $|\hat{f}(x, u) - \hat{f}(\tilde{x}, \tilde{u})| \leq L_f|(x - \tilde{x}, u - \tilde{u})|$ for all $x, \tilde{x} \in \overline{\mathcal{S}}$ and $u, \tilde{u} \in \mathbb{U}$. Then, for each $\theta \in \Theta$, we have

$$\begin{aligned}&|\hat{\phi}(k+1; x^+, \tilde{\mathbf{u}}(x)) - \hat{\phi}(k+1; \hat{x}^+, \tilde{\mathbf{u}}(x))| \\ &\qquad = |\hat{f}(\hat{\phi}(k; x^+, \tilde{\mathbf{u}}(x)), u^0(k; x)) - \hat{f}(\hat{\phi}(k; \hat{x}^+, \tilde{\mathbf{u}}(x)), u^0(k; x))| \\ &\qquad \leq L_f|\hat{\phi}(k; x^+, \tilde{\mathbf{u}}(x)) - \hat{\phi}(k; \hat{x}^+, \tilde{\mathbf{u}}(x))|\end{aligned}$$

for each $k \in \mathbb{I}_{0:N-1}$, and therefore

$$|\hat{\phi}(k; x^+, \tilde{\mathbf{u}}(x)) - \hat{\phi}(k; \hat{x}^+, \tilde{\mathbf{u}}(x))| \leq L_f^k |x^+ - \hat{x}^+|, \tag{54}$$

for each $k \in \mathbb{I}_{0:N}$, where $\hat{x}^+ := \hat{f}_c(x)$ and $x^+ := f_c(x, \theta)$. Finally, combining (51), (53), and (54), we have (50) for all $x \in \mathcal{S}$ and $\theta \in \Theta$, where $c_{a,1} := 2L_f^N \gamma_f c_x \overline{\sigma}(P_f)$ and $c_{a,2} := L_f^{2N} \overline{\sigma}(P_f)$.

Moving on to the proof of (49), we have, for each $x \in \mathcal{S}$ and $\theta \in \Theta$,

$$\begin{aligned} V_N(x^+, \tilde{\mathbf{u}}(x)) - V_N(N; \hat{x}^+, \tilde{\mathbf{u}}(x)) = \sum_{k=0}^{N-1} &|\hat{\phi}(k; x^+, \tilde{\mathbf{u}}(x)) - \hat{\phi}(k; \hat{x}^+, \tilde{\mathbf{u}}(x))|_Q^2 \\ &+ 2(\hat{\phi}(k; x^+, \tilde{\mathbf{u}}(x)) - \hat{\phi}(k; \hat{x}^+, \tilde{\mathbf{u}}(x)))^\top Q \hat{\phi}(k; \hat{x}^+, \tilde{\mathbf{u}}(x)) \\ &+ \tilde{V}_f(x^+, \tilde{\mathbf{u}}(x)) - \tilde{V}_f(\hat{x}^+, \tilde{\mathbf{u}}(x)) \end{aligned} \tag{55}$$

where $\hat{x}^+ := \hat{f}_c(x)$ and $x^+ := f_c(x, \theta)$, and combining (50), (52), (54), and (55), we have (49) with $c_{b,1} := c_{a,1} + 2\overline{\sigma}(Q) \sum_{k=0}^{N-1} L_f^k c_x$ and $c_{b,2} := c_{a,2} + \overline{\sigma}(Q) \sum_{k=0}^{N-1} L_f^{2k}$. □

Proof of Proposition 7. By Proposition 3, there exists $\tilde{\sigma}_f \in \mathcal{K}_\infty$ such that (28) for all $x \in \mathcal{S}$. Moreover, by Proposition 14, there exist $c_{b,1}, c_{b,2} > 0$ such that (49) for all $x \in \mathcal{S}$ and $\theta \in \mathbb{R}^{n_\theta}$, where $x^+ := f_c(x, \theta)$ and $\hat{f}_c(x)$. Finally, (28) and (49) imply (31) for all $x \in \mathcal{S}$ and $\theta \in \mathbb{R}^{n_\theta}$, where $\sigma_V(\cdot) := c_{b,1}\tilde{\sigma}_f(\cdot) + c_{b,2}[\tilde{\sigma}_f(\cdot)]^2 \in \mathcal{K}_\infty$. □

D.2 General nonlinear MPC

Next, we move on to the general nonlinear MPC results (Propositions 2, 5, 6 and 8). Again, several preliminary results are required.

Proposition 15. For each $\alpha \in \mathcal{K}$ and $\gamma \in \mathcal{K}^2$, let $\gamma_1(s,t) := \alpha(\gamma(s,t))$, $\gamma_2(s,t) := \gamma(\alpha(s), t)$, and $\gamma_3(s,t) := \gamma(s, \alpha(t))$ for each $s, t \geq 0$. Then $\gamma_1, \gamma_2, \gamma_3 \in \mathcal{K}^2$.

