Linear Model Predictive Control of Chemical Processes

by

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DISSERTATION

Presented to the Faculty of the Graduate School of
The University of Texas at Austin
in Partial Fulfillment
of the Requirements
for the Degree of

DOCTOR OF PHILOSOPHY

THE UNIVERSITY OF TEXAS AT AUSTIN
May 1995

This dissertation was prepared with the LaTeX document preparation system using additional macros written by John W. Eaton based on the thesis style written by Khe-Sing The for The University of Texas at Austin.

Printed in the United States of America.

10 9 8 7 6 5 4 3 2 1

To my parents, Karl and Barbara, for their love, encouragement, and support.

Acknowledgments

Long after the classes I took, the articles I struggled to understand, the insights I had that eventually led to progress, and the many dead ends that I encountered during the course of my research all fade into oblivion, it is my sincere belief that I will have a clear and pleasant memory of those I shared this experience with. During my graduate studies at The University of Texas at Austin, I was fortunate enough to interact with a number of people from whom I have benefited greatly. I gratefully acknowledge those who have offered their advise, assistance, encouragement, and, most importantly, friendship. The following people deserve special recognition.

I would first like to express my thanks and appreciation to my thesis advisor, Professor James B. Rawlings. I have thoroughly enjoyed our work together and consider myself very fortunate to have been under his direction. Our interaction has been an invaluable learning experience. Whatever accomplishments I have been able to make during my graduate studies are a direct result of his guidance, insight, and inspiration.

I would also like to acknowledge my undergraduate and graduate advisor at Northwestern University, Professor William F. Stevens. His enthusiasm, motivation, and commitment to teaching kindled the interest to continue my graduate studies, albeit after an extended industrial sabbatical.

I would like to thank Professors Thomas F. Edgar, David M. Himmelblau, Leon S. Lasdon, and Jeff S. Shama for their gracious service on my dissertation committee. I have gained much from their conversations, questions, and suggestions.

The support of my family from the beginning of my education all the way through to the completion of my graduate studies must be acknowledged. I am grateful for the sacrifices they made and for the environment that encouraged me to succeed. This work is just as much a product of their efforts as mine.

The help and encouragement of Babatunde Ogunnaike, Frank Kleinschrodt, James Stice, Ming Fang, Norm Jerome, Tom Badgwell, and Will Monin are gratefully acknowledged. Thanks to Chuck and Rosa Alsop for inviting me to stay with them during my summer at DuPont's Experimental Station and for graciously allowing me to cut the grass and help them fix their house.

I am grateful to Laurice, Jim, Dianne, Jean, John, Beth, Art, Paul, and Hae-kyung for their support and for making my trips back to Chicago and

Wisconsin so enjoyable. Thanks to Bill, Cheng, Frank, Sarah, Jean, Joel, John, and Pat for the times away from my research spent canoeing, kayaking, fishing, and just hanging around.

Finally, I would like to express my gratitude to the Rawlings research group: Missa Trueper, who quickly untangled every bureaucratic nightmare thrown my way and always with a smile; Walt Witkowski, who helped me get settled in when I first joined the group; Andy Fordyce, who pointed out why one should not end a sentence with a preposition and then graduated just so I could have his desk; John Eaton, who made my work possible by creating and maintaining a fabulous computing environment; Steve Miller, who always seemed to be able to lift my spirits with his fellowship, humor, and amazing grasp of covariance; Scott Meadows, who helped guide me through the mathematical obscurity and was always able to help with that last result just before the publication deadlines; Tony Mullins, who put up with me in the same office for five years, served as a part-time weightlifting and spelling coach, and graciously allowed me to help him fix his house; Rock Matthews, who could be counted on for a stimulating conversion about any number of outdoor and athletic activities and, more importantly, passed me the ball for the three-point shot during the intermural basketball games; John Campbell, who thoughtfully moved into an apartment two hundred yards away from mine so we could share Vicki's cooking; Scott Middlebrooks and Pierre Scokaert, who offered a number of welcome diversions during the preparation of this document; and Rahul Bindlish and Chris Rao, who reminded me what it was like during my first semester. This was a great group to be associated with both academically and personally.

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The University of Texas at Austin
May 1995

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Publication No. ______________

Kenneth Robert Muske, Ph.D.
The University of Texas at Austin, 1995

Supervisor: James B. Rawlings

Model predictive control has become one of the dominant methods of chemical process control in terms of successful industrial applications and as a focus of academic research. The relevant articles in the engineering literature, which range in scope from descriptions of industrial applications to theoretical analyses, clearly illustrate this fact. Most of the applications have been based on the implementations of linear model predictive control developed by the process industries to control constrained, multivariable chemical processes. The emphasis in the development of these controllers was a robust algorithm with acceptable performance that could be implemented on-line. Therefore, several aspects of these controllers were designed based on a heuristic approach with little theoretical justification. This design philosophy produced controllers that performed very well for certain processes, but were unable to adequately address others. After more than a decade of experience with this technology, these limitations have become apparent.

This study presents a theoretical analysis of linear model predictive control that addresses these limitations. By exploiting the features common to both linear model predictive control and linear quadratic regulator/estimator theory, many of the heuristic design features that have restricted the technology are removed. Linear state-space models are used to represent the process to address unstable as well as stable plants. The incorporation of a stabilizing, constrained receding horizon regulator guarantees nominal stability for all valid tuning parameters and eliminates the need to tune for nominal stability. Output feedback is performed using linear state estimation techniques. These techniques provide increased flexibility in the design of the disturbance model for the process within a well-established framework. Off-set free control also can be guaranteed with the use of these techniques. Target and reference trajectory tracking are obtained by using results from standard linear quadratic regulator theory.

The result of this research is the creation of a framework for the development of an industrially implementable controller that improves the current technology. This framework provides a rigorous and flexible theoretical basis that retains, and in several cases enhances, the features necessary to handle chemical process control applications.

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Nomenclature

Upper Case Symbols

A, B, C	—	state-space process model matrices
A^f, C^f	—	state-space measured disturbance model matrices
A^m, B^m, C^m	—	state-space feedforward model matrices
A^r, C^r	—	state-space reference trajectory model matrices
$\bar{A}, \bar{B}, \bar{C}$	—	augmented state-space trajectory tracking model matrices
$\tilde{A}, \tilde{B}, \tilde{C}$	—	augmented state-space disturbance model matrices
$\acute{C}$	—	state-space controlled variable matrix
D	—	input constraint matrix
E_j	—	receding horizon estimator quadratic program matrix
F	—	state constraint matrix
F_j	—	receding horizon estimator quadratic program matrix
G	—	receding horizon regulator quadratic program matrix
G_1	—	finite horizon contribution to matrix G
G_2	—	terminal state penalty contribution to matrix G
G_d	—	unmeasured state disturbance matrix
G_j	—	receding horizon estimator quadratic program matrix
G_p	—	unmeasured output disturbance matrix
H	—	receding horizon regulator quadratic program Hessian
H_1	—	finite horizon contribution to matrix H
H_2	—	terminal state penalty contribution to matrix H
I	—	identity matrix
J	—	eigenvalue matrix of A
J^r	—	trajectory tracking initial condition matrix
J_s	—	stable eigenvalue matrix of A
J_u	—	unstable eigenvalue matrix of A
$J(\epsilon)$	—	discrete time index
K	—	linear quadratic regulator state feedback gain
L	—	Kalman filter or deterministic observer gain matrix

L_d — constant state disturbance filter gain matrix
L_f — measured disturbance filter gain matrix
L_m — feedforward model state filter gain matrix
L_p — constant output disturbance filter gain matrix
L_x — process model state filter gain matrix
M — receding horizon regulator constraint matrix
N — control horizon length, observer horizon length
O — feasible state constraint set
P — discrete filtering Riccati matrix
Q — open-loop state penalty matrix
$\acute{Q}$ — open-loop controlled variable penalty matrix
Q_ν — covariance matrix of ν
Q_ω — covariance matrix of ω
Q_s — controlled variable tracking error penalty matrix
Q_w — covariance matrix of w
Q_N — open-loop terminal state penalty
$\acute{Q}_N$ — open-loop terminal controlled variable penalty
$\tilde{Q}_w$ — covariance matrix of augmented state-space model
R — open-loop input penalty matrix
R_s — input target tracking error penalty matrix
R_v — covariance matrix of v
$\tilde{R}_v$ — covariance matrix of augmented state-space model
S — open-loop input rate of change penalty matrix
U — receding horizon regulator quadratic program matrix
V — generalized eigenvector matrix of A
V_s — stable generalized eigenvector matrix of A
V_u — unstable generalized eigenvector matrix of A
W — receding horizon regulator state constraint matrix

Lower Case Symbols

$\hat{d}$ — estimated unmeasured state disturbance vector
e — reconstruction error vector

f	— state constraint vector, measured disturbance
i_1	— receding horizon regulator quadratic program constraint vector
i_2	— receding horizon regulator quadratic program constraint vector
m_1	— receding horizon regulator quadratic program constraint vector
m_2	— receding horizon regulator quadratic program constraint vector
$\hat{p}$	— estimated unmeasured output disturbance vector
u	— closed-loop input vector
$u_{\max}$	— maximum input constraint vector
$u_{\min}$	— minimum input constraint vector
u_s	— steady-state input vector
u_t	— steady-state input target vector
v	— open-loop input vector
$\hat{v}$	— estimated output disturbance vector
$\hat{w}$	— estimated state disturbance vector
$\hat{w}_{\max}$	— maximum state disturbance constraint vector
$\hat{w}_{\min}$	— minimum state disturbance constraint vector
x	— closed-loop process model state vector
x^f	— measured disturbance model state vector
x^m	— feedforward model state vector
x^r	— reference trajectory model state vector
x_s	— steady-state process model state vector
x_s^m	— steady-state feedforward model state vector
$\hat{x}$	— estimated state vector
$\hat{x}^f$	— estimated measured disturbance model state vector
$\hat{x}^m$	— estimated feedforward model state vector
$\hat{x}_{\max}$	— maximum estimated state constraint vector
$\hat{x}_{\min}$	— minimum estimated state constraint vector
y	— measured output vector

y^c — controlled variable vector
y_a^c — achievable controlled variable target
y_t^c — controlled variable target
$y_{\max}^c$ — maximum controlled variable constraint vector
$y_{\min}^c$ — minimum controlled variable constraint vector
z — open-loop state vector
z^s — stable modes of A
z^u — unstable modes of A

Greek Symbols

ΔI — receding horizon regulator quadratic program constraint matrix
Δi_1 — receding horizon regulator quadratic program constraint vector
Δi_2 — receding horizon regulator quadratic program constraint vector
Δu — change in the closed-loop input vector
$\Delta u_{\max}$ — maximum input rate of change constraint vector
$\Delta u_{\min}$ — minimum input rate of change constraint vector
Δv — change in the open-loop input vector
κ — matrix condition number
λ — eigenvalue
Φ^N — receding horizon regulator quadratic objective function
Ψ^N — moving horizon estimator quadratic objective function
ν — zero-mean, normal output disturbance noise vector
ω — zero-mean, normal state disturbance noise vector

Script Symbols

$\mathcal{C}$ — controllability matrix
$\mathcal{F}$ — feasible estimated state set
$\mathcal{O}$ — observability matrix
$\mathcal{U}$ — feasible input set

$\mathcal{V}$	— feasible unstable A initial state and reconstruction error set
$\mathcal{W}$	— feasible estimated state disturbance set

Chapter 1

Introduction

Linear model predictive control refers to a class of control algorithms that compute a manipulated variable profile by utilizing a linear process model. Using this model, a linear or quadratic open-loop performance objective is optimized over a future time horizon subject to linear constraints. The first move of this open-loop optimal manipulated variable profile is then implemented. This procedure is repeated at each control interval or sampling time with the process measurements used to update the optimization problem.

This class of control algorithms, which is also referred to as receding horizon control, moving horizon control, or predictive control, has several advantages for application in chemical processes. The controller uses a linear transfer function, state-space, or convolution model to represent the process. These models can be obtained from process tests using time series analysis techniques that do not require a significant fundamental modeling effort. Multivariable processes can easily be handled by superposition of these linear models. Optimization of the open-loop performance objective is performed by either linear or quadratic programming algorithms. These algorithms are efficient and robust, which is essential for on-line applications. Constraints on the manipulated and controlled variables are incorporated into the performance objective optimization, which allows for operation close to the process constraints.

The application of linear model predictive control began with the chemical process industries in the late 1970's in an attempt to improve the control of constrained, multivariable processes. The most efficient and profitable operation for the majority of these processes is at one or several constraints.

However, the interactions in highly constrained multivariable processes made it difficult to operate safely close to the constraints with the standard control technology of that time. With a multivariable process model to describe the interactions and the ability to incorporate future constraints, model predictive control was ideally suited to these problems and a number of formulations were developed and implemented by industrial researchers. In addition to the industrially developed controllers, there have also been several implementations of linear model predictive control presented by academic researchers. Most of these implementations were developed for adaptive control.

1.1 Linear Model Predictive Controller Summary

The industrial model predictive control implementations began with Model Algorithmic Control, MAC, developed by Richalet *et al.* [72] and Dynamic Matrix Control, DMC, developed by Cutler and Ramaker [16]. The implementation in [72] is also referred to as IDCOM. Linear Dynamic Matrix Control, LDMC, which uses a linear objective function and incorporates constraints explicitly, is outlined by Morshedi *et al.* [62]. García and Morshedi [28] discuss Quadratic Dynamic Matrix Control, QDMC, which is an extension of DMC incorporating a quadratic performance function and explicit incorporation of constraints. Grosdidier *et al.* [32] present IDCOM-M, which is an extension of IDCOM using a quadratic programming algorithm to replace the iterative solution technique of the original implementation. Marquis and Broustail [54] discuss Shell Multivariable Optimizing Control, SMOC.

The model predictive control implementations developed by academic researchers include a constrained, multivariable linear programming approach by Chang and Seborg [9], a constrained, multivariable algorithm similar to Quadratic Dynamic Matrix Control discussed by Ricker [73], Simplified Model Predictive Control, SMPC, presented by Arulalan and Deshpande [2], and Receding Horizon Tracking Control, RHTC, presented by Kwon and Byun [49]. Implementations developed for adaptive control include the MUSMAR approach presented by Mosca *et al.* [63], Extended Horizon Adaptive Control, EHAC, presented by Ydstie [86], Extended Prediction Self-Adaptive Control, EPSAC, presented by De Keyser and Van Caunwenberge [18], Multivariable Optimal Constrained Control Algorithm, MOCCA, presented by Sripada and Fisher [81], and Generalized Predictive Control, GPC, presented by Clarke *et al.* [13].

A more complete discussion of model predictive control implementations is contained in the review articles by De Keyser *et al.* [19], Byun and Kwon [8], and García *et al.* [29]. These papers present comparisons of several of the implementations listed previously. The differences between these implementations are in the form of the linear model and performance objective, the choice of horizon, and the tuning parameters.

Impulse or step response models, which are referred to as convolution models, are used in all of the implementations developed by industry except SMOC. Convolution models are also used in several of the other controllers mentioned previously. Although convolution models have historically been the model form of choice and can easily represent very complex stable process dynamics, these model forms are non-minimal representations and cannot model unstable processes. Hovd *et al.* [38] and Eaton and Rawlings [22] present a finite step response model that can represent an integrating process. In order to use convolution models to represent an unstable process, the process must be modeled as an integrator. This imposes a limitation on the performance that can be achieved by the controller due to the structural error in the model.

The nominal stability guarantees available for linear model predictive controllers require restrictions on either the tuning parameters or the plant models that can be considered. The following results are also limited to the unconstrained controller. Rouhani and Mehra [78] discuss stability of Model Algorithmic Control for stable systems. García and Morari [27] discuss stability of Dynamic Matrix Control in the framework of Internal Model Control for stable systems. Clarke *et al.* [14], Clarke and Mohtadi [12], and Clarke [11] discuss stability of the Generalized Predictive Control algorithm by the choice of both tuning parameters and horizon length. Scattolini and Bittanti [79] discuss stability of both Generalized Predictive Control and Extended Horizon Adaptive Control by the choice of horizon length for stable systems. Byun and Kwon [8] discuss sufficient conditions for stability of Generalized Predictive Control and Extended Horizon Adaptive Control based on tuning parameters. Maurath *et al.* [55] present a necessary condition for the stability of a SISO model predictive controller for stable systems.

For the constrained controller, there are fewer stability results in the literature. Gutman and Hagander [33] present a stabilizing saturated linear state feedback controller. Zafiriou [88] and Zafiriou and Marchal [89] discuss the contraction properties of Quadratic Dynamic Matrix Control subject to output constraints. Sznaier and Damborg [82] present a modified receding horizon formulation that is stable for certain classes of constraints. Clarke and Scattolini [15] and Rossiter and Kouvaritakis [77] present a stabilizing constrained Generalized Predictive Control algorithm.

The industrial implementations of model predictive control are all based on one simple output feedback method. In these controllers, the difference between the model prediction and the measured output is assumed to be caused by a step output disturbance that remains constant in the future. This disturbance model has the advantage of being very easy to implement with convolution models and also yields offset free control. The disadvantage is that it is unrealistic for many processes and, therefore, cannot adequately address many practical applications without external signal processing or controller detuning. This output feedback method also cannot be used for unstable plants. Li *et al.* [52], Ricker [74], and Morari and Lee [61] discuss more general disturbance modeling with Quadratic Dynamic Matrix Control while retaining the convolution model for the implementation.

1.2 Project Motivation

The model predictive controllers outlined previously share the same general structure with many of the features that originated in the industrial implementations. Consequently, they also share the same deficiencies. The most significant deficiency is that nominal stability is not guaranteed. Nominal stability refers to stability of the controller with an exact model of the process. These controllers either must be tuned to achieve nominal stability or the linear systems that can be considered must be restricted. A second deficiency of the implementations that use convolution models is that they are unable to adequately address unstable processes. Finally, the output feedback methods developed for the industrial controllers are generally not representative of the actual disturbances to the process. Their use requires external signal processing to obtain acceptable performance in many applications. The motivation for this work is toward the development of an industrially implementable controller that addresses these deficiencies of the current technology while retaining those features necessary for chemical process control applications.

A number of contributions to receding horizon control theory that have either motivated parts of this work or have been used directly are summarized here. Stability analysis of dynamic systems using Lyapunov stability theory is presented by Kalman and Bertram for continuous-time systems in [41] and for discrete-time systems in [42]. Bitmead *et al.* [7] demonstrate the poor nominal stability properties of the unconstrained finite horizon linear quadratic regulator and, based on this analysis, recommend abandoning the finite horizon regulator in favor of an infinite horizon approach. Thomas [83] presents a nominally stable receding horizon regulator for linear systems by penalizing only the input and imposing a terminal state constraint such that the state is zero at the end of the horizon. Kwon and Pearson [47] and Kwon *et al.* [46] extend this result to time-varying linear systems using the standard linear quadratic regulator performance objective. Mayne and Michalska [57] utilize a quadratic performance objective and a final time constraint for nonlinear systems and show that the resulting receding horizon regulator is stabilizing under certain conditions. Keerthi and Gilbert [44] discuss a constrained receding horizon regulator for nonlinear discrete-time systems and show that stability can be guaranteed under certain conditions on the objective function and constraints. Keerthi and Gilbert [43] and Isidori and Byrnes [39] consider the existence of solutions to the control problem. Gilbert and Tan [31] discuss the representation of an infinite set of linear inequalities by a finite set for discrete-time linear systems that allows for the practical implementation of an infinite constraint horizon.

1.3 Project Goals

In order to increase the functionality of model predictive control, many of the heuristic design features of the original implementations should be removed. Since linear model predictive control and the linear quadratic regulator/estimator share several common features, linear regulator and estimation theory is an obvious place to begin. However, many of the heuristics in model predictive control are due to the presence of constraints in the controller. The constraints make the linear model predictive controller a nonlinear regulator that results in a nonlinear closed-loop system. Therefore, it is necessary to extend this theory to constrained linear quadratic regulators. These extensions can then be used as the theoretical basis for a new linear model predictive control formulation. The goal of this work is the creation of a framework for the future development of linear model predictive control technology. This framework is intended to provide a rigorous theoretical basis and design philosophy from which an industrially implementable controller can be developed.

1.4 Dissertation Overview

Chapter 2 presents the constrained receding horizon regulator that forms the foundation of the linear model predictive controller developed in this work. This regulator is a constrained state feedback controller that uses a discrete state-space model of the process. The chapter begins with a discussion of discrete state-space models, receding horizon regulators, and the regulator constraints that are considered. Nominal asymptotic stability of the constrained receding horizon regulator is then demonstrated followed by a brief discussion of an alternative regulator formulation that also provides nominal asymptotic stability.

The receding horizon regulator developed in Chapter 2 is designed to control the state of the process model to the origin. Non-zero controlled variable target tracking is discussed in Chapter 3 in which a target tracking receding horizon regulator formulation is presented. The non-zero target is determined by an optimization that minimizes the controlled variable tracking error at steady state.

The target tracking regulator in Chapter 3 is a state feedback controller. If measurements of the full state vector of the process model are not available, estimates of the state must be used. Chapter 4 discusses state estimation from the available output measurements for linear state-space models. A constrained moving horizon estimator is also presented and shown to be an exponentially stable observer. The estimate of the state is then used in the output feedback receding horizon regulator developed in Chapter 5.

Chapter 6 discusses unmeasured constant disturbance models that are augmented with the process state-space model to provide offset free control with the output feedback regulator. Offset free control ensures that offset between the controlled variables and their targets is eliminated in the presence of unmeasured disturbances and mismatch between the process and the process model provided the closed loop is stable and reaches a steady state. Rejection of measured disturbances is discussed in Chapter 7 in which a feedforward/feedback receding horizon regulator is developed. The final chapter serves as an overview of the linear model predictive control framework presented in this document and discusses future research directions.

The constrained receding horizon regulator and moving horizon estimator require the solution of a quadratic programming optimization problem at each control interval. Appendices A and B present the quadratic program formulation for the regulator and estimator, respectively, and discuss the properties of the solution. Appendix C discusses the converse Lyapunov stability theorems presented in Halanay [35] used to demonstrate asymptotic stability for the output feedback receding horizon regulator in Chapter 5. The remaining appendices contain the proofs for the lemmas and theorems presented in each of the chapters in the text.

This work is based on the results presented in the following publications. Rawlings and Muske [71] present the nominally stable, constrained, state feedback receding horizon regulator discussed in Chapter 2. Muske and Rawlings [64] develop a linear model predictive controller based on this receding horizon regulator with output feedback and consider the control of unstable processes. Muske and Rawlings [65] discuss the features necessary for implementation of the linear model predictive controller such as target tracking of non-square systems, reference trajectory tracking, unmeasured disturbance modeling, and feedforward control. Offset free control is considered in Rawlings *et al.* [70]. Muske *et al.* [67] and Meadows *et al.* [59] present the constrained moving horizon estimator discussed in Chapter 4. References to the appropriate examples presented in these publications are used to illustrate the corresponding issues discussed in this document.

1.5 Mathematical Definitions

This work considers discrete-time dynamic systems of the form

$$x_{k+1} = \mathbf{f}(x_k) \tag{1.1}$$

in which $x_k \in \Re^n$ is the state of the system, $\mathbf{f} : \Re^n \to \Re^n$, and $k \geq 0$ is the discrete time index. The definitions for stability and convergence used in this work are based on those presented in Yoshizawa [87], Kwakernaak and Sivan [45], and Vidyasager [85]. Asymptotic stability in Definition 1.6 differs from the standard definition in that the neighborhood of convergence is specified.

Definition 1.1 *A point $x_e \in \Re^n$ is an* equilibrium point *of the system in Eq. 1.1 if $\mathbf{f}(x_e) = x_e$.*

Definition 1.2 *An equilibrium point x_e of the system in Eq. 1.1 is* stable *if, for every $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$, such that if $\|x_0 - x_e\| < \delta(\epsilon)$, then $\|x_k - x_e\| < \epsilon$ for all $k \geq 0$.*

Definition 1.3 *The sequence $\{x_k\}$* converges *to a point x_e if for every $\epsilon > 0$ there exists a $J(\epsilon)$ such that $\|x_k - x_e\| < \epsilon$ for all $k \geq J(\epsilon)$. The point x_e is the limit of the sequence $\{x_k\}$ denoted by $\lim_{k\to\infty} x_k = x_e$.*

Definition 1.4 *An equilibrium point x_e is an* asymptotically stable *solution of Eq. 1.1 if it is stable and there exists a $\delta > 0$ such that for $\|x_0 - x_e\| < \delta$, the sequence $\{x_k\}$ determined from the solution of Eq. 1.1 with initial condition x_0 converges to x_e.*

Definition 1.5 *An equilibrium point x_e is an* exponentially stable *solution of Eq. 1.1 if there exists a $\delta > 0$ such that for $\|x_0 - x_e\| < \delta$ there exists a $\rho > 0$ and $0 < \lambda < 1$ such that $\|x_k - x_e\| \leq \rho\lambda^k$ for all $k \geq 0$.*

Definition 1.6 *An equilibrium point x_e is an* asymptotically stable *solution of Eq. 1.1 for all $x_0 \in X^n$, in which X^n is a linear subspace of $\Re^n$, if it is stable and the sequence $\{x_k\}$ determined from the solution of Eq. 1.1 with initial condition x_0 converges to x_e for all $x_0 \in X^n$.*

Definition 1.7 *An equilibrium point* x_e *is a* globally asymptotically stable *solution of Eq. 1.1 if it is stable and the sequence* $\{x_k\}$ *determined from the solution of Eq. 1.1 with initial condition* x_0 *converges to* x_e *for all* $x_0 \in \Re^n$*.*

Definition 1.8 *The norm operator* $\| \; \|$ *represents the Euclidean or* l_2 *norm for a vector argument,* x*, and the induced matrix norm for a matrix argument,* A*.*

$$\|x\| = \sqrt{x^T x}$$
$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

Definition 1.9 *The matrix operator* Null(A) *represents the null space of the matrix* $A \in \Re^{m \times n}$ *which is defined as the following linear subspace.*

$$\mathrm{Null}(A) = \{x \in \Re^n \mid Ax = 0\}$$

Chapter 2

Constrained Receding Horizon Regulator

This chapter discusses the constrained receding horizon regulator used in the model predictive control formulation. The chapter begins with a presentation of the linear discrete state-space dynamic model followed by a brief review of the linear quadratic open-loop optimal and receding horizon regulators, which form the basis for the regulator in this work. Constraints on the input and state variables are discussed and then the constrained regulator is presented and shown to be nominally asymptotically stable. The discussion in this chapter assumes that the state is controlled to the origin and that perfect measurements of the state are available for feedback to the regulator. Controlled variable target tracking is discussed in Chapter 3. Output feedback is discussed in Chapter 5.

