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Stability of Robust Minmax Control

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Abstract

This paper establishes stability properties for robust minmax model predictive control (RMPC) under stagewise bounded disturbances, addressing nonlinear discrete time systems where disturbances are constrained independently at each time step. Building on the input-to-state practical stability framework for minmax model predictive control (Limon, Alamo, Salas, and Camacho, 2006; Lazar, De La Pena, Heemels, and Alamo, 2008; Raimondo, Limon, Lazar, Magni, and ndez Camacho, 2009), we provide a self-contained worst-case stability analysis for the stagewise-bounded formulation with policy parametrization. We introduce worst-case input-to-state stability Lyapunov functions and prove that the RMPC optimal cost satisfies the required Lyapunov properties on the robust feasible set. The resulting stability guarantee is practical rather than asymptotic: the closed-loop system converges to a neighborhood of the origin whose size depends on the assumed disturbance bound. Crucially, because the controller is designed for the worst-case disturbance, even if the actual disturbance realization is zero the system may converge only to this neighborhood rather than to the origin itself. The analysis provides a parallel deterministic worst-case counterpart to recent stochastic model predictive control (SMPC) stability theory. A numerical example illustrates nonintuitive behavior arising from worst-case optimization incentives.

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1 Introduction

The disturbance attenuation regulator formulates robust control as a deterministic minmax game between controller and disturbance. In this sequential dynamic noncooperative zero-sum game, the disturbance optimizes first and the control optimizes second, with the control objective to maintain low cost despite any admissible bounded disturbance. Two constraint structures bound disturbances differently and produce distinct control strategies. First one, the signal bound disturbance attenuation regulator, constrains total energy through a single squared signal norm bound over all time steps. Second one, the stage bound disturbance attenuation regulator, independently constrains energy at each time step through stagewise squared norm bounds.

The stage bound formulation addresses a fundamental limitation of the signal bound approach. Aggregate signal bound constraints impose temporal coupling wherein the admissible disturbance at each stage depends on all past disturbance realizations through a remaining energy budget. This coupling treats the disturbance as an adversarial agent that strategically allocates energy from a fixed budget to maximize system cost. While this game-theoretic perspective provides theoretical insights, it is less representative of physical disturbances in practical applications. Actual disturbances arising from environmental conditions, measurement noise, or model uncertainty do not possess knowledge of past realizations nor strategically coordinate to deplete an energy budget. Stage bound constraints eliminate this artificial coupling by independently bounding disturbance magnitude at each stage, naturally accommodating persistent disturbances that act consistently over time or time-varying disturbances whose bounds change with operating conditions. Consequently, the stage bound formulation appears more suitable for designing controllers robust to the types of uncertain disturbances encountered in practical control applications.

When the stage bound disturbance attenuation problem is solved within a receding horizon framework with constant disturbance bounds at each stage, the resulting control scheme is equivalent to RMPC. At each time step, the controller solves a finite horizon minmax optimization over all possible disturbance sequences satisfying stagewise bound constraints, applies the first control action, and repeats this process as new state measurements become available. This receding horizon implementation provides a natural bridge between the classical game-theoretic disturbance attenuation literature and modern RMPC.

Literature Review RMPC for systems with bounded disturbances has been extensively studied in both linear and nonlinear settings. Early work on linear RMPC includes Kothare, Balakrishnan, and Morari (1996), who developed linear matrix inequality based offline designs for polytopic uncertainty, and Chisci, Rossiter, and Zappa (2001), who addressed constraint tightening for systems with persistent bounded disturbances and established convergence to robust positively invariant sets. Tube-based approaches for linear systems, developed by Langson, Chryssochoos, Raković, and Mayne (2004) and Mayne, Seron, and Raković (2005), guarantee robust constraint satisfaction and practical stability by maintaining trajectories within invariant tubes around nominal predictions.

For nonlinear systems, Magni, Raimondo, and Scattolini (2006) developed early RMPC frameworks using regional input-to-state stability (ISS) concepts and dynamic game formulations. Critically, Limon et al. (2006) provided a rigorous input-to-state practical stability (ISpS) analysis for RMPC with general bounded uncertainties, proving that if terminal ingredients yield an ISpS-Lyapunov function, then the RMPC value function is also an ISpS-Lyapunov function and the closed loop achieves input-to-state practical stability. Lazar et al. (2008) further developed ISS/ISpS conditions for minmax nonlinear MPC, deriving explicit state bounds and dual-mode conditions for asymptotic stability. Raimondo et al. (2009) provided a comprehensive survey unifying the ISS/ISpS framework for RMPC, explicitly articulating that standard minmax formulations with persistent bounded disturbances guarantee only practical stability rather than asymptotic convergence to the origin. More recent developments include He, Ji, and Yu (2013) for continuous-time systems and Sasfi, Zeilinger, and Köhler (2023) using control contraction metrics for adaptive RMPC.

For SMPC, McAllister and Rawlings (2023) established that nonlinear SMPC renders the origin robustly asymptotically stable in expectation under basic regularity assumptions. Their analysis introduced stochastic input-to-state stability Lyapunov functions and proved that the SMPC optimal cost function satisfies the required decrease conditions, thereby guaranteeing convergence properties for the closed-loop stochastic system. To our knowledge, a self-contained treatment for the standard stagewise-bounded RMPC formulation in the policy parametrization framework, and its explicit parallel with SMPC stability theory, has not been worked out in the existing literature.

Contributions. Building on the ISS/ISpS framework for RMPC developed by Limon et al. (2006), Lazar et al. (2008), Raimondo et al. (2009), and the ISS comparison theory of Jiang and Wang (2001), this paper provides a specialized, self-contained worst-case stability analysis for nonlinear RMPC with stagewise bounded disturbances. The analysis parallels the SMPC framework of McAllister and Rawlings (2023) but addresses the deterministic worst-case setting, requiring careful treatment of maxima over compact disturbance sequence sets and deterministic selection of control laws from possibly set-valued optimal solutions. The main contributions are:

- Adaptation of input-to-state stability concepts to the stagewise-bounded disturbance setting through worst-case input-to-state stability (WISS) Lyapunov functions and robust asymptotic stability in worst-case sense (RASiW), making explicit the dependence on the disturbance set magnitude $\|\mathbb{W}\|$.

- Complete existence and regularity analysis for the RMPC value function under stagewise bounded disturbances, including lower semicontinuity and robust positive invariance of the feasible set.

- Building on the ISpS-Lyapunov results of Limon et al. and Lazar et al., proof that the RMPC optimal cost is a WISS-Lyapunov function under standard terminal region and terminal cost assumptions, yielding explicit $\mathcal{KL}$-$\mathcal{K}$ bounds on closed-loop trajectories.

- Demonstration that RMPC with stagewise bounded disturbances renders the origin robustly asymptotically stable in worst-case sense (a practical ISS-type property depending on $\|\mathbb{W}\|$), providing a parallel deterministic worst-case counterpart to SMPC stability results.

- Clarification of the connection between stagewise-bounded disturbance attenuation and RMPC, contrasting stagewise bounds with classical signal-energy formulations.
- Numerical example demonstrating nonintuitive behavior where RMPC, due to worst-case incentives and asymmetric stage costs, deliberately drives the state away from the origin.

Relationship to SMPC. The analysis framework of this paper parallels the SMPC stability results of McAllister and Rawlings (2023). However, the deterministic worst-case setting requires fundamentally different treatment that precludes direct application of stochastic results. Key distinctions include the use of suprema over compact disturbance sets rather than expectations with respect to probability measures, deterministic selection of single-valued control laws from possibly set-valued optimal solutions, and worst-case input-to-state stability Lyapunov functions in place of stochastic input-to-state stability Lyapunov functions. Consequently, while the structure of basic properties (existence, measurability, cost decrease) and stability definitions follows the stochastic framework, each result requires reformulation and proof for the worst-case setting. The proofs presented herein address these distinctions and establish the analogous stability guarantees for RMPC under stagewise bounded disturbances.

Organization. Section 2 formulates the RMPC problem with stagewise bounded disturbances and states the required assumptions. Section 3 establishes basic properties including existence of optimal solutions and regularity of the value function. Section 4 proves optimal cost decrease under worst-case disturbances. Section 5 defines robust asymptotic stability in worst-case sense, introduces worst-case input-to-state stability Lyapunov functions, and proves the main stability theorem for RMPC. Section 6 presents a numerical example demonstrating the nonintuitive behavior that RMPC may exhibit. Section 7 presents conclusions.

Notation. Let $\mathbb{I}$ and $\mathbb{R}$ denote the integers and reals. Let superscripts and subscripts denote dimensions and restrictions (e.g., $\mathbb{R}^n_{\geq 0}$ denotes nonnegative real-valued vectors of dimension n). Let $|\cdot|$ denote the Euclidean norm. For $x \in \mathbb{R}^n$, $|x|_\infty := \max_{i\in\{1,\ldots,n\}} |x_i|$ denotes the infinity norm. For product spaces we use the sum norm $|(x, w)| := |x| + |w|$. For a closed set $S \subseteq \mathbb{R}^d$ and $z \in \mathbb{R}^d$, $|z|_S := \inf_{y\in S} |z - y|$ denotes the Euclidean point-to-set distance. For a compact set $W \subseteq \mathbb{R}^p$, we define $\|W\| := \max_{w\in W} |w|$ as the maximum norm of any element in W. A function $f : \mathbb{R}^n \to \mathbb{R}$ is lower semicontinuous if and only if the set $\{x \in \mathbb{R}^n : f(x) \leq y\}$ is closed for every $y \in \mathbb{R}$.

The function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is in class $\mathcal{K}$ if it is continuous, strictly increasing, and $\alpha(0) = 0$. The function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is in class $\mathcal{K}_\infty$ if $\alpha(\cdot) \in \mathcal{K}$ and unbounded, i.e., $\lim_{s\to\infty} \alpha(s) = \infty$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{I}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is in class $\mathcal{KL}$ if for every $k \in \mathbb{I}_{\geq 0}$ the function $\beta(\cdot, k)$ is in class $\mathcal{K}$ and for fixed $s \in \mathbb{R}_{\geq 0}$ the function $\beta(s, \cdot)$ is nonincreasing and $\lim_{k \to \infty} \beta(s, k) = 0$.

2 Problem Set Up

2.1 Stage bound disturbance attenuation problem

Consider the discrete time system

$$x^+ = f(x, u, w) \tag{1}$$

in which $x \in \mathbb{X} \subseteq \mathbb{R}^n$ is the state, $u \in \mathbb{U} \subseteq \mathbb{R}^m$ is the control, $w \in \mathbb{R}^p$ is a disturbance, and x^+ is the successor state. The function $f : \mathbb{X} \times \mathbb{U} \times \mathbb{R}^p \to \mathbb{X}$ is continuous.

We assume no probabilistic description of the disturbance. Instead, the disturbance is constrained by a deterministic bound at each time step. For a control horizon $N \in \mathbb{I}_{\geq 1}$, denote the control sequence $\mathbf{u} := (u(0), \ldots, u(N-1))$ and disturbance sequence $\mathbf{w} := (w(0), \ldots, w(N-1))$. The stage bound disturbance attenuation regulator (StDAR) (Mannini and Rawlings, 2026) seeks a control sequence that minimizes the worst-case cost over all disturbances satisfying independent stagewise constraints $|w(k)| \leq \alpha_k$ for each $k \in \mathbb{I}_{[0,N-1]}$.

Define the stage cost $\ell : \mathbb{X} \times \mathbb{U} \to \mathbb{R}$ and terminal cost $V_f : \mathbb{X} \to \mathbb{R}_{\geq 0}$. The finite horizon StDAR is

$$\min_{\mathbf{u}} \max_{\mathbf{w}} \left[\sum_{k=0}^{N-1} \ell(x(k), u(k)) + V_f(x(N)) \right]$$

subject to system dynamics (1) and the stagewise constraints

$$|w(k)| \leq \alpha_k \quad \forall k \in \mathbb{I}_{[0,N-1]}$$

The StDAR problem defines the optimal control policy for a single finite horizon. When this problem is solved repeatedly in a receding horizon framework, applying the first control action, shifting the horizon, and re-optimizing with new measurements, the resulting closed-loop strategy is RMPC. To analyze the stability of this infinite horizon closed-loop system, we assume the disturbance magnitude bounds are time-invariant, i.e., $\alpha_k = \alpha$ for all k.