Proof. This fact follows directly from the closure of $\mathcal{K}$ under composition (Kellett, 2014). For example, for each $s \geq 0$, we have $\gamma_2(\cdot, s) = \gamma(\alpha(\cdot), s) \in \mathcal{K}$ by closure under composition, $\gamma_2(s, \cdot) = \gamma(\alpha(s), \cdot) \in \mathcal{K}$ trivially, and γ_2 is continuous as it is a composition of continuous functions. □

Proof of Proposition 5. Without loss of generality, assume $\mathcal{S}$ and Θ contain the origin. By assumption, $C := \mathcal{S} \times \mathbb{U} \times \Theta$ is compact, and by Proposition 11, there exists $\sigma_f \in \mathcal{K}_\infty$ such that

$$|f(x, u, \theta) - f(\tilde{x}, \tilde{u}, \tilde{\theta})| \leq \sigma_f(|(x, u, \theta) - (\tilde{x}, \tilde{u}, \tilde{\theta})|) \tag{56}$$

for all $(x, u, \theta), (\tilde{x}, \tilde{u}, \tilde{\theta}) \in C$. Specializing (56) to $(\tilde{x}, \tilde{u}, \tilde{\theta}) = (x, u, 0) \in C$ gives

$$|f(x, u, \theta) - \hat{f}(x, u)| \leq \sigma_f(|\theta|) \tag{57}$$

for all $(x, u, \theta) \in C$. On the other hand, specializing (56) to $(\tilde{x}, \tilde{u}, \tilde{\theta}) = (0, 0, \theta) \in C$ gives

$$|f(x, u, \theta)| = |f(x, u, \theta) - f(0, 0, \theta)| \leq \sigma_f(|(x, u)|)$$

and therefore

$$|f(x,u,\theta) - \hat{f}(x,u)| \leq |f(x,u,\theta)| + |\hat{f}(x,u)| \leq 2\sigma_f(|(x,u)|) \tag{58}$$

for all $(x,u,\theta) \in C$. Combining (57) and (58) gives

$$|f(x,u,\theta) - \hat{f}(x,u)| \leq \min\{2\sigma_f(|(x,u)|), \sigma_f(|\theta|)\}$$

for all $(x,u,\theta) \in C$, which is an upper bound that is clearly continuous, nondecreasing in each $|x|$ and $|\theta|$, and zero if either $|x|$ or $|\theta|$ is zero. To make the upper bound strictly increasing, pick any $\sigma_1, \sigma_2 \in \mathcal{K}$ and let $\gamma_f(s,t) := \min\{2\sigma_f(s), \sigma_f(t)\} + \sigma_1(s)\sigma_2(t)$ for each $s,t \geq 0$. Then $\gamma_f \in \mathcal{K}^2$, and (29) holds for all $(x,u,\theta) \in C$. □

Proof of Proposition 6. First, we have $\gamma_f \in \mathcal{K}^2$ satisfying (29) for all $x \in \mathcal{S}$, $u \in \mathbb{U}$, and $\theta \in \Theta$ by Proposition 5. Using the bounds (9) and (20a) with $u = \kappa_N(x)$, we have, for each $x \in \mathcal{X}_N$,

$$\alpha_1(|\kappa_N(x)|) \leq \ell(x, \kappa_N(x)) \leq V_N^0(x) \leq \alpha_2(|x|)$$

and thus $|\kappa_N(x)| \leq \alpha_\kappa(|x|)$, where $\alpha_\kappa := \alpha_1^{-1} \circ \alpha_2 \in \mathcal{K}_\infty$. Then, for each $x \in \mathcal{S}$ and $\theta \in \Theta$,

$$\begin{aligned} |f_c(x,\theta) - \hat{f}_c(x)| &\leq \gamma_f(|(x,\kappa_N(x))|, |\theta|) \\ &\leq \gamma_f(|x| + |\kappa_N(x)|, |\theta|) \\ &\leq \gamma_f(|x| + \alpha_\kappa(|x|), |\theta|) = \tilde{\gamma}_f(|x|, |\theta|). \end{aligned}$$

where $\tilde{\gamma}_f(s,t) := \gamma_f(s + \alpha_\kappa(s), t)$ for each $s,t \geq 0$. Then $(\cdot) + \alpha_\kappa(\cdot) \in \mathcal{K}_\infty$, and $\tilde{\gamma}_f \in \mathcal{K}^2$ by Proposition 15. Finally, (30) holds for all $x \in \mathcal{S}$ and $\theta \in \Theta$. □

Proof of Proposition 8. By Proposition 11, there exists $\alpha_b \in \mathcal{K}_\infty$ such that

$$V_N(x_1, \mathbf{u}_1) - V_N(x_2, \mathbf{u}_2) \leq \alpha_b(|(x_1 - x_2, \mathbf{u}_2 - \mathbf{u}_2)|) \tag{59}$$

for all $(x, \mathbf{u}), (\tilde{x}, \tilde{\mathbf{u}}) \in f(\mathcal{S}, \mathbb{U}, \Theta) \times \mathbb{U}^N$. Specializing (59) to $x_1 = x^+ := f_c(x,\theta)$, $x_2 = \hat{x}^+ := f_c(x)$, and $\mathbf{u}_1 = \mathbf{u}_2 = \tilde{\mathbf{u}}(x)$ gives

$$|V_N(x^+, \tilde{\mathbf{u}}(x)) - V_N(\hat{x}^+, \tilde{\mathbf{u}}(x))| \leq \alpha_b(|x^+ - \hat{x}^+|) \tag{60}$$

for each $x \in \mathcal{S}$ and $\theta \in \Theta$. By Proposition 6 there exists $\tilde{\gamma}_f \in \mathcal{K}^2$ satisfying (30) for all $x \in \mathcal{S}$ and $\theta \in \Theta$. Finally, combining (30) and (60) gives (32) with $\gamma_V(s,t) := \alpha_b(\tilde{\gamma}_f(s,t))$ for all $s,t \geq 0$, where $\gamma_V \in \mathcal{K}^2$ by Proposition 15. □

