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Robustness of MPC and Disturbance Models for Multivariable Ill-conditioned Processes

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Abstract

The problem of robust predictive control for multivariable ill-conditioned systems is addressed in this work. A technique for designing robust disturbance models is presented, which is based on an off-line min-max optimization problem. It is shown that the most common disturbance model – the output disturbance model – is not robust to modeling errors when the process model is ill-conditioned. On the other hand the input disturbance model shows robustness to uncertainties, and the optimal disturbance model obtained with the technique proposed is close to the input disturbance model. Application to a well-known ill-conditioned distillation column is presented.

Keywords

Model Predictive Control, Ill-conditioned systems, Disturbance modeling, Robust control, Distillation columns

1 Introduction

In the last two decades Model Predictive Control (MPC) has become one the most studied and applied control techniques both in academia and in the process industries. MPC arose from the pioneering industrial applications called Dynamic Matrix Control (DMC) [7] and Identification and Command (IDCOM) [23]. There are number of features that rendered MPC interesting:

- ability of operating on multivariable systems
- ability of dealing with non-minimum phase processes without requiring factorization in minimum and non-minimum phase, as required by other model based control technique (e.g. IMC)
- direct handling of input and output process constraints, using Quadratic Programming [8]

These control algorithms used finite impulse or step response models to predict the process behavior. Using past input information, the future control action is evaluated by minimizing a quadratic objective function in which the error between the predicted output and the reference signal appears together with a term related to the amplitude of the control action. Only the first control action is implemented and this optimization is repeated at each sampling time. In order to obtain offset-free control, the model is updated with feedback information. Comparing the current measured process output and the current predicted output, a constant bias term is added to the future model forecasts.

Despite its popularity, some of the assumptions on which the original formulation of MPC were based, limit the controller performance [14].

First of all, the use of a finite step (or impulse) response model may require a huge number of coefficients and does not allow to deal with open-loop unstable processes. Open-loop unstable and integrating dynamics are likely to occur in the process industries. Lee *et al.* [11] presented a state-space formulation of DMC that allowed to handle integrating processes, still using finite step response models. The most "natural" way to describe stable and unstable processes is, however, to use state-space models [16] or autoregressive models like in the Generalized Predictive Control (GPC) [4] [5].

Shinskey [27] clearly pointed out that DMC is able to outperform PID controllers on set-point changes but not on load changes introduced upstream of a dominant lag. Actually, the most efficient way to reject disturbances is to include them into a feed-forward scheme, and DMC can easily accomplish this end provided that the disturbance is measurable and its dynamic has been identified. The problem of a bad rejection of slow disturbances in DMC is strictly related to the addition of constant bias term based on feedback information. Lundström *et al.* [12] proposed a solution to this problem by modeling the disturbance as a double integrated white noise sequence. Under this assumption, therefore, the disturbance is assumed to be a ramp with piece-wise constant slope. In the GPC framework, the disturbance is modeled by the choice of the observer polynomial (usually called $T(q^{-1})$). Clarke and Mohtadi [3] clarify the role of this polynomial for enhancing the robustness and shaping the disturbance response. Typically, the polynomial T is chosen such that $1/T$ is a low-pass filter. Yoon and Clarke [31] provide useful consideration for the choice of T based on frequency analysis and they show how the disturbance polynomial can increase the robustness to plant-model mismatch. Prada *et al.* [19] present a comparative study of DMC and GPC, emphasizing the role of the polynomial T. Details for the multivariable extension of GPC can be found in [6] [13]. However, all the tuning considerations reported in the cited papers are given for SISO systems and they do not deal with ill-conditioned processes, which is the objective of this paper.

In the process industries (and in particular in chemical applications) ill-conditioned processes are rather frequent. A great effort of facing this problem has been carried out in the classical feedback framework [29], [1] [9], [24], [10], and in Internal Model Control theory [15], [2], [26]. Ill-conditioned processes arise when two (or more) manipulated inputs have almost the same effect to controlled variables. A tight control of processes with large interactions, like the ill-conditioned ones, requires an inversion of the process model. However, because of the ill-conditioning, this inversion becomes sensitive to input uncertainties and plant-model mismatch [29]. In the industrial implementations of MPC, this problem is faced in several ways [20]:

- In DMC, the input move suppression factor are adjusted to improve the conditioning of the matrix to be inverted.
- In SMC-IDCOM by Setpoint Inc, the user defines a priority ranking of controlled variables. When a high-condition number is detected, the controller drops low priority variables until a well-conditioned sub-process remains.
- In RMPCT (Robust Model Predictive Control Technology by Honeywell), the Singular Value Thresholding (SVT) method is used in which a singular value decomposition of the process model is carried out. Singular values below a threshold magnitude are discarded, and a process with a much lower ill-conditioning is reassembled and used for control.

Semino and Pannocchia [25], [17], propose to modify the nominal model in order to decrease the controller ill-conditioning. Application of "modified" DMC to a simulated binary distillation column can be found in [18].

In this paper, the problem of robust control of multivariable ill-conditioned processes is addressed in the general framework of MPC. The state-space formulation is chosen, and different disturbance models are selected and compared.

This paper is organized as follows. In Section 2 the MPC algorithm is summarized, which uses infinite horizon and state estimation. In Section 3 the choice of the disturbance model is addressed, by presenting tools useful for comparing different disturbance models. In Section 4 a case study, taken from the literature, is presented, in which the role of the disturbance model is emphasized by several simulations. Finally, in Section 5 we summarize the main results of this work.

2 MPC algorithm

2.1 Plant description

We assume that the plant is described by the following time-invariant, linear and discrete equations (the subscript p denotes "plant"):

$$\begin{aligned} x_{k+1} &= A_p x_k + B_p u_k + D_p \bar{d} \\ y_k &= C_p x_k + P_p \bar{d} \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$. $\bar{d}$ represents the plant disturbance vector while D_p and P_p are the dynamic matrices of the disturbance itself.

2.2 Model description and state estimation

The model used by the MPC controller is the following:

$$\begin{aligned} x_{k+1} &= A_m x_k + B_m u_k + D_m d_k \\ d_{k+1} &= d_k \\ y_k &= C_m x_k + P_m d_k \end{aligned} \tag{2}$$

where, in general, the dynamic matrices A_m, B_m and C_m can be different from the corresponding plant matrices. Here, the integrated disturbance vector $d_k \in \mathbb{R}^{n_d}$ is added in order to obtain offset-free control in the presence of plant-model mismatch and/or unmodeled disturbances. The choice of the disturbance model matrices D_m and P_m is a key-issue of this paper and, therefore, it is carefully discussed in the next section.