In this context, it is convenient to define a compact disturbance set $\mathbb{W} := \{w \in \mathbb{R}^p : |w| \leq \alpha\}$. The stagewise constraints on the sequence $\mathbf{w}$ are then equivalent to requiring $\mathbf{w}$ to lie in the Cartesian product set $\mathbb{W}^N$

$$|w(k)| \leq \alpha \quad \forall k \iff \mathbf{w} \in \mathbb{W} \times \cdots \times \mathbb{W} = \mathbb{W}^N$$

To characterize the size of the uncertainty for stability analysis, we define the magnitude of the disturbance set as the maximum norm of any single disturbance realization

$$\|\mathbb{W}\| := \max_{w \in \mathbb{W}} |w|$$

This definition ensures that $\|\mathbb{W}\| = \alpha$, providing a tight scalar bound that does not scale with the horizon length N. The RMPC problem is then formulated as the minmax optimization over $\mathbf{w} \in \mathbb{W}^N$.

2.2 Robust Minmax Model Predictive Control

We adopt the policy parameterization approach of McAllister and Rawlings (2023) cf. Rawlings, Mayne, and Diehl (2020, Ch. 3). Let the policy be $\pi : \mathbb{X} \times \mathbb{V} \to \mathbb{U}$ with $\mathbb{V} \subseteq \mathbb{R}^q$ compact and $\pi(\cdot, \cdot)$ continuous, and require

$$\pi(x, 0) = \kappa_f(x) \quad \forall x \in \mathbb{X}_f \tag{2}$$

Define $f_\pi(x, v, w) := f(x, \pi(x, v), w)$ and denote by $\hat{\phi}(k; x, \mathbf{v}, \mathbf{w})$ the state at time k generated by f_π from initial condition x, parameter sequence $\mathbf{v} = (v(0), \dots, v(N-1))$, and disturbance sequence $\mathbf{w} \in \mathbb{W}^N$.

We consider hard state and input constraints $(x, u) \in \mathbb{Z} \subseteq \mathbb{X} \times \mathbb{U}$. For RMPC, all constraints must be satisfied for every feasible disturbance sequence. The set of admissible parameter sequences for a given initial state x is

$$\begin{aligned} \mathcal{V}^r(x) := \big\{ \mathbf{v} \in \mathbb{V}^N : & (x(k), \pi(x(k), v(k))) \in \mathbb{Z} \\ & \forall \mathbf{w} \in \mathbb{W}^N \quad k \in \mathbb{I}_{[0,N-1]} \\ & x(N) \in \mathbb{X}_f \quad \forall \mathbf{w} \in \mathbb{W}^N \big\} \end{aligned}$$

in which $x(k) = \hat{\phi}(k; x, \mathbf{v}, \mathbf{w})$. The set of admissible initial states is $\mathcal{X}^r := \{x \in \mathbb{X} : \mathcal{V}^r(x) \neq \emptyset\}$.

Define the cost functional

$$J_N^r(x, \mathbf{v}, \mathbf{w}) = \sum_{k=0}^{N-1} \ell(x(k), \pi(x(k), v(k))) + V_f(x(N))$$

in which $x(k) := \hat{\phi}(k; x, \mathbf{v}, \mathbf{w})$ and $\mathbf{w} \in \mathbb{W}^N$. The worst-case cost is

$$V_N^r(x, \mathbf{v}) := \max_{\mathbf{w} \in \mathbb{W}^N} J_N^r(x, \mathbf{v}, \mathbf{w})$$

The RMPC problem for any $x \in \mathcal{X}^r$ is

$$\mathbb{P}_N^r(x) : V_N^{r0}(x) = \min_{\mathbf{v} \in \mathcal{V}^r(x)} V_N^r(x, \mathbf{v}) = \min_{\mathbf{v} \in \mathcal{V}^r(x)} \max_{\mathbf{w} \in \mathbb{W}^N} J_N^r(x, \mathbf{v}, \mathbf{w}) \tag{3}$$

The optimal solutions are defined by $\mathbf{v}^{r0}(x) := \arg\min_{\mathbf{v} \in \mathcal{V}^r(x)} V_N^r(x, \mathbf{v})$ for $x \in \mathcal{X}^r$.

Define the one-step control law

$$K^r(x) := \big\{ \pi(x, v(0)) : \mathbf{v} = (v(0), \dots, v(N-1)) \in \mathbf{v}^{r0}(x) \big\} \subseteq \mathbb{U}$$

and fix a deterministic selection rule $\kappa^r(x) \in K^r(x)$.

We make the following standard assumption for the disturbance.

Assumption 1 (Disturbance)**.** The not random support $\mathbb{W}$ is compact and contains the origin. The magnitude of the disturbance set is $\|\mathbb{W}\| := \max_{w \in \mathbb{W}} |w|$.

Remark 1. Requiring bounded $\mathbb{W}$ is essential for RMPC to ensure that worst-case constraints can be satisfied and that the terminal set remains robustly positive invariant. This is analogous to the bounded support Assumption 1 in SMPC (McAllister and Rawlings, 2023).

2.3 Assumptions

The following regularity assumptions are analogous to those used in SMPC (McAllister and Rawlings, 2023).

Assumption 2 (Continuity of system and cost)**.** The functions $f : \mathbb{X} \times \mathbb{U} \times \mathbb{R}^p \to \mathbb{X}$, $\pi : \mathbb{X} \times \mathbb{V} \to \mathbb{U}$, $\ell : \mathbb{X} \times \mathbb{U} \to \mathbb{R}$, and $V_f : \mathbb{X}_f \to \mathbb{R}_{\geq 0}$ are continuous. The function $\ell(x, u)$ is lower-bounded for all $(x, u) \in \mathbb{Z}$. Furthermore, $f(0, 0, 0) = 0$, $\ell(0, 0) = 0$, and $V_f(0) = 0$.

Assumption 3 (Properties of constraint sets)**.** The state constraint set $\mathbb{X} \subseteq \mathbb{R}^n$ is closed and contains the origin. The sets $\mathbb{Z} \subseteq \mathbb{X} \times \mathbb{U}$ and $\mathbb{X}_f \subseteq \mathbb{X}$ are compact and contain the origin. The input sets $\mathbb{U} \subseteq \mathbb{R}^m$ and $\mathbb{V} \subseteq \mathbb{R}^q$ are compact and contain the origin.

To ensure robust recursive feasibility and performance, the following terminal control law assumption is required.

Assumption 4 (Terminal control law)**.** There exists a continuous terminal control law $\kappa_f : \mathbb{X}_f \to \mathbb{U}$ such that for all $x \in \mathbb{X}_f$

$$f(x, \kappa_f(x), w) \in \mathbb{X}_f \quad \forall w \in \mathbb{W} \tag{4}$$

$$V_f(f(x, \kappa_f(x), 0)) \leq V_f(x) - \ell(x, \kappa_f(x)) \tag{5}$$

Furthermore, $(x, \kappa_f(x)) \in \mathbb{Z}$ for all $x \in \mathbb{X}_f$, and $\pi(x, 0) = \kappa_f(x)$ for all $x \in \mathbb{X}_f$.

Remark 2. In contrast to nominal MPC, the set $\mathbb{X}_f$ may be empty. For a given model, constraint sets, and disturbance set $\mathbb{W}$, there may be no nontrivial pair $(\mathbb{X}_f, \kappa_f)$ satisfying (4)–(5). In particular, if $\mathbb{W}$ is too large relative to the admissible constraints, there may be no compact robust positively invariant set $\mathbb{X}_f \subseteq \mathbb{X}$ with $(x, \kappa_f(x)) \in \mathbb{Z}$ and the terminal cost decrease property; in this case $\mathbb{X}_f$ (and hence $\mathcal{X}^r$) can be empty. All subsequent feasibility and stability results are therefore conditional on the existence of such a terminal pair $(\mathbb{X}_f, \kappa_f)$ for the chosen disturbance set.

For tracking problems, the following assumption is required.

Assumption 5 (Tracking cost bounds)**.** There exists $\alpha_\ell(\cdot) \in \mathcal{K}_\infty$ such that $\ell(x, u) \geq \alpha_\ell(|x|)$ for all $(x, u) \in \mathbb{Z}$. Furthermore, $\mathbb{X}_f$ contains the origin in its interior and $\mathcal{X}^r$ is bounded.

The following sets have been defined: $\mathbb{X} \subseteq \mathbb{R}^n$ is the state constraint set, $\mathbb{U} \subseteq \mathbb{R}^m$ is the input constraint set, $\mathbb{W} \subseteq \mathbb{R}^p$ is the disturbance set, $\mathbb{V} \subseteq \mathbb{R}^q$ is the policy parameter set, $\mathbb{Z} \subseteq \mathbb{X} \times \mathbb{U}$ is the state-input constraint set, $\mathbb{X}_f \subseteq \mathbb{X}$ is the terminal set, $\mathcal{V}^r(x) \subseteq \mathbb{V}^N$ is the set of admissible parameter sequences for initial state x, $\mathcal{X}^r \subseteq \mathbb{X}$ is the set of admissible initial states, and $K^r(x) \subseteq \mathbb{U}$ is the optimal one-step control law.

We establish that the terminal set is contained in the robust feasible set.

Lemma 1 (Terminal set contained in robust feasible set)**.** *Let Assumptions 1–4 hold. Then* $\mathbb{X}_f \subseteq \mathcal{X}^r$*.*

Proof. Pick any $x \in \mathbb{X}_f$ and consider the parameter sequence $\mathbf{v} := (0, \ldots, 0) \in \mathbb{V}^N$. By (2), $\pi(\xi, 0) = \kappa_f(\xi)$ for all $\xi \in \mathbb{X}_f$. For any disturbance sequence $\mathbf{w} = (w(0), \ldots, w(N-1)) \in \mathbb{W}^N$, the closed-loop trajectory under $(x, \mathbf{v}, \mathbf{w})$ satisfies

$$x(0) = x \qquad x(k+1) = f\big(x(k), \kappa_f(x(k)), w(k)\big) \quad k = 0, \ldots, N-1$$

Assumption 4 gives $f(x(k), \kappa_f(x(k)), w) \in \mathbb{X}_f$ for all $x(k) \in \mathbb{X}_f$ and all $w \in \mathbb{W}$, so by induction $x(k) \in \mathbb{X}_f$ for all $k = 0, \ldots, N$. Moreover, $(x(k), \kappa_f(x(k))) \in \mathbb{Z}$ for all $x(k) \in \mathbb{X}_f$ by Assumption 4, and $x(N) \in \mathbb{X}_f$ by construction. Thus, the pair $(x, \mathbf{v})$ satisfies all stage and terminal constraints for every $\mathbf{w} \in \mathbb{W}^N$, i.e., $\mathbf{v} \in \mathcal{V}^r(x)$.

Since we have found a $\mathbf{v} \in \mathcal{V}^r(x)$, the set $\mathcal{V}^r(x)$ is nonempty. By the definition $\mathcal{X}^r := \{z \in \mathbb{X} : \mathcal{V}^r(z) \neq \emptyset\}$, this implies $x \in \mathcal{X}^r$. As $x \in \mathbb{X}_f$ was chosen arbitrarily, we conclude $\mathbb{X}_f \subseteq \mathcal{X}^r$. □

3 Basic Properties of Robust Minmax Model Predictive Control

Before proceeding to stability guarantees for RMPC, we establish that a solution to $\mathbb{P}_N^r$ exists and verify regularity of the optimal cost function.

We begin with the following result for the feasible set.

Lemma 2. *Let Assumptions 1–3 hold. Then the set*

$$\mathcal{Z}_N^r := \{(x, \mathbf{v}) \in \mathbb{X} \times \mathbb{V}^N : \mathbf{v} \in \mathcal{V}^r(x)\}$$

is closed.