Proof of Proposition 2. Let

$$\tilde{\gamma}(s,t) := \sup_{\tilde{s} \in (0,s)} \frac{\gamma(\tilde{s},t)}{\alpha(\tilde{s})}$$

for each $s,t > 0$, so that

$$L := \limsup_{s \to 0^+} \frac{\gamma(s,\tau)}{\alpha(s)} = \lim_{s \to 0^+} \tilde{\gamma}(s,\tau).$$

Suppose $L < 1$. Then there exists $\delta_0 > 0$ such that $|\tilde{\gamma}(s,\tau) - L| < 1 - L$ for all $s \in (0, \delta_0]$. But $\tilde{\gamma}(s,t) \geq 0$ and $L \geq 0$ for all $s, t > 0$, so $\tilde{\gamma}(s,\tau) < 1$ for all $s \in (0,\delta_0]$ by the reverse triangle inequality. Therefore

$$\frac{\gamma(s,t)}{\alpha(s)} \leq \frac{\gamma(s,\tau)}{\alpha(s)} \leq \tilde{\gamma}(s,\tau) < 1$$

and $\gamma(s,t) < \alpha(s)$ for all $s \in (0,\delta_0]$ and $t \in [0,\tau]$.

If $\delta_0 \geq \rho$, the proof is complete with $\delta := \tau$. Otherwise, we must enlarge the interval in s by shrinking the interval in t. For each $t \in (0,\tau]$, let

$$\gamma_0(t) := \inf \{ s > 0 \mid \gamma(s,t) \geq \alpha(s) \} .$$

Since $\gamma(s,t) \leq \gamma(s,\tau) < \alpha(s)$ for each $s \in (0,\delta_0]$ and $t \in [0,\tau]$, we have $\gamma_0(t) > 0$. Then, by continuity of α and γ, $\gamma_0(t)$ must be equal to the first nonzero point of intersection if it exists. Otherwise $\gamma_0(t)$ is infinite. Note that γ_0 is a strictly decreasing function since, for any $t \in (0,\tau]$, we have $\gamma(\gamma_0(t), t') < \gamma(\gamma_0(t), t) = \alpha(\gamma_0(t))$ for all $t' \in (0,t)$. Moreover, $\lim_{t\to 0^+} \gamma_0(t) = \infty$ since, if γ_0 was upper bounded by some $\overline{\gamma} > 0$, we could take $\gamma(\overline{\gamma}, t) \geq \alpha(\overline{\gamma}) > 0$ for all $t \in (0,\tau]$, a contradiction of the fact that $\gamma(s,\cdot) \in \mathcal{K}$ for all $s > 0$. Then there must exist $\delta > 0$ such that $\gamma_0(\delta) > \rho$ and therefore $\gamma(s,t) < \alpha(s)$ for all $s \in (0,\rho]$ and $t \in [0,\delta]$. ☐

E Examples

E.1 Strong asymptotic stability counterexample

Consider the plant (33) and MPC defined in Section 6.1. We aim to show the closed-loop system $x^+ = f(x, \kappa_1(x), \theta), |\theta| \leq \delta$ is RES with $\delta = 3$, but not inherently strongly stabilizing for any $\delta > 0$. By Lipschitz continuity of x^2 on bounded sets and 1/2-Hölder continuity of $\sqrt{|x|}$,

$$|x^2 - y^2| \leq 4|x-y|, \qquad \forall\, x,y \in [-2,2], \tag{61}$$

$$|\sigma(x) - \sigma(y)| \leq 2\sqrt{|x-y|}, \qquad \forall\, x,y \in \mathbb{R}. \tag{62}$$

To show (61), note that, for each $\delta > 0$, we have

$$|x^2 - y^2| = |x+y||x-y| \leq 2\delta|x-y|$$

for all $x, y \in [-\delta, \delta]$, and take $\delta = 2$ to give (61). For (62), we first show $\sqrt{(\cdot)}$ is 1/2-Hölder continuous on $\mathbb{R}_{\geq 0}$:

$$|\sqrt{x} - \sqrt{y}| = \frac{|x-y|}{\sqrt{x}+\sqrt{y}} \leq \frac{|x-y|}{\sqrt{x}+\sqrt{y}} = \sqrt{|x-y|}\frac{\sqrt{|x-y|}}{\sqrt{x}+\sqrt{y}} \leq \sqrt{|x-y|}$$

for all $x, y \geq 0$, where the last inequality follows by the triangle inequality. Then we automatically get $|\sigma(x) - \sigma(y)| \leq \sqrt{|x-y|}$ if $x, y \geq 0$. On the other hand, if $x \geq 0$ and $y \leq 0$, we have

$$|\sigma(x) - \sigma(y)| = |\sqrt{x} + \sqrt{y}| \leq \sqrt{x} + \sqrt{-y} \leq 2\sqrt{x-y}.$$