2.1 Linear Dynamic Model

The time-invariant, discrete state-space formulation is used as the dynamic process model.

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k, \qquad k = 0, 1, 2, \dots \\ y_k &= Cx_k \end{aligned} \tag{2.1}$$

In this formulation, $y \in \Re^p$ is the output or measured variable vector, $u \in \Re^m$ is the input or manipulated variable vector, and $x \in \Re^n$ is the state vector.

The linear system matrices are $A \in \Re^{n \times n}$, $B \in \Re^{n \times m}$, and $C \in \Re^{p \times n}$.

Since model predictive control techniques are implemented on process control computers, only discrete measurements of the outputs are available. The difference equation in Eq. 2.1 normally represents a continuous-time system in which the outputs are measured at equally spaced time intervals or sampling periods. It is an exact representation when the system is linear, the input remains constant over the sampling period, and the output is sampled at the same time the input is applied [45]. The last two conditions can typically be assumed for chemical process model predictive control applications due to the long process response times that allow for sampling periods much larger than the computational time typically required by the controller.

There are several advantages of state-space models over discrete transfer function and convolution models. The state-space formulation provides an exact parametric representation for all linear system dynamics including unstable processes. The number of model parameters is not a function of the sample period or process dynamics as is the case for convolution models. Minimal state-space representations are obtained using standard minimum realization techniques [75]. Reduced order state-space representations can also be constructed using standard techniques [68]. Controller calculations are matrix operations that can be computed efficiently using standard linear algebra software packages such as LAPACK [1].

Multivariable systems described by discrete transfer function or convolution models are easily transformed into an equivalent state-space model as shown in [69]. A reduced order state-space form of the step response convolution model is presented in [52]. Dead time can be added to a discrete state-space model with state augmentation as shown in [26].

Stabilizability. The processes considered for the constrained receding horizon regulator are restricted to stabilizable systems. This restriction ensures that the unstable modes or subspace of A can be controlled by the input. If the model matrices (A, B) are not stabilizable, there is no regulator that can control all of the unstable modes of the system.

The unstable modes of the system span the unstable linear subspace of A. They can be determined by partitioning the Jordan form of the A matrix into stable and unstable parts in which the unstable eigenvalues of A are contained in J_u and the stable eigenvalues of A are contained in J_s.

$$A = VJV^{-1} = [V_u \; V_s] \begin{bmatrix} J_u & 0 \\ 0 & J_s \end{bmatrix} \begin{bmatrix} \tilde{V}_u \\ \tilde{V}_s \end{bmatrix} \tag{2.2}$$

The unstable and stable modes, z^u and z^s respectively, satisfy the following relationships.

$$x = [V_u \ V_s] \begin{bmatrix} z^u \\ z^s \end{bmatrix} \tag{2.3}$$

$$\begin{bmatrix} z^u_{k+1} \\ z^s_{k+1} \end{bmatrix} = \begin{bmatrix} J_u & 0 \\ 0 & J_s \end{bmatrix} \begin{bmatrix} z^u_k \\ z^s_k \end{bmatrix} + \begin{bmatrix} \tilde{V}_u \\ \tilde{V}_s \end{bmatrix} Bu_k \tag{2.4}$$

The unstable subspace is the linear subspace spanned by the columns of V_u. The stable subspace is the linear subspace spanned by the columns of V_s.

The linear system in Eq. 2.1 is stabilizable if the unstable subspace of A is contained within the controllable subspace. The controllable subspace consists of the states that can be reached from the origin within a finite number of input moves and is also referred to as the reachable subspace. This space is the linear subspace spanned by the columns of the controllability matrix $\mathcal{C}$ [45].

$$\mathcal{C} = \left[B,\ AB,\ A^2B,\ \ldots,\ A^{n-1}B\right] \tag{2.5}$$

When $\mathcal{C}$ is full rank, the system is referred to as controllable and the controllable subspace is $\Re^n$.

Detectability. When the output measurements are used to estimate the state of the system, as discussed in Chapters 4 and 5, the process models that can be considered are further restricted to detectable systems. This restriction ensures that the unstable modes or subspace of A can be observed from the output measurements. The linear system in Eq. 2.1 is detectable if its unobservable subspace is contained within the stable subspace of A. The unobservable subspace consists of the states x_0 for which $y_k = CA^k x_0 = 0$ for all $k \geq 0$ and is also referred to as the unreconstructable subspace. The unobservable subspace is the null space of the observability matrix $\mathcal{O}$ [45].

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{2.6}$$

When $\mathcal{O}$ is full rank, the system is referred to as observable and the observable subspace is $\Re^n$.

Discrete Transfer Function Realizations. State-space models derived from minimal realizations of discrete transfer function models are controllable and observable [10]. Therefore, all discrete transfer function matrix minimal realizations can be considered in this work.

2.2 Open-loop Optimal Regulator

Given the discrete state-space model in Eq. 2.1, an unconstrained N stage open-loop optimal linear quadratic regulator is constructed based on the minimization of the following quadratic cost or objective function.

$$\min_{\pi} \Phi^N(x_0, \pi) = z_N^T Q_N z_N + \sum_{j=0}^{N-1} \left(z_j^T Q z_j + v_j^T R v_j \right) \tag{2.7}$$

$$\text{Subject to:} \quad \begin{aligned} z_0 &= x_0 \\ z_{j+1} &= A z_j + B v_j, \quad j = 0, \ldots, N-1 \end{aligned} \tag{2.8}$$

The quadratic objective function Φ^N is minimized over the vector of N future inputs represented by π.

$$\pi = \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{N-1} \end{bmatrix} \tag{2.9}$$

In order to guarantee a unique solution to this least squares problem, Q and R are normally restricted to positive definite matrices.

This regulator determines the N future control moves that bring the state of the system close to the origin with minimal cost. The use of large input moves, which is normally not desirable in practice, is inhibited by the quadratic penalty on the future inputs. The optimal control law is the solution of the least squares problem obtained from the minimization of Eq. 2.7. As shown in Bertsekas [5], the result is a linear function of the state at each stage that can be computed recursively as follows.

$$v_j = -(B^T P_{N-j} B + R)^{-1} B^T P_{N-j} A z_j = -K_{N-j} z_j \tag{2.10}$$

$$P_{j+1} = A^T \left(P_j - P_j B (B^T P_j B + R)^{-1} B^T P_j \right) A + Q \tag{2.11}$$

$$P_1 = Q_N$$

The recursion in Eq. 2.11 is referred to as the Riccati difference equation [7].

2.3 Linear Quadratic Regulator

Since the optimal open-loop regulator computes a finite number of future control moves based on the initial state, it is not useful for controlling continuous chemical processes. However, a receding horizon approach based on the open-loop optimal regulator is applicable to these processes. In the unconstrained receding horizon regulator, the open-loop optimal control problem is solved and the first input in the vector π is applied to the process. At each successive sample time, this procedure is repeated using the current value of the state and computing all N future open-loop control moves in which only the first move in π is ever implemented. This technique takes advantage of feedback from the process in the determination of the current input by the use of the current state at each sample time. The result is the closed-loop state feedback control law

$$u_k = -K_N x_k \tag{2.12}$$

in which the feedback gain K_N is determined from the recursion in Eq. 2.10.

The state trajectory for the nominal system with no disturbances is determined from Eqs. 2.1 and 2.12 as follows.

$$x_{k+1} = (A - BK_N)\, x_k \tag{2.13}$$

The linear system in Eq. 2.13 is nominally stable if the eigenvalues of $(A - BK_N)$ have moduli strictly less than one. Nominal stability is not guaranteed for every positive definite choice of Q and R for all N. Kwon and Byun [49] and Bitmead *et al.* [7] discuss sufficient conditions on the horizon length N and terminal penalty matrix Q_N to ensure stability. Kwon and Pearson [48] present a nominally stabilizing regulator by the incorporation of a terminal state constraint that forces the state to the origin after the N input moves.

The most common method of obtaining nominal stability without tuning is to use an infinite open-loop horizon by allowing $N \to \infty$ in the objective function in Eq. 2.7. When (A, B) is stabilizable, $(A, Q^{1/2})$ is detectable, R is positive definite, and Q_N is non-negative definite, the recursion in Eq. 2.11 converges to a constant matrix P_∞ yielding a nominally stabilizing feedback gain [7].

$$P_\infty = A^T \left(P_\infty - P_\infty B(B^T P_\infty B + R)^{-1} B^T P_\infty\right) A + Q \tag{2.14}$$

$$K_\infty = -(B^T P_\infty B + R)^{-1} B^T P_\infty A \tag{2.15}$$

The disadvantage of this method is that the addition of constraints results in a nonlinear control law that is normally determined from the solution of a quadratic program. The quadratic program for an infinite open-loop horizon is an infinite-dimensional optimization problem. Since constraints are one of the most important issues that must be addressed in chemical process control applications, this infinite horizon regulator must be abandoned in favor of an approach that results in a finite-dimensional optimization problem.

2.4 Unconstrained Receding Horizon Regulator

The approach adopted in this work is an infinite horizon open-loop regulator with a finite number of decision variables. The infinite horizon is chosen to guarantee nominal stability for all valid tuning parameters. The finite number of decision variables is required to obtain a finite-dimensional quadratic programming problem when constraints are present. In the formulation considered here, the decision variables are the first N open-loop input moves represented by π. After this time, the open-loop input is set to zero. The result is a receding horizon regulator based on the minimization of the following open-loop quadratic objective function discussed in Gauthier and Bornard [30] and Rawlings and Muske [71].

$$\min_{\pi} \Phi_k^N(x_k, \pi) = \sum_{j=0}^{N-1} \left(z_j^T Q z_j + v_j^T R v_j \right) + \sum_{j=N}^{\infty} \left(z_j^T Q z_j \right) \tag{2.16}$$

$$\text{Subject to:} \quad \begin{array}{rcl} z_0 & = & x_k \\ z_{j+1} & = & A z_j + B v_j, \quad j = 0, 1, \ldots \\ v_j & = & 0, \quad j \geq N \end{array} \tag{2.17}$$

In this open-loop objective, Q is a symmetric, positive semidefinite penalty matrix on the states and R is a symmetric, positive definite penalty matrix on the input. At time $j = N$, the future input is set to zero and kept at this value for all time $j > N$.

Stable Process Model. For stable A, the infinite sum in the objective function in Eq. 2.16 can be transformed into a penalty on the terminal state.

$$\sum_{j=N}^{\infty} \left(z_j^T Q z_j \right) = z_N^T Q_N Z_N$$

The terminal state penalty matrix is computed from the following discrete Lyapunov equation.

$$Q_N = \sum_{j=0}^{\infty} A^{jT} Q A^j = A^T Q_N A + Q \tag{2.18}$$

There are standard methods available for the solution of this equation as discussed in Bartles and Stewart [4] and Laub [51].

Unstable Process Model. For unstable A, the unstable modes of the system must be brought to the origin at time N. If the unstable modes are not zero, they will evolve uncontrolled and not converge since the input is zero after this time. Therefore, the open-loop objective function in Eq. 2.16 is subject to the following equality constraint on the state at time N in which $\tilde{V}_u$ is determined as shown in Eq. 2.2.

$$z_N^u = \tilde{V}_u z_N = 0 \tag{2.19}$$

This equality constraint ensures that z_N has no component in the unstable subspace of A. Since only stable modes contribute to the value of Φ^N after time $N-1$, Q_N for unstable A can be computed from the stable modes of A in a manner similar to Eq. 2.18 in which $\tilde{V}_s$, V_s, and J_s are defined in Eq. 2.2.

$$Q_N = \tilde{V}_s^T \Sigma \tilde{V}_s \tag{2.20}$$

$$\Sigma = \sum_{j=0}^{\infty} J_s^{jT} V_s^T Q V_s J_s^j = J_s^T \Sigma J_s + V_s^T Q V_s \tag{2.21}$$

Implementation of the receding horizon regulator based on the quadratic program in Eqs. 2.16, 2.17, and 2.19 requires feasibility of the equality constraint for an optimal solution to exist. Therefore, the regulator must be restricted to stabilizable systems with $N \geq r$, in which r is the number of unstable modes in A. If the system is not stabilizable, there exist uncontrollable, unstable modes that cannot be brought to zero. If the number of control moves is less than the number of unstable modes, the unstable modes cannot all be brought to zero from an arbitrary initial condition. Both of these cases will result in the infeasibility of the equality constraint in Eq. 2.19, which allows the regulator to detect that the system cannot be stabilized. When $N = r$, the open-loop optimal regulator is a deadbeat regulator on the unstable modes of A.

The approach of Kwon and Pearson [48] is a nominally stabilizing receding horizon regulator based on a finite horizon objective subject to a terminal state constraint. The terminal constraint forces all of the modes of the system to be zero at the end of the horizon instead of only the unstable modes. This constraint leads to aggressive control action with small values of N for both stable and unstable systems since the regulator approaches a deadbeat controller. Feasibility of this terminal constraint also requires that the system be controllable rather than detectable. This point is illustrated by Example 1 in Muske and Rawlings [65].

2.5 Constraints

Linear input and state constraints of the following form are considered

$$Dv_j \leq \mathrm{d}, \qquad j = 0, 1, \ldots \tag{2.22}$$

$$Fz_j \leq \mathrm{f}, \qquad j = 1, 2, \ldots \tag{2.23}$$

in which $\mathrm{d} = [d_1, \ldots, d_s]^T$, $\mathrm{f} = [f_1, \ldots, f_q]^T$, and $d_i, f_i > 0$. The restrictions on d_i and f_i guarantee that the constraints specify a convex region in the input space containing a neighborhood of the origin. Maximum and minimum rate of change constraints on the input are also considered in which $\Delta v_j = v_j - v_{j-1}$, $\Delta_{\min} = [\Delta d_1^{\min}, \ldots, \Delta d_m^{\min}]^T$, $\Delta_{\max} = [\Delta d_1^{\max}, \ldots, \Delta d_m^{\max}]^T$, and $\Delta d_i^{\max}, \Delta d_i^{\max} > 0$.

$$-\Delta_{\min} \leq \Delta v_j \leq \Delta_{\max} \tag{2.24}$$

In order to guarantee that the origin can be reached from all u such that $Du \leq \mathrm{d}$, the following restrictions are imposed on the input constraints and control horizon length

$$\mathrm{d} \leq D(N+1)\min(\Delta_{\max}, \Delta_{\min}) \tag{2.25}$$

in which $\min(\Delta_{\max}, \Delta_{\min})$ is defined as

$$\min(\Delta_{\max}, \Delta_{\min}) = \begin{bmatrix} \min(\Delta d_1^{\min}, \Delta d_1^{\max}) \\ \vdots \\ \min(\Delta d_m^{\min}, \Delta d_m^{\max}) \end{bmatrix} \tag{2.26}$$

The convex constraint regions in the state and input space are then defined as follows.

$$\mathcal{X}^n = \{x \in \Re^n \mid Fx \leq \mathrm{f}\} \tag{2.27}$$

$$\mathcal{U}^m = \{u \in \Re^m \mid Du \leq \mathrm{d},\ \mathrm{d} \leq D(N+1)\min(\Delta_{\max}, \Delta_{\min})\} \tag{2.28}$$

The constraints are specified on an infinite horizon in order to guarantee nominal constrained stability. This point is illustrated by the example of Section D in Rawlings and Muske [71] and Example 2 in Muske and Rawlings [65]. The infinite horizon also ensures that feasibility of the constraints at time $k = 0$ implies that they are feasible for all time $k > 0$ for the nominal model with perfect state measurement and no disturbances.

Lemma 2.1 *For all $x_0 \in \Re^n$ and the nominal model with perfect state measurement and no disturbances, feasibility of the constraints in Eqs. 2.19, 2.22, 2.23, and 2.24 at time $k = 0$ implies feasibility of these constraints at every time $k > 0$.*

Proof: See Appendix D.1.

2.5.1 Finite Open-Loop Constraint Horizon

Although an infinite constraint horizon is necessary to ensure feasibility of the constraints, it leads to an infinite number of constraints in the corresponding quadratic program. The result is an optimization problem that cannot practically be solved. However, an equivalent finite constraint set for both the input and state can be determined. Because of the input parameterization $v_j = 0$ for all $j \geq N$, the input constraints are obviously a finite set.

$$D v_j \leq \mathrm{d}, \qquad j = 0, 1, \ldots, N-1 \tag{2.29}$$

$$\Delta v_j \leq \Delta_{\max}, \qquad j = 0, 1, \ldots, N \tag{2.30}$$

$$-\Delta v_j \leq \Delta_{\min}, \qquad j = 0, 1, \ldots, N \tag{2.31}$$

Rawlings and Muske [71] show that there exists a finite state constraint horizon, j_2, such that feasibility of the constraints on this finite horizon guarantees feasibility on the infinite horizon.

$$F z_j \leq \mathrm{f}, \qquad j = 1, 2, \ldots, j_2 \qquad \Longrightarrow \qquad F z_j \leq \mathrm{f}, \qquad \forall \ j > j_2$$

The existence of a finite value of j_2 is demonstrated as follows assuming, for simplicity, that the eigenvalues of A are distinct [71]. Let z_N be achieved by some feasible input sequence, π. In addition, let $f_{\min} = \min_i f_i$, $\kappa(V)$ be the condition number of V in Eq. 2.2, and $\lambda_{\max} = \max_i |\lambda_i(J_s)|$ in which J_s are the stable eigenvalues of A in Eq. 2.2. Because of the equality constraint in Eq. 2.19 for unstable A, the unstable modes of the system are zero at time $j = N$ and need not be considered. A bound on the state constraint is determined as follows.

$$\|F z_{N+j}\| \leq \|F\| \, \kappa(V) \lambda_{\max}^{j} \, \|z_N\| \tag{2.32}$$

Therefore, the following value of j_2 is sufficiently large.

$$j_2 \geq N + \max\left\{ \ln\left(\frac{f_{\min}}{\|F\| \, \kappa(V) \, \|z_N\|} \right) / \ln(\lambda_{\max}), 0 \right\} \tag{2.33}$$

The bound on j_2 in Eq. 2.33 is a function of z_N. The use of this bound requires an iterative procedure in which the quadratic program is solved for some j_2 and the corresponding value of z_N determined from the solution is used to check if the choice of j_2 is large enough. If the value of j_2 is not greater than or equal to the bound in Eq. 2.33, the procedure must be repeated using a larger value of j_2. In practice, the bound on j_2 in Eq. 2.33 is very conservative and usually requires the addition of more state constraints in the optimization problem than the minimum necessary. Because of these additional constraints and the iterative nature of the determination of j_2, this procedure is not acceptable for practical application.

As discussed in Meadows [58], Gilbert and Tan [31] provide an algorithm that does not depend on the value of z_N to determine the minimum value of j_2. The problem posed in [31] is to find the set

$$O_\infty^n (A, F, \mathrm{f}) = \left\{ x \in \Re^n \,\middle|\, FA^i x \leq \mathrm{f} \;\; \forall \;\; i = 0, 1, \ldots \right\} \tag{2.34}$$

in which O_j is defined as follows.

$$O_j^n (A, F, \mathrm{f}) = \left\{ x \in \Re^n \,\middle|\, FA^i x \leq \mathrm{f} \;\; \forall \;\; i = 0, 1, \ldots j \right\} \tag{2.35}$$

Gilbert and Tan show that if $O_j^n = O_{j+1}^n$ for some finite j, $O_\infty^n = O_j^n$. A numerical procedure to determine j_2 such that $O_\infty^n = O_{j_2}^n$ is presented in [31].

2.5.2 Initial Constraint Feasibility

In order to achieve a solution to the constrained receding horizon regulator for all time $k > 0$, the input and state constraints must be feasible at time $k = 0$. If these constraints are not initially feasible, they must be relaxed to achieve feasibility. Since the input constraints are intended to represent physical limits of the process, they must be strictly enforced and cannot be changed arbitrarily. Therefore, it is the state constraints that are relaxed in this formulation.

The state constraints may not be feasible in the open loop for all time $j \geq 1$. However, Rawlings and Muske [71] show that a feasible set can be obtained by removing the constraints at the beginning of the open-loop constraint horizon up to time $j = j_1$ in which $j_1 \geq 1$.

$$F z_j \leq \mathrm{f}, \qquad j = j_1, j_1 + 1, \ldots, j_1 + j_2 \tag{2.36}$$

A consequence of Lemma 2.1 is that after a value of j_1 sufficiently large to ensure feasibility is chosen, the open-loop constraint horizon can slide backward at each future sample time and still retain feasibility for the nominal model with no disturbances.

$$j_1(k + 1) = \max(j_1(k) - 1,\ 1) \tag{2.37}$$

Therefore, for the nominal model with no disturbances, the constraints will be satisfied in the closed loop for all time $k \geq j_1(0)$. As discussed in Chapter 5, this procedure is also used to implement the constrained regulator in the presence of disturbances.

Stable Process Model. For stable A, the input constraints are feasible for all $\{A, B\}$, N, $x_k \in \Re^n$, and $u_{k-1} \in \mathcal{U}^m$. Since A is stable, the states are exponentially asymptotically stable in all of $\Re^n$ for a zero input sequence and stay within any given neighborhood of the origin after a finite time. Therefore, the following bound for j_1 as a function of the initial state can be derived assuming that the eigenvalues of A are distinct [71]. Let z_0 be the initial state, $f_{\min} = \min_i f_i$, $\kappa(V)$ the condition number of V in Eq. 2.2, $\lambda_{\max} = \max_i |\lambda_i(A)|$, and Eq. 2.38 a bound on the state constraint Fz_j for a zero input.

$$\|Fz_j\| \leq \|F\| \, \kappa(V) \lambda_{\max}^j \, \|z_0\| \tag{2.38}$$

The following value of j_1 is then sufficiently large to ensure feasibility of the state constraints in Eq. 2.36.

$$j_1 \geq \max\left\{ \ln\left(\frac{f_{\min}}{\|F\| \, \kappa(V) \, \|z_0\|} \right) / \ln(\lambda_{\max}), 1 \right\} \tag{2.39}$$

The bound in Eq. 2.39 tends to be much larger than the minimum value of j_1 necessary to ensure feasibility. Since a large value of j_1 effectively removes the state constraints from the problem as shown by Example 2 in Muske and Rawlings [65], this value of j_1 may not be useful to obtain feasibility of the constrained regulator in practice. The use of the minimum value of j_1 ensures that the state constraints will be satisfied in the closed loop as soon as possible. The determination of the minimum value of j_1 necessary to ensure feasibility requires the solution of an integer programming problem as discussed in Meadows *et al.* [60].

Unstable Process Model. For unstable A, feasibility of the state constraints in Eq. 2.23 can also be obtained by removing the constraints at the beginning of the open-loop constraint horizon up to time $j = j_1$ as shown in Eq. 2.36. The existence of a finite value of j_1 can be shown in the same manner as the stable A case by considering the stable modes after time $j = N$ [71]. The result is the following value of j_1

$$j_1 \geq N + \max\left\{ \ln\left(\frac{f_{\min}}{\|F\| \, \kappa(V) \, \|z_N^s\|} \right) / \ln(\lambda_{\max}), 1 \right\} \tag{2.40}$$

in which $\lambda_{\max} = \max_i |\lambda_i(J_s)|$ and $z_N^s = \tilde{V}_s z_N$ are the stable modes at time $j = N$. The integer programming procedure for stable A also applies to determine the minimum value of j_1 for unstable A.

Feasibility of the equality constraint on the unstable modes in Eq. 2.19 is a function of the process model, input constraints in Eqs. 2.22 through 2.24, control horizon length, N, and the initial state, z_0. In this work, the process model and input constraints are considered to be constant parameters that cannot be changed. Therefore, feasibility can only be achieved by restricting the set of initial conditions that can be considered or increasing the horizon length. Rawlings and Muske [71] define the admissible set of initial states for a given control horizon length. Meadows [58] defines the minimum control horizon length necessary to achieve feasibility for a given initial state.