The augmented state $[x_k^T \quad d_k^T]^T$ is estimated from the plant measurements y_k by using the Linear Quadratic filtering theory. In this framework, the state is described as a stochastic variables and the model equations can be written as:

$$\begin{aligned} x_{k+1} &= A_m x_k + B_m u_k + D_m d_k + w_k^x \\ d_{k+1} &= d_k + w_k^d \\ y_k &= C_m x_k + P_m d_k + v_k \end{aligned} \tag{3}$$

where w_k^x, w_k^d and v_k are zero-mean uncorrelated random sequences that satisfy:

$$w^x \sim (0, Q_x) \qquad w^d \sim (0, Q_d) \qquad v \sim (0, R_v)$$

Thus, the model equations for the prediction become:

$$\begin{aligned} \hat{x}_{k+1|k} &= A_m \hat{x}_{k|k} + B_m u_k + D_m \hat{d}_{k|k} \\ \hat{d}_{k+1|k} &= \hat{d}_{k|k} \\ \hat{y}_k &= C_m \hat{x}_{k|k} + P_m \hat{d}_{k|k} \end{aligned} \tag{4}$$

Given the current measurement y_k, the filtering equations are:

$$\begin{aligned} \hat{x}_{k|k} &= \hat{x}_{k|k-1} + L_x(y_k - \hat{y}_{k|k-1}) \\ \hat{d}_{k|k} &= \hat{d}_{k|k-1} + L_d(y_k - \hat{y}_{k|k-1}) \end{aligned} \tag{5}$$

where L_x and L_d are the gain matrices of the steady-state Kalman filter. Several algorithms are available to compute this steady-state filter.

2.3 Regulator

The MPC control law used requires steady-state targets for the state and the input vectors. In the general case of different number of inputs and outputs, an optimization problem can be posed and solved with the respect of input and state constraints [21]. If we consider the unconstrained square case, a simpler solution exists.

The target calculation is based on the assumption that the disturbance estimate remains constant in the future, *i.e.* $d_s = \hat{d}_{k|k}$. Therefore, we can write:

$$\begin{aligned} x_{s|k} &= A_m x_{s|k} + B_m u_{s|k} + D_m \hat{d}_{k|k} \\ \bar{y} &= C_m x_{s|k} + P_m \hat{d}_{k|k} \end{aligned} \tag{6}$$

from which

$$\begin{aligned} u_{s|k} &= G_u^{-1}\bar{y} - G_u^{-1}(P_m + G_d)\hat{d}_{k|k} \\ x_{s|k} &= (I - A_m)^{-1}[B_m u_{s|k} + D_m \hat{d}_{k|k}] \end{aligned} \tag{7}$$

where:

$$G_u = C_m(I - A_m)^{-1}B_m \qquad G_d = C_m(I - A_m)^{-1}D_m$$

are the model steady-state gains of the input u and of the input disturbance estimate d.

The regulator objective function is defined over an infinite horizon [22] and is given by:

$$\begin{aligned} \Phi(\pi) = \frac{1}{2}\sum_{j=0}^{\infty} \big\{ &(\hat{y}_{k+j|k} - \bar{y})^T Q (\hat{y}_{k+j|k} - \bar{y}) + (u_{k+j} - u_{s|k})^T R (u_{k+j} - u_{s|k}) \\ &+ (u_{k+j} - u_{k+j-1})^T S (u_{k+j} - u_{k+j-1}) \big\} \end{aligned} \tag{8}$$

where $\bar{y}$ is the set-point reference vector, $\pi = \{u_k,\, u_{k+1},\, ,u_{k+2}, \ldots\}$ is the input sequence. Q, R and S are positive definite tuning matrices; usually, either R or S is zero. Eqn. 8 can be rewritten into the standard Linear-Quadratic formulation by augmenting the state vector:

$$\begin{gathered} x_j \leftarrow \begin{bmatrix} \hat{x}_{k+j|k} - x_{s|k} \\ u_{k+j-1} - u_{s|k} \end{bmatrix}, \quad u_j \leftarrow u_{k+j} - u_{s|k}, \\ Q \leftarrow \begin{bmatrix} C^T Q C & 0 \\ 0 & S \end{bmatrix}, \quad R \leftarrow R + S, \quad M = \begin{bmatrix} 0 \\ -S \end{bmatrix} \end{gathered}$$

where $x_{s|k}$ and $u_{s|k}$ are the state and the input target, as previously defined. Hence, the objective function becomes:

$$\Phi(\pi) = \frac{1}{2}\sum_{j=0}^{\infty} \left(x_j^T Q x_j + u_j^T R u_j + 2 x_j^T M u_j \right) \tag{9}$$

Then the regulation problem is defined in terms of finding the optimal sequence π, by solving:

$$\min_{\pi} \Phi(\pi) \tag{10}$$

subject to the model Eqn. 3 and, in general, subject to constraints on the input and the output. Only the first element of the optimal sequence π^* is injected into the plant and this optimization is repeated at each sampling time.

The unconstrained problem was formulated and solved by Kalman, and the solution is the well-known linear feedback control law:

$$u_k = K_x(\hat{x}_{k|k} - x_{s|k}) + K_u(u_{k-1} - u_{s|k}) + u_{s|k} \tag{11}$$

where K_x and K_u are the gain matrices of LQ regulator. Several numerical subroutines are available for solving this problem as well.

3 Choice of the disturbance model

3.1 Observability limitations

Using the Hautus lemma [30], we can obtained a number a conditions that have to be satisfied in order for the augmented system to be observable.

Lemma 3.1 (Hautus observability) *For a linear system defined by the matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, the Hautus observability matrix is:*

$$\mathcal{H} = \begin{bmatrix} \lambda I - A \\ C \end{bmatrix}$$

The following properties are equivalent:

- (A, C) *is an observable pair*
- $\operatorname{rank}\mathcal{H} = n \qquad \forall\lambda \in \mathbb{C}$
- $\operatorname{rank}\mathcal{H} = n \qquad \forall\lambda(A)$

In order to update the augmented state with (5) and therefore, to compute the Kalman gain matrices L_x and L_d, the augmented system has to be observable. The augmented system matrices are:

$$\tilde{A} = \begin{bmatrix} A_m & D_m \\ 0 & I_{n_d} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B_m \\ 0 \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} C_m & P_m \end{bmatrix}$$

Applying the Hautus lemma, we obtain that $(\tilde{A}, \tilde{C})$ is observable if and only if the following condition is satisfied

$$\operatorname{rank}\begin{bmatrix} \lambda I - A_m & -D_m \\ 0 & (\lambda - 1)I_{n_d} \\ C_m & P_m \end{bmatrix} = \operatorname{rank}\tilde{\mathcal{H}} = n + n_d \qquad \forall\lambda(\tilde{A}) \tag{12}$$

Since $\tilde{A}$ is block diagonal its eigenvalues are the union of the eigenvalues of A_m and of I_{n_d} (*i.e.* 1), the following theorem can be stated:

Theorem 3.1 *The augmented system $(\tilde{A}, \tilde{C})$ is observable if and only if the original system is observable and:*

$$\text{rank} \begin{bmatrix} I - A_m & -B_m \\ C_m & P_m \end{bmatrix} = n + n_d \tag{13}$$

From the previous theorem we also obtain a limitation on the dimension of the disturbance vector d, which is defined by the following lemma.