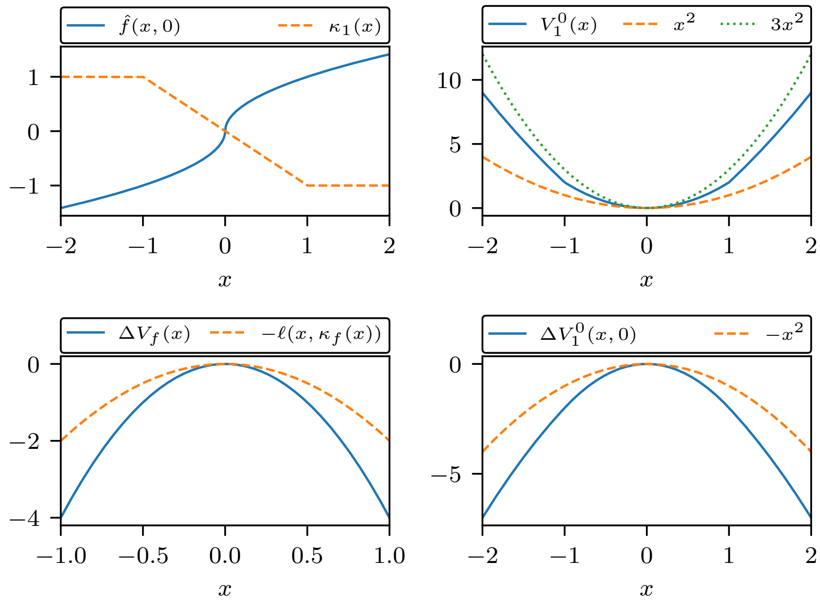


Figure 8: For the MPC of (33), plots of (top left) the open-loop dynamics and control law, (bottom left) the terminal cost difference, (top right) the optimal value function, and (bottom right) the cost difference, each with the relevant (nominal) bounds from Assumption 3 and (22).

Finally, flipping the signs of the prior arguments gives (62).

First, we derive the control law. The terminal set can be reached in a single move if and only if $|x| \leq 2$, so we have the steerable set $\mathcal{X}_1 = [-2, 2]$. Consider the problem *without* the terminal constraint. The objective is

$$V_1(x, u) = x^2 + u^2 + 4|x + u|$$

which is increasing in u if $x > 1$ and $|u| \leq 1$, and decreasing in u if $x < -1$ and $|u| \leq 1$. Thus $V_1(x, \cdot)$ is minimized (over $|u| \leq 1$) by $\mathbf{u}^0(x) = -\text{sgn}(x)$ for all $x \notin [-1, 1]$. On the other hand, if $|x| \leq 1$, then $V_1(x, \cdot)$ is decreasing on $[-1, -x)$ and increasing on $(-x, 1]$. Thus $V_1(x, \cdot)$ is minimized (over $|u| \leq 1$) by $\mathbf{u}^0(x) = -x$ so long as $|x| \leq 1$. In summary, we have the control law $\kappa_1(x) := -\text{sat}(x)$. But

$$|\hat{f}(x, \kappa_1(x))| = \begin{cases} 0, & |x| \leq 1 \\ |x - \text{sgn}(x)| = |x| - 1, & 0 < |x| \leq 2 \end{cases}$$

so $u = \kappa_1(x)$ drives each state in $\mathcal{X}_1 = [-2, 2]$ to the terminal constraint $\mathbb{X}_f = [-1, 1]$. Therefore κ_1 is also the control law of the problem *with* the terminal constraint. The control law κ_1 is plotted, along with the unforced dynamics $\hat{f}(\cdot, 0)$, against $x \in \mathcal{X}_1$ in Figure 8 (top left).

Assumptions 1 and 4 are satisfied by definition, Assumption 2 is satisfied with $c_f := 8$, and Assumption 3 is satisfied with $\kappa_f(x) := -x$ since $\hat{f}(x, \kappa_f(x)) = 0$ and

$$\Delta V_f(x) := V_f(\hat{f}(x, \kappa_f(x))) - V_f(x) = -4x^2 \leq -2x^2 = -\ell(x, \kappa_f(x))$$

for all $x \in \mathbb{X}_f$. See Figure 8 (bottom left) for plots of ΔV_f and $-\ell(\cdot, \kappa_f(\cdot))$. Therefore, by Theorem 5, the closed-loop system $x^+ = f(x, \kappa_1(x), \theta), |\theta| \leq \delta$ is RAS on $\mathcal{X}_1 = [-2, 2]$ with ISS Lyapunov function V_1^0 for some $\delta > 0$. Our next goal is to find such a $\delta > 0$.

First, however, let us establish that V_1^0 is a Lyapunov function for the modeled closed-loop $x^+ = \hat{f}(x, \kappa_1(x))$ in $\mathcal{X}_1 = [-2, 2]$. We already have $V_1^0(x) \geq x^2$ for all $|x| \leq 2$. For the upper bound, we have

$$V_1^0(x) = V_1(x, \kappa_1(x)) = \begin{cases} 2x^2, & |x| \leq 1, \\ x^2 + 4|x| - 3, & 1 < |x| \leq 2 \end{cases}$$

for each $|x| \leq 2$. But the polynomials $-2x^2 \pm 4x - 3$ have no real roots, so $4|x| - 3 < 2x^2$, and the above inequality gives $V_1^0(x) \leq 3x^2$ for all $|x| \leq 2$. Moreover, by (40), we have $\Delta V_1^0(x, \theta) \leq -x^2$, so $x^+ = \hat{f}(x, \kappa_1(x))$ is in fact *exponentially* stable on $\mathcal{X}_1 = [-2, 2]$. We plot V_1^0 and $\Delta V_1^0(\cdot, 0) := V_1^0(\hat{f}(\cdot, \kappa_1(\cdot))) - V_1^0(\cdot)$, along with their exponential Lyapunov bounds, in Figure 8 (right).