Proof. Define for each $\mathbf{w} \in \mathbb{W}^N$ the set

$$\mathcal{Z}_N^r(\mathbf{w}) := \{(x, \mathbf{v}) \in \mathbb{X} \times \mathbb{V}^N : \eta_k(x, \mathbf{v}, \mathbf{w}) \leq 0 \quad \forall k \in \mathbb{I}_{[0,N]}\}$$

with

$$\eta_k(x, \mathbf{v}, \mathbf{w}) := |\big(\hat{\phi}(k; x, \mathbf{v}, \mathbf{w}), \pi(\hat{\phi}(k; x, \mathbf{v}, \mathbf{w}), v(k))\big)|_{\mathbb{Z}} \quad (k \leq N-1)$$
$$\eta_N(x, \mathbf{v}, \mathbf{w}) := |\hat{\phi}(N; x, \mathbf{v}, \mathbf{w})|_{\mathbb{X}_f}$$

Continuity of f, π implies continuity of $\hat{\phi}$ in $(x, \mathbf{v})$ for fixed $\mathbf{w}$ (Rawlings et al., 2020, Prop. 2.1). Since point-to-set distance to closed sets is continuous, each η_k is continuous; hence $\mathcal{Z}_N^r(\mathbf{w})$ is closed. By the definition of $\mathcal{V}^r(x)$, constraints hold for all $\mathbf{w}$ and

$$\mathcal{Z}_N^r = \bigcap_{\mathbf{w} \in \mathbb{W}^N} \mathcal{Z}_N^r(\mathbf{w})$$

an intersection of closed sets, thus closed. □

Using Lemma 2, we establish that solutions to the RMPC optimization problem exist.

Proposition 3 (Existence of minima)**.** *Let Assumptions 1–3 hold. Then for each $x \in \mathcal{X}^r$, the function $V_N^r(x, \cdot)$ is continuous on $\mathcal{V}^r(x)$, the set $\mathcal{V}^r(x)$ is compact, and a solution to $\mathbb{P}_N^r(x)$ exists.*

Proof. For $(x, \mathbf{v}) \in \mathcal{Z}_N^r$, by continuity of f and π, $\hat{\phi}(k; x, \mathbf{v}, \mathbf{w})$ is continuous in $(x, \mathbf{v}, \mathbf{w})$ for all k (Rawlings et al., 2020, Prop. 2.1), hence $J_N^r(x, \mathbf{v}, \mathbf{w})$ is continuous in $(x, \mathbf{v}, \mathbf{w})$. Since $\mathbb{W}^N$ is compact, the map $(x, \mathbf{v}) \mapsto V_N^r(x, \mathbf{v}) := \max_{\mathbf{w} \in \mathbb{W}^N} J_N^r(x, \mathbf{v}, \mathbf{w})$ is continuous on $\mathcal{Z}_N^r$ by Berge's maximum theorem (Berge, 1963) cf. (Rockafellar and Wets, 1998, Theorem 1.17).

By Lemma 2, $\mathcal{Z}_N^r$ is closed. Hence for fixed x, $\mathcal{V}^r(x) = \{\mathbf{v} : (x, \mathbf{v}) \in \mathcal{Z}_N^r\} \subseteq \mathbb{V}^N$ is closed, and since $\mathbb{V}$ is compact, $\mathbb{V}^N$ is compact. Thus $\mathcal{V}^r(x)$ is compact. Since $\mathbf{v} \mapsto V_N^r(x, \mathbf{v})$ is continuous on the compact set $\mathcal{V}^r(x)$, a solution to $\mathbb{P}_N^r(x)$ exists by Weierstrass's theorem (Rawlings et al., 2020, Prop. A.7). □

We next establish regularity of the optimal cost function and closedness of the feasible set.

Proposition 4 (Lower semicontinuity of the optimal cost)**.** *Let Assumptions 1–3 hold. Then $V_N^{r0} : \mathcal{X}^r \to \mathbb{R}$ is lower semicontinuous, the set $\mathcal{X}^r$ is closed, and for every $x \in \mathcal{X}^r$ the minimizer set $\mathbf{v}^{r0}(x) = \arg\min_{\mathbf{v} \in \mathcal{V}^r(x)} V_N^r(x, \mathbf{v})$ is nonempty and compact.*

Proof. Closedness of $\mathcal{X}^r$. By Lemma 2, $\mathcal{Z}_N^r \subset \mathbb{X} \times \mathbb{V}^N$ is closed and, by Assumption 3, $\mathbb{X}$ is closed and $\mathbb{V}^N$ is compact. The projection $\pi_{\mathbb{X}} : \mathbb{X} \times \mathbb{V}^N \to \mathbb{X}$ is a closed map when the second factor is compact; hence

$$\mathcal{X}^r = \pi_{\mathbb{X}}(\mathcal{Z}_N^r) = \{x \in \mathbb{X} : \exists\, \mathbf{v} \in \mathbb{V}^N \text{ with } (x, \mathbf{v}) \in \mathcal{Z}_N^r\}$$

is closed in $\mathbb{X}$ and therefore closed in $\mathbb{R}^n$.

Lower semicontinuity of V_N^{r0}. By Proposition 3, the map

$$(x, \mathbf{v}) \mapsto V_N^r(x, \mathbf{v}) := \max_{\mathbf{w} \in \mathbb{W}^N} J_N^r(x, \mathbf{v}, \mathbf{w})$$

is continuous on $\mathbb{X} \times \mathbb{V}^N$. Define the feasible-set mapping

$$\mathcal{V}^r(x) := \{\mathbf{v} \in \mathbb{V}^N : (x, \mathbf{v}) \in \mathcal{Z}_N^r\} \qquad x \in \mathcal{X}^r$$

$\mathcal{V}^r(x)$ is nonempty for each $x \in \mathcal{X}^r$ by definition of $\mathcal{X}^r$. Since $\mathbb{V}^N$ is compact and $\mathcal{Z}_N^r$ is closed, $\mathcal{V}^r(x)$ is compact for each $x \in \mathcal{X}^r$ and the graph of $\mathcal{V}^r$ is closed; hence $\mathcal{V}^r(\cdot)$ is upper semicontinuous with compact values.

Let

$$G(x, \mathbf{v}) := -V_N^r(x, \mathbf{v}) \qquad (x, \mathbf{v}) \in \mathcal{X}^r \times \mathbb{V}^N$$

Then G is continuous and

$$\tilde{V}(x) := \sup_{\mathbf{v} \in \mathcal{V}^r(x)} G(x, \mathbf{v}) = -\inf_{\mathbf{v} \in \mathcal{V}^r(x)} V_N^r(x, \mathbf{v}) = -V_N^{r0}(x) \qquad x \in \mathcal{X}^r$$

By Berge's maximum theorem (Berge, 1963) cf. (Rockafellar and Wets, 1998, Theorem 1.17) (applied to the maximization of G over $\mathcal{V}^r(x)$), $\tilde{V}$ is upper semicontinuous on $\mathcal{X}^r$. Hence $V_N^{r0} = -\tilde{V}$ is lower semicontinuous on $\mathcal{X}^r$.

Nonempty compact argmin sets. For each fixed $x \in \mathcal{X}^r$, $\mathcal{V}^r(x)$ is compact and nonempty, and $\mathbf{v} \mapsto V_N^r(x, \mathbf{v})$ is continuous (Proposition 3). By Weierstrass's theorem, the minimum over $\mathcal{V}^r(x)$ is attained and the argmin set $\mathbf{v}^{r0}(x)$ is nonempty and compact. □

To define the closed-loop system, we fix a deterministic selection from the optimal control law. For each $x \in \mathcal{X}^r$, the optimal parameter set $\mathbf{v}^{r0}(x) \subset \mathbb{V}^N$ is nonempty and compact by Proposition 4. Choose a deterministic selection

$$\mathbf{v}^{r*}(x) := \big(v^{r*}(0; x), \dots, v^{r*}(N-1; x)\big) \in \mathbf{v}^{r0}(x) \qquad x \in \mathcal{X}^r$$

Define the single-valued feedback law

$$\kappa^r(x) := \pi\big(x, v^{r*}(0; x)\big) \qquad x \in \mathcal{X}^r$$

By construction, $\kappa^r(x) \in K^r(x)$ for all $x \in \mathcal{X}^r$.

We next establish robust positive invariance of the feasible set under this feedback law.

Lemma 5 (Robust positive invariance)**.** *Let Assumptions 1–4 hold and fix the deterministic selection $\kappa^r : \mathcal{X}^r \to \mathbb{U}$ defined above. Then $\mathcal{X}^r$ is robustly positive invariant for the closed loop*

$$x^+ = f\big(x, \kappa^r(x), w\big) \qquad w \in \mathbb{W}$$

and the closed-loop trajectory $\phi^r(k; x, \mathbf{w}_k)$ is well-defined for all $x \in \mathcal{X}^r$, all $\mathbf{w}_k \in \mathbb{W}^k$, and all $k \in \mathbb{I}_{\geq 0}$.

Proof. Step 1: Successor feasibility. Fix any $x \in \mathcal{X}^r$ and any $w(0) \in \mathbb{W}$. By definition of $\mathcal{X}^r$, $\mathcal{V}^r(x) \neq \emptyset$, and by construction of $\mathbf{v}^{r*}$ we can take

$$\mathbf{v}^* = \mathbf{v}^{r*}(x) = \big(v^*(0), \dots, v^*(N-1)\big) \in \mathbf{v}^{r0}(x) \quad \kappa^r(x) = \pi\big(x, v^*(0)\big)$$

Define the successor state

$$x^+ := f\big(x, \kappa^r(x), w(0)\big) = f\big(x, \pi(x, v^*(0)), w(0)\big)$$

We construct a candidate feasible parameter sequence at x^+ by the shift-append rule

$$\tilde{\mathbf{v}}^+ := \big(v^*(1), \dots, v^*(N-1), 0\big) \in \mathbb{V}^N$$

We claim $\tilde{\mathbf{v}}^+ \in \mathcal{V}^r(x^+)$, i.e., starting from x^+ this sequence satisfies all state-input constraints and the terminal constraint for every disturbance sequence of length N.

Step 2: State matching and constraint preservation. Let an arbitrary disturbance tail

$$\tilde{\mathbf{w}}^+ := \big(w(1), \dots, w(N)\big) \in \mathbb{W}^N$$

be given. Form the combined disturbance sequence

$$\mathbf{w}' := \big(w(0), w(1), \dots, w(N)\big) \in \mathbb{W}^{N+1}$$

Consider the state sequence generated from x under the optimal parameters $\mathbf{v}^*$ and disturbance $\mathbf{w}'$

$$x(0) := x \qquad x(k+1) := f\big(x(k), \pi(x(k), v^*(k)), w(k)\big) \quad k \in \mathbb{I}_{[0,N-1]}$$

By feasibility of $\mathbf{v}^* \in \mathcal{V}^r(x)$, we have for every $\mathbf{w} \in \mathbb{W}^N$, hence in particular for the prefix $(w(0), \ldots, w(N-1))$ of $\mathbf{w}'$, that

$$\big(x(k), \pi(x(k), v^*(k))\big) \in \mathbb{Z} \quad k \in \mathbb{I}_{[0,N-1]} \qquad x(N) \in \mathbb{X}_f \tag{6}$$

Next, consider the state sequence from x^+ with parameters $\tilde{\mathbf{v}}^+$ and disturbance tail $\tilde{\mathbf{w}}^+$

$$\begin{aligned} x^+(0) &:= x^+ \\ x^+(k+1) &:= f\big(x^+(k), \pi(x^+(k), v^*(k+1)), w(k+1)\big) \quad k \in \mathbb{I}_{[0,N-2]} \\ x^+(N) &:= f\big(x^+(N-1), \pi(x^+(N-1), 0), w(N)\big) \end{aligned}$$

We show by induction that

$$x^+(k) = x(k+1) \qquad \forall k \in \mathbb{I}_{[0,N-1]} \tag{7}$$

For $k = 0$, this follows directly from the definition of x^+

$$x^+(0) = x^+ = f\left(x, \pi(x, v^*(0)), w(0)\right) = x(1)$$

Assume $x^+(k) = x(k+1)$ for some $k \in \mathbb{I}_{[0,N-2]}$. Then

$$\begin{aligned} x^+(k+1) &= f\big(x^+(k), \pi(x^+(k), v^*(k+1)), w(k+1)\big) \\ &= f\big(x(k+1), \pi(x(k+1), v^*(k+1)), w(k+1)\big) \\ &= x(k+2) \end{aligned}$$

which completes the induction and proves (7).