Lemma 3.2 *Given a observable system with n states and p independent output, the maximum number of integrated disturbances that can be added without loosing observability is given by:*

$$n_d \leq p \tag{14}$$

Since each integrated disturbance is responsible for the removal of the offset in one of the outputs, and since we want to remove offset in all the output, we choose:

$$n_d = p \tag{15}$$

3.2 Closed-loop system evolution

In this paragraph we write the evolution of the closed-loop system by using the equations presented for the plant, the state estimator and the regulator. Again, we remark the assumptions over which this calculation is based:

- The system is square.
- Input and state constraints are not active.

Combining Eqs. 1, 5 and 4 we can write a relation between the plant and the model state. Thereafter, using Eqs. 11 and 7, we obtain an expression of the control action as a function of the plant and the model state. Finally, we obtain:

$$\begin{bmatrix} x_k \\ \hat{x}_{k|k} \\ \hat{d}_{k|k} \\ u_{k-1} \end{bmatrix} = \Lambda \begin{bmatrix} x_{k-1} \\ \hat{x}_{k-1|k-1} \\ \hat{d}_{k-1|k-1} \\ u_{k-2} \end{bmatrix} + \Xi \bar{y} + \Theta \bar{d} \tag{16}$$

where the matrices Λ, Ξ and Θ are reported in Appendix A.1. Let z_k be the augmented state:

$$z_k = \begin{bmatrix} x_k \\ \hat{x}_{k|k} \\ \hat{d}_{k|k} \\ u_{k-1} \end{bmatrix}$$

The closed-loop system evolution can be described by:

$$\begin{aligned} z_{k+1} &= \Lambda z_k + \Xi \bar{y} + \Theta \bar{d} \\ y_k &= \Gamma z_k + P_p \bar{d} \end{aligned} \tag{17}$$

where

$$\Gamma = \begin{bmatrix} C_p & 0 & 0 & 0 \end{bmatrix}$$

The set-point reference $\bar{y}$ and the plant disturbance $\bar{d}$ appear in Eqn. 17 as exogenous terms.

Eqn. 17 is the base of further analyses and, therefore, a number of comment are appropriate:

- The initial state z_0 depends on the initial plant and model state and the last input. If we assume that the plant and the model states are at the origin, the initial augmented state is zero.
- Using z-transforms, Eqn. 17 can be used to evaluate the transfer functions from the the exogenous terms (*i.e.* the set-point reference and the disturbance) to the plant output usually referred to as "Complementary Sensitivity Function" and "Sensitivity Function". With minor changes, analogue transfer functions from the exogenous terms to the manipulated input can be computed.
- Writing an appropriate Lyapunov equation, we can use Eqn. 17 in order to evaluate to plant objective function for any given plant and model.

The latter approach is used in the next paragraph for comparing and selecting different disturbance models.

3.3 Performance evaluation

By augmenting the state with the exogenous terms, Eqn. 17 can be written as:

$$\tilde{z}_{k+1} = \Psi \tilde{z}_k \tag{18}$$

where

$$\tilde{z}_k = \begin{bmatrix} x_k \\ \hat{x}_{k|k} \\ \hat{d}_{k|k} \\ u_{k-1} \\ \bar{y} \\ \bar{d} \end{bmatrix} \qquad \Psi = \begin{bmatrix} \Lambda & \Xi & \Theta \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

The initial augmented state is:

$$\tilde{z}_0 = \begin{bmatrix} x_0 \\ \hat{x}_0 + L_x[C_p x_0 + P_p \bar{d} - (C_m \hat{x}_0 + P_m \hat{d}_0)] \\ \hat{d}_0 + L_d[C_p x_0 + P_p \bar{d} - (C_m \hat{x}_0 + P_m \hat{d}_0)] \\ 0 \\ \bar{y} \\ \bar{d} \end{bmatrix} \tag{19}$$

The plant objective function is the measure of the controller performance and is given by Eqn. 8 in which the output estimate $\hat{y}$ are replaced by the real plant output y. The objective function can be written in terms of augmented state $\tilde{z}$ as:

$$\begin{aligned}\Phi &= \sum_{j=0}^{\infty} \tilde{z}_j^T \tilde{Q} \tilde{z}_j \\ &= z_0^T \left\{ \tilde{Q} + \Psi^T \tilde{Q} \Psi + (\Psi^2)^T \tilde{Q} \Psi^2 + \cdots \right\} z_0 \end{aligned} \tag{20}$$

where $\tilde{Q}$ is reported in Appendix A.2. The term within brackets can be evaluated as solution of the following Lyapunov equation:

$$\tilde{S} = \tilde{Q} + \Psi^T \tilde{S} \Psi \tag{21}$$

Finally, the objective function becomes:

$$\Phi = \tilde{z}_0^T \tilde{S} \tilde{z}_0 \tag{22}$$

A number of comments are appropriate:

- Given a plant and a model, and specified the set-point reference and the plant disturbance, Eqn. 22 can be used to evaluate the closed-loop performance, without requiring simulations.

- A number of tuning knobs (regulator and estimator penalty matrices, disturbance model matrices) can be adjusted in order to minimize the objective function. Let χ be the chosen set of tuning parameters. We can pose an optimization problem as shown below:

  $$\min_{\chi} \Phi \tag{23}$$

  Since the purpose of this paper is to understand the implications of the disturbance model, we regard regulator and estimator penalty matrices as fixed and we only change the matrices D_m and P_m in order to minimize the objective function.

- In general the plant matrices are not exactly known. The true plant lies in a (bounded) region around the nominal plant (*i.e.* the model). Hence, Eqn. 22 can be used to find the worst case of performance within the plant region, by maximizing the objective function:

  $$\max_{A_p, B_p, C_p} \Phi \tag{24}$$

- Combining the minimization and maximization, Eqn. 22 can be used to find the tuning parameters that guarantee the best performance in the worst case of plant-model mismatch:

  $$\min_{\chi} \max_{A_p, B_p, C_p} \Phi \tag{25}$$

3.4 Disturbance model matrices

The disturbance model is defined by the matrices D_m and P_m that appear in Eqn. 4. Among all the infinite combinations of these matrices, two of them have a particular physical interpretation:

1. *Output disturbance model*, obtained by setting $D_m = 0$ and $P_m = I$. With this choice the difference between the predicted and the actual plant output is assumed to be caused by an output step disturbance, which remains constant in the future. This is the choice of all industrial implementations of MPC [20].

2. *Input disturbance model*, obtained by setting $D_m = B_m$ and $P_m = 0$. With this choice the difference between the predicted and the actual plant output is assumed to be caused by an input step disturbance, which remains constant in the future.