For robust positive invariance, let $|x| \leq 2$, $\theta \in \mathbb{R}$, $x^+ := f(x, \kappa_1(x), \theta)$, $\hat{x}^+ := \hat{f}(x, \kappa_1(x))$ and note that

$$x^+ = \sigma(\sigma^{-1}(\hat{x}^+) - \theta \text{sat}(x))$$

where $\sigma^{-1}(x) = \text{sgn}(x)|x|^2$, and therefore

$$|x^+| \leq \sqrt{|\hat{x}^+|^2 + |\theta||\text{sat}(x)|} \leq \sqrt{1 + |\theta|}.$$

Then $|x^+| \leq 2$ so long as $|\delta| \leq 3$, so $\mathcal{X}_1 = [-2, 2]$ is RPI for $x^+ = f(x, \kappa_1(x), \theta), |\theta| \leq 3$.

By continuity of f, V_1^0, and κ_1 and Proposition 11, there exists $\sigma \in \mathcal{K}_\infty$ such that $|V_1^0(x^+) - V_1^0(\hat{x}^+)| \leq \sigma(|\theta|)$ and therefore $V_1^0(x^+) \leq V_1^0(\hat{x}^+) + |V_1^0(x^+) - V_1^0(\hat{x}^+)| \leq V_1^0(x) - x^2 + \sigma(|\theta|)$ for all $|x| \leq 2$ and $|\theta| \leq 3$, where $x^+ := f(x, \kappa_1(x), \theta)$ and $\hat{x}^+ := \hat{f}(x, \kappa_1(x))$. Therefore $x^+ = f(x, \kappa_1(x), \theta), |\theta| \leq 3$ is not only RAS, but *RES* on $\mathcal{X}_1$ by Theorem 3.

We now aim to show strong stability is *not* achieved. For simplicity, we consider $\mathcal{S} := \text{lev}_2 V_1^0 = [-1, 1] = \mathbb{X}_f$ as the candidate basin of attraction. Let $|x| \leq 1$, $|\theta| \leq 3$, $x^+ := f(x, \kappa_1(x), \theta)$, and $\hat{x}^+ := \hat{f}(x, \kappa_1(x))$. Moreover, $\ell(x, \kappa_1(x)) \geq 2|x|^2 =: \alpha_3(|x|)$. Next, we have $\kappa_1(x) = -x$, $x^+ = \sigma(x\theta)$, and $\hat{x}^+ = 0$. Therefore

$$\begin{aligned} |V_1(x^+, \tilde{\mathbf{u}}(x)) - V_1(\hat{x}^+, \tilde{\mathbf{u}}(x))| &= |(x^+)^2 + 4|x^+|| \leq |x^+|^2 + 4|x^+| \\ &\leq |x||\theta| + 4\sqrt{|x||\theta|} =: \gamma_V(|x|, |\theta|) \end{aligned}$$

where $\gamma_V \in \mathcal{K}^2$. For each $t > 0$, we have $\frac{\gamma_V(s,t)}{\alpha_3(s)} = (st + 4\sqrt{st})/(2s^2) = t/(2s) + 2\sqrt{t}/s^{3/2}$, so $\lim_{s \to 0^+} \frac{\gamma_V(s,t)}{\alpha_3(s)} = \infty$ for all $t > 0$, and (24) is not satisfied.

As mentioned in the main text, (24) is sufficient but not necessary. But the cost difference curve is positive definite, as

$$\Delta V_1^0(x,\theta) = 2[\sigma(\theta x)]^2 - 2x^2 = 2(|\theta| - |x|)|x| > 0$$

for any $0 < |x| < |\theta| \leq 1$. In other words, θ can be arbitrarily small but nonzero, and the cost difference curve will remain positive definite near the origin.

E.2 Nonlinearizable yet inherently strongly stabilizing

Consider the plant (34) and MPC defined in Section 6.2. We aim to show the closed-loop system $x^+ = f(x, \kappa_1(x), \theta), |\theta| \leq \delta$ is RES in $\mathcal{X}_1$ with $\delta = 1$, and SES with $\delta = 1/2$.

To derive the control law, we first consider the problem *without* the terminal constraint (i.e., $\mathbb{X}_f = \mathbb{R}$). We have the objective

$$V_1(x,u) = x^2 + u^2 + 4\left(x + (1/2)\gamma(x) + u\right)^2.$$

Taking the partial derivative in u,

$$\frac{\partial V_1}{\partial u}(x,u) = 8x + 4\gamma(x) + 10u$$

and setting that to zero gives the optimal input

$$\mathbf{u}^0(x) = -g(x) := -(4/5)x - (2/5)\gamma(x)$$

whenever $|g(x)| \leq 1$. Otherwise the solution saturates at $\mathbf{u}^0(x) = -\mathrm{sgn}(g(x))$, so we have $\mathbf{u}^0(x) = \kappa_1(x) := -\mathrm{sat}(g(x))$ for all $|x| \leq 2$.