From (6) and (7) we obtain, for $k \in \mathbb{I}_{[0,N-2]}$,

$$\big(x^+(k), \pi(x^+(k), v^*(k+1))\big) = \big(x(k+1), \pi(x(k+1), v^*(k+1))\big) \in \mathbb{Z}$$

so the stage constraints up to $k = N-2$ are satisfied for any tail $\tilde{\mathbf{w}}^+$.

At the last input stage $k = N-1$, the appended parameter 0 satisfies

$$\pi\big(x^+(N-1), 0\big) = \kappa_f\big(x^+(N-1)\big)$$

Using $x^+(N-1) = x(N) \in \mathbb{X}_f$ from (7) and Assumption 4, we have

$$\big(x^+(N-1), \pi(x^+(N-1), 0)\big) = \big(x(N), \kappa_f(x(N))\big) \in \mathbb{Z}$$

so the stage constraint at $k = N-1$ also holds.

Step 3: Terminal set invariance. Finally, terminal feasibility at the end of the shifted horizon follows from robust positive invariance of $\mathbb{X}_f$ under κ_f:

$$\begin{aligned} x^+(N) &= f\big(x^+(N-1), \pi(x^+(N-1), 0), w(N)\big) \\ &= f\big(x(N), \kappa_f(x(N)), w(N)\big) \in \mathbb{X}_f \qquad \forall w(N) \in \mathbb{W} \end{aligned}$$

by Assumption 4. Thus, for the arbitrary tail $\tilde{\mathbf{w}}^+ \in \mathbb{W}^N$, the sequence $\tilde{\mathbf{v}}^+$ satisfies all constraints and the terminal condition from x^+, i.e., $\tilde{\mathbf{v}}^+ \in \mathcal{V}^r(x^+)$ and therefore $x^+ \in \mathcal{X}^r$.

Since $x \in \mathcal{X}^r$ and $w(0) \in \mathbb{W}$ were arbitrary, we have shown

$$\forall x \in \mathcal{X}^r \quad \forall w \in \mathbb{W}: \quad f\big(x, \kappa^r(x), w\big) \in \mathcal{X}^r$$

so $\mathcal{X}^r$ is robustly positive invariant for the closed loop $x^+ = f(x, \kappa^r(x), w)$, $w \in \mathbb{W}$.

Step 4: Well-defined closed-loop trajectory. Given any $x \in \mathcal{X}^r$ and any disturbance sequence $\mathbf{w}_\infty = (w(0), w(1), \dots) \in \mathbb{W}^{\mathbb{I}_{\geq 0}}$, define inductively

$$\phi^r(0; x, \emptyset) := x \qquad \phi^r(k+1; x, \mathbf{w}_{k+1}) := f\big(\phi^r(k; x, \mathbf{w}_k), \kappa^r(\phi^r(k; x, \mathbf{w}_k)), w(k)\big)$$

By robust positive invariance, $\phi^r(k; x, \mathbf{w}_k) \in \mathcal{X}^r$ for all $k \in \mathbb{I}_{\geq 0}$, hence $\kappa^r(\phi^r(k; x, \mathbf{w}_k))$ is well-defined for all k. This proves that the closed-loop trajectory exists for all $k \in \mathbb{I}_{\geq 0}$ and all disturbance sequences, completing the proof. □

4 Optimal Cost Decrease in Worst-Case Sense

We now establish that the optimal cost for RMPC satisfies a cost decrease inequality along the closed-loop trajectory under worst-case disturbances. This parallels the expected-value decrease for SMPC (McAllister and Rawlings, 2023), but uses uniform bounds over all disturbances.

We use the following technical result from Allan, Bates, Risbeck, and Rawlings (2017, Prop. 20).

Proposition 6. *Let $C \subseteq D \subseteq \mathbb{R}^n$ with C compact and D closed. If $f : D \to \mathbb{R}^n$ is continuous, there exists $\tilde{\alpha}(\cdot) \in \mathcal{K}_\infty$ such that, for all $x \in C$ and $y \in D$, we have $|f(x) - f(y)| \leq \tilde{\alpha}(|x-y|)$*

The following result establishes a uniform bound on the terminal cost increase due to disturbances.

Lemma 7. *Let Assumptions 1–4 hold. Then there exists $\sigma(\cdot) \in \mathcal{K}$ such that, for all $x \in \mathbb{X}_f$ and all $w \in \mathbb{W}$*

$$V_f\big(f(x, \kappa_f(x), w)\big) \leq V_f(x) - \ell\big(x, \kappa_f(x)\big) + \sigma(\|\mathbb{W}\|)$$

Proof. Define $F : \mathbb{X}_f \times \mathbb{W} \to \mathbb{R}$ by $F(x, w) := V_f(f(x, \kappa_f(x), w))$. By continuity of V_f, f, and κ_f, the map F is continuous. Apply Proposition 6 with $C = D = \mathbb{X}_f \times \mathbb{W}$ (compact, hence closed) to obtain $\tilde{\alpha}(\cdot) \in \mathcal{K}_\infty$ such that for all $x \in \mathbb{X}_f$ and $w \in \mathbb{W}$

$$|F(x, w) - F(x, 0)| \leq \tilde{\alpha}\big(\,|(x, w) - (x, 0)|\,\big)$$

We endow $\mathbb{X}_f \times \mathbb{W}$ with the sum norm $|(x, w)| := |x| + |w|$, hence $|(x, w) - (x, 0)| = |w|$ and

$$F(x, w) \leq F(x, 0) + \tilde{\alpha}(|w|)$$

Using the terminal cost decrease (5), $V_f(f(x, \kappa_f(x), 0)) \leq V_f(x) - \ell(x, \kappa_f(x))$, we obtain

$$V_f(f(x, \kappa_f(x), w)) \leq V_f(x) - \ell(x, \kappa_f(x)) + \tilde{\alpha}(|w|) \leq V_f(x) - \ell(x, \kappa_f(x)) + \tilde{\alpha}(\|\mathbb{W}\|)$$

since $|w| \leq \|\mathbb{W}\|$ for all $w \in \mathbb{W}$. Define $\sigma(s) := \tilde{\alpha}(s)$ to obtain the claim. □

We now extend the terminal region cost decrease to the entire robust feasible set. This result constitutes the central stability argument of the paper, as it establishes the dissipation inequality required to characterize the optimal value function as a Lyapunov function.

Proposition 8. *Let Assumptions 1–4 hold. Then $\mathcal{X}^r$ is robustly positive invariant for the control law κ^r, and there exists $\sigma(\cdot) \in \mathcal{K}$ such that, for all $x \in \mathcal{X}^r$ and all $w \in \mathbb{W}$*

$$V_N^{r0}(f(x, \kappa^r(x), w)) \leq V_N^{r0}(x) - \ell(x, \kappa^r(x)) + \sigma(\|\mathbb{W}\|)$$

Proof. By Lemma 5, $\mathcal{X}^r$ is robustly positive invariant. Fix $x \in \mathcal{X}^r$ and $w(0) \in \mathbb{W}$. Pick $\mathbf{v}^* = (v^*(0), \ldots, v^*(N-1)) \in \mathbf{v}^{r0}(x)$ with $\kappa^r(x) = \pi(x, v^*(0))$.

Case $N = 1$: For $N = 1$, we have $V_1^{r0}(x) \geq J_1^r(x, \mathbf{v}^*, w(0)) = \ell(x, \kappa^r(x)) + V_f(x^+)$. Since $x^+ \in \mathbb{X}_f$ and $0 \in \mathcal{V}^r(x^+)$, Lemma 7 gives $V_1^{r0}(x^+) \leq V_1^r(x^+, 0) = \max_{w' \in \mathbb{W}}[\ell(x^+, \kappa_f(x^+)) + V_f(f(x^+, \kappa_f(x^+), w'))] \leq V_f(x^+) + \sigma(\|\mathbb{W}\|)$. Thus $V_1^{r0}(x^+) \leq V_1^{r0}(x) - \ell(x, \kappa^r(x)) + \sigma(\|\mathbb{W}\|)$.

Case $N \geq 2$: Define the successor state $x^+ = f(x, \kappa^r(x), w(0))$. Define the shifted parameter sequence $\tilde{\mathbf{v}}^+ := (v^*(1), \ldots, v^*(N-1), 0) \in \mathbb{V}^N$. As established in the proof of Lemma 5, this specific shifted sequence is feasible, i.e., $\tilde{\mathbf{v}}^+ \in \mathcal{V}^r(x^+)$.

We first relate the cost of an arbitrary disturbance sequence at the next step to a specific sequence at the current step (see Figure 1 for an illustration of this construction). Let $\tilde{\mathbf{w}} = (\tilde{w}(0), \ldots, \tilde{w}(N-1)) \in \mathbb{W}^N$ be any disturbance sequence for the shifted problem starting at x^+. Construct the disturbance sequence $\mathbf{w}^\dagger := (w(0), \tilde{w}(0), \ldots, \tilde{w}(N-2)) \in \mathbb{W}^N$ for the problem starting at x. Denote $x(k) := \hat{\phi}(k; x, \mathbf{v}^*, \mathbf{w}^\dagger)$ and $x^+(k) := \hat{\phi}(k; x^+, \tilde{\mathbf{v}}^+, \tilde{\mathbf{w}})$. As established in Lemma 5 (Step 2), we have $x^+(k) = x(k+1)$ for $k = 0, \ldots, N-1$ and $x^+(N-1) = x(N) \in \mathbb{X}_f$.

Evaluating J_N^r along the two trajectories yields

$$\begin{aligned}
J_N^r(x^+, \tilde{\mathbf{v}}^+, \tilde{\mathbf{w}}) &= \sum_{k=0}^{N-2} \ell\big(x^+(k), \pi(x^+(k), v^*(k+1))\big) + \ell\big(x^+(N-1), \kappa_f(x^+(N-1))\big) \\
&\qquad + V_f\big(f(x^+(N-1), \kappa_f(x^+(N-1)), \tilde{w}(N-1))\big) \\
&= \Big(J_N^r(x, \mathbf{v}^*, \mathbf{w}^\dagger) - \ell(x, \kappa^r(x)) - V_f(x(N))\Big) \\
&\qquad + \ell(x(N), \kappa_f(x(N))) + V_f\big(f(x(N), \kappa_f(x(N)), \tilde{w}(N-1))\big)
\end{aligned}$$

By Lemma 7 (applied at $x(N) \in \mathbb{X}_f$ with disturbance $\tilde{w}(N-1)$), and noting that $|\tilde{w}(N-1)| \leq \|\mathbb{W}\|$, we obtain

$$-V_f(x(N)) + \ell(x(N), \kappa_f(x(N))) + V_f\big(f(x(N), \kappa_f(x(N)), \tilde{w}(N-1))\big) \leq \sigma(\|\mathbb{W}\|)$$

We substitute this bound into the previous expansion for $J_N^r(x^+, \tilde{\mathbf{v}}^+, \tilde{\mathbf{w}})$ to obtain

$$J_N^r(x^+, \tilde{\mathbf{v}}^+, \tilde{\mathbf{w}}) \leq J_N^r(x, \mathbf{v}^*, \mathbf{w}^\dagger) - \ell(x, \kappa^r(x)) + \sigma(\|\mathbb{W}\|) \tag{8}$$

Now we consider the worst-case disturbance at x^+. Since $\mathbb{W}^N$ is compact and $J_N^r(x^+, \tilde{\mathbf{v}}^+, \cdot)$ is continuous, a maximizer exists. Let $\tilde{\mathbf{w}}^* \in \mathbb{W}^N$ be such a sequence, i.e.,

$$J_N^r(x^+, \tilde{\mathbf{v}}^+, \tilde{\mathbf{w}}^*) = \max_{\mathbf{w}' \in \mathbb{W}^N} J_N^r(x^+, \tilde{\mathbf{v}}^+, \mathbf{w}') = V_N^r(x^+, \tilde{\mathbf{v}}^+)$$