Despite its large use, the output disturbance model is rather unrealistic in the process industry. In fact, disturbances always enter upstream of the dominant process time constant and, in many cases, at the same point as the manipulated variables [27]. For ill-conditioned processes, moreover, input uncertainties act like disturbances entering at the input, and this is the reason why DMC (and similar MPC controllers) are sensitive to input uncertainties [12]. On the other hand, the input disturbance model is able to reject quickly disturbances with slow dynamics (sometimes called "ramp-like"). Moreover, the input disturbance model renders the MPC controller not sensitive to input uncertainty and, therefore, it increases the controller robustness for ill-conditioned processes.

In general, we can choose a disturbance model that has both D_m and P_m by solving an optimization problem like Eqn. 25. In order to simplify the problem, we can assume the following disturbance model:

$$D_m = B_m \Omega_D; \qquad P_m = I \Omega_P \tag{26}$$

where $\Omega_D = \mathrm{diag}[\omega_1^D, \ldots, \omega_m^D]$ and $\Omega_D = \mathrm{diag}[\omega_1^P, \ldots, \omega_p^P]$, in which ω_i^D and ω_i^P are coefficients in $[0, 1]$. Since we consider square systems only, Ω_D and Ω_P are diagonal matrices of the same dimension (equal to the number of inputs or outputs). Thus, the optimal disturbance model can be found by solving the following:

$$\min_{\Omega_D, \Omega_P} \max_{A_p, B_p, C_p} \Phi \tag{27}$$

It is important to point out that the result of this optimization depends on the exogenous signals, *i.e.* the set-point reference and the plant disturbance. For different plant disturbances, for example, different optimal tuning parameters are obtained. In the case study, we provide a further clarification of this point.

4 Case study

4.1 Plant and model description

The case analyzed here is a well-known binary distillation column [28] with the modifications applied by Lundström *et al.* [12] in order to include a measurement delay. The nominal model is:

$$\begin{bmatrix} y_D \\ x_B \end{bmatrix} = e^{-\theta s} \begin{bmatrix} \frac{0.878}{1+\tau_1 s} & \left(\frac{0.014}{1+\tau_2 s} - \frac{0.878}{1+\tau_1 s}\right) \\ \frac{1.082}{1+\tau_1 s} g_L(s) & \left(\frac{-0.014}{1+\tau_2 s} - \frac{1.082}{1+\tau_1 s}\right) \end{bmatrix} \begin{bmatrix} L_t \\ V_b \end{bmatrix} \tag{28}$$

where y_D and x_D are the top and the bottom logarithmic compositions, respectively, L_t and V_b are the reflux rate and the boil-up rate, respectively. The characteristic time constants are $\tau_1 = 194$min and $\tau_2 = 15$min and the delay time is $\theta = 1$min. $g_L(s)$ expresses the liquid flow dynamic:

$$g_L(s) = \frac{1}{(1 + \theta_L / n_T s)^{n_T}} \tag{29}$$

where $\theta_L = 2.46$min and n_T should be equal to the number of trays. As Lundström *et al.* [12], we choose $n_T = 5$ in order to avoid a model of unnecessary high order. Choosing a sampling time of 1min, a state-space realization of this model has been found. The system has 12 states, 2 inputs and 2 outputs. This column is a typical example of multivariable ill-conditioned system.

Independent input uncertainty is considered for robustness analysis, in which the magnitude of the uncertainty is bounded:

$$\begin{bmatrix} \Delta L_t \\ \Delta V_b \end{bmatrix}_{\text{actual}} = \begin{bmatrix} 1+\delta_1 & 0 \\ 0 & 1+\delta_2 \end{bmatrix} \begin{bmatrix} \Delta L_t \\ \Delta V_b \end{bmatrix}_{\text{computed}}$$
$$-0.2 \leq \delta_1, \delta_2 \leq 0.2 \tag{30}$$

4.2 Disturbance models and tuning

Output and input disturbance models, as described in Section 3.4, are used and compared. We denote MPC_1 the controller based on the output disturbance model and MPC_2 the controller based on the input disturbance model. Moreover, when an optimal disturbance model is found by solving Eqn. 27, we denote MPC_3 the controller based on this disturbance model. For this example, the optimization problem of Eqn. 27 becomes:

$$\min_{\omega_1^D, \omega_2^D, \omega_1^P, \omega_2^P} \max_{\delta_1, \delta_2} \Phi \tag{31}$$

s.t.

$$0 \leq \omega_1^D, \omega_2^D, \omega_1^P, \omega_2^P \leq 1$$
$$-0.2 \leq \delta_1, \delta_2 \leq 0.2$$

For all controllers, same tuning parameters have been chosen:

$$Q = 50I; \quad R = 0; \quad S = I$$

For all estimators, the following covariance matrices have been chosen:

$$Q_x = 10^{-6} I; \quad Q_d = I; \quad R_v = 10^{-6} I$$

4.3 Set-point change

A set-point change in the output unfavorable direction is considered in this section (obtained via Singular Value Decomposition, SVD):

$$\bar{y} = \begin{bmatrix} -0.78094 \\ 0.62460 \end{bmatrix}$$

In Figure 1 the value of the objective function *vs* the input uncertainties δ_1 and δ_2 is reported for MPC_1 (based on the output disturbance model). The corresponding plot for MPC_2 (based on the input disturbance model) is reported in Figure 2. From Figures 1 and 2 it appears that the robustness of MPC_1 is poor when the input uncertainties have opposite sign. On the other hand, the input disturbance model guarantees a robust performance over the entire input uncertainty region. Moreover, for this set-point change, the solution of the optimization in Eqn. 31 is:

$$\Omega_D^* = \begin{bmatrix} 0.1963 & 0 \\ 0 & 1.0000 \end{bmatrix}; \quad \Omega_P^* = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence, the optimal disturbance model for this set-point change is “almost” equal to the input disturbance model. In Figure 3 a comparative simulation of MPC_1, MPC_2 and MPC_3 (based on the optimal disturbance model) is reported for the uncertain case with:

$$\delta_1 = 0.2; \quad \delta_2 = -0.2$$

From Figure 3 it is clear that, despite the input uncertainty, MPC_2 guarantees a good performance because the input disturbance model quickly corrects the plant-model mismatch. MPC_1, instead, provides a sluggish and slow set-point change because the output disturbance model does not properly correct the input uncertainty. Moreover, the responses of MPC_2 and MPC_3 are almost identical.