Let $\mathcal{Z}_N^n$ denote the set of z_0 for which there exists a feasible input sequence $\{v_0, v_1, \ldots, v_{N-1}\}$ such that $v_j \in \mathcal{U}^m$ and Eq. 2.19 is satisfied.

$$\mathcal{Z}_N^n = \left\{ z_0 \in \Re^n \;\middle|\; \tilde{V}_u \left(A^N z_0 + \left[A^{N-1}B \;\; A^{N-2}B \;\ldots\; B \right] \pi \right) = 0, \; \pi \in \mathcal{U}^{N \cdot m} \right\} \tag{2.41}$$

The constrained linear system is constrained stabilizable if and only if $z_0 \in \mathcal{Z}_\infty^n$ [71]. A similar definition is presented in Sznaier and Damborg [82]. The admissible set of initial states for feasibility of the equality constraint on the unstable modes subject to the input constraints is $z_0 \in \mathcal{Z}_N^n$. Since $\mathcal{Z}_N^n \subseteq \mathcal{Z}_\infty^n$, $z_0 \in \mathcal{Z}_N^n$ is a sufficient condition for constrained stabilizability. Note that for A stable, $\mathcal{Z}_N^n = \Re^n$ for all $N \geq 1$.

If there exists some $\mathcal{Z}_N^n$, $N < \infty$, such that $\mathcal{Z}_N^n = \Re^n$, the system is controllable to the origin and the stabilizability index N_{cs} is defined to be N [58]. Therefore, an input horizon length $N = N_{cs}$ is sufficiently long to obtain feasibility of the equality and input constraints for all $x_0 \in \Re^n$. Meadows [58] presents an iterative procedure to determine the minimum value of N to achieve feasibility for a given initial state in which the constrained dynamic system is controllable to the origin.

If the constrained linear system is not constrained stabilizable, the input constraints are too restrictive to control the unstable modes for the given initial state. A stabilizable system that is not constrained stabilizable has the same implications as an unstabilizable system since there are unstable modes that cannot be controlled by any constrained regulator. Example 1 in Muske and Rawlings [64] and Example 3 in Muske and Rawlings [65] demonstrate this point. The constrained receding horizon regulator in this work is limited to constrained stabilizable systems by restricting the initial conditions that are considered such that $z_0 \in \mathcal{Z}_N^n$.

2.6 Constrained Receding Horizon Regulator

The constrained receding horizon regulator is based on the minimization of the following open-loop quadratic objective function in which Q is a symmetric, positive semidefinite penalty matrix on the states, R is a symmetric, positive definite penalty matrix on the input, and S is a symmetric, positive semidefinite penalty matrix on the rate of change of the input.

$$\min_{\pi} \Phi_k^N(x_k, u_{k-1}, \pi) = z_N^T Q_N z_N + \Delta v_N^T S \Delta v_N + \sum_{j=0}^{N-1} \left(z_j^T Q z_j + v_j^T R v_j + \Delta v_j^T S \Delta v_j \right) \quad (2.42)$$

$$\text{Subject to:} \quad \begin{array}{rcl} z_0 & = & x_k \\ z_{j+1} & = & A z_j + B v_j, \quad j = 0, 1, \ldots, j_1 + j_2 \\ v_{-1} & = & u_{k-1} \\ \Delta v_j & = & v_j - v_{j-1} \\ F z_j & \leq & \mathrm{f}, \quad j = j_1, j_1 + 1, \ldots, j_1 + j_2 \\ D v_j & \leq & \mathrm{d}, \quad j = 0, 1, \ldots, N-1 \\ \Delta v_j & \leq & \Delta_{\max}, \quad j = 0, 1, \ldots, N \\ -\Delta v_j & \leq & \Delta_{\min}, \quad j = 0, 1, \ldots, N \\ \tilde{V}_u z_N & = & 0 \\ v_j & = & 0, \quad j \geq N \end{array} \quad (2.43)$$

The terminal state penalty, Q_N, is computed as shown in Eq. 2.18 for stable A or Eq. 2.20 for unstable A. This objective consists of an input rate of change penalty added to the objective function in Eq. 2.16 and the state and input constraints added to Eq. 2.17. The Δv penalty is used to inhibit rapid movement of the input.

This constrained open-loop optimal control problem results in a quadratic objective function in π subject to linear constraints. The solution is typically found by quadratic programming optimization techniques. The quadratic program formulated in π and a discussion of the properties of its solution are presented in Appendix A. The closed-loop system can be expressed as follows

$$x_{k+1} = A x_k + B \mu(x_k) \quad (2.44)$$

in which $\mu(x_k)$ represents the closed-loop input u_k determined as v_0 from the solution to the quadratic program in Eqs. 2.42 and 2.43. Since constraints are present in the receding horizon regulator formulation, $\mu(x_k)$ is a nonlinear function of the state x_k. De Oliveira and Biegler [20] discuss the nonlinearity properties of the solution to the linear model predictive control quadratic program.

2.7 Nominal Constrained Stability

Nominal asymptotic stability of the closed-loop system in Eq. 2.44 is demonstrated for a non-zero initial state and perfect state measurement. Since the input is determined from the solution of a quadratic program, the receding horizon regulator is nonlinear. Consequently, the standard linear system techniques used to demonstrate asymptotic stability are not appropriate. Asymptotic stability for this system is demonstrated by showing that the closed-loop state trajectory converges to the origin and that the origin is stable. Convergence is demonstrated by showing that the optimal objective function value of the receding horizon quadratic program is monotonically convergent and a Lyapunov function for the closed-loop system. These results are based on those presented in Muske and Rawlings [64].

2.7.1 Convergence

The following lemmas demonstrate convergence of the input and state to the origin. The proof of convergence for stable A is based on showing that the closed-loop input sequence $\{u_k\}$ converges to zero which then implies that the closed-loop state sequence $\{x_k\}$ converges to zero. For unstable A, the convergence of $\{u_k\}$ to zero is not sufficient to show that the unstable modes converge to zero. In this case, it is necessary to show that the sequence of optimal solutions to the quadratic program, $\{\pi_k^*\}$, converges to zero. The convergence of these sequences depends on the monotonic convergence of the optimal objective function sequence $\{\Phi_k^*\}$ to some non-negative value, which is demonstrated first.

Lemma 2.2 *The sequence of open-loop optimal objection function values for the feasible quadratic program in Eqs. 2.42 and 2.43, $\{\Phi_k^*\}$, is nonincreasing and converges to some $\Phi_\infty^* \in \Re^+$ for the nominal system with no disturbances.*

Proof: See Appendix D.2.

Lemma 2.3 *The closed-loop input sequence $\{u_k\}$ determined as v_0 from the solution of the feasible quadratic program in Eqs. 2.42 and 2.43 at each time k converges to zero for the nominal system with no disturbances.*

Proof: See Appendix D.3.

Lemma 2.4 *The sequence of optimal solutions to the feasible quadratic program in Eqs. 2.42 and 2.43, $\{\pi_k^*\}$, converges to zero for the nominal system with no disturbances.*

Proof: See Appendix D.4.

Lemma 2.5 *The closed-loop state trajectory in Eq. 2.44 converges to zero for the nominal system with no disturbances.*

Proof: See Appendix D.5.

2.7.2 Stability

Stability of the constrained receding horizon regulator is demonstrated by computing a maximum bound on the state trajectory for a given bound on the initial state and input. To simplify the stability proof, the the sequence of optimal open-loop objective function values, $\{\Phi_k^*\}$, is first shown to converge to zero.

Lemma 2.6 *The limit of the open-loop optimal objection function sequence in Lemma 2.2 is zero.*

Proof: See Appendix D.6

Lemma 2.7 *The origin is a stable equilibrium point for the closed-loop system in Eq. 2.44.*

Proof: See Appendix D.7

2.7.3 Asymptotic Stability

The following theorem demonstrates nominal asymptotic stability for a non-zero initial state without disturbances. Feasibility of the receding horizon regulator quadratic program is ensured by restricting the initial state such that $x_0 \in \mathcal{Z}_N^n$ in which $\mathcal{Z}_N^n = \Re^n$ for stable A. Nominal asymptotic stability then follows from convergence and stability.

Theorem 2.1 *For (A, B) stabilizable, $N \geq r$, in which r is the number of unstable modes of A, and the nominal system with no disturbances, $x_k = 0$ is an asymptotically stable solution of the closed-loop system in Eq. 2.44 for all $x_0 \in \mathcal{Z}_N^n$.*

Proof: See Appendix D.8.

2.8 Time-Varying Penalty Matrices

If the penalty matrices Q, R, and S in Eq. 2.42 are time-varying matrices, the result is the following quadratic objective function

$$\min_{\pi} \Phi^N(x_k, u_{k-1}, \pi) = \sum_{j=0}^{N-1} \left(z_j^T Q_j z_j + v_j^T R_j v_j + \Delta v_j^T S_j \Delta v_j \right) + \sum_{j=N}^{\infty} \left(z_j^T Q_j z_j \right) \tag{2.45}$$

in which $\{R_j\}$ is a sequence of symmetric positive definite matrices, $\{S_j\}$ is a sequence of symmetric positive semidefinite matrices, and $\{Q_j\}$ is a bounded sequence of symmetric positive semidefinite matrices. In order to have a bounded objective function value, the infinite series used to determine the terminal state penalty matrix must converge. For stable A, this series is computed as follows.

$$Q_N = \sum_{j=0}^{\infty} A^{jT} Q_{N+j} A^j \tag{2.46}$$

For unstable A, the terminal state penalty matrix is computed from the stable modes by the following series.

$$\Sigma = \sum_{j=0}^{\infty} J_s^{jT} V_s^T Q_{N+j} V_s J_s^j \tag{2.47}$$

2.8.1 Nominal Asymptotic Stability

A sufficient condition for nominal asymptotic stability of the constrained receding horizon regulator with time-varying penalty matrices is that each of the matrix sequences are nondecreasing

$$Q_{j+1} \geq Q_j, \quad R_{j+1} \geq R_j, \quad S_{j+1} \geq S_j, \quad \forall\ j \geq 0$$

in which $M \geq N$ implies the matrix $M - N$ is positive semidefinite.

Theorem 2.2 *For (A, B) stabilizable, $N \geq r$, in which r is the number of unstable modes of A, and the nominal system with no disturbances, $x_k = 0$ is an asymptotically stable solution of the closed-loop system in Eq. 2.44 for all $x_0 \in \mathcal{Z}_N^n$ in which $\mu(x_k)$ is determined from the solution of the quadratic program in Eqs. 2.45 and 2.43 with $\{R_j\}$ a symmetric positive definite nondecreasing sequence, $\{S_j\}$ a symmetric positive semidefinite nondecreasing sequence, $\{Q_j\}$ a symmetric positive semidefinite nondecreasing bounded sequence, and the infinite series in Eq. 2.46 for stable A or Eq. 2.47 for unstable A convergent.*

Proof: See Appendix D.9.

2.8.2 Terminal State Penalty

For certain implementations, it may be desirable to change the penalty matrix on the open-loop states by some δQ after time $j = N - 1$ when the open-loop input is set to zero. The resulting infinite series for the terminal state penalty are then determined as follows.

$$Q_N = \sum_{j=0}^{\infty} A^{jT}(Q + \delta Q)A^j \tag{2.48}$$

$$\Sigma = \sum_{j=0}^{\infty} J_s^{jT} V_s^T (Q + \delta Q) V_s J_s^j \tag{2.49}$$

A consequence of Theorem 2.2 is that increasing the penalty on the open-loop states at time $j \geq N$ by some positive semidefinite matrix δQ does not effect the nominal asymptotic stability properties of the receding horizon regulator. Therefore, any positive semidefinite matrix that is the solution of a discrete Lyapunov equation can be directly added to the terminal state penalty computed from Eqs. 2.18 or 2.20 without loss of nominal asymptotic stability.

2.9 Feedback Gain Input Parameterization

In the closed-loop, the input is not brought to the origin after N moves as parameterized in the previous approach. A more realistic parameterization is to employ a stabilizing feedback gain, K, to compute the open-loop input moves from the state after time $j = N - 1$. This approach is similar to that proposed for nonlinear systems in Mayne and Michalska [57]. The result is the following open-loop state and input trajectories after time $j = N - 1$.

$$z_{N+j} = (A - BK)^j z_N, \qquad j = 0, 1, \ldots \tag{2.50}$$

$$v_{N+j} = K(A - BK)^j z_N, \qquad j = 0, 1, \ldots \tag{2.51}$$

$$\Delta v_{N+j+1} = K(A - BK - I)(A - BK)^j z_N, \qquad j = 0, 1, \ldots \tag{2.52}$$

The receding horizon regulator open-loop objective function with this approach is the same as that presented in Eq. 2.42 for the previous parameterization. The terminal state penalty is computed from the following discrete Lyapunov equation.

$$Q_N = \sum_{j=0}^{\infty} (A - BK)^{jT} \Pi (A - BK)^j = (A - BK)^T Q_N (A - BK) + \Pi \tag{2.53}$$

$$\Pi = Q + K^T R K + K^T (A - BK - I)^T S (A - BK - I) K$$

Since $A - BK$ is stable, there are no constraints on the unstable modes of the system, which are brought to the origin by the open-loop inputs. Therefore, this approach is also restricted to stabilizable systems.

The advantage of this parameterization is that the open-loop input trajectory is a better representation of the actual closed-loop input trajectory. Since the open-loop input is a better approximation to what can actually be implemented in the closed loop, better regulator performance is expected with this approach. However, constraints on the input, which must always be respected since they represent physical limits of the process, are more difficult to handle with this input parameterization.

2.9.1 Constraints

The linear state constraints in Eq. 2.23 are expressed as follows for time $j \geq N$ in this parameterization.

$$F(A - BK)^{j-N} z_N \leq \mathrm{f}, \qquad j = N, N+1, \ldots, j_2^f \tag{2.54}$$

Since the input parameterization is nonzero after time $j = N$, the constraints on the input must also be considered after this time. The input position and rate of change constraints in Eqs. 2.22 through 2.24 are specified as follows for time $j \geq N$.

$$\begin{aligned} DK(A - BK)^{j-N} z_N &\leq \mathrm{d}, & j &= N, N+1, \ldots, j_2^d & (2.55)\\ K(A - BK - I)(A - BK)^{j-N} z_N &\leq \Delta_{\max}, & j &= N, N+1, \ldots, j_2^{\Delta_x} & (2.56)\\ -K(A - BK - I)(A - BK)^{j-N} z_N &\leq \Delta_{\min}, & j &= N, N+1, \ldots, j_2^{\Delta_n} & (2.57) \end{aligned}$$

A finite constraint set is determined in the same manner as that for the state constraints in Section 2.5.1. The algorithm in Gilbert and Tan [31], used to compute j_2, can also be used to compute j_2^d, j_2^f, $j_2^{\Delta_x}$, and $j_2^{\Delta_n}$ with this approach.

Feasibility of the state constraint at time $k = 0$ can also be guaranteed by relaxing the constraint starting at the beginning of the horizon in this approach. The value of j_1 can be determined in the same manner as in Section 2.5.2 using $A - BK$ as the system matrix after time $j = N$. Since $A - BK$ is stable by construction, there are no unstable modes and the algorithm discussed in Section 2.5.2 can be applied.

Initial feasibility of the input constraint set is more complex with this input parameterization since the input constraints after time $j = N$ must also be considered. These constraints cannot be relaxed, as with the state constraints, to achieve feasibility. Therefore, the terminal state z_N must be restricted such that all of the future input constraints are feasible. Define $\mathcal{K}^n$ as the set of all terminal states such that the input constraints in Eqs. 2.55 through 2.57 are feasible.

$$\mathcal{K}^n = \left\{ z_N \in \Re^n \;\middle|\; \begin{bmatrix} DK \\ K(A-BK-I) \\ -K(A-BK-I) \end{bmatrix} (A-BK)^j z_N \leq \begin{bmatrix} \mathrm{d} \\ \Delta_{\max} \\ \Delta_{\min} \end{bmatrix} \right\} \quad (2.58)$$

The constrained stabilizability restriction on the initial state becomes

$$\mathcal{Z}_N^n = \left\{ z_0 \in \Re^n \;\middle|\; \left(A^N z_0 + \left[A^{N-1}B \;\; A^{N-2}B \;\; \ldots \;\; B\right] \pi\right) \in \mathcal{K}^n,\; \pi \in \mathcal{U}^{N\cdot m} \right\} \quad (2.59)$$

in which π is defined in Eq. 2.9. In this approach, $\mathcal{Z}_N$ is not necessarily equal to $\Re^n$ for stable A. Therefore, there is a restriction on the initial state for both stable and unstable A requiring the determination of the sets $\mathcal{K}^n$ and $\mathcal{Z}_N^n$.

The constrained receding horizon regulator is based on the minimization of the open-loop quadratic objective function in Eq. 2.42 subject to the following constraints.

$$\begin{aligned} z_0 &= x_k \\ z_{j+1} &= Az_j + Bv_j, \quad j = 0, 1, \ldots \\ v_{-1} &= u_{k-1} \\ \Delta v_j &= v_j - v_{j-1} \\ Fz_j &\leq \mathrm{f}, \quad j = j_1^f, j_1 + 1, \ldots, j_1^f + j_2^f \\ Dv_j &\leq \mathrm{d}, \quad j = 0, 1, \ldots, j_2^d \\ \Delta v_j &\leq \Delta_{\max} \quad j = 0, 1, \ldots, j_2^{\Delta_x} \\ -\Delta v_j &\leq \Delta_{\min} \quad j = 0, 1, \ldots, j_2^{\Delta_n} \\ v_j &= -Kz_j, \quad j \geq N \end{aligned} \quad (2.60)$$

Feasibility of the constraints at time $k = 0$ also implies feasibility for all time $k > 0$ for the nominal model with perfect state measurement and no disturbances in this approach.

Lemma 2.8 *For all $x_0 \in \mathcal{Z}_N^n$, defined in Eq. 2.59, and the nominal model with perfect state measurement and no disturbances, feasibility of the constraints in Eq. 2.60 at time $k = 0$ implies feasibility of these constraints at every time $k > 0$.*

Proof: The proof follows in the same manner as that for Lemma 2.1.

2.9.2 Nominal Asymptotic Stability

Nominal asymptotic stability of the constrained receding horizon regulator discussed in this section is demonstrated for a non-zero initial state and perfect state measurement. Since the feedback gain input parameterization only changes the form of the constraints and the terminal state penalty, nominal asymptotic stability follows in the same manner as Theorem 2.1 in which the objective function was shown to be a Lyapunov function for the closed-loop system.

Theorem 2.3 *For (A, B) stabilizable, all K such that $(A - BK)$ is stable, and the nominal system with no disturbances, $x_k = 0$ is an asymptotically stable solution of the closed-loop system in Eq. 2.44 for all $x_0 \in \mathcal{Z}_N^n$, in which $\mathcal{Z}_N$ is defined in Eq. 2.59 and $\mu(x_k)$ is determined from the solution of the quadratic program in Eqs. 2.42 and 2.60.*

Proof: See Appendix D.10.

2.9.3 Implementation

This input parameterization is a more realistic open-loop approximation to the actual closed-loop input trajectory. Consequently, it can provide improved closed-loop controller performance for small N since the open-loop input is not forced to zero after a small number of input moves. Because the computational requirements of the regulator increase with N, reducing the size of N without a significant loss in performance is an important implementation issue. This parameterization is also a more natural way to handle unstable process models since it does not require a constraint on the unstable modes. However, constraints on the input are more difficult to handle with this parameterization. In addition, the set of feasible initial states can be significantly reduced by the input constraints even for stable A. This input parameterization is a topic for future research and will not be considered further in this work.

Chapter 3

Controlled Variable Target Tracking

The constrained receding horizon regulator discussed in the previous chapter controls the state of the system to the origin. For practical applications, the regulator must be able to handle non-zero controlled variable targets. It must also be able to handle non-square systems, in which the number of controlled variables does not equal the number of manipulated variables, in a consistent manner. This chapter presents a target tracking receding horizon regulator that addresses these issues. A controlled variable reference trajectory tracking receding horizon regulator formulation is then discussed.

3.1 Controlled Variables

The controlled variables that are considered in this work are determined linearly from the state of the system. These variables can consist of some or all of the measured outputs. They may also be other linear combinations of the state vector. These controlled variables are defined as

$$y_k^c = \acute{C} x_k \tag{3.1}$$

in which $\acute{C}$ is not necessarily equal to or the same row dimension as C. If the regulator is to bring the controlled variables to a nonzero target y_t, steady-state state and input vectors, x_s and u_s, are required such that the system reaches y_t at steady state.

$$\begin{aligned} x_s &= Ax_s + Bu_s \\ y_t^c &= \acute{C} x_s \end{aligned} \tag{3.2}$$

In order to have a well-posed target tracking regulator, x_s and u_s must be uniquely determined from the linear system matrices $(A, B, \acute{C})$, and the controlled variable target y_t. If the intersection of the null spaces of $(I - A)$ and $\acute{C}$ is a vector space containing more than the zero vector, x_s cannot be determined uniquely as shown from rearranging Eq. 3.2.

$$\begin{bmatrix} (I - A) \\ \acute{C} \end{bmatrix} x_s = \begin{bmatrix} Bu_s \\ y_t^c \end{bmatrix} \tag{3.3}$$

Therefore, a necessary restriction on $\acute{C}$ is that the matrix $\acute{O}$ is full rank.

$$\acute{O} = \begin{bmatrix} (I - A) \\ \acute{C} \end{bmatrix} \tag{3.4}$$

This matrix is full rank under the conditions stated in Theorem 3.1.

Theorem 3.1 *The matrix $\acute{O}$ in Eq. 3.4 is full rank if and only if the integrating modes of A are in the observable subspace of $(\acute{C}, A)$.*

Proof: See Appendix E.1.

3.2 Target Tracking

The determination of the steady-state state and input vectors x_s and u_s is discussed in this section. Non-square systems with more controlled variables than inputs are handled by minimizing the steady-state deviation from the controlled variable target in a least squares sense. In order to guarantee a unique steady state for non-square systems with more inputs than controlled variables, input targets are specified to remove any additional degrees of freedom.

3.2.1 Perfect Target Tracking

Steady-state state and input vectors that track the controlled variable target exactly can be determined from the solution of the following quadratic program.

$$\min_{[x_s, u_s]^T} (u_s - u_t)^T R_s (u_s - u_t) \tag{3.5}$$

$$\text{Subject to:} \quad \begin{bmatrix} I - A & -B \\ \acute{C} & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} 0 \\ y_t^c \end{bmatrix} \tag{3.6}$$

In this quadratic program, u_t is the desired value of the input at steady state and R_s is a positive definite penalty matrix for the deviation of the steady-state input from the input target. The input target is used to determine the steady-state input when there are degrees of freedom present in the system. The penalty matrix specifies the relative importance of deviations from the steady-state input target. The equality constraints in Eq. 3.6 guarantee a steady-state solution, offset free tracking of the controlled variable target such that $\acute{C}x_s = y_t^c$, and provide a unique x_s and u_s.

Theorem 3.2 *The feasible quadratic program in Eq. 3.5 with the equality constraint in Eq. 3.6 has a unique solution for $\acute{\mathcal{O}}$ in Eq. 3.4 full rank and R_s positive definite.*

Proof: See Appendix E.2.

3.2.2 Least Squares Target Tracking

It may not always be possible to achieve a given y_t^c as required by the equality constraint in Eq. 3.6. If the system is non-square with more controlled variables than inputs, there will normally not be enough degrees of freedom in the system to bring the controlled variables exactly to the target. The quadratic program in Eqs. 3.5 and 3.6 will then be infeasible due to the equality constraint. In this case, the controlled variable target is tracked in a least squares sense. The solution of the following quadratic program can be used to determine x_s and u_s in which Q_s is a positive definite penalty matrix on the tracking error that specifies the relative importance of each controlled variable.

$$\min_{[x_s,u_s]^T} (y_t^c - \acute{C}x_s)^T Q_s (y_t^c - \acute{C}x_s) + (u_s - u_t)^T R_s (u_s - u_t) \tag{3.7}$$

$$\text{Subject to:} \quad \begin{bmatrix} I - A & -B \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = 0 \tag{3.8}$$

A steady-state solution is guaranteed by the equality constraint in Eq. 3.8. The actual steady-state value of the controlled variables is $y_a^c = \acute{C}x_s$ in which x_s and u_s are determined uniquely.

Theorem 3.3 *The quadratic program in Eq. 3.7 with the equality constraint in Eq. 3.8 has a unique solution for $\acute{\mathcal{O}}$ in Eq. 3.4 full rank and R_s, Q_s positive definite.*

Proof: See Appendix E.3.

3.2.3 Perfect Least Squares Target Tracking

The preceding quadratic program penalizes the deviations of both the controlled variables and the inputs from their steady-state targets. When R_s is a positive definite matrix, this formulation may not result in the perfect tracking solution when perfect target tracking is possible. In order to bring the controlled variables as close as possible to the target, in a least squares sense, the penalty on the inputs must be determined such that only the input space that does not influence the controlled variable tracking error is penalized. This input space can be determined from the null space of the reduced Hessian for the quadratic program in Eqs. 3.7 and 3.8 in which $R_s = 0$.

$$\mathcal{N} = \text{Null}\left([(I - A) \quad - B]\right) = \begin{bmatrix} \mathcal{N}_x \\ \mathcal{N}_u \end{bmatrix} \tag{3.9}$$

$$\alpha = \text{Null}\left(\mathcal{N}^T \begin{bmatrix} \acute{C}^T Q_s \acute{C} & 0 \\ 0 & 0 \end{bmatrix} \mathcal{N}\right) = \text{Null}\left(\mathcal{N}_x^T \acute{C}^T Q_s \acute{C} \mathcal{N}_x\right) \tag{3.10}$$

$$R_s = \mathcal{N}_u \alpha \alpha^T \mathcal{N}_u^T \tag{3.11}$$

The input penalty in Eq. 3.11, which is not a positive definite matrix in general, is required in order to achieve a unique solution. The solution of the quadratic program in Eqs. 3.7 and 3.8 using this input penalty results in a unique x_s and u_s that minimizes the deviation from the controlled variable target in a least squares sense. If possible, this formulation reaches the controlled variable target exactly.