Let $\mathbf{w}^\ddagger := (w(0), \tilde{w}^*(0), \ldots, \tilde{w}^*(N-2)) \in \mathbb{W}^N$ be the specific sequence constructed by prepending $w(0)$ to the first $N-1$ elements of $\tilde{\mathbf{w}}^*$. Applying (8) with $\tilde{\mathbf{w}} = \tilde{\mathbf{w}}^*$ and $\mathbf{w}^\dagger = \mathbf{w}^\ddagger$, we obtain

$$V_N^r(x^+, \tilde{\mathbf{v}}^+) \leq J_N^r(x, \mathbf{v}^*, \mathbf{w}^\ddagger) - \ell(x, \kappa^r(x)) + \sigma(\|\mathbb{W}\|)$$

By the definition of the value function at x, we have

$$J_N^r(x, \mathbf{v}^*, \mathbf{w}^\ddagger) \leq \max_{\mathbf{w} \in \mathbb{W}^N} J_N^r(x, \mathbf{v}^*, \mathbf{w}) = V_N^r(x, \mathbf{v}^*)$$

Since $\mathbf{v}^* \in \mathbf{v}^{r0}(x)$, we have $V_N^r(x, \mathbf{v}^*) = V_N^{r0}(x)$. Thus we have

$$V_N^r(x^+, \tilde{\mathbf{v}}^+) \leq V_N^{r0}(x) - \ell(x, \kappa^r(x)) + \sigma(\|\mathbb{W}\|)$$

Since $\tilde{\mathbf{v}}^+ \in \mathcal{V}^r(x^+)$, suboptimality implies $V_N^{r0}(x^+) \leq V_N^r(x^+, \tilde{\mathbf{v}}^+)$. Therefore

$$V_N^{r0}(x^+) \leq V_N^{r0}(x) - \ell(x, \kappa^r(x)) + \sigma(\|\mathbb{W}\|)$$

Because $w(0) \in \mathbb{W}$ was arbitrary, the bound holds for all $w \in \mathbb{W}$. □

5 Worst-Case Stability: RASiW

We adapt classical input-to-state stability concepts to the stagewise-bounded disturbance setting. The robust asymptotic stability in worst-case sense (RASiW) notion parallels input-to-state practical stability (ISpS) from the RMPC literature (Limon et al., 2006; Lazar et al., 2008; Raimondo et al., 2009), specialized to our stagewise-bounded disturbance formulation with explicit dependence on the disturbance set magnitude $\|\mathbb{W}\|$. Similarly, worst-case input-to-state stability (WISS) Lyapunov functions are natural adaptations of ISS-Lyapunov concepts to the worst-case setting.

5.1 Robust Asymptotic Stability in Worst-Case Sense

Discrete time Lyapunov stability theory for nominal systems relies on comparison lemmas to convert Lyapunov decrease conditions into explicit stability estimates (Sontag, 1989). Jiang and Wang (2001) extended this framework to input-to-state stability for discrete time nonlinear systems subject to bounded inputs. The results below adapt these ISS comparison arguments to scalar recursions with a constant perturbation parameter arising from a fixed stagewise disturbance bound.

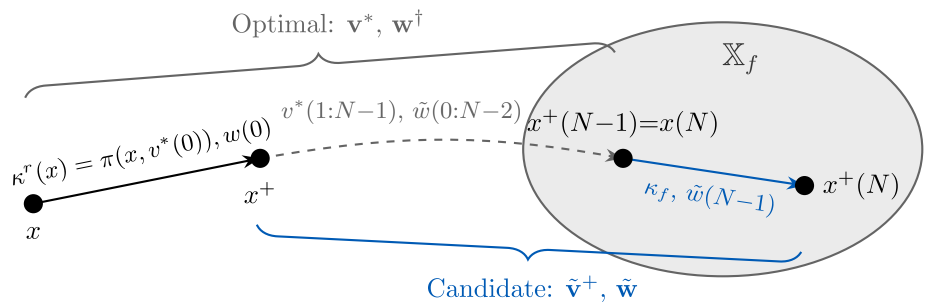

Figure 1: Horizon-shifting argument in Proposition 8 (for $N \geq 2$). For any disturbance sequence $\tilde{\mathbf{w}} = (\tilde{w}(0), \ldots, \tilde{w}(N{-}1)) \in \mathbb{W}^N$ at x^+, define $\mathbf{w}^\dagger := (w(0), \tilde{w}(0), \ldots, \tilde{w}(N{-}2)) \in \mathbb{W}^N$ at x. The optimal parameters $\mathbf{v}^* = (v^*(0), \ldots, v^*(N{-}1))$ satisfy $x(N) \in \mathbb{X}_f$ for all $\mathbf{w} \in \mathbb{W}^N$ by feasibility. The shifted candidate $\tilde{\mathbf{v}}^+ = (v^*(1), \ldots, v^*(N{-}1), 0)$ satisfies $x^+(k) = x(k{+}1)$ for $k = 0, \ldots, N{-}1$, so both trajectories reach $x^+(N{-}1) = x(N) \in \mathbb{X}_f$. Comparing trajectories shows that the first-stage cost $\ell(x, \kappa^r(x))$ is dropped from the horizon while the terminal-step mismatch is bounded by $\sigma(\|\mathbb{W}\|)$ via Lemma 7, yielding $V_N^{r0}(f(x, \kappa^r(x), w)) \leq V_N^{r0}(x) - \ell(x, \kappa^r(x)) + \sigma(\|\mathbb{W}\|)$.

Definition 1 (RASiW)**.** The origin is robustly asymptotically stable in worst-case sense (RASiW) for the system $x^+ = f(x, \kappa^r(x), w)$, $w \in \mathbb{W}$ on the robustly positive invariant set $\mathcal{X}^r$ if there exist $\beta(\cdot,\cdot) \in \mathcal{KL}$ and $\gamma(\cdot) \in \mathcal{K}$ such that the closed-loop trajectory satisfies

$$|\phi^r(k; x, \mathbf{w}_k)| \leq \beta(|x|, k) + \gamma(\|\mathbb{W}\|) \tag{9}$$

for all $x \in \mathcal{X}^r$, $k \in \mathbb{I}_{\geq 0}$, and $\mathbf{w}_k \in \mathbb{W}^k$.

Definition 2 (WISS-Lyapunov Function)**.** A lower semicontinuous function $V : \mathcal{X}^r \to \mathbb{R}_{\geq 0}$ is a worst-case input-to-state stability (WISS) Lyapunov function on the robustly positive invariant set $\mathcal{X}^r$ for the system $x^+ = f(x, \kappa^r(x), w)$, $w \in \mathbb{W}$ if there exist $\alpha_1(\cdot), \alpha_2(\cdot), \alpha_3(\cdot) \in \mathcal{K}_\infty$ and $\sigma_2(\cdot), \sigma_3(\cdot) \in \mathcal{K}$ such that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) + \sigma_2(\|\mathbb{W}\|) \tag{10}$$
$$V(f(x, \kappa^r(x), w)) \leq V(x) - \alpha_3(|x|) + \sigma_3(\|\mathbb{W}\|) \tag{11}$$

for all $x \in \mathcal{X}^r$ and $w \in \mathbb{W}$.

The connection between WISS-Lyapunov functions and RASiW bounds follows from standard ISS comparison arguments. We first establish a discrete time comparison lemma adapted from the ISS literature (Jiang and Wang, 2001; Sontag, 1989).

Lemma 9 (discrete time comparison)**.** *Let $\mu(\cdot) \in \mathcal{K}_\infty$ and $c \geq 0$. Suppose the nonnegative sequence $y : \mathbb{I}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfies*

$$y(k+1) \leq y(k) - \mu(y(k)) + c \tag{12}$$

for all $k \in \mathbb{I}_{\geq 0}$. Then there exist $\beta_Y(\cdot,\cdot) \in \mathcal{KL}$ and $\gamma_Y(\cdot) \in \mathcal{K}$ such that

$$y(k) \leq \beta_Y(y(0), k) + \gamma_Y(c)$$

for all $k \in \mathbb{I}_{\geq 0}$.

Proof. Since $\mu(\cdot) \in \mathcal{K}_\infty$, the function $\mu : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is continuous, strictly increasing, unbounded, and satisfies $\mu(0) = 0$. Hence μ is a bijection and admits a continuous inverse $\mu^{-1} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $\mu^{-1} \in \mathcal{K}_\infty$ (Khalil, 2002, Lemma 4.2).

Step 1: Reduce to a max-type recursion. For the given $c \geq 0$, define

$$s_c := \mu^{-1}(2c)$$

If $c = 0$ then $s_c = \mu^{-1}(0) = 0$. For any $s \geq s_c$ we have $\mu(s) \geq 2c$.

Define

$$\rho_1(s) := s - \frac{1}{2}\mu(s)$$

for $s \geq 0$. Since μ is continuous, so is ρ_1. Moreover $\rho_1(0) = 0$ and for all $s > 0$

$$\rho_1(s) = s - \frac{1}{2}\mu(s) < s$$

because $\mu(s) > 0$ for $s > 0$.

We define the nondecreasing envelope of ρ_1

$$\rho_2(s) := \max_{0 \leq \tau \leq s} \rho_1(\tau)$$

for $s \geq 0$. By construction $\rho_2(0) = \rho_1(0) = 0$, the function ρ_2 is nondecreasing and continuous, and for $s > 0$ we have $\rho_1(\tau) < \tau \leq s$ for all $\tau \in (0, s]$ and $\rho_1(0) = 0 < s$, hence $\rho_2(s) = \max_{0 \leq \tau \leq s} \rho_1(\tau) < s$. Thus, we obtain

$$\rho_2(0) = 0 \quad 0 \leq \rho_2(s) < s \text{ for all } s > 0 \quad \rho_2 \text{ nondecreasing and continuous} \tag{13}$$

Note also that $\rho_1(s) \leq \rho_2(s)$ for all $s \geq 0$ by construction.

Consider the recursion (12). We distinguish two cases.

Case 1: $y(k) \geq s_c$. Then $\mu(y(k)) \geq 2c$ and

$$y(k+1) \leq y(k) - \mu(y(k)) + c \leq y(k) - \frac{1}{2}\mu(y(k)) = \rho_1(y(k)) \leq \rho_2(y(k))$$

Case 2: $y(k) \leq s_c$. In this case

$$y(k+1) \leq y(k) - \mu(y(k)) + c \leq y(k) + c \leq s_c + c$$

Define

$$\gamma(c) := s_c + c = \mu^{-1}(2c) + c$$

Since $\mu^{-1} \in \mathcal{K}_\infty$ and $c \mapsto c$ is in $\mathcal{K}_\infty$, their sum is $\gamma(\cdot) \in \mathcal{K}_\infty \subset \mathcal{K}$ and $\gamma(0) = 0$.

Combining the two cases yields

$$y(k+1) \leq \max\{\rho_2(y(k)), \gamma(c)\} \tag{14}$$

for all $k \in \mathbb{I}_{\geq 0}$.

Step 2: Construct a contraction map dominating ρ_2. Define

$$\lambda(s) := \frac{1}{2}(s + \rho_2(s))$$

for $s \geq 0$. From (13) we have $0 \leq \rho_2(s) \leq s$ for all $s \geq 0$, hence $0 \leq \lambda(s) \leq s$. For $s > 0$, we have

$$\lambda(s) = \frac{s + \rho_2(s)}{2} < \frac{s + s}{2} = s$$

since $\rho_2(s) < s$. Moreover $\lambda(0) = 0$ and since $s \mapsto s$ and ρ_2 are continuous and nondecreasing with at least one strictly increasing component, $\lambda(\cdot)$ is continuous and strictly increasing. Thus $\lambda(\cdot) \in \mathcal{K}$ and

$$\lambda(0) = 0 \quad 0 \leq \lambda(s) < s \text{ for } s > 0 \quad \lambda(\cdot) \text{ strictly increasing} \tag{15}$$

By construction $\rho_2(s) \leq \lambda(s)$ for all $s \geq 0$.