4.4 Rejection of an output disturbance

The rejection of an output disturbance is considered in this section. Again, we consider a disturbance in the unfavorable direction of the plant, because this is the most difficult direction to reject:

$$\bar{d} = \begin{bmatrix} -0.78094 \\ 0.62460 \end{bmatrix}$$

In Figure 4 the value of the objective function *vs* the input uncertainties δ_1 and δ_2 is reported for MPC_1, while the corresponding plot for MPC_2 is reported in Figure 5. From Figures 4 and 5 it appears that MPC_1 rejects the disturbance better than MPC_2 in nominal conditions and when the input uncertainties have the same sign. However, when the input uncertainties have opposite sign, the output disturbance model is not able to correct properly this plant-model mismatch and the performance rapidly degrades. On the other hand, MPC_2 is able to face the input uncertainties and it guarantees a more robust performance

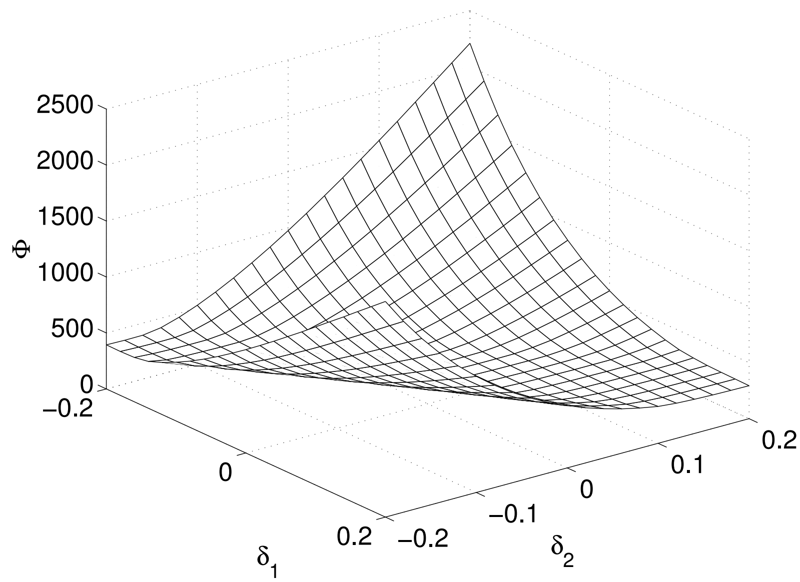

Figure 1: Objective function *vs* uncertainty for MPC_1 for the set-point change

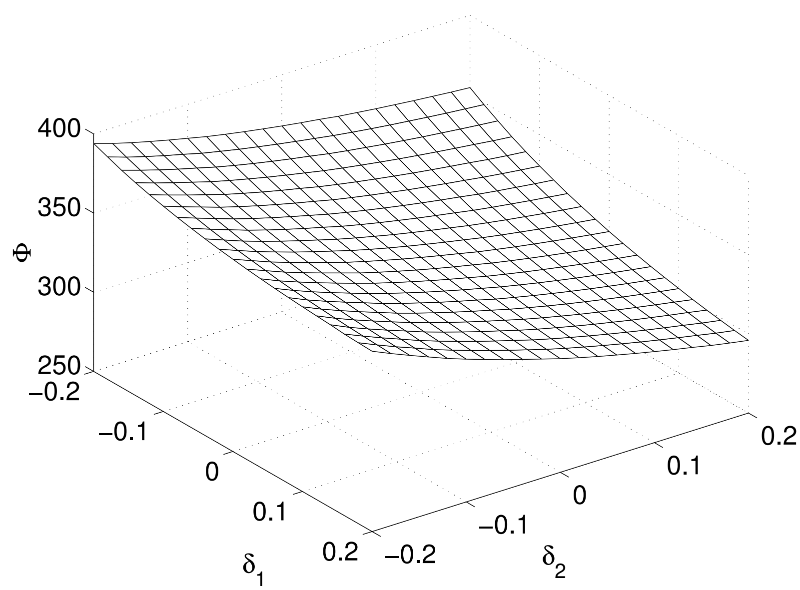

Figure 2: Objective function *vs* uncertainty for MPC_2 for the set-point change

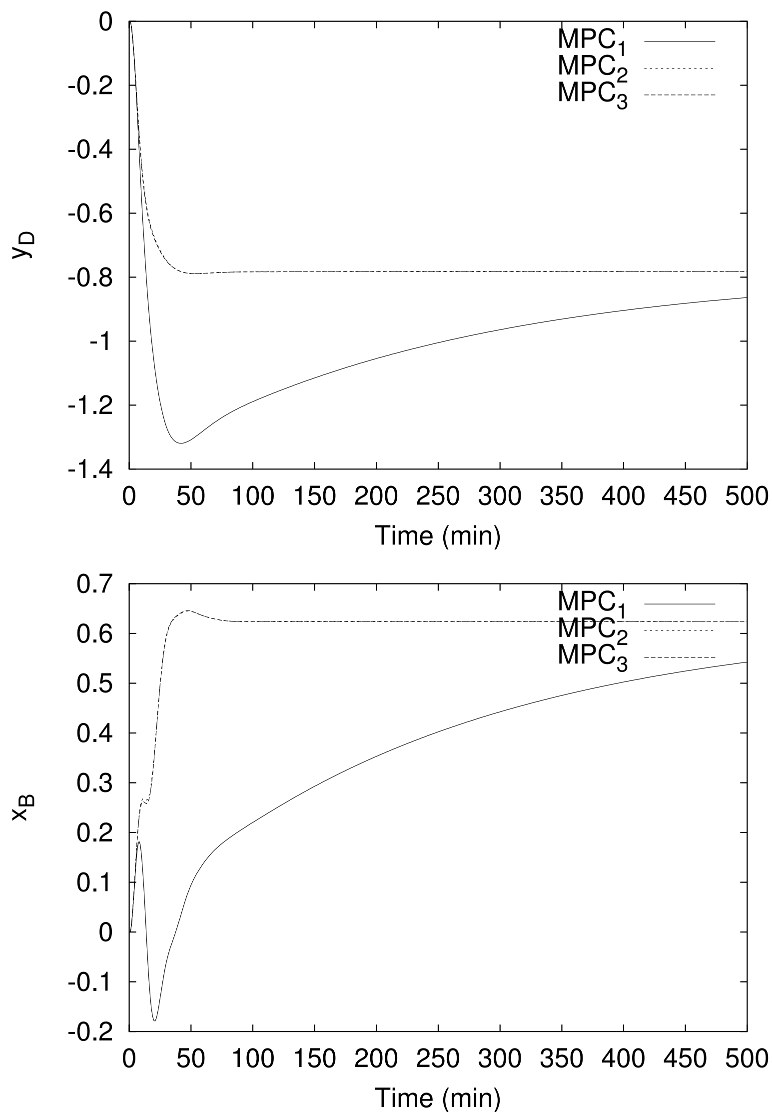

Figure 3: Comparative set-point change in the uncertain case

in the uncertain plant region. Moreover, for this disturbance rejection, the solution of the optimization in Eqn. 31 is:

$$\Omega_D^* = \begin{bmatrix} 0.8312 & 0 \\ 0 & 0.0005 \end{bmatrix}; \quad \Omega_P^* = \begin{bmatrix} 0.0421 & 0 \\ 0 & 0.9599 \end{bmatrix}$$

In Figure 6 a comparative simulation of MPC_1, MPC_2 and MPC_3 is reported for the uncertain case with:

$$\delta_1 = -0.2; \quad \delta_2 = 0.2$$

From 6 it appears that MPC_1 does not reject quickly the output disturbance when input uncertainty is present, while MPC_2 is faster even though the plant disturbance acts at the output. Finally, MPC_3 guarantees a performance that is slightly better than MPC_2.