Theorem 3.4 *The quadratic program in Eq. 3.7 with the equality constraint in Eq. 3.8 has a unique solution for $\acute{O}$ in Eq. 3.4 full rank, Q_s positive definite, and R_s computed by Eq. 3.11.*

Proof: See Appendix E.4.

3.2.4 Minimal Transfer Function Realizations

An important class of state-space process models that can be considered are those determined from a minimal realization of a transfer function matrix in which the controlled variables are taken as the measured outputs. In this case, $\acute{C} = C$ and the state-space model is both controllable and observable [10]. The target tracking quadratic program in Section 3.2.1 can then be used for systems in which the number of outputs is less than or equal to the number of inputs, $p \leq m$. For systems in which the number of outputs is greater than or equal to the number of inputs, $p \geq m$, the quadratic program in Section 3.2.2 with $R_s = 0$ can be used. This is the method presented by Muske and Rawlings [65] and provides a unique solution for both cases without requiring the solution of Eq. 3.11 to determine R_s. Example 4 in [65] demonstrates this target tracking quadratic program.

Theorem 3.5 *The perfect target tracking quadratic program in Eqs. 3.5 and 3.6 with R_s positive definite has a unique solution for all state-space models determined from a minimal realization of a discrete transfer function matrix with a full rank steady-state gain matrix, no integrating modes, and $p \leq m$.*

Proof: See Appendix E.5.

Theorem 3.6 *The least squares target tracking quadratic program in Eqs. 3.7 and 3.8 with Q_s positive definite and $R_s = 0$ has a unique solution for all state-space models determined from a minimal realization of a discrete transfer function matrix with a full rank steady-state gain matrix, no integrating modes, and $p \geq m$.*

Proof: See Appendix E.6.

3.3 Constraints

The target tracking receding horizon regulator quadratic program is subject to constraints on the input and controlled variables similar to those presented for the zero target regulator in the previous chapter. In this discussion, the constraints are specified as maximum and minimum position limits on the input and controlled variables and not the general form discussed in Chapter 2.

$$u_{\min} \leq v_j \leq u_{\max} \tag{3.12}$$
$$-\Delta u_{\min} \leq v_j - v_{j-1} \leq \Delta u_{\max} \tag{3.13}$$
$$y^c_{\min} \leq \acute{C} z_j \leq y^c_{\max} \tag{3.14}$$

A consistent constraint set is specified when the following restrictions are imposed on the position and rate of change limits.

$$\begin{bmatrix} u_{\max} \\ -u_{\min} \\ y^c_{\max} \\ -y^c_{\min} \\ \Delta u_{\max} \\ \Delta u_{\min} \end{bmatrix} > \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{3.15}$$

These restrictions guarantee a convex feasible region containing the origin. The constraints can always be constructed in this form by scaling the linear system such that the nominal operating point of the process model is the origin. In order to guarantee that the open-loop input trajectory can reach any feasible steady-state input u_s, the following restriction is imposed on the input constraints and control horizon length in which $\min(\Delta_{\max}, \Delta_{\min})$ is defined in Eq. 2.26.

$$(N+1)\min(\Delta_{\max}, \Delta_{\min}) \geq u_{\max} - u_{\min} \tag{3.16}$$

This constraint formulation is discussed in more detail in Appendix A.1.

The constraints on the input and controlled variable must also be respected in the target tracking quadratic program for the receding horizon regulator quadratic program to have a feasible solution. The following constraints are included in the quadratic program for the determination of x_s and u_s in Eqs. 3.7 and 3.8.

$$u_{\min} < \ u_s \ < u_{\max} \tag{3.17}$$

$$y^c_{\min} < \acute{C}x_s < y^c_{\max} \tag{3.18}$$

Since these constraints contain a neighborhood of the origin, the origin is a feasible solution of the target tracking quadratic program with objective Eq. 3.7 and constraints Eqs. 3.8, 3.17, and 3.18. Therefore, a solution to the target tracking quadratic program exists with the additional constraints in Eqs. 3.17 and 3.18. However, perfect tracking may not be attainable with these constraints. If the input at time $k-1$ does not satisfy the position constraints in Eq. 3.12, an additional constraint is required on the steady-state input to ensure feasibility of the receding horizon regulator quadratic program.

$$-(N+1)\Delta u_{\min} < \ u_s - u_{k-1} \ < (N+1)\Delta u_{\max} \tag{3.19}$$

In the sequel, it will be assumed that the previous input always satisfies the input position constraints which makes the constraint in Eq. 3.19 unnecessary. A discussion of this constrained target tracking quadratic program in which disturbances are considered is presented in Chapter 6.

3.4 Target Tracking Receding Horizon Regulator

When tracking a non-zero controlled variable target, the state penalty in Eq. 2.42 must be modified such that the deviation between the controlled variables and the achievable controlled variable target, $y^c_j - y^c_a$, is minimized in the open-loop objective. The penalty matrix $\acute{Q}$ is a positive semidefinite penalty matrix on the controlled variable target tracking error in the following controlled variable target tracking open-loop quadratic objective function subject to the constraints in Eqs. 3.12 through 3.14.

$$\min_{\pi} \Phi_k^N(x_k, u_{k-1}, \pi) = (z_N - x_s)^T \acute{Q}_N (z_N - x_s) + \Delta v_N^T S \Delta v_N \quad (3.20)$$

$$+ \sum_{j=0}^{N-1} \Big((z_j - x_s)^T \acute{C}^T \acute{Q} \acute{C} (z_j - x_s) + (v_j - u_s)^T R (v_j - u_s) + \Delta v_j^T S \Delta v_j \Big)$$

$$\text{Subject to:} \quad \begin{array}{rcl} z_0 & = & x_k \\ z_{j+1} & = & A z_j + B v_j, \quad j = 0, 1, \ldots, j_1 + j_2 \\ v_{-1} & = & u_{k-1} - u_s \\ \Delta v_j & = & v_j - v_{j-1} \\ \acute{C} z_j & \leq & y^c_{\max}, \quad j = j_1, j_1 + 1, \ldots, j_1 + j_2 \\ -\acute{C} z_j & \leq & y^c_{\min}, \quad j = j_1, j_1 + 1, \ldots, j_1 + j_2 \\ v_j & \leq & u_{\max}, \quad j = 0, 1, \ldots, N-1 \\ -v_j & \leq & u_{\min}, \quad j = 0, 1, \ldots, N-1 \\ \Delta v_j & \leq & \Delta_{\max} \quad j = 0, 1, \ldots, N \\ -\Delta v_j & \leq & \Delta_{\min} \quad j = 0, 1, \ldots, N \\ \tilde{V}_u (z_N - x_s) & = & 0 \\ v_j & = & u_s, \quad j \geq N \end{array} \quad (3.21)$$

The terminal state penalty, $\acute{Q}_N$, is determined in a manner similar to Eq. 2.18 for stable A.

$$\acute{Q}_N = \sum_{j=0}^{\infty} A^{jT} \acute{C}^T \acute{Q} \acute{C} A^j = A^T \acute{Q}_N A + \acute{C}^T \acute{Q} \acute{C} \quad (3.22)$$

For unstable A, $\acute{Q}_N$ is determined similar to Eq. 2.20.

$$\acute{Q}_N = \tilde{V}_s^T \acute{\Sigma} \tilde{V}_s \quad (3.23)$$

$$\acute{\Sigma} = \sum_{j=0}^{\infty} J_s^{jT} V_s^T \acute{C}^T \acute{Q} \acute{C} V_s J_s^j = J_s^T \Sigma J_s + V_s^T \acute{C}^T \acute{Q} \acute{C} V_s \quad (3.24)$$

The steady-state vectors x_s and u_s are computed from the quadratic program in Eqs. 3.7, 3.8, 3.17, and 3.18 with the input penalty R_s computed as shown in Eq. 3.11. The result is similar to the discrete nonzero setpoint optimal regulator discussed in Kwakernaak and Sivan [45].

The input and state penalties in this objective function penalize deviations from the steady-state target values. Therefore, the $v_j - u_s$ input penalty term is required along with the $z_j - x_s$ state penalty term to prevent offset in the regulator. This is equivalent to shifting the origin of the system to the steady state described by x_s and u_s. Stability of the target tracking regulator then follows in the same manner as the zero target regulator shown in Chapter 2.

Tuning of the target tracking regulator represents a tradeoff between deviation of the state and of the input from their steady-state values. In the limit as $Q \to 0$ and $S \to 0$, the regulator approaches a steady-state controller since only the input deviation is penalized. In the limit as $R \to 0$ and $S \to 0$, the regulator approaches a deadbeat controller since only the state deviation is penalized. The Δu penalty matrix, S, is used to penalize rapid movement of the input. This prevents the regulator from taking overly aggressive control action whenever the controlled variable target is changed as demonstrated by Example 5 in Muske and Rawlings [65].

3.5 Reference Trajectory Tracking Regulator

The target tracking receding horizon regulator presented in Section 3.4 is designed to track step changes in the controlled variable target vector. In order to track a reference trajectory from the current state of the system to the target, the following open-loop reference trajectory tracking quadratic objective function is used in the receding horizon regulator.

$$\begin{aligned} \min_{\pi} \Phi_k^N(x_k, u_{k-1}, \pi) = & \sum_{j=0}^{\infty} \left(\acute{C} z_j - \acute{C}^r z_j^r \right)^T \acute{Q} \left(\acute{C} v_j - \acute{C}^r z_j^r \right) \\ & + \sum_{j=0}^{N} \left((v_j - u_s)^T R (v_j - u_s) + \Delta v_j^T S \Delta v_j \right) \end{aligned} \tag{3.25}$$

In this objective function, the controlled variable penalty matrix $\acute{Q}$ penalizes deviations from the specified reference trajectory over the infinite horizon. The reference states, z_j^r, are computed from the following linear dynamic system in which A^r and $\acute{C}^r$ describe the desired trajectory of the controlled variables from the initial state at time k to the achievable target y_a^c.

$$\begin{aligned} z_{k+1}^r &= A^r z_k^r \\ y_k^r &= \acute{C}^r z_k^r \end{aligned} \tag{3.26}$$

The initial condition for the system and reference state vectors are as follows in which the matrix J^r is specified when the order of the reference trajectory model differs from the plant model.

$$\begin{aligned} z_0 &= x_k - x_s \\ z_0^r &= -J^r (x_k - x_s) \end{aligned}$$

The target tracking regulator of Section 3.4 can be recovered from this formulation by setting both A^r and C^r to zero.

As discussed in Kwakernaak and Sivan [45] and Bitmead *et al.* [7], the reference trajectory dynamics can be combined with the plant dynamics to form an augmented system model. The reference trajectory tracking regulator can then be implemented as the zero target regulator presented in Chapter 2 for the following augmented system.

$$\bar{A} = \begin{bmatrix} A & 0 \\ 0 & A^r \end{bmatrix}, \qquad \bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \qquad \bar{C} = \begin{bmatrix} \acute{C} & \acute{C}^r \end{bmatrix}$$

The input constraints are those presented in Eqs. 3.12 and 3.13. The controlled variable constraints on the augmented state are specified as follows.

$$y^c_{\min} \leq \begin{bmatrix} \acute{C} & 0 \end{bmatrix} \begin{bmatrix} z_j \\ z^r_j \end{bmatrix} \leq y^c_{\max} \tag{3.27}$$

This constraint is similar to that in Eq. 3.18 except the augmented system matrix $\bar{C}$ is not used since the reference states are not considered for the controlled variable constraints. The equality constraint on the unstable modes for unstable A is specified as follows in which the reference states are also not considered.

$$\begin{bmatrix} \tilde{V}_u & 0 \end{bmatrix} \begin{bmatrix} z_j \\ z^r_j \end{bmatrix} = 0 \tag{3.28}$$

Stabilizability of the augmented system requires that A^r be a stable matrix. This restriction prevents the regulator from tracking reference trajectories, such as ramps, that are not bounded. However, an unbounded reference trajectory specified on an infinite horizon cannot actually be implemented. A ramp is used in a finite horizon regulator to move the process from one operating point to another. Once the process has reached the new operating point, the ramp is replaced with some bounded reference. In this implementation, the reference trajectory that the process is to follow to the new operating point is specified on the infinite horizon by A^r and $\acute{C}^r$. Example 6 in Muske and Rawlings [65] demonstrates this reference trajectory tracking receding horizon regulator formulation.

Chapter 4

State Estimation

The implementation of the receding horizon regulator discussed in the previous chapters requires knowledge of the current state of the system in order to compute the solution to the open-loop optimal control problem formulated at each control interval. Feedback in the regulator comes from this update of the current state. In most applications, however, the states are not directly measured. If the state-space model came from a discrete transfer function, the states will usually have no physical meaning and not be measurable. Even if the states are physically meaningful, sensors may not be available to measure each state. In these cases, output feedback is performed by using an estimator to reconstruct the state from the available output measurements. Since the controller presented in this work utilizes a linear state-space model, it can take direct advantage of the results from linear estimation theory. In this chapter, state estimation for linear systems is reviewed and a constrained moving horizon state estimation formulation for linear systems is presented.

4.1 Optimal State Estimation

State estimation is the process of determining the state of the system from the output measurements in the presence of disturbances to both the state and measurement. These disturbances are represented by additive terms to the state-space model in Eq. 2.1 as follows

$$x_{k+1} = Ax_k + Bu_k + w_k \tag{4.1}$$

$$y_k = Cx_k + v_k$$

in which $w \in \Re^n$ is the process or state disturbance vector and $v \in \Re^p$ is the measurement disturbance vector. The process disturbance vector models the unmeasured disturbances to the process. The measurement disturbance vector represents output disturbances and error in the measuring device caused by instrument noise.

The discrete Kalman filter recursively estimates the state of the linear system in Eq. 4.1 from the output measurements at each sampling time as follows [3].

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + L_k(y_k - C\hat{x}_{k|k-1}) \tag{4.2}$$

$$\hat{x}_{k+1|k} = A\hat{x}_{k|k} + Bu_k \tag{4.3}$$

$$\hat{x}_{0|0} = \bar{x}_0$$

In this expression, $\hat{x}_{k|k}$ is the estimate of the state at sample time k given k output measurements and $\bar{x}_0$ is an *a priori* estimate of the initial state. The filter gain matrix, L_k, multiplies the difference between the measured output, y_k, and the predicted output, $C\hat{x}_{k|k-1}$, to produce a filtered state estimate by correcting the predicted state estimate at the previous sample time.

For the stochastic linear system in Eq. 4.1, it can be shown that when w_k and v_k are independent, zero mean, normally distributed random variables with covariances Q and R, respectively, and $\bar{x}_0$ is an independent, normally distributed random variable with covariance Q_0, the discrete Kalman filter produces the optimal or minimum variance estimate of the state [40]. For linear Gaussian systems, this estimate is also the maximum likelihood estimate. The filter is stable provided (C, A) is detectable, $(A, Q^{1/2})$ is stabilizable, R is positive definite, and Q_0 is non-negative definite [7]. Stability of the estimator implies that the reconstruction error, e_k, converges to zero for the nominal system with no state or measurement noise. The reconstruction error is defined as the difference between the actual state and the predicted state.

$$e_{k|k-1} = x_k - \hat{x}_{k|k-1} \tag{4.4}$$

Stability of the discrete Kalman filter is also discussed in Bimead *et al.* [6] and De Souza [21].

The filter gain is computed at each sample time from the following expression in which P_k is the covariance of the predicted state estimate in Eq. 4.3.

$$L_k = P_kC^T(CP_kC^T + R)^{-1} \tag{4.5}$$

The covariance of the state estimate is propagated at each sampling time using the discrete filtering Riccati equation.

$$\begin{aligned} P_{k+1} &= A(P_k - P_k C^T (C P_k C^T + R)^{-1} C P_k) A^T + Q \\ P_0 &= Q_0 \end{aligned} \tag{4.6}$$

This recursion tends to a constant matrix at large k that is referred to as the steady-state discrete filtering Riccati matrix [7]. Using this matrix to compute the filter gain in Eq. 4.5 results in the time-invariant steady-state discrete Kalman filter.

The Kalman filter is the optimal estimator when the assumptions concerning the stochastic process are valid and the covariance matrices of w_k and v_k are known. The covariances specify the expected magnitude of the disturbances to the output measurement and the state. If the covariance of the measurement noise, R, is small relative to the covariance of the process noise, Q, then the measurements are relatively noise free and the deviations between the measured output and the predicted output should be made small. If the measurement noise covariance is large relative to the process noise covariance, then the measurements are relatively poor and the feedback correction to the model prediction should be small.

4.2 Deterministic Observer

Another method of constructing a recursive estimator is to select a constant gain matrix L based on some desired performance criteria instead of determining the gain from the covariance matrices of a stochastic process. This method is referred to as the Luenberger observer [53]. It is a deterministic estimator that is not optimal in any probabilistic sense. The performance criterion is based on the stability of the observer and dynamic behavior of the reconstruction error. The observer is stable if and only if the eigenvalues of the $n \times n$ matrix $A - ALC$ have moduli strictly less than one. This result comes from the dynamic response of the reconstruction error from an initial nonzero value with no disturbances present.

$$e_{k+1|k} = (A - ALC) e_{k|k-1} \tag{4.7}$$

The gain matrix can be determined by choosing the eigenvalues of $A - ALC$, which are the closed-loop observer poles. This technique is called pole placement and requires that the system be observable. If the gain is chosen such that all of the eigenvalues are zero, the result is a deadbeat observer. The choice of the observer poles is a compromise between rapid decay of the reconstruction error, which requires that the poles be placed close to the origin, and sensitivity to measurement noise and modeling error, which increases as the poles are moved toward the origin.

4.3 Batch State Estimation

If the assumption that w_k and v_k are Gaussian random variables is relaxed and they are considered as process and measurement disturbances with unknown statistics, an optimal state estimate in the probabilistic sense cannot be obtained. However, an estimator that provides the best state estimate in a deterministic sense can be based on the solution of the following least squares problem.

$$\min_{\{\hat{w}_{-1|k},\ldots,\hat{w}_{k-1|k}\}} \Psi_k = \hat{w}_{-1|k}^T Q_0^{-1} \hat{w}_{-1|k} + \sum_{j=0}^{k-1} \hat{w}_{j|k}^T Q^{-1} \hat{w}_{j|k} + \sum_{j=0}^{k} \hat{v}_{j|k}^T R^{-1} \hat{v}_{j|k} \tag{4.8}$$

$$\text{Subject to:} \quad \begin{aligned} \hat{x}_{0|k} &= \bar{x}_0 + \hat{w}_{-1|k} \\ \hat{x}_{j+1|k} &= A\hat{x}_{j|k} + Bu_j + \hat{w}_{j|k} \\ y_j &= C\hat{x}_{j|k} + \hat{v}_{j|k} \end{aligned} \tag{4.9}$$

In this problem, $\hat{x}_{0|k}$ is the estimate of x_0 given k output measurements, $\hat{w}_{j|k}$ are the estimated process disturbances, and $\hat{v}_{j|k}$ are the estimated measurement disturbances. The coefficients in the weighting matrices, Q_0^{-1}, Q^{-1}, and R^{-1}, specify the relative contribution of each of the terms in the quadratic objective and are the tuning parameters for the batch estimator.

This approach attempts to minimize the estimated process and measurement disturbances in a least squares sense. The choice of the weighting matrices is based on a compromise between minimizing the estimated process disturbances and minimizing the estimated measurement disturbances in a manner similar to the measurement and state noise covariance matrices used in the Kalman filter. If the process measurements are very reliable, R^{-1} is chosen to be large relative to Q^{-1} to penalize estimated measurement disturbances. On the other hand, if the process measurements are poor, R^{-1} is chosen to be small relative to Q^{-1} to prevent uncertain measurements from causing large estimated process disturbances.

The solution to the least squares problem, $\hat{w}_{j|k}^*$, is used to compute the state estimate at time j given k output measurements, $\hat{x}_{j|k}$, as follows.

$$\hat{x}_{j|k} = A^j \bar{x}_0 + \sum_{i=0}^{j} A^{j-i} \hat{w}^*_{i-1|k} + \sum_{i=1}^{j} A^{j-i} B u_{i-1} \tag{4.10}$$

This expression computes a smoothed state estimate when $j < k$, a filtered state estimate when $j = k$, and a predicted state estimate when $j > k$. It can be shown that the filtered estimate from the batch least squares estimator is the optimal filtered estimate for the stochastic system in Eq. 4.1 when v_k, w_k and $\bar{x}_0$ follow the same assumptions made in the Kalman filter and the weighting matrices are chosen as the inverse of the corresponding covariance matrices [40]. Since the inverses are used in this formulation, Q, Q_0, and R must be restricted to nonsingular matrices.

4.4 Moving Horizon State Estimation

The batch state estimator requires the solution of a least squares problem using all of the k previous output measurements to obtain a filtered state estimate. As time progresses, the number of decision variables in this approach becomes prohibitive making it a rather cumbersome procedure that cannot be implemented in practice. A recursive form of the batch state estimator can be constructed using a moving horizon formulation. In this approach, the state is estimated from a horizon of the most recent $N+1$ output measurements that moves forward at each sampling time when a new measurement is available.

The state estimate at time k is determined from the solution of the following least squares problem in which $\hat{x}_{k-N|k-N-1}$ is the predicted estimate of the state at time $k-N$ given measurements up to time $k-N-1$ and P_{k-N} is the covariance of this estimate computed using the recursion in Eq. 4.6.

$$\begin{aligned} \min_{\{\hat{w}_{k-N-1|k},\ldots,\hat{w}_{k-1|k}\}} \Psi_k^N = {} & \hat{w}^T_{k-N-1|k} P^{-1}_{k-N} \hat{w}_{k-N-1|k} \\ & + \sum_{j=k-N}^{k-1} \hat{w}^T_{j|k} Q^{-1} \hat{w}_{j|k} + \sum_{j=k-N}^{k} \hat{v}^T_{j|k} R^{-1} \hat{v}_{j|k} \end{aligned} \tag{4.11}$$

$$\text{Subject to:} \quad \begin{aligned} \hat{x}_{k-N|k} &= \hat{x}_{k-N|k-N-1} + \hat{w}_{k-N-1|k} \\ \hat{x}_{j+1|k} &= A\hat{x}_{j|k} + Bu_j + \hat{w}_{j|k} \\ y_j &= C\hat{x}_{j|k} + \hat{v}_{j|k} \end{aligned} \tag{4.12}$$

The moving horizon allows for a finite number of decision variables at each sampling time. The first N estimates are computed using the batch estimator to initialize the observer horizon.

The state estimate at time $k - N + j$ given k output measurements, $\hat{x}_{k-N+j|k}$, is computed from the solution of the least squares problem in a manner similar to the batch estimator.

$$\begin{aligned}\hat{x}_{k-N+j|k} = {} & A^j \hat{x}_{k-N|k-N-1} \\ & + \sum_{i=0}^{j} A^{j-i} \hat{w}^*_{k-N-1+i|k} + \sum_{i=1}^{j} A^{j-i} B u_{k-N-1+i}\end{aligned} \tag{4.13}$$

This expression computes a smoothed state estimate when $j < N$, a filtered state estimate when $j = N$, and a predicted state estimate when $j > N$.

Muske *et al.* [67] show the predicted state estimate $\hat{x}_{k+1|k}$ computed from the moving horizon estimator to be equivalent to the Kalman filter estimate in Eq. 4.3 for the autonomous system

$$\begin{aligned} x_{k+1} &= A x_k + w_k \\ y_k &= C x_k + v_k \end{aligned} \tag{4.14}$$

in which v_k, w_k and $\bar{x}_0$ follow the same assumptions made in the Kalman filter and the weighting matrices P^{-1}_{k-N}, Q^{-1}, and R^{-1}, are the inverses of the nonsingular covariance matrices for $\hat{x}_{k-N|k-N-1}$, w_k, and v_k, respectively. This result is demonstrated in Appendix B.1. The autonomous system is used to simplify the presentation. There is no loss of generality since the model is linear and the contribution from the input can simply be added to the estimate as in Eq. 4.13.

4.5 Constraints

Since the moving horizon estimator produces the same estimate as the Kalman filter, there is no incentive to implement this approach due to the additional computational effort required to solve the least squares problem and compute the filtered state estimate. The motivation for employing the moving horizon formulation is the addition of constraints on the estimated states and state disturbances. The estimated state constraints specify maximum and minimum limits on the estimate of the states. These constraints are applied to prevent physically unrealistic state estimates that can be due to spurious output measurements. The estimated state disturbance constraints specify limits on the estimated state disturbances. These constraints can be viewed as altering the probability distribution function of the state disturbances such that the probability of any state disturbance outside of the constraints is zero. This prevents estimated state disturbances that cannot realistically occur in the process.

These constraints are imposed based on a heuristic argument with no probabilistic justification. Therefore, the constrained estimator is not necessarily optimal in any probabilistic sense even for linear, Gaussian systems. However, the constraints allow for the implementation of a reasonably simple nonlinear estimator that can handle complex stochastic systems as demonstrated by the following examples. Improved performance of the constrained moving horizon estimator compared to the Kalman filter for a gross output measurement error is demonstrated by the example in Muske *et al.* [67]. Improved closed-loop performance of a combined constrained estimator/regulator compared to a linear quadratic regulator with a Kalman filter for the same measurement error is demonstrated by the example of Section 1 in Meadows *et al.* [59].