To see where the control law $\kappa_1(x)$ saturates, first note

$$\frac{d^2 g}{dx^2}(x) = \frac{2}{5}\frac{d^2\gamma}{dx^2}(x) = -\frac{8\pi^2 \sin(2\pi/x)}{5|x|^3}$$

for all $x \neq 0$, so $g(x)$ is strictly concave on $x \in [1/(n-1/2), 1/n]$ and strictly convex on $x \in [1/n, 1/(n+1/2)]$ for each $n \in \mathbb{I}$. Therefore $g(x)$ achieves a local maximum on each $x \in [1/(n-1/2), 1/n]$, and the maximum is strictly decreasing with n. The last, and greatest, of these local maxima on $|x| \leq 2$ is achieved on $2/3 \leq x \leq 1$. Through numerical optimization, we find $\max_{0\leq x\leq 1} g(x) = \max_{2/3\leq x\leq 1} g(x) \approx 0.9849$. By strict convexity of $g(x)$ on $x \in [1,2]$, $g(1) = 4/5$, and $g(2) = 8/5$, we have $\max_{1\leq x\leq 2} g(x) = g(2) = 8/5$. Therefore $g(x)$ intersects the horizontal line at $u = 1$ exactly once over $x \in [-2,2]$, and it does so at some $x^* \in [1,2]$, which we can numerically verify is $x^* \approx 1.6989$. By symmetry, $g(x)$ intersects $u = -1$ at $-x^*$. Finally, because $g(x)$ is strictly convex (concave) on $[1,2]$ ($[-2,-1]$), it saturates on $(x^*, 2]$ (and $[-2, -x^*)$) and we have

$$\kappa_1(x) = \begin{cases} -(4/5)x - (2/5)\gamma(x), & |x| \leq x^*, \\ -\mathrm{sgn}(x), & x^* < |x| \leq 2. \end{cases}$$

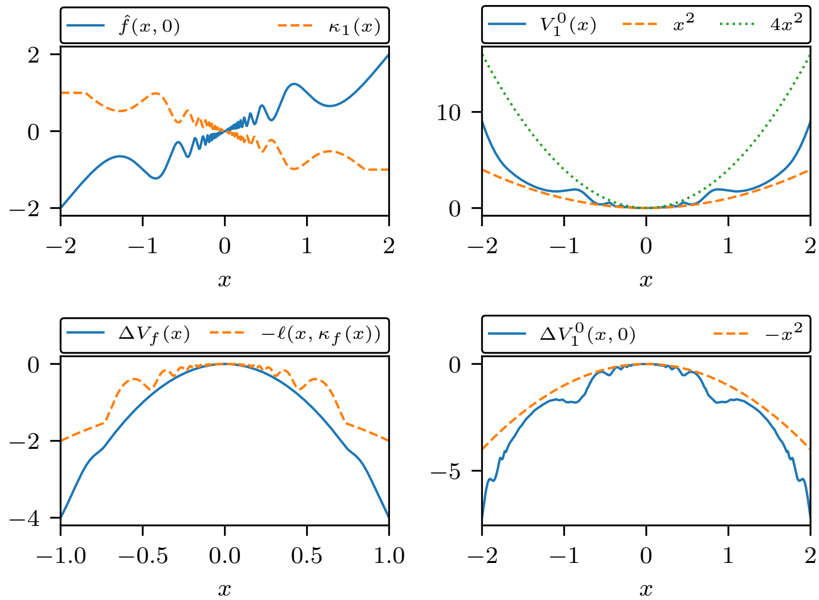


Figure 9: For the MPC of (34), we plot as a function of x (top left) the open-loop dynamics and control law, (bottom left) the terminal cost difference, (top right) the optimal value function, and (bottom right) the cost difference, each with the relevant (nominal) bounds from Assumption 3 and (22).

For the problem *with* the terminal constraint, we have

$$|\hat{f}(x,\kappa_1(x))| = |(1/5)x + (3/5)\gamma(x)| \leq (1/5)x + (3/5)|\gamma(x)| \leq 4/5$$

for each $x \in [0,1]$,

$$|\hat{f}(x,\kappa_1(x))| = |(1/5)x + (3/5)\gamma(x)| \leq |(1/5)x - (3/5)|\gamma(x)|| \leq (2/5)|x| \leq 4/5$$

for each $x \in [1,x^*]$, and

$$|\hat{f}(x,\kappa_1(x))| = |x + (1/2)\gamma(x) - 1| \leq |x| - 1 - (1/2)|\gamma(x)| \leq |x| - 1 \leq 1$$

for each $x \in [x^*,2]$, where we have used the fact that $\gamma(x) \leq 0$ for all $x \in [1,2]$. Therefore $|\hat{f}(x,\kappa_1(x))| \leq 1$ for all $x \in [0,2]$, and the same holds for all $x \in [-2,0]$ by symmetry. Therefore the terminal constraint $\mathbb{X}_f = [-1,1]$ is automatically satisfied by the unconstrained control law, so $\kappa_1(x)$ is also the control law for the MPC *with* the terminal constraint. In Figure 9 (top left), we plot κ_1 and $\hat{f}(\cdot,0)$ on $\mathcal{X}_1$.