4.5 Rejection of an input disturbance

In this section the rejection of an input disturbance in the favorable direction is considered:

$$\bar{d} = 10 \begin{bmatrix} -0.70655 \\ 0.70766 \end{bmatrix}$$

In Figure 7 the value of the objective function *vs* the input uncertainty δ_1 and δ_2 is reported for MPC_1, while the corresponding plot for MPC_2 is reported in Figure 8. From Figures 7 and 8 it appears that the performance of MPC_1 is poor even in nominal conditions. When input uncertainty is present, the disturbance rejection performance is even worse. MPC_2, instead, guarantees a good performance, which is almost insensitive to the plant-model mismatch. Moreover, for this disturbance rejection, the solution of Eqn. 31 is:

$$\Omega_D^* = \begin{bmatrix} 1.0000 & 0 \\ 0 & 1.0000 \end{bmatrix}; \quad \Omega_P^* = \begin{bmatrix} 0.0014 & 0 \\ 0 & 0.0013 \end{bmatrix}$$

In Figure 9 a comparative simulation of MPC_1, MPC_2 and MPC_3 is reported for the uncertain case with:

$$\delta_1 = -0.2; \quad \delta_2 = 0.2$$

Figures 9 clearly shows the the MPC_1 is not adequate to reject disturbance with slow dynamics. This behavior is amplified by the plant ill-conditioning in the presence of input uncertainty. On the other hand, MPC_2 guarantees a robust performance that is almost equal to the performance of MPC_3 based on the optimal disturbance model.

5 Conclusions

In this paper the problem of robust predictive control of multivariable ill-conditioned processes has been addressed by analyzing the implications of the choice of the disturbance model. A state-space realization of the closed-loop system has been obtained, which permits to evaluate the performance of the MPC controller. This tools has been used to analyze the robustness of different disturbance models in a bounded region where the

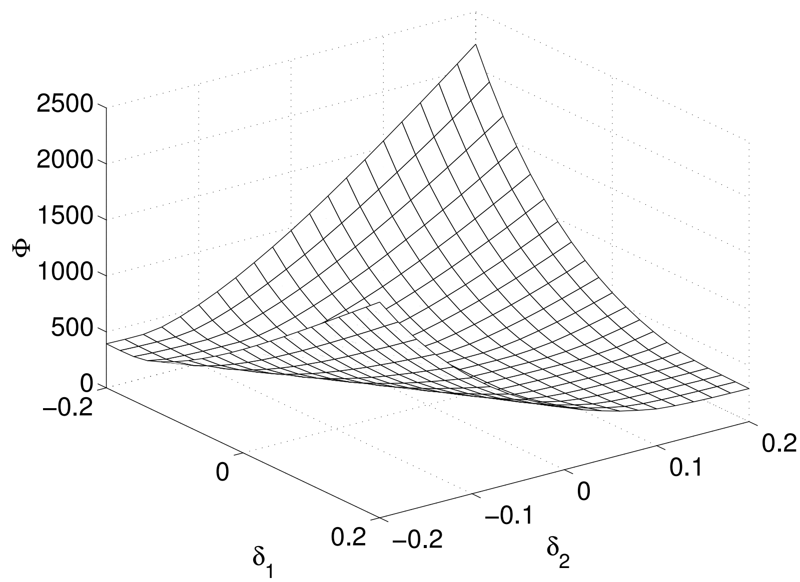

Figure 4: Objective function *vs* uncertainty for MPC_1 for the output disturbance rejection

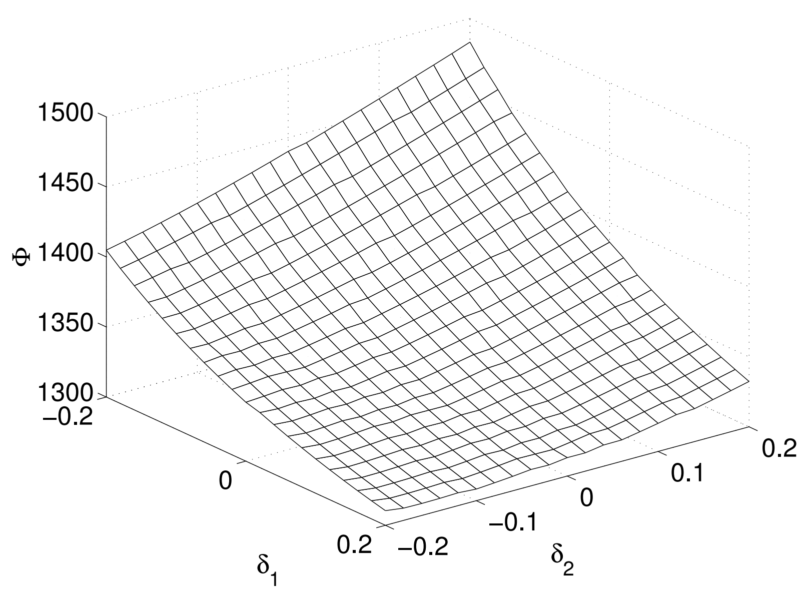

Figure 5: Objective function *vs* uncertainty for MPC_2 for the output disturbance rejection