4.5.1 Constrained Estimator

The constrained least squares estimator is a nonlinear deterministic estimator requiring the solution of a quadratic program that attempts to minimize the output prediction errors and state disturbances subject to the following constraints.

$$\hat{f}_{\min} \leq \hat{F}\hat{x}_{j|k} \leq \hat{f}_{\max} \tag{4.15}$$

$$\hat{w}_{\min} \leq \hat{w}_{i|k} \leq \hat{w}_{\max} \tag{4.16}$$

The constrained batch state estimator consists of the least squares objective in Eqs. 4.8 and 4.9 subject to these constraints in which $j = 1, 2, \ldots, k$ in Eq. 4.15 and $i = 0, 1, \ldots, k-1$ in Eq. 4.16. The constrained moving horizon state estimator consists of the least squares objective in Eqs. 4.11 and 4.12 subject to these constraints in which $j = k-N, k-N+1, \ldots, k$ in Eq. 4.15 and $i = k-N, k-N+1, \ldots, k-1$ in Eq. 4.16. These constraints are specified for the autonomous system in Eq. 4.14. For systems forced by an input, the state constraints are shifted by the nominal trajectory due to the input. In order to ensure that a consistent constraint set is specified, the following restrictions are imposed.

$$\begin{bmatrix} \hat{f}_{\max} \\ -\hat{f}_{\min} \\ \hat{w}_{\max} \\ -\hat{w}_{\min} \end{bmatrix} > \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{4.17}$$

These restrictions guarantee that the constraints specify a convex region in the $\hat{w}$ space containing a neighborhood of the origin. The constraint regions in $\hat{w}$ and $\hat{x}$ space are defined as follows.

$$\mathcal{W} = \{\hat{w} \in \Re^n \mid \hat{w}_{\min} \leq \hat{w} \leq \hat{w}_{\max}\} \tag{4.18}$$

$$\mathcal{F} = \left\{\hat{x} \in \Re^n \mid \hat{f}_{\min} \leq \hat{F}\hat{x} \leq \hat{f}_{\max}\right\} \tag{4.19}$$

4.5.2 Constraint Feasibility

Since the first estimated state and state disturbance is not constrained in Eqs. 4.15 and 4.16, feasibility of the constraints is guaranteed for both the batch and moving horizon estimators for any *a priori* estimate of the initial state, $\bar{x}_0$. Feasibility is demonstrated for stable A and then for unstable A without considering the estimated state constraints.

Stable Process Model. For stable A, the following feasibility results apply.

Lemma 4.1 *For all $\bar{x}_0, x_0 \in \Re^n$ and the nominal model with stable A, the estimated state and state disturbance constraints in Eqs. 4.15 and 4.16 are feasible at every time $k \geq 0$ for the constrained batch state estimator.*

Proof: See Appendix F.1.

Lemma 4.2 *For all $\bar{x}_0, x_0 \in \Re^n$, $N \geq 1$, and the nominal model with stable A, the estimated state and state disturbance constraints in Eqs. 4.15 and 4.16 are feasible at every time $k \geq 0$ for the constrained moving horizon state estimator.*

Proof: See Appendix F.2.

Unstable Process Model. For unstable A, the estimates of the state for the autonomous system in Eq. 4.14 become unbounded. Therefore, the estimated state constraints in Eq. 4.15 are not considered. The following feasibility results then apply.

Lemma 4.3 *For all $\bar{x}_0, x_0 \in \Re^n$ and the nominal model with unstable A, the estimated state disturbance constraints in Eq. 4.16 are feasible at every time $k \geq 0$ for the constrained batch state estimator.*

Proof: See Appendix F.3.

Lemma 4.4 *For all $\bar{x}_0, x_0 \in \Re^n$, $N \geq 1$, and the nominal model with unstable A, the estimated state disturbance constraints in Eq. 4.16 are feasible at every time $k \geq 0$ for the constrained moving horizon state estimator.*

Proof: See Appendix F.4.

4.6 Constrained Batch State Estimation

The nominal stability properties of the constrained batch estimator are demonstrated in this section. The constrained estimator is asymptotically stable provided the reconstruction error of the estimate converges to zero from an arbitrary non-zero initial condition without state or measurement disturbances. The reconstruction error for the autonomous system in Eq. 4.14 with no disturbances is determined by the following dynamic equation.

$$e_{j+1|k} = x_{j+1} - \hat{x}_{j+1|k} = Ae_{j|k} - \hat{w}^*_{j|k}, \quad j = 0, \ldots, k-1 \tag{4.20}$$

$$\hat{v}_{j|k} = Ce_{j|k} \tag{4.21}$$

$$e_{0|k} = x_0 - (\bar{x}_0 + \hat{w}^*_{-1|k})$$

in which $\hat{w}^*_{j|k}$ is the optimal solution to the constrained batch state estimation quadratic program. The state estimates are computed as follows.

$$\hat{x}_{j|k} = A^j \bar{x}_0 + \sum_{i=0}^{j} A^{j-i} \hat{w}^*_{i-1|k} \tag{4.22}$$

The constrained batch state estimator for the autonomous system in Eq. 4.14 consists of the objective function in Eq. 4.8 and equality constraints in Eq. 4.9, expressed in terms of the reconstruction error in Eq. 4.20, and the inequality constraints in Eqs. 4.15 and 4.16 as follows.

$$\min_{\{\hat{w}_{-1|k},\ldots,\hat{w}_{k-1|k}\}} \Psi_k = \hat{w}^T_{-1|k} Q_0^{-1} \hat{w}_{-1|k} + \sum_{j=0}^{k-1} \hat{w}^T_{j|k} Q^{-1} \hat{w}_{j|k} + \sum_{j=0}^{k} e^T_{j|k} C^T R^{-1} C e_{j|k} \tag{4.23}$$

$$\text{Subject to:} \quad \begin{array}{rcll} e_{0|k} & = & x_0 - (\bar{x}_0 + \hat{w}_{-1|k}) & \\ e_{j+1|k} & = & Ae_{j|k} - \hat{w}_{j|k}, & j = 0, 1, \ldots, k-1 \\ \hat{w}_{j|k} & \leq & \hat{w}_{\max}, & j = 0, 1, \ldots, k-1 \\ -\hat{w}_{j|k} & \leq & \hat{w}_{\min}, & j = 0, 1, \ldots, k-1 \end{array} \tag{4.24}$$

For stable A, the estimated state constraints are considered.

$$\begin{array}{rcll} \hat{F}(A^j x_0 - e_{j|k}) & \leq & \hat{f}_{\max}, & j = 1, 2, \ldots, k \\ -\hat{F}(A^j x_0 - e_{j|k}) & \leq & \hat{f}_{\min}, & j = 1, 2, \ldots, k \end{array} \tag{4.25}$$

4.6.1 Nominal Convergence

The following lemmas demonstrate convergence of the sequence of reconstruction errors $\{e_{k|k}\}$ to zero for the state estimate computed from the constrained batch state estimator consisting of the quadratic program in Eqs. 4.23

through 4.25 for stable A or Eqs. 4.23 and 4.24 for unstable A. Feasibility of this quadratic program follows from Lemma 4.1 for stable A and Lemma 4.3 for unstable A.

Stable Process Model. For stable A, the following convergence results apply.

Lemma 4.5 *The reconstruction error sequence* $\{e_{k|k}\}$ *determined at each time* k *from Eq. 4.20 converges to zero for stable* A.

Proof: See Appendix F.5.

Corollary 4.1 *The estimated state trajectory* $\{\hat{x}_{k|k}\}$ *determined at each time* k *from Eq. 4.22 converges to zero for stable* A.

Proof: This result follows directly from Lemma 4.5 and stable A.

Unstable Process Model. For unstable A, (C, A) must be detectable for the reconstruction error to converge. The following convergence results then apply.

Lemma 4.6 *The reconstruction error sequence* $\{e_{k|k}\}$ *determined at each time* k *from Eq. 4.20 converges to zero for unstable* A *and* (C, A) *detectable.*

Proof: See Appendix F.6.

4.6.2 Nominal Asymptotic Stability

Since the constraints contain a neighborhood of the origin, there exists some neighborhood of the origin in which the inequality constraints in Eqs. 4.24 and 4.25 are not active and the constrained batch state estimator is equivalent to the Kalman filter. Stability of the constrained batch estimator then follows from the stability of the Kalman filter. Asymptotic stability follows from stability and convergence.

Lemma 4.7 *The origin* $e_{k|k} = 0$ *is a stable equilibrium point for the reconstruction error in Eq. 4.20.*

Proof: See Appendix F.7.

Theorem 4.1 *For* (C, A) *detectable, the origin* $e_{k|k} = 0$ *is a globally asymptotically stable solution of the reconstruction error in Eq. 4.20.*

Proof: See Appendix F.8.

Theorem 4.2 *For (C, A) detectable, the origin $e_{k|k} = 0$ is an exponentially stable solution of the reconstruction error in Eq. 4.20.*

Proof: See Appendix F.9.

4.7 Constrained Moving Horizon State Estimation

An implementable moving horizon constrained estimator is discussed in this section. The constrained estimator is asymptotically stable provided the reconstruction error of the state estimate converges to zero from an arbitrary non-zero initial condition without state and measurement disturbances. The addition of constraints requires that the initial state estimate in the moving horizon come from the Kalman filter estimate in Eq. 4.3 and not from the moving horizon estimate at time $k - N - 1$. The Kalman filter estimate will be referred to as $\hat{z}_{k-N|k-N-1}$ in this section. This modification is sufficient to guarantee nominal constrained stability. The reconstruction error for the autonomous system in Eq. 4.14 with no disturbances is determined by the following dynamic equation.

$$
\begin{aligned}
e_{j+1|k} &= x_{j+1} - \hat{x}_{j+1|k} = Ae_{j|k} - \hat{w}^*_{j|k}, \quad j = k - N, \ldots, k - 1 \\
e_{k-N|k} &= x_{k-N} - (\hat{z}_{k-N|k-N-1} + \hat{w}^*_{k-N-1|k})
\end{aligned}
\tag{4.26}
$$

in which $\hat{w}^*_{j|k}$ is the optimal solution to the constrained moving horizon state estimation quadratic program. The state estimates are computed as follows.

$$
\hat{x}_{k-N+j|k} = A^j \hat{z}_{k-N|k-N-1} + \sum_{i=0}^{j} A^{j-i} \hat{w}^*_{k-N-1+i|k} \tag{4.27}
$$

The constrained moving horizon state estimator for the autonomous system in Eq. 4.14 consists of the objective function in Eq. 4.11, expressed in terms of the reconstruction error, subject to the inequality constraints in Eqs. 4.15 and 4.16 as follows.

$$
\begin{aligned}
\min_{\{\hat{w}_{k-N-1|k}, \ldots, \hat{w}_{k-1|k}\}} \Psi_k^N &= \hat{w}^T_{k-N-1|k} P^{-1}_{k-N} \hat{w}_{k-N-1|k} \\
&\quad + \sum_{j=k-N}^{k-1} \hat{w}^T_{j|k} Q^{-1} \hat{w}_{j|k} + \sum_{j=k-N}^{k} e^T_{j|k} C^T R^{-1} C e_{j|k}
\end{aligned}
\tag{4.28}
$$

$$
\begin{array}{rrcll}
& \hat{z}_{k-N|k-N-1} & = & \left(\prod_{j=0}^{k-N}(A-L_jC)\right)\bar{x}_0 & \\
& & & +\left(\sum_{j=0}^{k-N}\left(\prod_{i=j+1}^{k-N}(A-L_iC)\right)L_jCA^j\right)x_0 & \\
\text{Subject to:} & e_{k-N|k} & = & A^{k-N}x_0-(\hat{z}_{k-N|k-N-1}+\hat{w}_{k-N-1|k}) & \\
& e_{j+1|k} & = & Ae_{j|k}-\hat{w}_{j|k}, & j=k-N,\ldots,k-1 \\
& \hat{w}_{j|k} & \leq & \hat{w}_{\max}, & j=k-N,\ldots,k-1 \\
& -\hat{w}_{j|k} & \leq & \hat{w}_{\min}, & j=k-N,\ldots,k-1
\end{array}
\tag{4.29}
$$

For stable A, the estimated state constraints are considered.

$$
\begin{array}{rcll}
\hat{F}(A^jx_0-e_{j|k}) & \leq & \hat{f}_{\max}, & j=k-N+1,\ldots,k \\
-\hat{F}(A^jx_0-e_{j|k}) & \leq & \hat{f}_{\min}, & j=k-N+1,\ldots,k
\end{array}
\tag{4.30}
$$

4.7.1 Nominal Convergence

The following lemmas demonstrate convergence of the sequence of reconstruction errors $\{e_{k|k}\}$ to zero for the state estimate computed from the constrained moving horizon state estimator consisting of the quadratic program in Eqs. 4.28 through 4.30 for stable A or Eqs. 4.28 and 4.29 for unstable A. Feasibility of this quadratic program follows from Lemma 4.2 for stable A and Lemma 4.4 for unstable A.

Stable Process Model. For stable A, the following convergence results apply.

Lemma 4.8 *The reconstruction error sequence $\{e_{k|k}\}$ determined at each time k from Eq. 4.26 converges to zero for stable A.*

Proof: See Appendix F.10.

Corollary 4.2 *The estimated state trajectory $\{\hat{x}_{k|k}\}$ determined at each time k from Eq. 4.27 converges to zero for stable A.*

Proof: This result follows directly from Lemma 4.8 and stable A.

Unstable Process Model. For unstable A, (C, A) must be detectable for the reconstruction error to converge. The following convergence results then apply.

Lemma 4.9 *The reconstruction error sequence $\{e_{k|k}\}$ determined at each time k from Eq. 4.26 converges to zero for unstable A and (C, A) detectable.*

Proof: See Appendix F.11.

4.7.2 Nominal Asymptotic Stability

Since the constraints contain a neighborhood of the origin, there exists some neighborhood of the origin in which the inequality constraints in Eqs. 4.29 and 4.30 are not active and the constrained moving horizon state estimator is equivalent to the Kalman filter. Stability of the constrained moving horizon estimator then follows from the stability of the Kalman filter. Asymptotic stability follows from stability and convergence.

Lemma 4.10 *The origin $e_{k|k} = 0$ is a stable equilibrium point for the reconstruction error in Eq. 4.26.*

Proof: The proof follows in the same manner as that for Lemma 4.7.

Theorem 4.3 *For (C, A) detectable and $N \geq 1$, the origin $e_{k|k} = 0$ is a globally asymptotically stable solution of the reconstruction error in Eq. 4.26.*

Proof: See Appendix F.12.

Theorem 4.4 *For (C, A) detectable and $N \geq 1$, the origin $e_{k|k} = 0$ is an exponentially stable solution of the reconstruction error in Eq. 4.26.*

Proof: The proof follows in the same manner as that for Theorem 4.2.

4.8 Unpenalized Initial Estimate

Muske and Rawlings [66] present a method of implementing the constrained moving horizon estimation problem that does not require the Kalman filter estimate. In this method, the penalty on the initial estimated state disturbance, $\hat{w}_{k-N|k-N-1}$, is removed which allows the initial state estimate in the horizon to be chosen freely. The probabilistic interpretation is that that the state estimate used at the beginning of the horizon is completely uncertain. The moving horizon state estimator objective function is expressed in terms of the reconstruction error for this method as follows.

$$\min_{\{\hat{z}_{k-N|k}, \hat{w}_{k-N|k}, \ldots, \hat{w}_{k-1|k}\}} \Psi_k^N = \sum_{j=k-N}^{k-1} \hat{w}_{j|k}^T Q^{-1} \hat{w}_{j|k} + \sum_{j=k-N}^{k} e_{j|k}^T C^T R^{-1} C e_{j|k} \tag{4.31}$$

$$\text{Subject to:} \quad \begin{array}{rcll} e_{k-N|k} & = & A^{k-N} x_0 - \hat{z}_{k-N|k} & \\ e_{j+1|k} & = & A e_{j|k} - \hat{w}_{j|k}, & j = k-N, \ldots, k-1 \\ \hat{w}_{j|k} & \leq & \hat{w}_{\max}, & j = k-N, \ldots, k-1 \\ -\hat{w}_{j|k} & \leq & \hat{w}_{\min}, & j = k-N, \ldots, k-1 \end{array} \tag{4.32}$$

For stable A, the estimated state constraints are considered.

$$\begin{array}{rcll} \hat{F}(A^j x_0 - e_{j|k}) & \leq & \hat{f}_{\max}, & j = k - N + 1, \ldots, k \\ -\hat{F}(A^j x_0 - e_{j|k}) & \leq & \hat{f}_{\min}, & j = k - N + 1, \ldots, k \end{array} \tag{4.33}$$

4.8.1 Observability Restriction

Since the initial estimated state disturbance is not penalized, there is a possibility of unbounded or nonunique solutions to the quadratic program. If this estimator is restricted to systems in which (C, A) is observable and the moving horizon length N is greater than or equal to $n - 1$, a unique solution can be guaranteed.

Lemma 4.11 *For (C, A) observable and $N \geq n$, the quadratic program with objective function Eq. 4.31 and constraints Eq. 4.32 and, for stable A, Eq. 4.33 has a unique global solution.*

Proof: See Appendix F.13.

4.8.2 Nominal Asymptotic Stability

Convergence of the reconstruction error sequence $\{e_{k|k}\}$ can be proved in the same manner as Lemma 4.8 for stable A or Lemma 4.9 for unstable A. Since this formulation is a deadbeat estimator for the nominal system in Eq. 4.20, stability can be shown in the same manner as Lemma 4.7 in which equivalence to the deadbeat estimator instead of the Kalman filter is used. Global asymptotic stability then follows as shown in Theorem 4.3 and exponential asymptotic stability follows as shown in Theorem 4.4.

Chapter 5

Output Feedback

The discussion in Chapter 3 assumed that the states are perfectly measured at each time k. For applications in which all of the states cannot be perfectly measured, the receding horizon regulator is implemented using state estimates. These estimates are determined from an estimator or observer that reconstructs the state of a linear system from the output measurements as discussed in Chapter 4. This technique is referred to as output feedback. The implementation of the output feedback receding horizon regulator is presented followed by a discussion of its stability properties.

5.1 Output Feedback Receding Horizon Regulator

An output feedback receding horizon regulator can be constructed from the target tracking regulator presented in Section 3.4 by setting the initial state in the open-loop quadratic objective function equal to the estimate of the current state at each time k.

$$z_0 = \hat{x}_{k|k} \tag{5.1}$$

As discussed in Chapter 4, the estimate of the current state can be determined by a Kalman filter, deterministic observer, or optimization based state estimation technique. Tracking a non-zero controlled variable target is performed in the same manner described in Chapter 3. The controlled variables are determined from the state estimate as shown in Eq. 3.1.

$$\hat{y}^c_k = \acute{C}\hat{x}_{k|k} \tag{5.2}$$

The steady-state state and input vectors that minimize the controlled variable tracking error are determined by the constrained target tracking quadratic program discussed in Section 3.3.

5.1.1 Output Feedback Regulator Quadratic Program

The output feedback receding horizon regulator is based on the minimization of the following target tracking open-loop quadratic objective function.

$$\min_{\pi} \Phi_k^N(x_k, u_{k-1}, \pi) = (z_N - x_s)^T \acute{Q}_N (z_N - x_s) + \Delta v_N^T S \Delta v_N \tag{5.3}$$

$$+ \sum_{j=0}^{N-1} \Big((z_j - x_s)^T \acute{C}^T \acute{Q} \acute{C} (z_j - x_s) + (v_j - u_s)^T R (v_j - u_s) + \Delta v_j^T S \Delta v_j \Big)$$

$$\text{Subject to:} \quad \begin{array}{rcl} z_0 & = & \hat{x}_{k|k} \\ z_{j+1} & = & Az_j + Bv_j, \quad j = 0, 1, \ldots, j_1 + j_2 \\ v_{-1} & = & u_{k-1} - u_s \\ \Delta v_j & = & v_j - v_{j-1} \\ \acute{C} z_j & \leq & y^c_{\max}, \quad j = j_1, j_1 + 1, \ldots, j_1 + j_2 \\ -\acute{C} z_j & \leq & y^c_{\min}, \quad j = j_1, j_1 + 1, \ldots, j_1 + j_2 \\ v_j & \leq & u_{\max}, \quad j = 0, 1, \ldots, N - 1 \\ -v_j & \leq & u_{\min}, \quad j = 0, 1, \ldots, N - 1 \\ \Delta v_j & \leq & \Delta_{\max} \quad j = 0, 1, \ldots, N \\ -\Delta v_j & \leq & \Delta_{\min} \quad j = 0, 1, \ldots, N \\ \tilde{V}_u (z_N - x_s) & = & 0 \\ v_j & = & u_s, \quad j \geq N \end{array} \tag{5.4}$$

The terminal state penalty, $\acute{Q}_N$, is computed as show in Eq. 3.22 for stable A or Eq. 3.23 for unstable A. The steady-state state and input, x_s and u_s, are determined from the target tracking quadratic program in Eqs. 3.7 and 3.8.

5.1.2 Initial Constraint Feasibility

At any time k, a feasible controlled variable constraint set can be obtained by removing the constraints at the beginning of the open-loop constraint horizon up to time $j = j_1$ in which $j_1 \geq 1$ as discussed in Section 2.5.2. Since the reconstruction error introduces a perturbation into the nominal state feedback trajectory, Lemma 2.1 is no longer applicable and there is no guarantee that the algorithm in Eq. 2.37 remains feasible with output feedback. In this algorithm, the open-loop constraint horizon slides back to $j_1(k) - 1$ at time $k + 1$. When reconstruction error is present, this value of j_1 may have to be increased to retain feasibility. This will be the case when, for example, the actual state is outside of the feasible region of the controlled variable constraint, but the state estimate is inside of this region inducing control action that causes a constraint violation at a later time. Example 2 in Muske and Rawlings [64] demonstrates the non-monotonic response of $j_1(k)$ in the presence of reconstruction error.

The state feedback algorithm for $j_1(k)$ can be modified such that if $j_1(k)-1$ results in an infeasible constraint set, the minimum value of $j_1(k+1)$ needed to ensure feasibility is used. This procedure ensures feasibility of the controlled variable constraints and recovers the state feedback algorithm in the case of no reconstruction error. Since large reconstruction errors can significantly effect the minimum value of j_1, it also produces a less aggressive controller than that obtained from the use of the minimum value of $j_1(k)$ at every time k. For the nominal model with output feedback, there is no guarantee that $j_1(k) = 1$ for all time $k \geq j_1(0)$ so that the controlled variable constraints are satisfied in the closed loop after time $j_1(0)$. This point is revisited later in this chapter when stability of the output feedback receding horizon regulator with state constraints is considered.

5.1.3 Unstable Process Model

For unstable A with output feedback, the set of initial conditions is restricted such that $x_k \in \mathcal{Z}_N^n$ and $\hat{x}_{k|k} \in \mathcal{Z}_N^n$ for all $k \geq 0$ in which the set $\mathcal{Z}_N^n$ is defined in Eq. 2.41. The first restriction ensures constrained stabilizability for all $k > 0$. The second restriction ensures a that feasible solution to the equality constraint on the unstable modes in Eq. 5.4 of the output feedback regulator quadratic program exists for all $k \geq 0$. Therefore, an unstable process model controlled by output feedback can be stabilized if $(x_0, e_{0|0}) \in \mathcal{V}_N^n$ in which $\mathcal{V}_N^n$ denotes the set of $(x_0, e_{0|0})$ satisfying these restrictions.

$$\mathcal{V}_N^n = \left\{ x_0,\, e_{0|0} \in \Re^n \;\middle|\; x_k,\, \hat{x}_{k|k} \in \mathcal{Z}_N^n \;\; \forall \;\; k \geq 0 \right\} \tag{5.5}$$

This restriction is a sufficient condition for constrained stability with output feedback since $\hat{x}_{k|k} \in \mathcal{Z}_N^n$ for all $k \geq 0$ is stronger than is necessary to ensure that the unstable modes of x_k can be brought asymptotically to the origin with a feasible input sequence for every $k \geq 0$. The set of initial states in $\mathcal{V}_N^n$ is a subset of $\mathcal{Z}_N^n$ that depends on the value of the initial reconstruction error $e_{0|0}$. As the initial reconstruction error approaches zero, the set of initial states in $\mathcal{V}_N^n$ approaches the set $\mathcal{Z}_N^n$. For stable A, $\mathcal{Z}_N^n = \Re^n$ which implies $\mathcal{V}_N^n = (\Re^n, \Re^n)$.

5.1.4 Closed-loop State Trajectory

The closed-loop state trajectory of the output feedback receding horizon regulator can be expressed as

$$x_{k+1} = Ax_k + Bu_s + B\mu(\hat{x}_{k|k}) \tag{5.6}$$

in which $\mu(\hat{x}_{k|k})$ represents the nonlinear control law from the solution of the quadratic program in Eqs. 5.3 and 5.4 and u_s is the steady-state input computed from the target tracking quadratic program in Eqs. 3.7 and 3.8. The estimated state at time k is related to the actual state as follows

$$\hat{x}_{k|k} = x_k + e_{k|k} \tag{5.7}$$

in which $e_{k|k}$ is the reconstruction error at time k given output measurements up to time k. The difference between the input computed with the state estimate and that computed with the actual state is due to this reconstruction error. The closed-loop state trajectory in Eq. 5.8 can be rearranged to show that it evolves as the nominally asymptotically stable state feedback system discussed in Chapter 2 subject to additive perturbations due to the reconstruction error.

$$x_{k+1} = Ax_k + Bu_s + B\mu(x_k) + B\left(\mu(x_k + e_{k|k}) - \mu(x_k)\right) \tag{5.8}$$

The following norm bound on these perturbations can be obtained from the difference between the receding horizon regulator quadratic program solutions.

$$\|B\| \; \|\mu(x_k + e_{k|k}) - \mu(x_k)\| \tag{5.9}$$

Example 2 in Muske and Rawlings [64] demonstrates the closed-loop input and output response of the output feedback regulator in the presence of reconstruction error.