Assumptions 1 and 5 are satisfied by definition, and Assumption 2 is satisfied with $c_f := 4$. Let $\kappa_f(x) := -(1/2)(x+\gamma(x))$ for all $|x| \leq 1$. Then

$$|\kappa_f(x)| \leq (1/2)(|x| + |\gamma(x)|) \leq |x| \leq 1$$

for all $|x| \leq 1$, so $u = \kappa_f(x)$ is feasible in the terminal constraint. Moreover, $\hat{f}(x, \kappa_f(x)) = (1/2)x$, so

$$\Delta V_f(x) := V_f(\hat{f}(x, \kappa_f(x))) - V_f(x) + \ell(x, \kappa_f(x)) = -2x^2 + |\kappa_f(x)|^2 \leq -x^2 \leq 0$$

and Assumption 3 is satisfied. See Figure 9 (bottom left) for plots of ΔV_f and $-\ell(\cdot, \kappa_f(\cdot))$. By Theorem 6, the closed-loop system $x^+ = f(x, \kappa_1(x), \theta), |\theta| \leq \delta$ is RES on $\mathcal{X}_1$ with the ISS Lyapunov function V_1^0 for some $\delta > 0$. Our next aim is to find such a $\delta > 0$.

Let $|x| \leq 2$, $\theta \in \mathbb{R}$, $x^+ := f(x, \kappa_1(x), \theta)$, and $\hat{x}^+ := \hat{f}(x, \kappa_1(x))$. Then $x^+ = \hat{x}^+ + \theta\kappa_1(x)$, and we have

$$|x^+| \leq |\hat{x}^+| + |\theta||\kappa_1(x)| \leq 1 + |\theta|$$

for all $\theta \in \mathbb{R}$. But this means $|x^+| \leq 2$ for all $|\theta| \leq 1$, so $\mathcal{X}_1$ is RPI for $x^+ = f(x, \kappa_1(x), \theta), |\theta| \leq 1$. Continuity of f, ℓ, V_f, and κ_1 implies continuity of $V_1^0(f_c(\cdot, \cdot))$, at least for all $|x| \leq 2$ and $|\theta| \leq 1$ on which the function is well-defined. Then, by Proposition 11, there exists $\sigma \in \mathcal{K}_\infty$ such that, if $|\theta| \leq 1$, we have $|V_1^0(x^+) - V_1^0(\hat{x}^+)| \leq \sigma(|\theta|)$, and therefore $V_1^0(x^+) \leq V_1^0(\hat{x}^+) + |V_1^0(x^+) - V_1^0(\hat{x}^+)| \leq V_1^0(x) - x^2 + \sigma(|\theta|)$. Finally, $x^+ = f(x, \kappa_1(x), \theta), |\theta| \leq 1$ is RES in $\mathcal{X}_1 = [-2, 2]$ by Theorem 3.

Next, we aim to show the MPC is inherently strongly stabilizing via Assumption 9 and Theorem 8. Consider the candidate Lyapunov function $V(x) := x^2$ for all $|x| \leq 2$ and $V(x) := \infty$ otherwise, and let $\rho \geq 4$, $\mathcal{S} := \text{lev}_\rho V = [-2, 2] = \mathcal{X}_1$, and $\delta_0 := 1$. If we can show Assumption 9(a,b) hold with these ingredients, then Assumption 9 will hold for all $\rho > 0$. Assumption 9(a) and (25a) are already satisfied with $|\theta| \leq \delta_0 = 1$, and $\mathcal{S}$ is RPI, but it remains to construct the bound (25b). Throughout this derivation, let $x^+ := f(x, \kappa_1(x), \theta)$ and $\hat{x}^+ := \hat{f}(x, \kappa_1(x))$.

First, suppose $|x| \leq x^*$ and $|\theta| \leq 1$. Then the controller does not saturate, i.e., $\kappa_1(x) = -0.8x - 0.4\gamma(x)$, and we have in the nominal case $\hat{x}^+ = 0.2x + 0.1\gamma(x)$, $|\hat{x}^+| \leq 0.3|x|$, and

$$V(\hat{x}^+) - V(x) = |\hat{x}^+|^2 - |x|^2 \leq -0.91|x|^2. \tag{63}$$

Next, consider the identity

$$y^2 - z^2 = 2z(y - z) + (y - z)^2 \tag{64}$$

for all $y, z \in \mathbb{R}$. We have $x^+ = (0.2 - 0.8\theta)x + (0.1 - 0.4\theta)\gamma(x)$, so $|x^+ - \hat{x}^+| = |0.8\theta x + 0.4\theta\gamma(x)| \leq 1.2|\theta||x|$, and (64) implies

$$|V(x^+) - V(\hat{x}^+)| \leq 0.72|\theta||x|^2 + 1.44|\theta|^2|x|^2. \tag{65}$$