Using (14) and $\rho_2(\cdot) \leq \lambda(\cdot)$ yields

$$y(k+1) \leq \max\{\lambda(y(k)), \gamma(c)\} \tag{16}$$

for all $k \in \mathbb{I}_{\geq 0}$.

Step 3: Comparison with the contraction iterates. Define the iterates of $\lambda(\cdot)$ by

$$\lambda^0(s) := s \quad \lambda^{k+1}(s) := \lambda(\lambda^k(s))$$

for $k \in \mathbb{I}_{\geq 0}$. For each fixed k, the function $\lambda^k(\cdot)$ is continuous and strictly increasing with $\lambda^k(0) = 0$. From (15) and induction we have $\lambda^k(s) < s$ for all $s > 0$ and $k \geq 1$. For any fixed $s > 0$, define the sequence $s_0 := s$ and $s_{k+1} := \lambda(s_k)$ for $k \in \mathbb{I}_{\geq 0}$. From (15) we have $0 \leq s_{k+1} = \lambda(s_k) < s_k$ whenever $s_k > 0$, so the sequence is strictly decreasing and bounded below by zero, hence converges to some $L \geq 0$. By continuity of $\lambda(\cdot)$ we have $L = \lim_{k\to\infty} s_{k+1} = \lim_{k\to\infty} \lambda(s_k) = \lambda(L)$. From $\lambda(s) < s$ for all $s > 0$ it follows that the only fixed point is $L = 0$. Thus $s_k = \lambda^k(s) \to 0$ as $k \to \infty$ for every $s > 0$.

We claim that

$$y(k) \leq \max\{\lambda^k(y(0)), \gamma(c)\} \tag{17}$$

for all $k \in \mathbb{I}_{\geq 0}$. We proceed by induction. For $k = 0$, (17) reduces to $y(0) \leq \max\{y(0), \gamma(c)\}$, which holds. Assume (17) holds for some $k \in \mathbb{I}_{\geq 0}$. Using (16) we have

$$y(k+1) \leq \max\{\lambda(y(k)), \gamma(c)\}$$

By the induction hypothesis

$$y(k) \leq \max\{\lambda^k(y(0)), \gamma(c)\}$$

Since λ is strictly increasing

$$\lambda(y(k)) \leq \lambda(\max\{\lambda^k(y(0)), \gamma(c)\}) = \max\{\lambda^{k+1}(y(0)), \lambda(\gamma(c))\}$$

From (15) we have $\lambda(s) < s$ for $s > 0$, hence $\lambda(\gamma(c)) \leq \gamma(c)$ for all $c \geq 0$. Therefore

$$\lambda(y(k)) \leq \max\{\lambda^{k+1}(y(0)), \gamma(c)\}$$

Combining with (16) gives

$$\begin{aligned} y(k+1) &\leq \max\{\lambda(y(k)), \gamma(c)\} \\ &\leq \max\{\max\{\lambda^{k+1}(y(0)), \gamma(c)\}, \gamma(c)\} \\ &= \max\{\lambda^{k+1}(y(0)), \gamma(c)\} \end{aligned}$$

which is (17) with $k+1$. Thus the claim holds for all k.

Using the elementary inequality $\max\{a, b\} \leq a + b$ for $a, b \geq 0$ yields

$$y(k) \leq \lambda^k(y(0)) + \gamma(c)$$

for all $k \in \mathbb{I}_{\geq 0}$.

Step 4: Definition of $\beta_Y(\cdot,\cdot)$ and $\gamma_Y(\cdot)$. Define

$$\beta_Y(s,k) := \lambda^k(s) \quad \gamma_Y(c) := \gamma(c) = \mu^{-1}(2c) + c$$

For each fixed k, the map $s \mapsto \beta_Y(s,k)$ is continuous, strictly increasing, and satisfies $\beta_Y(0,k) = 0$, hence $\beta_Y(\cdot,k) \in \mathcal{K}$. For each fixed $s \geq 0$, the map $k \mapsto \beta_Y(s,k)$ is nonincreasing and $\lim_{k\to\infty} \beta_Y(s,k) = 0$ because $\lambda(s) < s$ for $s > 0$. Thus $\beta_Y(\cdot,\cdot) \in \mathcal{KL}$. The function γ_Y is the sum of two $\mathcal{K}_\infty$ functions $c \mapsto \mu^{-1}(2c)$ and $c \mapsto c$, hence $\gamma_Y(\cdot) \in \mathcal{K}_\infty \subset \mathcal{K}$. From the inequality $y(k) \leq \lambda^k(y(0)) + \gamma(c)$ we conclude

$$y(k) \leq \beta_Y(y(0), k) + \gamma_Y(c)$$

for all $k \in \mathbb{I}_{\geq 0}$, which completes the proof. □

We establish that WISS-Lyapunov functions ensure RASiW. This result adapts standard ISS-Lyapunov theory (Jiang and Wang, 2001; Limon et al., 2006) to the stagewise-bounded disturbance setting.

Proposition 10. *If the system $x^+ = f(x, \kappa^r(x), w)$, $w \in \mathbb{W}$ admits a WISS-Lyapunov function $V : \mathcal{X}^r \to \mathbb{R}_{\geq 0}$ on the robustly positive invariant set $\mathcal{X}^r$, and $\mathcal{X}^r$ is bounded, then the origin is RASiW on $\mathcal{X}^r$.*

Proof. Fix $x(0) \in \mathcal{X}^r$ and a disturbance sequence $\mathbf{w}_\infty = (w(0), w(1), \dots)$ with $w(k) \in \mathbb{W}$ and $|w(k)| \leq \|\mathbb{W}\|$ for all $k \in \mathbb{I}_{\geq 0}$. Let the closed-loop trajectory be

$$x(k+1) = f(x(k), \kappa^r(x(k)), w(k))$$

for $k \in \mathbb{I}_{\geq 0}$. Since $\mathcal{X}^r$ is robustly positive invariant, $x(k) \in \mathcal{X}^r$ for all $k \in \mathbb{I}_{\geq 0}$, so $V(x(k))$ and the WISS inequalities are well-defined for all k. Define $Y(k) := V(x(k))$.

Step 1: Lyapunov inequalities in scalar ISS form. From the WISS bounds (10) we have

$$\alpha_1(|x(k)|) \leq Y(k) \tag{18}$$
$$Y(k) \leq \alpha_2(|x(k)|) + \sigma_2(\|\mathbb{W}\|) \tag{19}$$

for all $k \in \mathbb{I}_{\geq 0}$. The decrease condition (11) gives

$$Y(k+1) \leq Y(k) - \alpha_3(|x(k)|) + \sigma_3(\|\mathbb{W}\|) \tag{20}$$

for all k and all admissible $w(k)$.

Since $\alpha_1(\cdot) \in \mathcal{K}_\infty$, it admits an inverse $\alpha_1^{-1}(\cdot) \in \mathcal{K}_\infty$ (Khalil, 2002, Lemma 4.2). Define $\mu(s) := \alpha_3(\alpha_1^{-1}(s)) \in \mathcal{K}_\infty$. From (18) we obtain

$$\alpha_1^{-1}(Y(k)) \geq |x(k)| \quad \Rightarrow \quad \mu(Y(k)) \geq \alpha_3(|x(k)|)$$

Define $\delta(x) := \mu(V(x)) - \alpha_3(|x|)$ for $x \in \mathcal{X}^r$, which satisfies $\delta(x) \geq 0$ by the lower WISS bound in (10). Since $\mathcal{X}^r$ is bounded, let $\|\mathcal{X}^r\| := \sup\{|x| : x \in \mathcal{X}^r\} < \infty$. From the upper WISS bound in (10), for all $x \in \mathcal{X}^r$,

$$V(x) \leq \alpha_2(|x|) + \sigma_2(\|\mathbb{W}\|) \leq \alpha_2(\|\mathcal{X}^r\|) + \sigma_2(\|\mathbb{W}\|)$$

Hence, by monotonicity of $\mu(\cdot) \in \mathcal{K}_\infty$, we have

$$\mu(V(x)) \leq \mu(\alpha_2(\|\mathcal{X}^r\|) + \sigma_2(\|\mathbb{W}\|)) \quad x \in \mathcal{X}^r$$

Thus the constant

$$c(\|\mathbb{W}\|) := \sigma_3(\|\mathbb{W}\|) + \sup_{x \in \mathcal{X}^r} \delta(x) \leq \sigma_3(\|\mathbb{W}\|) + \mu(\alpha_2(\|\mathcal{X}^r\|) + \sigma_2(\|\mathbb{W}\|)) < \infty$$

is well-defined. Since $\mathcal{X}^r$ is fixed, we write $c(\|\mathbb{W}\|)$ suppressing the dependence on $\mathcal{X}^r$. Combining (20) yields

$$Y(k+1) \leq Y(k) - \mu(Y(k)) + c(\|\mathbb{W}\|) \tag{21}$$

for $k \in \mathbb{I}_{\geq 0}$.

Step 2: Scalar ISS estimate for $Y(k)$. By Lemma 9 applied to (21), there exist $\beta_Y(\cdot,\cdot) \in \mathcal{KL}$ and $\gamma_Y(\cdot) \in \mathcal{K}$ such that

$$Y(k) \leq \beta_Y(Y(0), k) + \gamma_Y(c(\|\mathbb{W}\|)) \tag{22}$$

for all $k \in \mathbb{I}_{\geq 0}$.

Step 3: Convert from V-bound to a state bound. From (18) and (22) we obtain

$$|x(k)| \leq \alpha_1^{-1}\Big(\beta_Y(Y(0), k) + \gamma_Y(c(\|\mathbb{W}\|))\Big)$$

From (19) at $k = 0$ we have $Y(0) \leq \alpha_2(|x(0)|) + \sigma_2(\|\mathbb{W}\|)$. Using property (B.15) from Rawlings et al. (2020, App. B), for any $a, b \geq 0$ and $k \in \mathbb{I}_{\geq 0}$ we have

$$\beta_Y(a + b, k) \leq \beta_Y(2a, k) + \beta_Y(2b, 0)$$

Applying this with $a = \alpha_2(|x(0)|)$ and $b = \sigma_2(\|\mathbb{W}\|)$ yields

$$\beta_Y(Y(0), k) \leq \beta_Y(2\alpha_2(|x(0)|), k) + \beta_Y(2\sigma_2(\|\mathbb{W}\|), 0)$$

Using property (B.14) from Rawlings et al. (2020, App. B) for sums of $\mathcal{K}$ functions, we have

$$\alpha_1^{-1}(A + B) \leq \alpha_1^{-1}(2A) + \alpha_1^{-1}(2B)$$

Let $A = \beta_Y(2\alpha_2(|x(0)|), k)$ and $B = \beta_Y(2\sigma_2(\|\mathbb{W}\|), 0) + \gamma_Y(c(\|\mathbb{W}\|))$, then

$$|x(k)| \leq \alpha_1^{-1}(2\beta_Y(2\alpha_2(|x(0)|), k)) + \alpha_1^{-1}\big(2\beta_Y(2\sigma_2(\|\mathbb{W}\|), 0) + 2\gamma_Y(c(\|\mathbb{W}\|))\big)$$

Define $\beta(s, k) := \alpha_1^{-1}(2\beta_Y(2\alpha_2(s), k)) \in \mathcal{KL}$. Then

$$|x(k)| \leq \beta(|x(0)|, k) + \alpha_1^{-1}\big(2\beta_Y(2\sigma_2(\|\mathbb{W}\|), 0) + 2\gamma_Y(c(\|\mathbb{W}\|))\big)$$

If $\|\mathbb{W}\| = 0$, the nominal case follows directly from the standard Lyapunov theorem (Rawlings et al., 2020, Theorem B.15) using (10)–(11), so $|x(k)| \leq \beta(|x(0)|, k)$; since any $\gamma(\cdot) \in \mathcal{K}$ satisfies $\gamma(0) = 0$, the RASiW bound holds. For $\|\mathbb{W}\| > 0$, define

$$\gamma(s) := \frac{1}{\|\mathbb{W}\|}\alpha_1^{-1}\big(2\beta_Y(2\sigma_2(\|\mathbb{W}\|), 0) + 2\gamma_Y(c(\|\mathbb{W}\|))\big)\, s \quad s \geq 0$$

Then $\gamma(\cdot) \in \mathcal{K}$ and $\gamma(\|\mathbb{W}\|) = \alpha_1^{-1}\big(2\beta_Y(2\sigma_2(\|\mathbb{W}\|), 0) + 2\gamma_Y(c(\|\mathbb{W}\|))\big)$, so

$$|x(k)| \leq \beta(|x(0)|, k) + \gamma(\|\mathbb{W}\|)$$

for all $k \in \mathbb{I}_{\geq 0}$ and all disturbance sequences with $|w(k)| \leq \|\mathbb{W}\|$. This is exactly the RASiW property (9) on $\mathcal{X}^r$. □

5.2 RASiW of RMPC

The remainder of this section establishes that the RMPC optimal cost function is a WISS-Lyapunov function, thereby proving RASiW of the closed loop. We first establish the upper bound for the optimal cost function.