5.2 Unconstrained Controlled Variables

In this section, nominal asymptotic stability of the output feedback receding horizon regulator is demonstrated for a non-zero initial state and reconstruction error without the following controlled variable constraints in Eq. 5.4.

$$\begin{aligned} \acute{C}z_j &\leq y^c_{\max}, \quad j = j_1, j_1+1, \ldots, j_1+j_2 \\ -\acute{C}z_j &\leq y^c_{\min}, \quad j = j_1, j_1+1, \ldots, j_1+j_2 \end{aligned}$$

In this case, the solution of the receding horizon regulator quadratic program is a globally Lipschitz continuous function of the initial state, $\hat{x}_{k|k}$, as discussed in Appendix A.2.3. The perturbation in Eq. 5.9 is then bounded as follows

$$\|B\| \, \|\mu(x_k + e_{k|k}) - \mu(x_k)\| < \rho \|B\| \, \|e_{k|k}\| \tag{5.10}$$

for all $x_k, e_{k|k} \in \Re^n$.

5.2.1 Convergence

Provided the estimator is stable and the norm of the reconstruction error is a summable sequence, the convergence of the state to the steady-state state target x_s for the output feedback receding horizon regulator follows from the converse Lyapunov stability theorems in Halanay [35] and discussed in Appendix C. The restrictions on the estimator apply for the Kalman filter and deterministic observer presented in Chapter 4, which are nominally exponentially asymptotically stable. They also apply in a neighborhood of the origin for the constrained moving horizon estimator presented in Section 4.4.

Lemma 5.1 *The closed-loop state trajectory in Eq. 5.8 converges to x_s for the nominal system with no disturbances and the state estimate computed from an exponentially asymptotic stable state estimator.*

Proof: See Appendix G.1.

5.2.2 Nominal Asymptotic Stability

A necessary and sufficient condition for stability of a dynamic system with perturbations is Lipschitz continuity of the system function in a neighborhood of the equilibrium point and asymptotic stability of the unperturbed system. Under these conditions, LaSalle [50] refers to the equilibrium point as strongly stable under perturbations. Stability of an equilibrium point for an asymptotically stable discrete dynamic system with perturbations is demonstrated in Halanay [35]. Nominal asymptotic stability of the closed-loop system in Eq. 5.8 for a non-zero initial state and reconstruction error then follows from stability and convergence of the closed-loop state trajectory. Feasibility of the output feedback receding horizon regulator quadratic program is ensured by restricting the initial state and reconstruction error such that $(x_0, e_0) \in \mathcal{V}_N^n$.

Lemma 5.2 *The state $x_k = x_s$ is a stable equilibrium point for the closed-loop system in Eq. 5.8.*

Proof: The result follows directly from the stability proof of Section 3 in [35].

Theorem 5.1 *For (A, B) stabilizable, $N \geq r$, in which r is the number of unstable modes of A, the nominal system with no disturbances, and the state estimate computed from an exponentially stable estimator, $x_k = x_s$ is an asymptotically stable solution of the closed-loop system in Eq. 5.8 for all $(x_0, e_0) \in \mathcal{V}_N^n$.*

Proof: See Appendix G.2.

5.3 Controlled Variable Constraints

When the controlled variable constraints are included in the output feedback receding horizon regulator, the solution is no longer guaranteed to be a Lipschitz continuous function of the state as discussed in Appendix A. Discontinuous state feedback can result when the controlled variable constraint horizon increases to retain feasibility of the quadratic program. An example of discontinuous feedback for the receding horizon regulator presented in Chapter 2 with state constraints is presented in Meadows [58]. In this case, the converse Lyapunov stability theorems in Appendix C become local results confined to the neighborhood of the origin $\|x\| < \varepsilon$ in which ε is defined in Appendix D.7. Stability of the origin, which is a local property, still holds from the stability proof in Halanay [35]. However, convergence of the closed-loop state sequence $\{x_k\}$ cannot be shown in the same manner as Lemma 5.1.

The receding horizon regulator with perfect state measurement determines the beginning of the open-loop constraint horizon, j_1, by decrementing the previous value at each time k. This sequence decreases monotonically until $j_1(k) = 1$ for all time $k > j_1(0)$. When the estimated state is used in the receding horizon regulator, the monotonicity of j_1 is no longer guaranteed as discussed previously. A necessary condition for convergence of the state sequence is that $j_1(k) = 1$ for all $k \geq k^*$ in which k^* is some finite time. Since the nonlinear control law $\mu(\hat{x}_{k|k})$ is Lipschitz continuous for all $\hat{x}_{k|k}$ such that $j_1 = 1$, convergence of the state sequence under perturbations follows as shown in Lemma 5.1. Nominal asymptotic stability of the output feedback receding horizon regulator can the be shown as in Theorem 5.1. With an exponentially convergent reconstruction error, it is not unreasonable to assume that the $j_1(k)$ trajectory from the output feedback receding horizon regulator converges to the monotonic $j_1(k)$ trajectory from perfect state feedback. Numerical evidence supports this supposition, but it has not been proven. If convergence of the $j_1(k)$ trajectories can be established, nominal asymptotic stability of the output feedback receding horizon regulator with controlled variable constraints can be demonstrated as discussed previously.

Chapter 6

Offset Free Control

The preceding discussion of output feedback in Chapter 5 considered zero mean state and measurement disturbances to the nominal model. If the disturbances entering the process are constant or nonzero mean, steady-state offset between the controlled variables and their achievable targets can result with this output feedback controller. Mismatch between the model and the process can also lead to steady-state offset. An offset free controller is used to eliminate the possibility of steady-state offset with constant disturbances and model mismatch provided the closed loop is stable. The addition of a constant or step disturbance model to the state-space process model is used to obtain offset free control in the receding horizon regulator with output feedback. In this chapter, several unmeasured constant disturbance models are discussed and estimators are constructed to provide estimates of these constant disturbances. A target tracking receding horizon regulator with output feedback that eliminates the constant disturbances is presented and shown to provide offset free control.

6.1 Constant Disturbance Models

The most commonly implemented method to obtain offset free control in model predictive control is the incorporation of a constant step disturbance to the output. This is the standard method in the industrial model predictive control applications such as IDCOM [72], DMC [16],QDMC [28], IDCOM-M [32], and SMOC [54]. For the state-space model formulation in this work, the constant output disturbance is represented as the following augmented system in which s_p is the number of output disturbance states added to the system and $G_p \in \Re^{p \times s_p}$ determines the effect of these states on the output.

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k, \qquad k = 0, 1, 2, \ldots \\ p_{k+1} &= p_k \\ y_k &= Cx_k + G_p p_k \end{aligned} \tag{6.1}$$

Shinskey [80] criticizes industrial model predictive control implementations because they are unable to adequately handle load disturbances due to the assumption of a constant disturbance at the output. This assumption leads to poor disturbance rejection in the case of an input disturbance to a system with a slow dominant lag. Morari and Lee [61] suggest representing the output disturbance as a ramp to improve the rejection of input disturbances. In this work, the incorporation of a constant state disturbance is used to address this issue. This disturbance model formulation is a standard technique to remove steady-state offset in the linear quadratic regulator as shown by Davison and Smith [17] and Kwakernaak and Sivan [45]. The constant state disturbance is represented as the following augmented system in which s_d is the number of disturbance states added to the system and $G_d \in \Re^{n \times s_d}$ determines the effect of the disturbance on the state.

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + G_d d_k, \qquad k = 0, 1, 2, \ldots \\ d_{k+1} &= d_k \\ y_k &= Cx_k \end{aligned} \tag{6.2}$$

If it is assumed that the disturbance enters the system through the input, the constant state disturbance model in Eq 6.2 is used to represent the input disturbances by setting $G_d = B$. This assumption then results in the following constant input disturbance model.

$$\begin{aligned} x_{k+1} &= Ax_k + B(u_k + d_k), \qquad k = 0, 1, 2, \ldots \\ d_{k+1} &= d_k \\ y_k &= Cx_k \end{aligned} \tag{6.3}$$

The linear quadratic framework upon which this work is based allows for very general disturbance modeling. In the discussion of target tracking, output feedback, and offset free control that follows, both state and output disturbances will be considered forming the following combined disturbance model augmented system.

$$x_{k+1} = Ax_k + G_d d_k + Bu_k, \qquad k = 0, 1, 2, \ldots \tag{6.4}$$

$$
\begin{aligned}
d_{k+1} &= d_k \\
p_{k+1} &= p_k \\
y_k &= Cx_k + G_p p_k
\end{aligned}
$$

Output or state disturbance models can be recovered from this formulation by removing the unnecessary disturbance states without loss of generality.

The closed-loop controller performance is directly related to how well the disturbance model represents the actual disturbances entering the process. A suitable model of the dynamic structure of the disturbances to the system is required in order to accurately reject those disturbances and get acceptable performance. This subject is discussed by Francis and Wonham [25] for linear multivariable systems and has become known as the internal model principle. Example 7 in Muske and Rawlings [65] and Example 3 in Muske and Rawlings [64] illustrate the poor performance that can result from an incorrect disturbance model.

6.2 Detectability of the Augmented System

A limitation in constructing the constant disturbances model formulations presented in the preceding section is that the augmented disturbance states must be able to be estimated from the output measurements. Since the augmented states are not asymptotically stable, detectability of the augmented system implies that these states are observable. In this section, the conditions for detectability of each of the constant disturbance augmented systems is presented.

6.2.1 Constant Output Disturbance

The augmented state-space matrices for the output disturbance model in Eq. 6.1 are as follows.

$$
\tilde{A} = \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} C & G_p \end{bmatrix}
$$

The following conditions are necessary and sufficient for detectability of the augmented $(\tilde{A}, \tilde{C})$ system.

i) (A, C) detectable,

ii) G_p full column rank,

iiia) A has no integrating modes, *i.e.* eigenvalues at 1, or iiib) $G_p p \neq C v_i \;\; \forall \;\; p \in \Re^{s_p}$ in which v_i are the integrating modes of A.

The consequence of Condition ii) is that the number of augmented states can not exceed the number of outputs. However, full column rank of G_p alone is not sufficient for detectability of the augmented system. The output disturbances must also be restricted such that the range of G_p does not contain the output space spanned by any of the integrating modes of A. With this restriction, it is always possible to distinguish the contribution of the output disturbance model from that of the integrators in the process model. These conditions are proved in Appendix H.1.

6.2.2 Constant State Disturbance

The augmented state-space matrices for the state disturbance model in Eq. 6.2 are as follows.

$$\tilde{A} = \begin{bmatrix} A & G_d \\ 0 & I \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} C & 0 \end{bmatrix}$$

The following conditions are necessary and sufficient for detectability of the augmented $(\tilde{A}, \tilde{C})$ system.

i) (A, C) detectable,

ii) G_d full column rank,

iii) Column dimension of G_d less than or equal to the row dimension of C,

iv) $G_d d \neq \text{Null}(\mathcal{O})x \;\; \forall \;\; d, x \neq 0$ in which $\mathcal{O}$ is the observability matrix of (C, A).

The consequence of Conditions ii) and iii) is that the number of augmented states cannot exceed the number of states or outputs of the original system. The state disturbances must also be restricted such that the range of G_d does not contain any unobservable modes of (C, A), which are spanned by the nonzero null space of the observability matrix $\mathcal{O}$. This restriction ensures that the contribution of the state disturbance model is observable from the output of the augmented process model. These conditions are proved in Appendix H.2.

6.2.3 Constant Input Disturbance

The augmented state-space matrices for the input disturbance augmented system in Eq. 6.3 are a special case of the state disturbance model as follows.

$$\tilde{A} = \begin{bmatrix} A & B \\ 0 & I \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} C & 0 \end{bmatrix}$$

Dectectability of the constant input disturbance formulation follows under the same conditions given previously for the state disturbance augmented system with $G_d = B$.

6.3 State Estimation

A general stochastic model formulation for the constant disturbance system can be created by appending the constant disturbance states of the augmented system in Eq. 6.4 to the stochastic plant model presented in Eq. 4.1.

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + G_d d_k + w_k, \qquad k = 0, 1, 2, \ldots \\ d_{k+1} &= d_k + \omega_k \\ p_{k+1} &= p_k + \nu_k \\ y_k &= Cx_k + G_p p_k + v_k \end{aligned} \tag{6.5}$$

In this representation, $w_k \in \Re^n$, $\omega_k \in \Re^{s_d}$, $\nu_k \in \Re^{s_p}$, and $v_k \in \Re^p$ are zero mean, uncorrelated, normally distributed, disturbance vectors with covariances Q_w, Q_ω, Q_ν, and R_v respectively. The dynamics of the output step disturbance vector, p_k, are contained in G_p. When $G_p = I$, this disturbance becomes the standard model predictive control output step disturbance. The dynamics of the state step disturbance vector, d_k, are contained in G_d. When $G_d = B$, the disturbance becomes the input step disturbance presented in Eq. 6.3. This system is represented by the following augmented state-space matrices.

$$\tilde{A} = \begin{bmatrix} A & G_d & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B \\ 0 \\ 0 \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} C & 0 & G_p \end{bmatrix}, \quad \tilde{x}_k = \begin{bmatrix} x_k \\ d_k \\ p_k \end{bmatrix} \tag{6.6}$$

An augmented stochastic system in which either the output or the state disturbance is not considered is constructed by removing the corresponding disturbance from the system model in Eq. 6.5.

Assuming the augmented system is detectable, the Kalman filter gain is computed as shown in Eq. 4.5 using the augmented model matrices with the following augmented covariance matrix.

$$\tilde{Q} = \begin{bmatrix} Q_w & 0 & 0 \\ 0 & Q_\nu & 0 \\ 0 & 0 & Q_\omega \end{bmatrix}, \qquad \tilde{R} = R_v$$

The Kalman filter state estimate in Eq. 4.2 is determined as follows for the augmented system

$$\begin{bmatrix} \hat{x}_{k|k} \\ \hat{d}_{k|k} \\ \hat{p}_{k|k} \end{bmatrix} = \begin{bmatrix} \hat{x}_{k|k-1} \\ \hat{d}_{k|k-1} \\ \hat{p}_{k|k-1} \end{bmatrix} + \begin{bmatrix} L_x \\ L_d \\ L_p \end{bmatrix} \left(y_k - \tilde{C} \begin{bmatrix} \hat{x}_{k|k-1} \\ \hat{d}_{k|k-1} \\ \hat{p}_{k|k-1} \end{bmatrix} \right) \tag{6.7}$$

in which the Kalman filter gain is partitioned into a state filter gain, L_x, a state disturbance filter gain, L_d, and an output disturbance filter gain, L_p.

The observer gain for a deterministic observer can also be partitioned in the same manner as Eq. 6.7. The optimization based estimation techniques discussed in Chapter 4 are implemented using the augmented system in Eq. 6.5. With these techniques, constraints can be applied to the estimated disturbance states in addition to the estimated model states. Stability of the estimator in Eq. 6.7 implies the full rank condition in Lemma 6.1 on the filter or observer gain. This result is used in the discussion of offset free control in Section 6.7. Since the optimization based techniques without active constraints result in the same estimator as Eq. 6.7, the following lemma also applies to these techniques when no constraints are active.

Lemma 6.1 *Stability of the estimator in Eq. 6.7 for the augmented system in Eq. 6.6 with the number of disturbance states equal to the number of outputs, $s_d + s_p = p$, implies full rank of the filter gain $L_2 = [L_d^T \; L_p^T]^T$.*

Proof: See Appendix H.3.

6.4 Controlled Variable Target Tracking

Augmenting the system with the constant disturbances discussed in the previous section includes states that are not asymptotically stable. Since these additional states are also not controllable, the augmented system is not stabilizable. Therefore, the output feedback regulator presented in Chapter 5 cannot be implemented on the augmented system. However, the estimate of these states can be used to remove the disturbance with a constant disturbance regulator formulation similar to that in Kwakernaak and Sivan [45]. In this formulation, the constant disturbance is removed at steady state by a target tracking quadratic program that determines steady-state state and input vectors for the original process model that minimizes the controlled variable tracking error subject to the estimated constant disturbances.

6.4.1 Controlled Variables

The controlled variables are determined linearly from the state of the augmented system similar to those presented in Chapter 3. They are defined as

$$\hat{y}_k^c = \left[\acute{C}_x \; \acute{C}_d \; \acute{C}_p\right] \begin{bmatrix} \hat{x}_{k|k} \\ \hat{d}_{k|k} \\ \hat{p}_{k|k} \end{bmatrix} \tag{6.8}$$

in which $\acute{C} = [\acute{C}_x \; \acute{C}_d \; \acute{C}_p]$ is not necessarily equal to or the same row dimension as $\tilde{C}$. If the regulator is to bring the controlled variables to a nonzero target y_t^c, steady-state state and input vectors, x_s and u_s, are required such that the system reaches y_t^c at steady state in the presence of the constant disturbances.

$$\begin{bmatrix} I-A & -B \\ \acute{C}_x & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} -G_d \hat{d}_{k|k} \\ y_t^c - \acute{C}_d \hat{d}_{k|k} - \acute{C}_p \hat{p}_{k|k} \end{bmatrix} \tag{6.9}$$

In order to have a well-posed target tracking problem, x_s and u_s must be uniquely determined from the augmented system matrices $(\tilde{A}, \tilde{B}, \acute{C})$, the controlled variable target y_t^c, and the constant disturbances $\hat{d}$, $\hat{p}$. If the intersection of the null spaces of $(I-A)$ and $\acute{C}_x$ is a vector space containing more than the zero vector, x_s cannot be determined uniquely as discussed in Chapter 3. A necessary restriction on $\acute{C}_x$ is that the matrix $\acute{\mathcal{O}}_x$ is full rank.

$$\acute{\mathcal{O}}_x = \begin{bmatrix} (I-A) \\ \acute{C}_x \end{bmatrix} \tag{6.10}$$

This matrix is full rank under the conditions stated in Theorem 6.1.

Theorem 6.1 *The matrix $\acute{\mathcal{O}}_x$ in Eq. 6.10 is full rank if and only if the integrating modes of A are in the observable subspace of $(\acute{C}_x, A)$.*

Proof: The proof follows in the same manner as that for Theorem 3.1.

6.4.2 Least Squares Target Tracking

The input and state target vectors, u_s and x_s, that remove the constant disturbance at steady state can be determined from the solution of the following quadratic program in which y_t^c is the controlled variable target, $\hat{p}_{k|k}$ is the estimate of the constant output disturbance, $\hat{d}_{k|k}$ is the estimate of the constant state disturbance, and R_s is computed using Eq. 3.11.

$$\begin{aligned} \min_{[x_s, u_s]^T} & \left(y_t^c - [\acute{C}_x \; \acute{C}_d \; \acute{C}_p] \begin{bmatrix} x_s \\ \hat{d}_{k|k} \\ \hat{p}_{k|k} \end{bmatrix} \right)^T Q_s \left(y_t^c - [\acute{C}_x \; \acute{C}_d \; \acute{C}_p] \begin{bmatrix} x_s \\ \hat{d}_{k|k} \\ \hat{p}_{k|k} \end{bmatrix} \right) \\ & + (u_s - u_t)^T R_s (u_s - u_t) \end{aligned} \tag{6.11}$$

$$\begin{bmatrix} (I-A) & -B \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = G_d \hat{d}_k \tag{6.12}$$

Subject to:

$$u_{\min} < u_s < u_{\max} \tag{6.13}$$

$$y^c_{\min} < \acute{C}_x x_s + \acute{C}_d \hat{d}_{k|k} + \acute{C}_p \hat{p}_{k|k} < y^c_{\max} \tag{6.14}$$

Depending on which constant disturbance model is chosen, one of the disturbance vectors may not be present in the system and is set to zero. If both disturbance vectors are set to zero, this quadratic program reduces to the quadratic program of Eqs. 3.7 and 3.8 in Chapter 3. The achievable controlled variable target is determined from the solution of the target tracking quadratic program as follows.

$$y^c_a = \acute{C}_x x_s + \acute{C}_d \hat{d}_{k|k} + \acute{C}_p \hat{p}_{k|k} \tag{6.15}$$

If the position constraints on the steady-state input and controlled variable in Eqs. 6.13 and 6.14 are not considered, only the equality constraint must be feasible for the target tracking quadratic program to have a solution. Since (A, B) is stabilizable, the matrix $[(I-A) \;\; -B]$ is full rank. This is sufficient to guarantee feasibility of the equality constraint as shown in Appendix H.4.1.

6.4.3 Constraints

When the input and controlled variable position constraints are considered, feasibility of the target tracking quadratic program can no longer be guaranteed. For example, it may not be possible to find a feasible steady-state input that prevents a large estimated state disturbance from causing the steady-state controlled variable constraints to be violated. If the target tracking quadratic program is infeasible, the steady-state controlled variable constraints are removed in order to obtain a solution. They must also be removed in the receding horizon regulator quadratic program to ensure that it can also find a feasible solution.

Stable Process Model. If the controlled variable constraints in Eq. 6.14 are removed for stable A, a feasible solution to the equality constraint in Eq. 6.12 that respects the input position constraints in Eq. 6.13 can be determined as follows.

$$\begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} (I-A)^{-1} G_d d_k \\ 0 \end{bmatrix} \tag{6.16}$$

Therefore a feasible solution to the target tracking quadratic program always exists as shown in Appendix H.4.2.

Unstable Process Model. For unstable A, removing the controlled variable constraints in Eq. 6.14 does not guarantee a feasible steady-state solution that is a constrained stabilizable state target. If A has integrating modes that are contained within the range of the state disturbance matrix G_d, the steady-state input must be moved from the origin to cancel the state disturbance and achieve a bounded steady-state state vector. The definition of $\mathcal{V}_N^n$ in Eq. 5.5 must be extended for constant disturbance models such that a constrained stabilizable steady-state state vector is achievable at each time k.

$$\mathcal{V}_N^n = \left\{ x_0,\, e_{0|0} \in \Re^n \;\middle|\; x_k,\, \hat{x}_{k|k},\, x_s(k) \in \mathcal{Z}_N^n \;\; \forall \;\; k \geq 0 \right\} \tag{6.17}$$

With this restriction, a feasible constrained stabilizable solution to the target tracking quadratic program always exists as shown in Appendix H.4.3.

6.5 Output Feedback

Output feedback with a constant disturbance model is implemented in the same manner as presented in Chapter 5 in which the input is obtained from the solution of the output feedback receding horizon regulator quadratic program in Section 5.1. In the case of an augmented constant disturbance model, the state estimate and the steady-state state and input are determined as shown in this chapter. The estimate of the model and disturbance states is obtained from the constant disturbance model augmented system estimator in Eq. 6.7. The steady-state state and input vectors, x_s and u_s, are determined from the constant disturbance model target tracking quadratic program in Eqs. 6.11 through 6.14. The ability to use any detectable subset of the combined constant disturbance and noise augmented system presented in Eq. 6.5 allows for a great deal of flexibility in the design of the unmeasured disturbance model for the process.

The output feedback method of the industrial implementations of model predictive control is within the state-space constant disturbance modeling framework discussed in this chapter. Output feedback in the industrial implementations of model predictive control is normally based on the assumption of a constant output disturbance with no measurement noise. This output disturbance is determined as the difference between the model prediction and the measured output at each sample time. For the approach in this work, the state estimates are updated as follows in which $G_p = I$ in Eq. 6.1 and $\hat{p}_{k|k}$ is the estimate of the constant output disturbance.

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} \tag{6.18}$$

$$\hat{p}_{k|k} = y_k - C\hat{x}_{k|k-1} \tag{6.19}$$

The observer gain for this system is $L = [0 \; I]^T$ and results in a deadbeat observer for the output disturbance states and an open-loop observer for the model states. Although this is the standard method for output feedback in model predictive control formulations, it is optimal only for output step disturbances. Since measurement noise and load disturbances are present in most processes, this observer is often unrealistic and, therefore, cannot adequately address many practical applications. This observer also cannot be used with unstable process models since the observer poles contain the plant poles as discussed in Muske and Rawlings [64]. Example 8 in Muske and Rawlings [65] demonstrates the poor performance of this output feedback method in the presence of measurement noise and the improvement that can be obtained by taking the measurement noise into account in design of the disturbance model.

6.6 Nominal Asymptotic Stability

A nominal constant disturbance model implies that the disturbances entering the system are exactly represented by the disturbance model. When estimation error is present with the use of the augmented disturbance model, the perturbation to the input computed by the output feedback receding horizon regulator is due to the error in both the model states and the disturbance states. The perturbation from reconstruction error in the model states is due to the reconstruction error in the state estimate as shown in the discussion of output feedback in Chapter 5. The perturbation from reconstruction error in the estimate of the constant disturbance states is due to the error in the determination of the steady-state input u_s computed from the constant disturbance model target tracking quadratic program. The perturbed closed-loop state trajectory for the output feedback regulator with constant disturbances is then

$$\begin{aligned} x_{k+1} = {} & Ax_k + B\mu(\tilde{x}_k) + B\left(\mu(\tilde{x}_k + \tilde{e}_{k|k}) - \mu(\tilde{x}_k)\right) \\ & + Bu_s + B\left(\nu(\tilde{x}_k + \tilde{e}_{k|k}) - \nu(\tilde{x}_k)\right) \end{aligned} \tag{6.20}$$

in which $\tilde{x}_k$ is the augmented state in Eq. 6.6, $\tilde{e}_{k|k}$ is the augmented reconstruction error, $\mu(\tilde{x}_k)$ represents the input computed from the solution of the output feedback receding horizon regulator quadratic program, and $\nu(\tilde{x}_k)$ represents the steady-state input computed from the constant disturbance target tracking quadratic program. The expression in Eq. 6.20 is in the same form as the perturbed system of Eq. 5.8 in Chapter 5. Stability of the equilibrium point $x_k = x_s$ for the closed-loop system in Eq. 6.20 follows from nominal asymptotic stability of the target tracking receding horizon regulator in Chapter 3 and Lemma 5.2.