Next, suppose $x^* < x \leq 2$ and $|\theta| \leq 1$. Then the controller always saturates, i.e., $\kappa_1(x) = -1$. Since $\gamma(\tilde{x}) \leq 0$ for all $1 \leq \tilde{x} \leq 2$, we have $0 \leq 0.5x + 0.5\gamma(x) \leq 0.5x \leq 1$ and $\hat{x}^+ = x + 0.5\gamma(x) - 1 \leq 0.5x$. Moreover, $x - 1 > x^* - 1 > 0$, so $\hat{x}^+ = x + 0.5\gamma(x) - 1 > 0.5\gamma(x) \geq -0.5x$. Then we have $|\hat{x}^+| \leq 0.5|x|$ and

$$V(\hat{x}^+) - V(x) = |\hat{x}^+|^2 - |x|^2 \leq -0.75|x|^2. \tag{66}$$

Moreover, $|x^+ - \hat{x}^+| = |\theta|$ and (64) implies

$$|V(x^+) - V(\hat{x}^+)| \leq (1/x^*)|\theta||x|^2 + (1/x^*)^2|\theta|^2|x|^2 \tag{67}$$

where we have used the fact that $|x|/x^* > 1$. By symmetry, (66) and (67) also hold for $-2 \leq x < -x^*$.

Combining (63) and (65)–(67), we have

$$V(x^+) \leq V(x) - a_3|x|^2 + \sigma_V(|\theta|)|x|^2$$

for all $x \in \mathcal{X}_N$, where $a_3 := 0.75$ and $\sigma_V(t) := \max\{\, 0.72t + 1.44t^2, (2/x^*)t + (1/x^*)^2t^2 \,\}$ and Assumption 9 is satisfied. Finally, by Theorem 8 (and its proof), the closed-loop system $x^+ = f(x, \kappa_1(x), \theta), |\theta| \leq \delta$ is SES in $\mathcal{X}_N = [-2, 2]$ for any $\delta \in (0, \sigma_V^{-1}(a_3))$. Thus, it suffices to take $|\theta| \leq \delta = 0.5$ since

$$\sigma_V(0.5) = \max\{\, 0.72, 0.3809\ldots \,\} = 0.72 < 0.75 = a_3.$$

E.3 Upright pendulum

Consider the plant (36) and MPC defined in Section 6.3. It is noted in the main text that Assumptions 1, 2 and 5 to 7 are automatically satisfied. To design P_f and show Assumption 3 holds, consider the linearization

$$x^+ = \underbrace{\begin{bmatrix} 1 & 0.1 \\ 0.1 & 1 \end{bmatrix}}_{=:A} x + \underbrace{\begin{bmatrix} 0 \\ 5 \end{bmatrix}}_{=:B} u \tag{68}$$

and the feedback gain $K := \begin{bmatrix} 2 & 2 \end{bmatrix}$, which stabilizes (68) because $A_K := A - BK = \begin{bmatrix} 1 & 0.1 \\ -0.9 & 0 \end{bmatrix}$ has eigenvalues of 0.9 and 0.1. Numerically solving the Lyapunov equation

$$A_K^\top P_f A_K - P_f = -2Q_K$$

where $Q_K := Q + K^\top RK = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$, we have a unique positive definite solution $P_f := \begin{bmatrix} 31.133\ldots & 10.196\ldots \\ 10.196\ldots & 10.311\ldots \end{bmatrix}$. Using the inequality $|\sin x_1 - x_1| \leq (1/6)|x_1|^3$ for all $x_1 \in \mathbb{R}$, we have

$$\begin{aligned} &|V_f(\hat{f}(x, -Kx)) - V_f(A_K x)| \\ &\quad = 2x^\top A_K^\top P_f \begin{bmatrix} 0 \\ \Delta(\sin x_1 - x_1) \end{bmatrix} + [P_f]_{22}\Delta^2(\sin x_1 - x_1)^2 \\ &\quad \leq b|x|^4 + a|x|^6 \end{aligned}$$

for all $x \in \mathbb{R}^2$, where $a := \frac{[P_f]_{22}\Delta^2}{36} = 2.8643\ldots \times 10^{-3}$ and $b := \frac{\Delta|A_K^\top P_f \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]|}{3} = 0.045675\ldots$. Moreover, $\underline{\sigma}(Q_K) = 1$, so

$$\begin{aligned} &V_f(\hat{f}(x, -Kx)) - V_f(x) + \ell(x, -Kx) \\ &\quad = |A_K x|_{P_f}^2 - |x|_{P_f}^2 + |x|_{Q_K}^2 + V_f(\hat{f}(x, -Kx)) - V_f(A_K x) \\ &\quad = -|x|_{Q_K}^2 + V_f(\hat{f}(x, -Kx)) - V_f(A_K x) \\ &\quad \leq -[1 - b|x|^2 - a|x|^4]|x|^2 \end{aligned}$$

for all $x \in \mathbb{R}^2$. The polynomial inside the brackets has roots at $x_* = -1.0231\ldots$ and $x^* = 0.9774\ldots$ and is positive in between. Recall $c_f := \underline{\sigma}(P_f)/8$. Then $\underline{\sigma}(P_f)|x|^2 \leq V_f(x) \leq c_f = \underline{\sigma}(P_f)/8$ implies $|x| \leq \frac{1}{2\sqrt{2}} < x^*$ and $|u| = |Kx| = 2(|x_1| + |x_2|) \leq 2\sqrt{2}|x| \leq 1$, so Assumption 3 is satisfied with $\kappa_f(x) := -Kx = -2x_1 - 2x_2$, and P_f and $\mathbb{X}_f$ as defined.

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