Lemma 11. *Let Assumptions 1–5 hold. Then there exist $\alpha_2(\cdot) \in \mathcal{K}_\infty$ and $\sigma_2(\cdot) \in \mathcal{K}$ such that*

$$V_N^{r0}(x) \leq \alpha_2(|x|) + \sigma_2(\|\mathbb{W}\|)$$

for all $x \in \mathcal{X}^r$.

Proof. Fix $x \in \mathbb{X}_f$ and apply the terminal control law for N steps, i.e., choose the feasible parameter sequence $\mathbf{v} = \mathbf{0} \in \mathbb{V}^N$ so that $\pi(\cdot, 0) = \kappa_f(\cdot)$ by (2). By Assumption 4 and Lemma 7, for each step and every $w \in \mathbb{W}$, we have

$$V_f(x(k+1)) - V_f(x(k)) \leq -\ell(x(k), \kappa_f(x(k))) + \sigma(\|\mathbb{W}\|)$$

Summing $k = 0$ to $N-1$ yields, for every $\mathbf{w} \in \mathbb{W}^N$

$$\sum_{k=0}^{N-1} \ell(x(k), \kappa_f(x(k))) + V_f(x(N)) \leq V_f(x) + N\sigma(\|\mathbb{W}\|)$$

The left-hand side equals $J_N^r(x, \mathbf{0}, \mathbf{w})$, hence

$$\max_{\mathbf{w} \in \mathbb{W}^N} J_N^r(x, \mathbf{0}, \mathbf{w}) \leq V_f(x) + N\sigma(\|\mathbb{W}\|)$$

By optimality, $V_N^{r0}(x) \leq V_f(x) + N\sigma(\|\mathbb{W}\|)$ for $x \in \mathbb{X}_f$.

Since V_f is continuous on the compact set $\mathbb{X}_f$ and $V_f(0) = 0$, we apply Rawlings and Risbeck (2015, Prop. 14) to obtain $\tilde{\alpha}_f(\cdot) \in \mathcal{K}$ such that $V_f(x) \leq \tilde{\alpha}_f(|x|)$ for all $x \in \mathbb{X}_f$. Let $\|\mathbb{X}_f\| := \sup_{x \in \mathbb{X}_f} |x|$. We extend $\tilde{\alpha}_f(\cdot)$ to $\alpha_f(\cdot) \in \mathcal{K}_\infty$ by defining

$$\alpha_f(s) := \begin{cases} \tilde{\alpha}_f(s) & 0 \leq s \leq \|\mathbb{X}_f\| \\ \tilde{\alpha}_f(\|\mathbb{X}_f\|) + (s - \|\mathbb{X}_f\|) & s > \|\mathbb{X}_f\| \end{cases}$$

Thus $V_N^{r0}(x) \leq \alpha_f(|x|) + N\sigma(\|\mathbb{W}\|)$ for all $x \in \mathbb{X}_f$.

Define

$$W(x) := \max\{V_N^{r0}(x) - N\sigma(\|\mathbb{W}\|), 0\}$$

for $x \in \mathcal{X}^r$. For $x \in \mathbb{X}_f$ we have $W(x) \leq V_f(x) \leq \alpha_f(|x|)$. By Proposition 4, V_N^{r0} is lower semicontinuous on $\mathcal{X}^r$, hence W is also lower semicontinuous. Assumption 5 states that $\mathbb{X}_f$ contains the origin in its interior, so there exists $\rho > 0$ such that $\{x : |x| \leq \rho\} \subset \mathbb{X}_f$. For $|x| \leq \rho$ we have $W(x) \leq \alpha_f(|x|)$. Therefore

$$\limsup_{x \to 0} W(x) \leq \limsup_{x \to 0} \alpha_f(|x|) = \alpha_f(0) = 0$$

On the other hand, lower semicontinuity and $W(0) = 0$ give

$$\liminf_{x \to 0} W(x) \geq W(0) = 0$$

Hence $\lim_{x \to 0} W(x) = 0$, i.e., W is continuous at zero.

By Assumption 5 and Proposition 4, $\mathcal{X}^r$ is closed and bounded, hence compact. Since $\mathcal{Z}_N^r$ is closed (Lemma 2) and contained in the compact set $\mathcal{X}^r \times \mathbb{V}^N$, it is compact. The cost function $J_N^r(x, \mathbf{v}, \mathbf{w})$ is continuous on the compact set $\mathcal{Z}_N^r \times \mathbb{W}^N$ (where $x(N) \in \mathbb{X}_f$, so V_f is well-defined). Thus J_N^r is bounded on this set. Since $V_N^{r0}(x) = \min_{\mathbf{v} \in \mathcal{V}^r(x)} \max_{\mathbf{w} \in \mathbb{W}^N} J_N^r(x, \mathbf{v}, \mathbf{w})$, it follows that $V_N^{r0}(x)$ is bounded on $\mathcal{X}^r$. Consequently, $W(x)$ is bounded (and thus locally bounded) on $\mathcal{X}^r$. Since W is lower semicontinuous, continuous at zero, $W(0) = 0$, and locally bounded on the closed set $\mathcal{X}^r$, we apply Rawlings and Risbeck (2015, Prop. 14) to obtain $\tilde{\alpha}_2(\cdot) \in \mathcal{K}$ such that $W(x) \leq \tilde{\alpha}_2(|x|)$ for all $x \in \mathcal{X}^r$.

Let $\|\mathcal{X}^r\| := \sup_{x \in \mathcal{X}^r} |x|$. We extend $\tilde{\alpha}_2(\cdot)$ to $\alpha_2(\cdot) \in \mathcal{K}_\infty$ by

$$\alpha_2(s) := \begin{cases} \tilde{\alpha}_2(s) & 0 \leq s \leq \|\mathcal{X}^r\| \\ \tilde{\alpha}_2(\|\mathcal{X}^r\|) + (s - \|\mathcal{X}^r\|) & s > \|\mathcal{X}^r\| \end{cases}$$

Define $\sigma_2(s) := N\sigma(s) \in \mathcal{K}$. Then

$$V_N^{r0}(x) - \sigma_2(\|\mathbb{W}\|) = V_N^{r0}(x) - N\sigma(\|\mathbb{W}\|) \leq W(x) \leq \alpha_2(|x|)$$

so

$$V_N^{r0}(x) \leq \alpha_2(|x|) + \sigma_2(\|\mathbb{W}\|)$$

for all $x \in \mathcal{X}^r$, which completes the proof. □

We establish that the stage cost is dominated by the value function. For any $x \in \mathcal{X}^r$, any $\mathbf{v} \in \mathcal{V}^r(x)$ and any $\mathbf{w} \in \mathbb{W}^N$

$$\begin{aligned} J_N^r(x, \mathbf{v}, \mathbf{w}) &= \ell(x, \pi(x, v(0))) + \sum_{k=1}^{N-1} \ell(x(k), \pi(x(k), v(k))) + V_f(x(N)) \\ &\geq \ell(x, \pi(x, v(0))) \end{aligned}$$

Hence

$$V_N^r(x, \mathbf{v}) = \max_{\mathbf{w} \in \mathbb{W}^N} J_N^r(x, \mathbf{v}, \mathbf{w}) \geq \ell(x, \pi(x, v(0)))$$

Taking the minimum over $\mathbf{v} \in \mathbf{v}^{r0}(x)$ and using $\kappa^r(x) \in K^r(x)$ gives

$$\ell(x, \kappa^r(x)) \leq V_N^{r0}(x) \tag{23}$$

for all $x \in \mathcal{X}^r$.

We establish the main stability result for RMPC.

Theorem 12. *Let Assumptions 1–5 hold. Then the origin is RASiW for the system $x^+ = f(x, \kappa^r(x), w)$, $w \in \mathbb{W}$ on the robustly positive invariant set $\mathcal{X}^r$*

Proof. By Lemma 5, $\mathcal{X}^r$ is robustly positive invariant under κ^r, and by Assumption 5 the set $\mathcal{X}^r$ is bounded. Moreover, V_N^{r0} is lower semicontinuous on $\mathcal{X}^r$ by Proposition 4.

We show that V_N^{r0} is a WISS-Lyapunov function on $\mathcal{X}^r$. From Assumption 5 and (23), we obtain the lower bound $\alpha_1(|x|) \leq V_N^{r0}(x)$ with $\alpha_1(\cdot) := \alpha_\ell(\cdot) \in \mathcal{K}_\infty$. Lemma 11 gives the upper bound $V_N^{r0}(x) \leq \alpha_2(|x|) + \sigma_2(\|\mathbb{W}\|)$ for some $\alpha_2(\cdot) \in \mathcal{K}_\infty$ and $\sigma_2(\cdot) \in \mathcal{K}$. Finally, Proposition 8 together with $\ell(x, \kappa^r(x)) \geq \alpha_\ell(|x|)$ yields

$$V_N^{r0}(f(x, \kappa^r(x), w)) \leq V_N^{r0}(x) - \alpha_\ell(|x|) + \sigma(\|\mathbb{W}\|)$$

i.e., (11) with $\alpha_3(\cdot) = \alpha_\ell(\cdot) \in \mathcal{K}_\infty$ and $\sigma_3(\cdot) = \sigma(\cdot) \in \mathcal{K}$.

Thus V_N^{r0} is a WISS-Lyapunov function on the robustly positive invariant and bounded set $\mathcal{X}^r$. Applying Proposition 10 gives the RASiW property on $\mathcal{X}^r$. □

We establish that the stage cost satisfies analogous worst-case asymptotic bounds.

Corollary 13 (ℓ-RASiW)**.** *Let Assumptions 1–5 hold. Then there exist* $\beta_\ell(\cdot,\cdot) \in \mathcal{KL}$ *and* $\gamma_\ell(\cdot) \in \mathcal{K}$ *such that*

$$\ell(\phi^r(k; x, \mathbf{w}_k), \kappa^r(\phi^r(k; x, \mathbf{w}_k))) \leq \beta_\ell(|x|, k) + \gamma_\ell(\|\mathbb{W}\|)$$

for all $x \in \mathcal{X}^r$*,* $k \in \mathbb{I}_{\geq 0}$*, and* $\mathbf{w}_k \in \mathbb{W}^k$*.*

Proof. As shown in the proof of Theorem 12, V_N^{r0} is a WISS-Lyapunov function on $\mathcal{X}^r$. Along the closed loop $x(k+1) = f(x(k), \kappa^r(x(k)), w(k))$, set $Y(k) := V_N^{r0}(x(k))$. By Proposition 10 (specifically Step 2 of its proof), $Y(k)$ satisfies the bound

$$Y(k) \leq \tilde{\beta}(Y(0), k) + \tilde{\gamma}(\|\mathbb{W}\|)$$

for some $\tilde{\beta}(\cdot,\cdot) \in \mathcal{KL}$ and $\tilde{\gamma}(\cdot) \in \mathcal{K}$. Using the WISS upper bound $Y(0) \leq \alpha_2(|x|) + \sigma_2(\|\mathbb{W}\|)$ and property (B.15) from Rawlings et al. (2020, App. B), we have

$$Y(k) \leq \tilde{\beta}(\alpha_2(|x|) + \sigma_2(\|\mathbb{W}\|), k) + \tilde{\gamma}(\|\mathbb{W}\|) \leq \tilde{\beta}(2\alpha_2(|x|), k) + \tilde{\beta}(2\sigma_2(\|\mathbb{W}\|), 0) + \tilde{\gamma}(\|\mathbb{W}\|)$$

Since $\ell(x, \kappa^r(x)) \leq V_N^{r0}(x) = Y(k)$ along the trajectory, we obtain

$$\ell(x(k), \kappa^r(x(k))) \leq \tilde{\beta}(2\alpha_2(|x|), k) + \tilde{\beta}(2\sigma_2(\|\mathbb{W}\|), 0) + \tilde{\gamma}(\|\mathbb{W}\|)$$

Define

$$\beta_\ell(s, k) := \tilde{\beta}(2\alpha_2(s), k), \qquad \gamma_\ell(s) := \tilde{\beta}(2\sigma_2(s), 0) + \tilde{\gamma}(s)$$

Then $\beta_\ell(\cdot,\cdot) \in \mathcal{KL}$, $\gamma_\ell(\cdot) \in \mathcal{K}$, and the claimed bound follows. □

6 Numerical Example

We illustrate the theoretical results with a liquid level control problem adapted from McAllister and Rawlings (2023). The system consists of two tanks in series as shown in Figure 2, where the objective is to regulate the liquid height in each tank. Tank 1 drains into tank 2 by gravity driven flow at a rate proportional to the height in tank 1. We extend the nominal setup of McAllister and Rawlings (2023) to the worst-case disturbance setting by treating the proportionality constant as an uncertain parameter subject to bounded disturbances, and we apply the RMPC design developed in Section to guarantee worst-case stability properties.