Expressing the target tracking quadratic program in Eqs. 6.11 through 6.14 in the form discussed in Appendix A.2.3 and applying the Lipschitz continuity result of Hager [34] in Eq. A.8 results in the following bound on the perturbation due to the error in the steady-state input for all $\tilde{x}_k, \tilde{e}_{k|k} \in \Re^{n+s_d+s_p}$.

$$\|B\| \, \|\nu(\tilde{x}_k + \tilde{e}_{k|k}) - \nu(\tilde{x}_k)\| < \rho_\nu \|B\| \, \|\tilde{e}_{k|k}\| \tag{6.21}$$

This perturbation is a globally Lipschitz continuous function of the state and, therefore, is exponentially convergent for an exponentially asymptotically stable estimator. If the solution of the receding horizon regulator quadratic program is also a globally Lipschitz continuous function of the state, the perturbation in Eq. 6.20 due to the reconstruction error in the state estimate is bounded as follows for all $\tilde{x}_k, \tilde{e}_{k|k} \in \Re^{n+s_d+s_p}$.

$$\|B\| \, \|\mu(\tilde{x}_k + \tilde{e}_{k|k}) - \mu(\tilde{x}_k)\| < \rho_\mu \|B\| \, \|\tilde{e}_{k|k}\| \tag{6.22}$$

In this case, provided the estimator is exponentially asymptotically stable, convergence of the closed-loop state trajectory in Eq. 6.20 to x_s is demonstrated in the same manner as in the proof of Lemma 5.1. For $(x_0, e_0) \in \mathcal{V}_N^n$, defined by Eq. 6.17, nominal asymptotic stability of the perturbed system in Eq. 6.20 then follows in the same manner as shown in Theorem 5.1.

Theorem 6.2 *For (A, B) stabilizable, $N \geq r$, in which r is the number of unstable modes of A, the nominal system with no disturbances, and the state estimate computed from an exponentially stable estimator, $x_k = x_s$ is an asymptotically stable solution of the closed-loop system in Eq. 6.20 for all $(x_0, e_0) \in \mathcal{V}_N^n$ in which $\mathcal{V}_N^n$ is defined by Eq. 6.17*

Proof: The proof follows in the same manner as that for Theorem 5.1.

6.7 Offset Free Control

The output feedback controller with constant disturbances presented in this chapter is shown to be an offset free controller. The industrial implementations of linear model predictive control were designed for offset free control by using the simple constant output disturbance model presented in Eqs. 6.18 and 6.19. However, no analysis proving the elimination of steady-state offset was given. The constant state disturbance model is a standard technique in linear quadratic regulator design [45]. This model has been shown by Davison and Smith [17] to be stabilizing and to eliminate steady-state offset in the presence of constant disturbances for the nominal model only. In this section, the disturbance models discussed in this chapter are shown to eliminate steady-state offset in the presence of constant disturbances and model mismatch provided the closed-loop system is stable. The proof of the elimination of steady-state offset does not assume that the plant output is determined from the process model or that the disturbances entering the system are represented by the disturbance model.

In this analysis, two aspects of offset free control are considered. The first is regulator offset free control and refers to the ability of the closed-loop receding horizon regulator to reach the achievable controlled variable target, y_a^c in Eq. 6.15, without offset at steady state. The second is target tracking offset free control and refers to the ability to track the controlled variable target exactly at steady state such that the achievable controlled variable target is the actual target, $y_a^c = y_t^c$.

Regulator Offset Free Control.

Theorem 6.3 *The closed-loop output feedback receding horizon regulator with objective function Eq. 5.3, constraints Eq. 5.4, state estimator Eq. 6.7, and target tracking quadratic program Eqs. 6.11 through 6.14 reaches the achievable controlled variable steady-state target in Eq. 6.15 under the following conditions.*

i) *The closed-loop system is stable and reaches a steady-state output y_s.*

ii) *The matrix $\acute{O}_x$ in Eq. 6.10 is full rank.*

iii) *The constant disturbance augmented state-space model is detectable.*

iv) *The number of constant disturbance states equals the number of outputs.*

v) *No constraints are active in the regulator or the estimator.*

Proof: See Appendix H.5.

Condition i) requires that the closed-loop system actually reach a steady state. The integral control formulation does not guarantee robust stability of the controller. With significant model mismatch, the closed-loop system can become unstable. Condition ii) ensures that x_s and u_s can be determined uniquely. Conditions iii) and iv) ensure that a stable estimator can be constructed and that the disturbance model effects each of the outputs. Condition v) requires that the input determined by the regulator and the estimate determined by the estimator can actually be implemented. The proof is an extension of that presented in Rawlings *et al.* [70] for a constant output disturbance to an observable system.

Target Tracking Offset Free Control.

Theorem 6.4 *The closed-loop output feedback receding horizon regulator with objective function Eq. 5.3, constraints Eq. 5.4, state estimator Eq. 6.7, and target tracking quadratic program Eqs. 6.11 through 6.14 reaches the controlled variable steady-state target y_t^c under the following conditions.*

i) The conditions i) through v) in Theorem 6.3 are satisfied.

ii) No constraints are active in the target tracking quadratic program.

iii) The following matrix is full rank.

$$\begin{bmatrix} I - A & -B \\ \acute{C}_x & 0 \end{bmatrix}$$

Proof: See Appendix H.6.

Condition i) ensures that the steady-state state and input vectors computed by the target tracking quadratic program can be achieved by the receding horizon regulator in the closed loop. Conditions ii) and iii) ensure that the controlled variable target can be achieved at steady state.

6.8 Velocity Form of the State-Space Model

Another method to obtain offset free control is to form an augmented plant model as follows

$$\begin{bmatrix} \Delta x_{k+1} \\ z_{k+1} \end{bmatrix} = \begin{bmatrix} A & 0 \\ CA & I \end{bmatrix} \begin{bmatrix} \Delta x_k \\ z_k \end{bmatrix} + \begin{bmatrix} B \\ CB \end{bmatrix} \Delta u_k \tag{6.23}$$
$$y_k = \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} \Delta x_k \\ z_k \end{bmatrix}$$

in which the state and input represent the change in the original state and input and the augmented states are the output of the original system. This representation is discussed by Prett and García [69] and is obtained by differencing the state-space model in Eq. 2.1 and augmenting with the output. Unlike the previous formulation in which the augmented process model is used only to estimate the model and disturbance states and is not used by the regulator, this formulation uses the augmented system model in both the regulator and estimator. Therefore, the velocity form is restricted to systems in which the augmented process model in Eq. 6.23 is both stabilizable and detectable. Muske and Rawlings [64] present a nominally stabilizing receding horizon regulator for this velocity form state-space model and discuss the conditions under which the controlled variable target can be achieved exactly when the controlled variables are taken as the outputs. Example 4 in Muske and Rawlings [64] compares the closed-loop performance of this model formulation to that of the previous model form.

Chapter 7

Feedforward Control

The preceding chapter discussed offset free control in the presence of unmeasured disturbances to the process that were modeled as constant state and output disturbances. That discussion was concerned with the construction of a disturbance model and the estimation of the corresponding unmeasured disturbance states. In this chapter, feedforward control of measured disturbances to the process is discussed. A linear system is first presented to represent the measured disturbance. This system allows for the optimal estimation of the measured disturbance. The feedforward linear system model that describes the effect of the measured disturbance on the output of the process is then presented and feedforward control of the measured disturbance is discussed for stable disturbances.

7.1 Measured Disturbance Models

The measured disturbance is described by the following linear system in which f_k is the measured disturbance and x_k^f is the measured disturbance model state vector.

$$\begin{aligned} x_{k+1}^f &= A^f x_k^f \\ f_k &= C^f x_k^f \end{aligned} \tag{7.1}$$

The measured disturbance models considered in this work are restricted to observable linear systems. The standard assumption that the measured disturbance remains constant in the future can be represented in this framework by setting $A^f = I$ and $C^f = I$. The effect of the measured disturbance on the output of the process is modeled by the following feedforward model linear dynamic system.

$$\begin{aligned} x_{k+1}^m &= A^m x_k^m + B^m f_k \\ y_k &= C^m x_k^m \end{aligned} \tag{7.2}$$

The input to this feedforward model is the measured disturbance determined from the linear system in Eq. 7.1. For the feedforward model, A^m is taken to be stable. These two linear systems result in the following augmented system that describes the measured disturbance and its corresponding effect on the process output.

$$\begin{aligned} \begin{bmatrix} x_{k+1}^m \\ x_{k+1}^f \end{bmatrix} &= \begin{bmatrix} A^m & B^m C^f \\ 0 & A^f \end{bmatrix} \begin{bmatrix} x_k^m \\ x_k^f \end{bmatrix} \\ \begin{bmatrix} y_k \\ f_k \end{bmatrix} &= \begin{bmatrix} C^m & 0 \\ 0 & C^f \end{bmatrix} \begin{bmatrix} x_k^m \\ x_k^f \end{bmatrix} \end{aligned} \tag{7.3}$$

7.2 Measured Disturbance Regulator

Combining the feedforward model in Eq. 7.1 with the process model in Eq. 2.1 results in the following augmented system that describes the output response to the input and the measured disturbance.

$$\begin{aligned} \begin{bmatrix} x_{k+1} \\ x_{k+1}^m \\ x_{k+1}^f \end{bmatrix} &= \begin{bmatrix} A & 0 & 0 \\ 0 & A^m & B^m C^f \\ 0 & 0 & A^f \end{bmatrix} \begin{bmatrix} x_k \\ x_k^m \\ x_k^f \end{bmatrix} + \begin{bmatrix} B \\ 0 \\ 0 \end{bmatrix} u_k \\ \begin{bmatrix} y_k \\ f_k \end{bmatrix} &= \begin{bmatrix} C & C^m & 0 \\ 0 & 0 & C^f \end{bmatrix} \begin{bmatrix} x_k \\ x_k^m \\ x_k^f \end{bmatrix} \end{aligned} \tag{7.4}$$

When A^f is stable, this augmented system is stabilizable. Feedforward control of the measured disturbance, along with feedback control, is then implemented using the receding horizon regulator formulation presented in the previous chapters on this augmented process model.

7.3 Feedforward Control Target Tracking

If it is assumed that the measured disturbance remains constant in the future, $A^f = I$ and $C^f = I$. The augmented system in Eq. 7.4 is not stabilizable and a modification of the target tracking regulator presented in Chapter 6 is required to control the system. The input and state target vectors, u_s and x_s, that remove the constant unmeasured and measured disturbances at steady state can be determined from the quadratic program in Eqs. 6.11 and 6.12 in which the controlled variable target tracking error is computed as follows.

$$y_t^c - \left[\acute{C}_x \; \acute{C}_m \; \acute{C}_f \; \acute{C}_d \; \acute{C}_p \right] \begin{bmatrix} x_s \\ x_s^m \\ \hat{f}_k \\ \hat{d}_k \\ \hat{p}_k \end{bmatrix} \tag{7.5}$$

In the feedforward control formulation, the controlled variables are a linear function of the process model state, feedforward model state, and the measured and unmeasured disturbance estimates as shown in Eq. 7.5. Since A^m is stable and the measured disturbance remains constant, the steady-state value of the feedforward model state used in the determination of the controlled variable tracking error is computed as follows.

$$x_s^m = (I - A^m)^{-1} \, B^m \hat{f}_k \tag{7.6}$$

Feedforward control of the measured disturbance and feedback control is then implemented using the target tracking receding horizon regulator formulation presented in the previous chapters on the augmented process model in Eq. 7.4 with the measured disturbance model states, x_k^f, removed. Integral control in this case follows in the same manner as shown in Chapter 6. Example 9 in Muske and Rawlings [65] demonstrates the the combined feedforward/feedback regulator discussed in this chapter. This example presents the closed-loop performance for both A^f stable and a constant future measured disturbance in which $A^f = I$ and $C^f = I$.

7.4 State Estimation

State estimates are obtained from an estimator applied to the augmented system in Eq. 7.3 using the techniques described in Chapter 4. An unmeasured constant disturbance model can be added to this augmented system to provide integral control as discussed in Chapter 6. The estimator in Eq. 6.7 applied to the constant disturbance stochastic system in Eq. 6.5 in which the process model is the augmented feedforward system in Eq. 7.3 results in the following state estimates.

$$\begin{bmatrix} \hat{x}_{k|k} \\ \hat{x}^m_{k|k} \\ \hat{x}^f_{k|k} \\ \hat{d}_{k|k} \\ \hat{p}_{k|k} \end{bmatrix} = \begin{bmatrix} \hat{x}_{k|k-1} \\ \hat{x}^m_{k|k-1} \\ \hat{x}^f_{k|k-1} \\ \hat{d}_{k|k-1} \\ \hat{p}_{k|k-1} \end{bmatrix} + \begin{bmatrix} L_x \\ L_m \\ L_f \\ L_d \\ L_p \end{bmatrix} \left(\begin{bmatrix} y_k \\ f_k \end{bmatrix} - \tilde{C} \begin{bmatrix} \hat{x}_{k|k-1} \\ \hat{x}^m_{k|k-1} \\ \hat{x}^f_{k|k-1} \\ \hat{d}_{k|k-1} \\ \hat{p}_{k|k-1} \end{bmatrix} \right) \tag{7.7}$$

in which the augmented matrices $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ are as follows.

$$\tilde{A} = \begin{bmatrix} A & 0 & 0 & G_d & 0 \\ 0 & A^m & B^m C^f & 0 & 0 \\ 0 & 0 & A^f & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}, \; \tilde{B} = \begin{bmatrix} B \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \; \tilde{C} = \begin{bmatrix} C & C^m & 0 & 0 & G_p \\ 0 & 0 & C^f & 0 & 0 \end{bmatrix}$$

This augmented system is detectable provided A^f is stable and the conditions in Chapter 6 are met. When A^f is taken to be the identity matrix, detectability of the augmented system is not necessarily implied by observability of (C^f, A^f). In this case, detectability of the augmented system must be verified directly.

Chapter 8

Conclusions

This document presents a framework for the future development and improvement of linear model predictive control technology for chemical process control applications. In order to enhance the technology beyond the current industrial implementations, the features that are necessary to control chemical processes are included in this framework. A linear model predictive controller that cannot address these issues is of little practical value. By bringing together concepts and results from linear and nonlinear system theory, linear estimation theory, optimization theory, and Lyapunov stability theory, a rigorous theoretical basis and design philosophy is created.

The major theoretical contributions of this work are constrained nominal stability and offset free control of the linear model predictive controller. Constrained nominal stability guarantees that the constrained controller will be stable with the nominal model for all valid tuning parameters. Offset free control guarantees that the controlled variables will reach their targets exactly at steady state provided no constraints are active. An additional contribution of this work is the development of a nominally stable constrained moving horizon state estimator that can be viewed as the dual estimator to the constrained controller.

The design of the linear model predictive controller in this work is based on a nominally stable, constrained, receding horizon regulator with state feedback. The features necessary for chemical process control, which include target tracking for non-square systems, reference trajectory tracking, output feedback, offset free control, and feedforward control, are implemented by modifications and extensions to this receding horizon regulator. These modifications and extensions are summerized in the following.

The choice of an infinite prediction horizon and a finite parameterization of the input yielded the nominally stable, constrained, receding horizon linear quadratic regulator constructed in Chapter 2. This receding horizon regulator was based on a linear state-space dynamic model of the process. The use of state-space models provided an exact parametric representation for all linear system dynamics including unstable processes. Nominal stability came from the use of an infinite horizon in the regulator. The finite parameterization of the input resulted in a finite dimensional quadratic programming optimization problem that was solved to determine the input. This quadratic program was no more computationally expensive than those resulting from the existing technology. Since the input was determined from the solution of a quadratic program, the receding horizon regulator was a nonlinear controller and stability was proved using Lyapunov stability theory.

The regulator presented in Chapter 2 was a zero state target regulator that brought the state of the system to the origin. A target tracking regulator, in which nonzero controlled variable targets are specified, was constructed in Chapter 3 from this regulator by shifting the origin to the steady state representing the controlled variable target. The nominal stability properties of the regulator were not changed by this procedure. Since a desired controlled variable target may not be achievable, a steady-state optimization was used to determine the origin of the receding horizon regulator such that the desired controlled variable target was tracked in a least squares sense at steady state. This steady-state optimization required the solution of a quadratic program, also constructed in Chapter 3, that determined the achievable steady-state controlled variable target.

When perfect measurements of the current state are not available, an estimate of the state was used to initialize the receding horizon regulator optimization problem. Chapter 4 reviewed linear state estimation techniques and presented an exponentially stable constrained moving horizon state estimator. An output feedback receding horizon regulator, in which a state estimator was used to obtain the current state from the output measurements, was constructed in Chapter 5. Since the reconstruction error in the state estimates introduced perturbations to the closed-loop system, the nominal stability proofs presented in Chapter 2 were no longer applicable. Stability of the output feedback receding horizon regulator was shown using the converse Lyapunov stability theorems of Halanay [35]. The stability proof required exponential asymptotic stability of the state estimator and Lipschitz continuity of the state feedback receding horizon regulator.

Offset free control of the output feedback regulator was obtained in Chapter 6 by the introduction of constant disturbance models to represent unmeasured disturbances to the system. The state-space model formulation allowed for a great deal of flexibility in the design of these unmeasured disturbance models and the application of a linear state estimator to determine estimates of these disturbances from the output measurements. The disturbance estimates were used in the target tracking steady-state optimization problem to determine the achievable controlled variable target. Provided that the closed-loop system reached steady state, no constraints were active in the regulator or the estimator, and a valid constant disturbance model was designed, it was proved that the output feedback regulator tracked the achievable controlled variable target exactly at steady state. Feedforward control of measured disturbances was obtained in Chapter 7 by the addition of a measured disturbance model.

This work raises a number of theoretical and implementation issues that are currently unresolved. The future research directions from this work concern these issues.

- This work considered only nominal stability. An analysis of robust stability is of great practical importance since model mismatch is present in every chemical process control application.

- Efficient implementation of this controller on large-scale systems is also of great practical importance. Efficient large-scale implementation would require an analysis of large-scale, sparse optimization techniques for the regulator optimization problem.

- The input parameterization in Section 2.9 appears to have several advantages over the parameterization that this work is based on with the exception of handling input constraints easily. The major advantage is the improvement in performance without a corresponding increase in the computational requirements. More study into alternate input parameterizations that provide an infinite horizon open-loop objective with a finite number of decision variables would be worthwhile.

- Nominal stability for unstable process models required restrictions on the initial conditions that can be considered. An analysis of $\mathcal{Z}_N^n$ in Eq. 2.41 and $\mathcal{V}_N^n$ in Eq. 5.5 is needed to determine the properties of these sets. If the set of constrained stabilizable states is too small, normal process disturbances can cause the closed-loop system to loose constrained stabilizability for a given regulator horizon length. In this case, some method of adapting the horizon length on-line is required to implement the controller on unstable processes.

- The convergence of the closed-loop state trajectory for the output feedback receding horizon regulator with controlled variable constraints was conjectured in Chapter 5 and not proven. A proof of this conjecture is needed.

- The computational issues involved in implementing this technology have not been considered in this work. These issues include scaling of the controlled and manipulated variables to avoid numerical errors in the optimizations, efficient computation techniques for the quadratic program matrices when tuning parameters are changed on-line, and the enforcement of consistent constraints on the controlled and manipulated variables.

Appendix A

Receding Horizon Regulator Quadratic Program

This appendix discusses the quadratic program solved at each execution of the constrained receding horizon regulator presented in Chapter 2. It begins with the construction of the least squares and constraint matrices that comprise the quadratic programming problem. The uniqueness and continuity properties of the solution, used in the analysis of the stability of the regulator, are then discussed.

A.1 Quadratic Program Formulation

Straightforward algebraic manipulation of the quadratic objective presented in Eq. 2.42 results in the following quadratic program for the open-loop input trajectory π defined in Eq. 2.9.

$$\Phi_k(x_k, u_{k-1}, \pi) = \pi^T H \pi + 2\pi^T (G x_k - U u_{k-1}) + x_k^T Q x_k \tag{A.1}$$

For stable A, the matrices H, G, and U are computed as follows.

$$H = \begin{bmatrix} B^T Q_N B + R + 2S & B^T A^T Q_N B - S & \cdots & B^T {A^T}^{N-1} Q_N B \\ B^T Q_N A B - S & B^T Q_N B + R + 2S & \cdots & B^T {A^T}^{N-2} Q_N B \\ \vdots & \vdots & \ddots & \vdots \\ B^T Q_N A^{N-1} B & B^T Q_N A^{N-2} B & \cdots & B^T Q_N B + R + 2S \end{bmatrix}$$

$$G = \begin{bmatrix} B^T Q_N A \\ B^T Q_N A^2 \\ \vdots \\ B^T Q_N A^N \end{bmatrix}, \qquad U = \begin{bmatrix} S \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The terminal state penalty, Q_N, is computed as shown in Eq. 2.18 for the state penalty in Chapter 2 or Eq. 3.22 for the controlled variable penalty in Chapter 3.

The matrices H and G in Eq. A.1 for unstable A consist of the sum of the contribution from the finite horizon terms in Eq. 2.42 and the contribution from the terminal state penalty on the stable modes. The contribution from the finite horizon terms, H_1 and G_1, is computed as follows.

$$H_1 = \begin{bmatrix} B^T Q_{N-1} B + R + 2S & B^T A^T Q_{N-2} B - S & \cdots & B^T A^{T^{N-1}} Q_0 B \\ B^T Q_{N-2} A B - S & B^T Q_{N-2} B + R + 2S & \cdots & B^T A^{T^{N-2}} Q_0 B \\ \vdots & \vdots & \ddots & \vdots \\ B^T Q_0 A^{N-1} B & B^T Q_0 A^{N-2} B & \cdots & B^T Q_0 B + R + 2S \end{bmatrix}$$

$$G_1 = \begin{bmatrix} B^T Q_{N-1} A \\ \vdots \\ B^T Q_0 A^N \end{bmatrix}, \qquad Q_j = \sum_{i=0}^{j} A^{T^i} \check{Q} A^i \tag{A.2}$$

Computation of the contribution from the terminal state penalty on the stable modes, H_2 and G_2, is shown below with Q_N determined from Eq. 2.20.

$$H_2 = \begin{bmatrix} B^T \bar{Q}_{N-1} B & B^T A^T \bar{Q}_{N-2} B & \cdots & B^T A^{T^{N-1}} \bar{Q}_0 B \\ B^T \bar{Q}_{N-2} A B & B^T \bar{Q}_{N-2} B & \cdots & B^T A^{T^{N-2}} \bar{Q}_0 B \\ \vdots & \vdots & \ddots & \vdots \\ B^T \bar{Q}_0 A^{N-1} B & B^T \bar{Q}_0 A^{N-2} B & \cdots & B^T \bar{Q}_0 B \end{bmatrix}$$

$$G_2 = \begin{bmatrix} B^T \bar{Q}_{N-1} A \\ \vdots \\ B^T \bar{Q}_0 A^N \end{bmatrix}, \qquad \bar{Q}_j = A^{T^j} Q_N A^j \tag{A.3}$$

For a state penalty, $\check{Q} = Q$ and Q_N is computed as shown in Eq. 2.20. For a controlled variable penalty, $\check{Q} = \acute{C}^T \acute{Q} \acute{C}$ and Q_N is computed as shown in Eq. 3.23. The matrix U is the same as that presented for stable A.