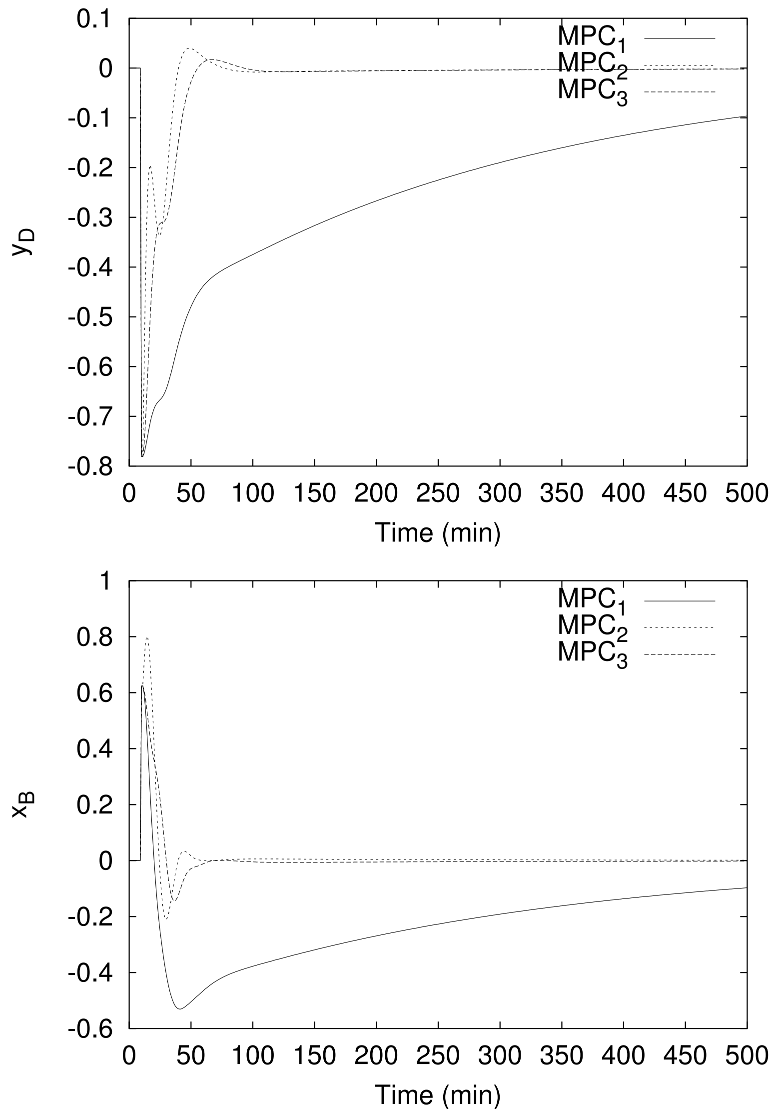

Figure 6: Comparative output disturbance rejection in the uncertain case

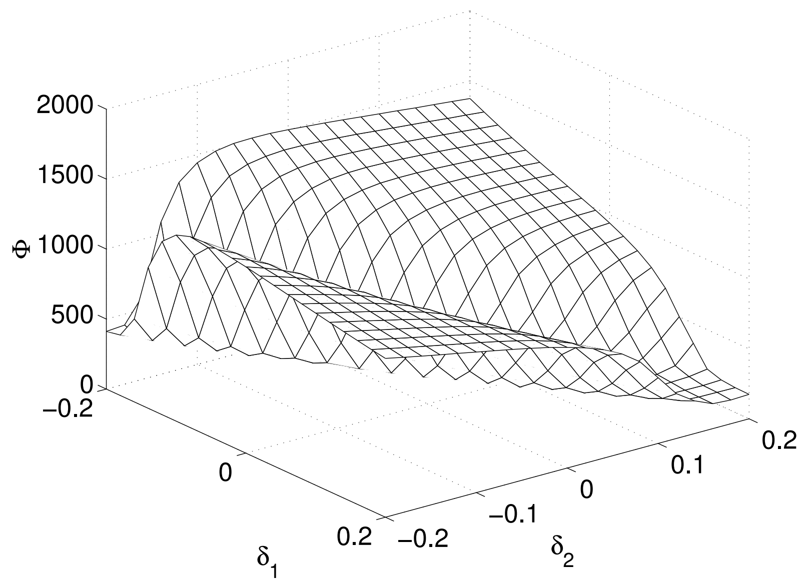

Figure 7: Objective function *vs* uncertainty for MPC_1 for the input disturbance rejection

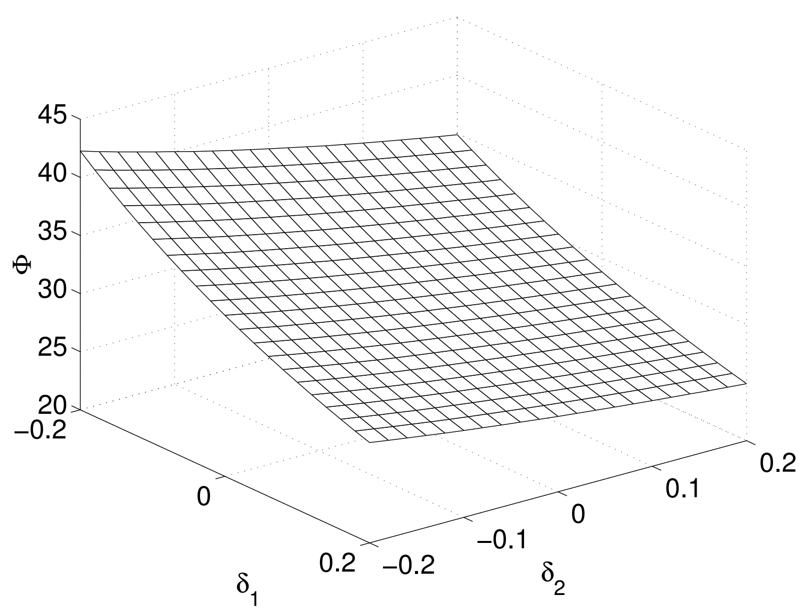

Figure 8: Objective function *vs* uncertainty for MPC_2 for the input disturbance rejection