Let h_i denote the height in tank i and $F_i \in [0, 2]$ the flow rate, where F_1 is the inlet to tank 1 and F_2 is the outlet from tank 2. The flow from tank 1 to tank 2 is $(1+w)h_1$ where $w \in \mathbb{W} := \{-0.3, 0, 0.3\}$ represents uncertainty in the proportionality constant. The target steady state is $h_1^s = h_2^s = F_1^s = F_2^s = 1$. Define state and input deviation variables $x = [h_1 - h_1^s, h_2 - h_2^s]'$ and $u = [F_1 - F_1^s, F_2 - F_2^s]'$. The continuous-time dynamics are

$$\frac{dx_1}{dt} = -(1+w)x_1 + u_1 + w$$
$$\frac{dx_2}{dt} = (1+w)x_1 - u_2 + w$$

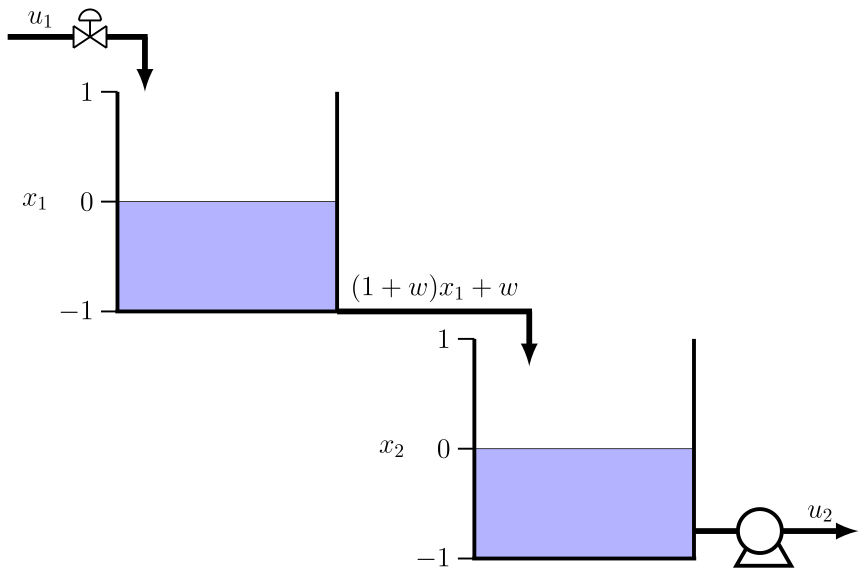

Figure 2: Liquid level control with two tanks. Tank 1 drains into tank 2 by gravity-driven flow with uncertain proportionality constant. Taken from McAllister and Rawlings (2023).

The nominal system with $w = 0$ is linear. The parametric uncertainty produces multiplicative terms $(1 + w)x_1$ that render the system nonlinear. Moreover, the additive term w ensures the disturbance effect does not vanish at the origin.

We discretize these equations exactly for all $w \in \mathbb{W}$ using zero-order hold with sampling time $\Delta = 1$. The finite support permits exact worst-case evaluation by enumerating all disturbance sequences. Flow rate constraints produce $u_i \in [-1, 1]$. The stage cost is $\ell(x, u) = x'Qx + u'Ru$ with

$$Q = \begin{bmatrix} 0.1 & 0 \\ 0 & 20 \end{bmatrix} \qquad R = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$$

The large penalty on x_2 strongly discourages height deviations in tank 2.

The terminal cost is $V_f(x) = x'Px$ where P is the H_∞ solution, and the terminal constraint is $\mathbb{X}_f := \{x : |x|_\infty \leq 0.4\}$. We solve both the nominal and RMPC problems via dynamic programming over discretized state and control grids.

Figure 3 shows closed-loop trajectories for horizon $N = 5$ from $x(0) = 0$. Thin lines show all disturbance realizations and thick lines with markers show the nominal trajectory with $w = 0$. Nominal MPC (left) regulates both states near the origin. RMPC (right) drives x_1 significantly negative, moving away from the origin and violating the terminal constraint $|x|_\infty \leq 0.4$ during the transient. This counterintuitive behavior arises because the large penalty on x_2 makes it optimal in the worst case to reduce tank 1 height, thereby limiting disturbance propagation to tank 2 through gravity-driven flow. RMPC sacrifices regulation of x_1 to achieve superior worst-case performance on the heavily weighted state x_2.

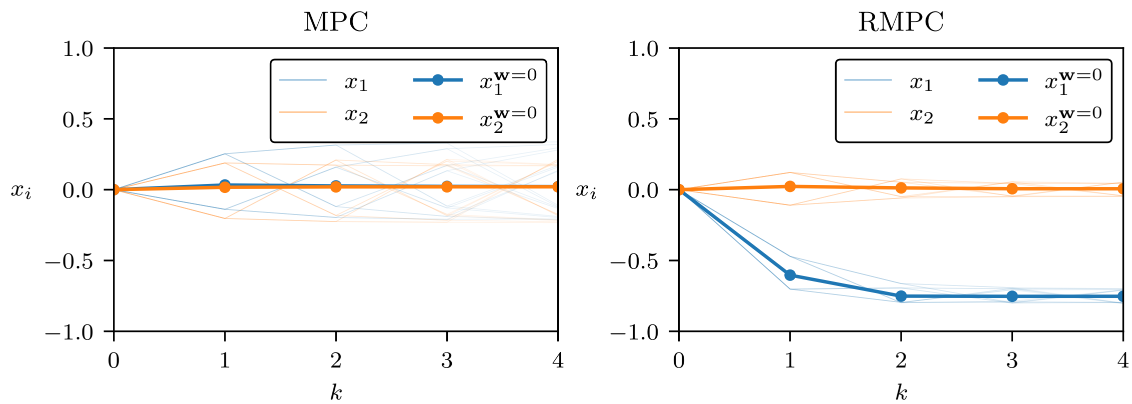

Figure 3: Closed-loop trajectories for nominal MPC (left) and RMPC (right) with horizon $N = 5$. Thin lines show all disturbance realizations; thick lines with markers show nominal trajectory with $\mathbf{w} = 0$

Figure 4 compares the worst-case closed-loop performance of nominal MPC and RMPC. The top panel shows $\max_{\mathbf{w}\in\mathbb{W}^k} |x(k)|_\infty$, the maximum state norm over all disturbance realizations at each time step. The bottom panel shows $\max_{\mathbf{w}\in\mathbb{W}^k} \ell(x(k), u(k))$, the maximum stage cost over all disturbance realizations. While RMPC achieves lower worst-case stage costs as expected from the minmax formulation, the worst-case state norm under RMPC exceeds that of nominal MPC. This occurs because RMPC deliberately drives x_1 negative to limit disturbance propagation to the heavily weighted state x_2, sacrificing state regulation to optimize worst-case cost performance.

7 Conclusion

This paper established robust asymptotic stability in worst-case sense for RMPC under stagewise bounded disturbances. Building on the input-to-state practical stability framework for min-max MPC (Limon et al., 2006; Lazar et al., 2008; Raimondo et al., 2009), we provided a self-contained analysis by defining worst-case input-to-state stability Lyapunov functions and proving that the RMPC optimal cost satisfies the required Lyapunov properties.

The main result (Theorem 12) guarantees that for any initial state $x(0) \in \mathcal{X}^r$ and any disturbance sequence satisfying $|w(k)| \leq \|\mathbb{W}\|$, the robust feasible set $\mathcal{X}^r$ is robustly positive invariant and the state satisfies the RASiW bound

$$|x(k)| \leq \beta(|x(0)|, k) + \gamma(\|\mathbb{W}\|)$$

with $\beta(\cdot,\cdot) \in \mathcal{KL}$ and $\gamma(\cdot) \in \mathcal{K}$. The $\mathcal{KL}$ function $\beta(\cdot,\cdot)$ ensures that the effect of the initial state decays with time, while $\gamma(\|\mathbb{W}\|)$ characterizes the size of the ultimate neighborhood. When $\|\mathbb{W}\| = 0$, the system converges to the origin.

The stability guarantee is practical rather than asymptotic. Because the controller is designed for the worst-case disturbance magnitude $\|\mathbb{W}\|$, even if the actual disturbance

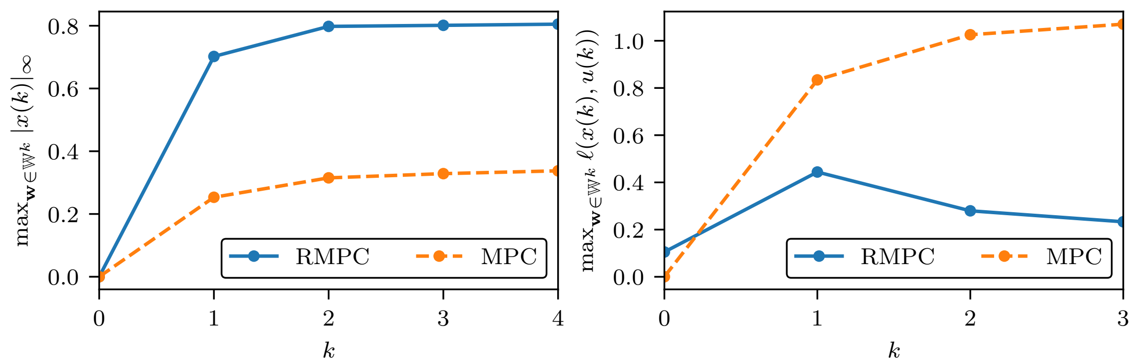

Figure 4: Worst-case closed-loop performance for nominal MPC and RMPC with horizon $N = 5$. Top: maximum state norm $\max_{\mathbf{w}\in\mathbb{W}^k} |x(k)|_\infty$ over all disturbance realizations. Bottom: maximum stage cost $\max_{\mathbf{w}\in\mathbb{W}^k} \ell(x(k), u(k))$ over all disturbance realizations.

realization is identically zero, the closed-loop system may converge only to a neighborhood of the origin rather than to the origin itself. The controller has no mechanism to exploit favorable disturbance realizations; it commits to a policy that performs well under the worst case, and this conservatism persists regardless of the actual disturbance experienced.

The analysis parallels the SMPC framework of McAllister and Rawlings (2023) but addresses the deterministic worst-case setting. Key differences include the use of suprema over compact disturbance sets rather than expectations, deterministic selection of control laws from possibly set-valued optimal solutions, and WISS-Lyapunov functions in place of SISS-Lyapunov functions. The results establish a parallel deterministic worst-case counterpart to SMPC stability theory, demonstrating that RMPC achieves practical input-to-state stability under standard regularity and terminal constraint assumptions.

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