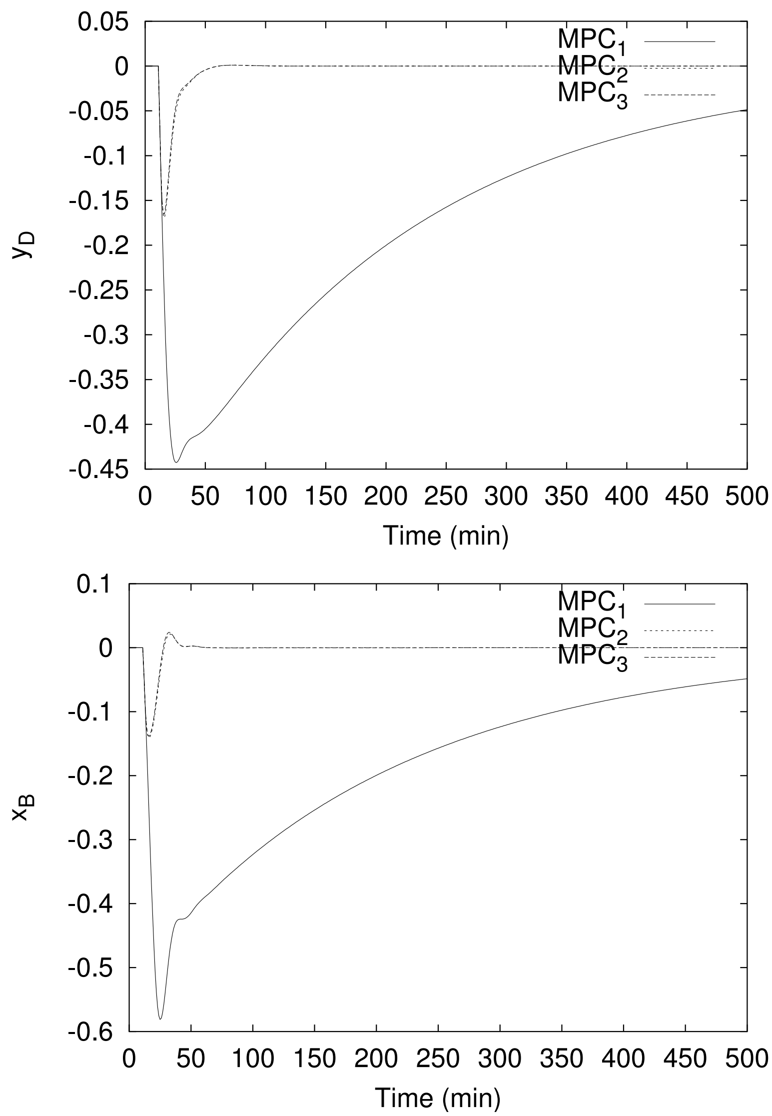

Figure 9: Comparative input disturbance rejection in the uncertain case

plant could lay, for any given set-point change or disturbance rejection. Moreover, a min-max optimization can be posed in order to find the disturbance model that guarantees the best performance in the worst case of plant-model mismatch. In particular, the output disturbance model and the input disturbance model have been compared for the control of a well-known ill-conditioned distillation column. For different cases of set-point change or disturbance rejection the optimal disturbance model has been found.

Industrial implementations of Model Predictive Control (like DMC) use an output disturbance model to correct the model prediction in the presence of plant-model mismatch. However, for ill-conditioned systems, the output disturbance model is not robust to input uncertainty and does not guarantee a good performance. Moreover, the output disturbance model is too slow when rejecting disturbances with slow dynamics ("ramp-like" disturbances). This problem becomes more dramatic when input uncertainty is present. It is important to point out that when there is input uncertainty the output disturbance model is not able to reject properly even an output disturbance for which it is optimally designed.

The input disturbance model, on the other hand, is able to overcome both the problems of robustness for ill-conditioned processes and the rejection of slow-dynamics disturbances. The increment in robustness clearly appears by comparing the objective function over the plant-model mismatch region. Moreover, in many cases the optimal disturbance model and the input disturbance model guarantee almost the same robust performance.

The output disturbance model shows a better performance than the input disturbance model only in the rejection of a "pure" output disturbance, which is unlike to occur in the process industries [27]. However, when input uncertainty is considered the weak robustness of the output disturbance model appears and the performance rapidly degrades.

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A Matrices

A.1 Matrices of Eqn. 16

$$\Lambda = \begin{bmatrix} \Lambda_1 & \Lambda_2 & \Lambda_3 & \Lambda_4 \end{bmatrix}$$

$$\Lambda_1 = \begin{bmatrix} A_p \\ L_x C_p A_p \\ L_d C_p A_p \\ 0 \end{bmatrix}; \quad \Lambda_2 = \begin{bmatrix} B_p \beta B_p K_x \\ (I - L_x C_m)(A_m + B_m K_x) + L_x C_p B_p K_x \\ -L_d C_m (A_m + B_m K_x) + L_d C_p B_p K_x \\ K_x \end{bmatrix}$$

$$\Lambda_3 = \begin{bmatrix} B_p \beta \\ (I - L_x C_m)(D_m + B_m \beta) - L_x P_m + L_x C_p B_p \beta \\ I - L_d C_m (D_m + B_m \beta) - L_d P_m + L_d C_p B_p \beta \\ \beta \end{bmatrix}$$

$$\Lambda_4 = \begin{bmatrix} B_p K_u \\ (B_m - L_x C_m B_m + L_x C_p B_p) K_u \\ (-L_d C_m B_m + L_d C_p B_p) K_u \\ K_u \end{bmatrix}$$

$$\Xi = \begin{bmatrix} B_p\alpha \\ B_m\alpha - L_xC_mB_m\alpha + L_xC_pB_p\alpha \\ -L_dC_mB_m\alpha + L_dC_pB_p\alpha \\ \alpha \end{bmatrix}; \quad \Theta = \begin{bmatrix} D_p \\ L_x(C_pD_p + P_p) \\ L_d(C_pD_p + P_p) \\ 0 \end{bmatrix}$$

$$\alpha = (I - K_u - K_x(I - A_m)^{-1}B_m)G_u^{-1}; \quad \beta = -\alpha(P_m + G_d) - K_x(I - A_m)^{-1}D_m$$

A.2 Matrices of Eqn. 20

$$\tilde{Q} = \begin{bmatrix} Q_1 & Q_2 & Q_3 & Q_4 & Q_5 & Q_6 \end{bmatrix}$$

$$Q_1 = \begin{bmatrix} C_p^TQC_p \\ 0 \\ 0 \\ 0 \\ -QC_p \\ P_p^TQC_p \end{bmatrix}; \quad Q_2 = \begin{bmatrix} 0 \\ K_x^T(R+S)K_x \\ (\beta-\delta)^TRK_x + \beta^TSK_x \\ K_u^TRK_x + (K_u - I)^TSK_x \\ (\alpha-\gamma)^TRK_x + \alpha^TSK_x \\ 0 \end{bmatrix}$$

$$Q_3 = \begin{bmatrix} 0 \\ [(\beta-\delta)^TRK_x + \beta^TSK_x]^T \\ (\beta-\delta)^TR(\beta-\delta) + \beta^TS\beta \\ K_u^TR(\beta-\delta) + (K_u - I)^TS\beta \\ (\alpha-\gamma)^TR(\beta-\delta) + \alpha^TS\beta \\ 0 \end{bmatrix}; \quad Q_4 = \begin{bmatrix} 0 \\ [K_u^TRK_x + (K_u - I)^TSK_x]^T \\ [K_u^TR(\beta-\delta) + (K_u - I)^TS\beta]^T \\ K_u^TRK_u + (K_u - I)^TS(K_u - I) \\ (\alpha-\gamma)^TRK_u + \alpha^TS(K_u - I) \\ 0 \end{bmatrix}$$

$$Q_5 = \begin{bmatrix} -C_p^TQ \\ [(\alpha-\gamma)^TRK_x + \alpha^TSK_x]^T \\ [(\alpha-\gamma)^TR(\beta-\delta) + \alpha^TS\beta]^T \\ [(\alpha-\gamma)^TRK_u + \alpha^TS(K_u - I)]^T \\ Q + (\alpha-\gamma)^TR(\alpha-\gamma) + \alpha^TS\alpha \\ -P_p^T \end{bmatrix}; \quad Q_6 = \begin{bmatrix} C_p^TQP_p \\ 0 \\ 0 \\ 0 \\ -P_p \\ P_p^TQP_p \end{bmatrix}$$

$$\gamma = G_u^{-1}; \quad \delta = G_u^{-1}(P_m + G_d)$$