The input and controlled variable constraints discussed in Chapter 3 are considered here.

$$u_{\min} \le v_j \le u_{\max}, \qquad j = 0, 1, \ldots, N-1 \tag{A.4}$$

$$y^c_{\min} \le \acute{C} z_j \le y^c_{\max}, \qquad j = j_1, j_1 + 1, \ldots, j_1 + j_2 \tag{A.5}$$

$$-\Delta u_{\min} \le \Delta v_j \le \Delta u_{\max}, \qquad j = 0, 1, \ldots, N \tag{A.6}$$

The constraints in Eqs. A.4, A.5, and A.6 can be expressed as the following constraint on π.

$$\begin{bmatrix} I \\ -I \\ \Delta I \\ -\Delta I \\ M \\ -M \end{bmatrix} \pi \leq \begin{bmatrix} \mathrm{i}_1 \\ \mathrm{i}_2 \\ \Delta \mathrm{i}_1 \\ \Delta \mathrm{i}_2 \\ \mathrm{m}_1 \\ \mathrm{m}_2 \end{bmatrix} \tag{A.7}$$

The matrices ΔI and M are computed as shown below in which A^{j-i} is defined to be 0 for all $j < i$.

$$\Delta I = \begin{bmatrix} I & 0 & \cdots & 0 \\ -I & I & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & -I & I \\ 0 & \cdots & 0 & -I \end{bmatrix}, \qquad M = \begin{bmatrix} \acute{C}A^{j_1-1}B & \cdots & \acute{C}A^{j_1-N}B \\ \vdots & & \vdots \\ \acute{C}A^{j_1+j_2-1}B & \cdots & \acute{C}A^{j_1+j_2-N}B \end{bmatrix}$$

The values of the right hand side vectors in Eq. A.7 are the following.

$$\mathrm{i}_1 = \begin{bmatrix} u_{\max} \\ \vdots \\ u_{\max} \end{bmatrix}, \quad \Delta \mathrm{i}_1 = \begin{bmatrix} \Delta u_{\max} + u_{k-1} \\ \Delta u_{\max} \\ \vdots \\ \Delta u_{\max} \end{bmatrix}, \quad \mathrm{m}_1 = \begin{bmatrix} y^c_{\max} - \acute{C}A^{j_1}x_k \\ \vdots \\ y^c_{\max} - \acute{C}A^{j_1+j_2}x_k \end{bmatrix}$$

$$\mathrm{i}_2 = \begin{bmatrix} -u_{\min} \\ \vdots \\ -u_{\min} \end{bmatrix}, \quad \Delta \mathrm{i}_2 = \begin{bmatrix} \Delta u_{\min} - u_{k-1} \\ \Delta u_{\min} \\ \vdots \\ \Delta u_{\min} \end{bmatrix}, \quad \mathrm{m}_2 = \begin{bmatrix} -y^c_{\min} + \acute{C}A^{j_1}x_k \\ \vdots \\ -y^c_{\min} + \acute{C}A^{j_1+j_2}x_k \end{bmatrix}$$

The equality constraint on the unstable modes in Eq. 2.19 is expressed as the following equality constraint on π.

$$\tilde{V}_u \begin{bmatrix} A^{N-1}B & A^{N-2}B & \ldots & B \end{bmatrix} \pi = -\tilde{V}_u A^N x_k$$

The basis $\tilde{V}_u$ for the unstable subspace of A determined from the Jordan form in Eq. 2.2 would not be used in practice since the computation of the Jordan form is numerically unstable. Meadows *et al.* [60] present a numerically stable procedure using the Schur form of A to determine a basis for the unstable subspace that is used in the equality constraint on the unstable modes.

A.2 Properties of the Solution

A.2.1 Uniqueness

The Hessian of the quadratic program in Eq. A.1, H, is the sum of a matrix constructed from the input penalty matrix R, the state penalty matrix, Q, and the input rate of change penalty matrix, S. The matrix constructed from the input penalty is a block diagonal matrix with R on the diagonal. Since R is restricted to symmetric positive definite matrices, this block diagonal matrix is positive definite. Since Q and S are restricted to symmetric positive semidefinite matrices, the matrices constructed from the state and input rate of change penalties are positive semidefinite. Therefore, the Hessian is positive definite. The constraints in Eqs. A.4 through A.6 specify a convex set containing the origin. Since the Hessian is positive definite and the constraint set is convex, the quadratic program is a strictly convex programming problem. This property ensures that the solution is unique and global [24].

A.2.2 Perturbation of the Solution

Any feasible perturbation to the optimal solution of the receding horizon regulator quadratic program is bounded by the change in the objective function value. The proof is based on that proposed by Mayne [56].

Lemma A.1 *Any feasible perturbation $\Delta\pi$ to the optimal solution π^* of the feasible receding horizon regulator quadratic program with objective function Eq. A.1, constraints Eq. A.7, Q and S symmetric positive semidefinite, and R symmetric positive definite is bounded as follows in which $\gamma > 0$.*

$$\|\Delta\pi\| \leq \gamma \left|\Phi(\pi^* + \Delta\pi) - \Phi(\pi^*)\right|^{1/2}$$

Proof: Express the change in objective function by the following expansion.

$$\Phi(\pi^* + \Delta\pi) - \Phi(\pi^*) = \nabla\Phi(\pi^*)^T \Delta\pi + \Delta\pi^T H \Delta\pi$$

Since π^* is the optimal solution, it satisfies the first order Kuhn–Tucker conditions. For any direction vector s, a consequence of the first order conditions is

$$\nabla\Phi(\pi^*)^T s \geq 0$$

in which s is a feasible incremental step direction [24]. Therefore, the first term in the expansion must be nonnegative and the following inequality holds in which $\sigma_{\min}(H)$ is the minimum singular value of H.

$$\Phi(\pi^* + \Delta\pi) - \Phi(\pi^*) \geq \Delta\pi^T H \Delta\pi \geq \sigma_{\min}(H)\,\|\Delta\pi\|^2$$

Since H is positive definite, $\sigma_{\min}(H) > 0$ and the result is obtained from of the preceding inequality.

A.2.3 Lipschitz Continuity

The following theorem, presented in Hager [34], demonstrates that the solution of a quadratic program is a Lipschitz continuous function of the data of the problem. The notation in the theorem is taken from [34].

Theorem A.1 (Hager [34]) *Given the quadratic program*

$$\min_{v \in \Re^n} \frac{1}{2} v^T R v + r^T v$$

$$\textit{Subject to:} \quad \begin{array}{rcl} Av + a & \leq & 0 \\ Bv + b & = & 0 \end{array}$$

and defining the data set $d = (R, r, A, b, B, b)$, *and data subsets* $\delta(d) = (r, a, b)$ *and* $\Delta(d) = (R, A, B)$. *Let* $\mathcal{D}$ *be any convex set of data satisfying the following for all* $d \in \mathcal{D}$*:*

1. *There exists a unique solution* $u(d)$ *to the quadratic program.*
2. *There exist* $\Gamma_1, \Gamma_2 < \infty$ *such that* $\|R\| < \Gamma_1$ *and* $\|M(d)^T\| < \Gamma_2$, *in which* $M(d)$ *is the matrix of binding constraints from* A *and* B.
3. *There exists* α *such that* $v^T R v \geq \alpha \|v\|^2$ *for all* v *satisfying* $M(d)v = 0$.
4. *There exists* β *such that* $\|M(d)^T \lambda\| \geq \beta \|\lambda\|$ *for all* λ.

Then there exists a constant $\rho < \infty$ *such that for all* $d_1, d_2 \in \mathcal{D}$, *we have*

$$\|v(d_1) - v(d_2)\| \leq \rho \|\delta(d_1) - \delta(d_1)\| + \rho^2 \|\Delta(d_1) - \Delta(d_2)\| (\|\delta(d_1)\| + \|\delta(d_2)\|) \tag{A.8}$$

For the constrained receding horizon regulator quadratic program in this appendix, Meadows [58] shows that the necessary conditions enumerated in Theorem A.1 are satisfied. When the start of the controlled variable constraint horizon, j_1, is the same for both $d(1)$ and $d(2)$, Eq. A.8 can be simplified to the following relationship that demonstrates Lipschitz continuity of the solution of the receding horizon regulator quadratic program to the current state and previous input.

$$\|\pi_k(d_1) - \pi_k(d_2)\| \leq \rho_1 \|x_k(d_1) - x_k(d_2)\| + \rho_2 \|u_{k-1}(d_1) - u_{k-1}(d_2)\| \tag{A.9}$$

This relationship will also hold when no controlled variable constraints are present. When the controlled variable constraint horizons are different for $d(1)$ and $d(2)$, Eq. A.9 is no longer valid and the bound must be computed using the expression in Eq. A.8 which can result in a discontinuous solution of the quadratic program as shown in Meadows [58].

Appendix B

Moving Horizon State Estimator

This appendix discusses the moving horizon state estimator introduced in Chapter 4. It begins with the construction of the least squares matrices that comprise the unconstrained moving horizon estimator. The equivalence of the unconstrained estimator to the Kalman Filter is then demonstrated.

B.1 Unconstrained Moving Horizon State Estimator

The optimal series of estimated state noise vectors, $\hat{w}^*_{k-N-1+j|k}$, from the solution of the autonomous least squares problem can be determined as follows.

$$\begin{bmatrix} \hat{w}^*_{k-N-1|k} \\ \vdots \\ \hat{w}^*_{k-1|k} \end{bmatrix} = (E_N + Q_N^{k-N})^{-1} G_N Y_{N|k} \tag{B.1}$$

The matrices E_N, G_N, and Q_N^{k-N} are defined by the following recursions.

$$E_0 = F_0, \qquad E_{j+1} = \begin{bmatrix} A_j^T E_j A_j + F_j & \tilde{A}_j^T F_0 \\ F_0 \tilde{A}_j & F_0 \end{bmatrix}$$

$$F_0 = C^T R^{-1} C, \qquad F_{j+1} = \begin{bmatrix} F_j & \tilde{A}_j^T F_0 \\ F_0 \tilde{A}_j & F_0 \end{bmatrix}$$

$$G_0 = C^T R^{-1}, \qquad G_{j+1} = \begin{bmatrix} G_j & \tilde{A}_j^T G_0 \\ 0 & G_0 \end{bmatrix}$$

$$Q_0^{k-N} = P_{k-N}^{-1}, \qquad Q_{j+1}^{k-N} = \begin{bmatrix} Q_j^{k-N} & 0 \\ 0 & Q^{-1} \end{bmatrix}$$

The matrix A_j is the $n(j+1) \times n(j+1)$ block diagonal matrix with A on the diagonal. The matrix $\tilde{A}_j$ is defined as $[A^{j+1} \; A^j \; \dots \; A]$. The measurement vector $Y_{N|k}$ is defined as

$$Y_{N|k} = \begin{bmatrix} y_{k-N} & - & C\hat{x}_{k-N|k-N-1} \\ & \vdots & \\ y_k & - & CA^N \hat{x}_{k-N|k-N-1} \end{bmatrix}$$

The predicted state estimate is computed from the following expression.

$$\begin{aligned} \hat{x}_{k+1|k} &= A^{N+1}\hat{x}_{k-N|k-N-1} + L_{N|k}Y_{N|k} \\ L_{N|k} &= \tilde{A}_N (E_N + Q_N^{k-N})^{-1} G_N \end{aligned} \tag{B.2}$$

This estimate is shown to be equivalent to the Kalman filter estimate in Eq. 4.3 in Theorem B.1 after stating the following preliminary result required for the proof.

Lemma B.1 *The discrete filtering Riccati matrix in Eq. 4.6 can be computed from the following expression for all $N \geq 0$ and all $k > N$.*

$$\begin{aligned} P_k &= Q + \tilde{A}_N (E_N + \tilde{Q}_N^{k-N-1})^{-1} \tilde{A}_N^T \\ P_0 &= Q_0 \end{aligned}$$

Proof: See Appendix B.2.

Theorem B.1 *The state estimate $\hat{x}_{k+1|k}$ computed in Eq. B.2 from the solution of the moving horizon least squares problem in Eq. B.1 is the Kalman filter estimate in Eq. 4.3 for all $N \geq 0$ and all k.*

Proof: The proof is by induction. For $N = 0$, the solution of the least squares problem in Eq. B.1 yields the following estimate for all $k \geq 0$.

$$\begin{aligned} \hat{x}_{k+1|k} &= A\hat{x}_{k|k-1} + L_k(y_k - C\hat{x}_{k|k-1}) \\ L_k &= A(E_0 + P_k^{-1})^{-1} G_0 \end{aligned}$$

This estimate is the predicted state estimate from the Kalman filter in Eq. 4.3 for all $k \geq 0$ with $P_0 = Q_0$. For $N > 0$, the estimate at time k from the solution of the least squares problem for a horizon length of $M+1$ can be obtained by replacing N with $M+1$ in the expression in Eq. B.2. Partitioning the result yields

$$\hat{x}_{k+1|k} = \begin{bmatrix} A\tilde{A}_M & A \end{bmatrix} \begin{bmatrix} A_M^T E_M A_M + F_M + Q_M^{k-M-1} & \tilde{A}_M^T F_0 \\ F_0 \tilde{A}_M & \tilde{E}_0 \end{bmatrix}^{-1}$$
$$\times \begin{bmatrix} G_M & \tilde{A}_M^T G_0 \\ 0 & G_0 \end{bmatrix} \begin{bmatrix} Y_{M|k-1} \\ y_k - CA^{M+1}\hat{x}_{M-1} \end{bmatrix} + A^{M+2}\hat{x}_{k-M-1}$$

in which $\tilde{E}_0 = F_0 + Q^{-1}$. Performing the partitioned matrix inversion and multiplying.

$$\begin{aligned} \hat{x}_{k+1|k} &= A^{M+2}\hat{x}_{k-M-1|k-M-2} + A\tilde{E}_0^{-1}Q^{-1}\tilde{A}_M W_M^{-1} G_M Y_{M|k-1} + \\ &\qquad A\left(\tilde{E}_0^{-1} + \tilde{E}_0^{-1}Q^{-1}\tilde{A}_M W_M^{-1}\tilde{A}_M^T Q^{-1}\tilde{E}_0^{-1}\right) G_0 (y_k - CA^{M+1}\hat{x}_{k-M-1}) \\ W_M &= E_M + Q_M^{k-M-1} + \tilde{A}_M^T\left(Q^{-1} - Q^{-1}\tilde{E}_0^{-1}Q^{-1}\right)\tilde{A}_M \end{aligned}$$

The expression multiplying the current measurement term can be simplified by performing the matrix inversion lemma first with $\tilde{E}_0^{-1}$ and then with Q^{-1} and substituting P_k for the expression in Lemma B.1.

$$A\left(\tilde{E}_0^{-1} + \tilde{E}_0^{-1}Q^{-1}\tilde{A}_M W_M^{-1}\tilde{A}_M^T Q^{-1}\tilde{E}_0^{-1}\right) G_0 = A\left(E_0 + P_k^{-1}\right)^{-1} G_0 = L_k$$

The expression multiplying the previous measurement vector, $Y_{M|k-1}$, can be rearranged using the matrix inversion lemma on W_{k-1}.

$$\begin{aligned} A\tilde{E}_0^{-1}Q^{-1}\tilde{A}_{k-1}W_{k-1}^{-1}G_{k-1} &= (A - L_k C)\,\tilde{A}_M (E_M + Q_M^{k-M-1})^{-1} G_M \\ &= (A - L_k C)\, L_{M|k-1} \end{aligned}$$

The result is the following simplified expression.

$$\begin{aligned} \hat{x}_{k+1|k} = {} & A^{M+2}\hat{x}_{k-M-1|k-M-2} + (A - L_k C) L_{M|k-1} Y_{M|k-1} + \\ & L_k (y_k - CA^{M+1}\hat{x}_{k-M-1|k-M-2}) \end{aligned}$$

Assuming Eq. B.2 is the Kalman filter result for horizon of length M for all $k \geq M$, the preceding expression can be simplified as follows.

$$\hat{x}_{k+1|k} = A\hat{x}_{k|k-1} + L_k (y_k - C\hat{x}_{k|k-1})$$

This is the predicted estimate at time $k+1$ from the Kalman filter for all $k \geq M$ which proves the theorem.

Scholium B.1 *The state estimate* $\hat{x}_{k+1|k}$ *computed in Eq. B.2 from the solution of the moving horizon least squares problem in Eq. B.1 in which* P_∞^{-1} *is the first matrix on the diagonal of* Q_N *for all time* k *is the steady-state Kalman filter state estimate at time* k.

Proof: The proof is by induction. For $N = 0$, the solution of the least squares problem in Eq. B.1 yields the following estimate for all $k \geq 0$.

$$\begin{aligned}
\hat{x}_{k+1|k} &= A\hat{x}_{k|k-1} + L_\infty(y_k - C\hat{x}_{k|k-1}) \\
L_\infty &= A(E_0 + P_\infty^{-1})^{-1}G_0
\end{aligned}$$

This is the steady-state Kalman filter predicted state estimate for all $k \geq 0$. The remainder of this proof proceeds in the same manner as the proof by induction for Theorem B.1 making use of the following result from Lemma B.1.

$$P_\infty = Q + \tilde{A}_N(E_N + Q_N^\infty)^{-1}\tilde{A}_N^T, \qquad N \geq 0$$

B.2 Proof of Lemma B.1

The proof is by induction. For $N = 0$, the matrix inversion lemma on the discrete filtering Riccati recursion in Eq. 4.6 yields the following expression for all $k \geq 1$ which is the result in Lemma B.1.

$$P_k = Q + A\left(E_0 + P_{k-1}^{-1}\right)^{-1} A^T$$

For $N > 0$, change the index k in the expression in Lemma B.1 to $j + N$.

$$P_{j+N} = Q + \tilde{A}_N\left(E_N + \tilde{Q}_N^{j-1}\right)^{-1}\tilde{A}_N^T$$

Assume that this relation is true for $N - 1$ and all $j \geq 1$ and partition as follows in which $\tilde{E}_0 = F_0 + Q^{-1}$.

$$\begin{aligned}
P_{j+N} = &\begin{bmatrix} A\tilde{A}_{N-1} & A \end{bmatrix} \begin{bmatrix} A_{N-1}^T E_{N-1} A_{N-1} + F_{N-1} + \tilde{Q}_{N-1}^{j-1} & \tilde{A}_{N-1}^T F_0 \\ F_0\tilde{A}_{N-1} & \tilde{E}_0 \end{bmatrix}^{-1} \\
&\times \begin{bmatrix} \tilde{A}_{N-1}^T A^T \\ A^T \end{bmatrix} + Q
\end{aligned}$$

Performing the partitioned matrix inversion and multiplying results in the following relation.

$$\begin{aligned}
P_{j+N} &= Q + A\left[\tilde{E}_0^{-1}Q^{-1}\tilde{A}_{N-1}W_{N-1}^{-1}\tilde{A}_{N-1}^T Q^{-1}\tilde{E}_0^{-1} + \tilde{E}_0^{-1}\right]A^T \\
W_{N-1} &= E_{N-1} + Q_{N-1}^{j-1} + \tilde{A}_{N-1}^T\left(Q^{-1} - Q^{-1}\tilde{E}_0^{-1}Q^{-1}\right)\tilde{A}_{N-1}
\end{aligned}$$

This expression can be simplified by performing the matrix inversion lemma first with $\tilde{E}_0^{-1}$ and then with Q^{-1}. Substitution of the assumed relation

$$P_{j+N-1} = Q + \tilde{A}_{N-1}(E_{N-1} + \tilde{Q}_{N-1}^{j-1})^{-1}\tilde{A}_{N-1}^T$$

results in

$$P_{j+N} = Q + A\left(E_0 + P_{j+N-1}^{-1}\right)^{-1} A^T$$

This recurrence can be shown to be the discrete filtering Riccati matrix recursion by using the matrix inversion lemma which proves the lemma.

Appendix C

Converse Lyapunov Stability Theorems

This appendix discusses the converse Lyapunov stability theorems for discrete systems presented by Halanay [35]. It begins with the statement of the converse theorems in [35]. These theorems are then shown to apply globally for the closed-loop system in this work provided the receding horizon regulator quadratic program satisfies a Lipschitz continuity condition.

C.1 Converse Lyapunov Theorems

The following results, presented in Halany [35] and translated from the French by Valax [84], demonstrate the existence of a Lyapunov function of the state in the neighborhood of the origin provided the origin is stable. The notation in the theorem is taken from [35].

Theorem C.1 (Halanay [35]) *Consider the following system*

$$x(t_{k+1}) = h[t_k, x(t_k)]$$

and suppose that $h[t_k, 0] = 0$ *and the zero solution is uniformly stable. This means that there exists a* $\delta(\epsilon)$ *such that* $\|x(t_k)\| < \delta(\epsilon)$ *implies* $\|x(t_l; t_k, x(t_k))\| < \epsilon$ *for all* $l \geq k$*. There exists a function* $V(t_k, x)$ *such that*

$$\begin{aligned} & a(\|x\|) \leq V(t_k, x) \leq b(\|x\|) \\ & V(t_{k+1}, x(t_{k+1}; t_0, x_0)) - V(t_k, x(t_k; t_0, x_0)) \leq 0 \ \ \forall \ \ k \end{aligned} \tag{C.1}$$

Suppose now that the zero solution is uniformly asymptotically stable. This means that there exist two functions $\delta(\epsilon)$, $N(\epsilon)$, *and* $\delta_0 > 0$ *such that* $\|x(t_k)\| < \delta(\epsilon)$ *implies* $\|x(t_{k+l}; t_k, x_0)\| < \epsilon$ *for* $l \geq 0$ *and* $\|x(t_k)\| < \delta_0$, $l \geq N(\epsilon)$ *implies* $\|x(t_{k+l}; t_k, x_0)\| < \epsilon$. *If we admit the additional condition that there exists a function* m *satisfying* $m(r) > 0$ *for* $r > 0$ *and* $m(0) = 0$ *such that*

$$\|h[t_k, x]\| \geq m(\|x\|)$$

then we can prove the existence of a function $V(t_k, x)$ *such that*

$$\begin{array}{l} a(\|x\|) \leq V(t_k, x) \leq b(\|x\|) \\ V(t_{k+1}, x(t_{k+1}; t_0, x_0)) - V(t_k, x(t_k; t_0, x_0)) \leq -c(\|x(t_k; t_0, x_0)\|) \;\; \forall \;\; k \end{array} \tag{C.2}$$

Now suppose that

$$\|h[t_k, x_1] - h[t_k, x_2]\| \leq L_r \|x_1 - x_2\|$$

for $\|x_1\| \leq r$ *and* $\|x_2\| \leq r$. *We will prove that for a suitable choice of function* G, *one can obtain a function* V *with the following property.*

$$|V(t_k, x_1) - V(t_k, x_2)| \leq M \|x_1 - x_2\|, \quad \|x_1\| < \delta(\delta_0) \text{ and } \|x_2\| < \delta(\delta_0) \tag{C.3}$$

The Lyapunov function in [35] is taken as

$$V(t_k, x) = \sup_{l \geq 0} G\left(\|x\left(t_{k+l}; t_k, x\right)\|\right) \frac{1 + \alpha l}{1 + l}$$

in which $G(r)$ is defined for $r \geq 0$ with $G(0) = 0$,$G'(0) = 0$, $G'(r) > 0$,$G''(r) > 0$ and $\alpha > 1$. Halanay shows that

$$G\left(\|x\|\right) \leq V(t_k, x) \leq \alpha G\left[\epsilon\left(\|x\|\right)\right]$$

in which $\epsilon(\|x\|)$ is the inverse $\delta(\epsilon)$ function used to demonstrate stability of the origin for $h[t_k, x]$ and $\delta_0 = \sup \delta(\epsilon)$. A clearer explanation of these functions and a more detailed proof for continuous-time systems can be found in [36]. The restriction $\|h[t_k, x]\| \geq m(\|x\|)$ on the relationship in Eq. C.2 is required to prevent $h[t_k, x] = 0$ for $x \neq 0$. This is necessary since the proof of Eq. C.2 requires $N(\|h[t_k, x]\|)$ in which $N(\epsilon)$ is the discrete time index used to demonstrate asymptotic stability. The relation in Eq. C.3 is demonstrated by the construction of a specific G function.

C.2 Global Converse Lyapunov Theorems

The theorems stated in the previous section are local results. The application of these theorems globally requires the existence of an inverse $\delta(\epsilon)$ function, $\epsilon(\delta)$, for all $\delta \in \Re^+$ implying an unbounded $\delta_0 = \sup \delta(\epsilon)$. For the δ–ϵ relationship determined in the proof of stability of the receding horizon regulator in Lemma 2.7,

$$\delta(\epsilon) = \begin{cases} \epsilon/\Lambda, & \epsilon < \varepsilon \\ \varepsilon/\Lambda, & \epsilon \geq \varepsilon \end{cases}$$

$\delta_0 = \varepsilon/\Lambda$ in which ε represents some neighborhood of the origin in which the state constraints are not active. This restriction on δ is required when state constraints are present in order to guarantee Lipschitz continuity of the solution of the receding horizon regulator quadratic program since Lipschitz continuity is necessary in the determination of Λ. If there are no state constraints in the receding horizon regulator quadratic program, the solution is globally Lipschitz continuous as shown in Appendix A.2.3. In this case, $\delta(\epsilon) = \epsilon/\Lambda$ for all $\delta \in \Re^+$ which implies δ_0 is unbounded and the inverse function, $\epsilon(\delta) = \Lambda\delta$, exists for all $\delta \in \Re^+$. Therefore, Eqs. C.1 and C.3 apply for all $x \in \Re^n$.

The proof of Eq. C.2 proceeds as follows in which the notation is taken from [35]. For $l \geq N(\epsilon)$, $\|x\left(t_{k+l}; t_k, x\right)\| < \epsilon$. Therefore, if $l \geq N\left(\frac{1}{\alpha}\|x\|\right)$, it follows that $\|x\left(t_{k+l}; t_k, x\right)\| < \frac{1}{\alpha}\|x\|$ and

$$G\left(\|x\left(t_{k+l}; t_k, x\right)\|\right)\frac{1+\alpha l}{1+l} \leq \alpha G\left(\|x\left(t_{k+l}; t_k, x\right)\|\right) < \alpha G\left(\frac{1}{\alpha}\|x\|\right) < V\left(t_k, x\right)$$

The Lyapunov function can then be expressed as

$$V\left(t_k, x\right) = \sup_{l \leq N\left(\frac{1}{\alpha}\|x\|\right)} G\left(\|x\left(t_{k+l}; t_k, x\right)\|\right)\frac{1+\alpha l}{1+l} = G\left(\|x\left(t_{k+l_1}; t_k, x\right)\|\right)\frac{1+\alpha l_1}{1+l_1}$$

in which l_1 depends on x. Let $x = x\left(t_k; t_0, x_0\right)$ and $x^* = x\left(t_{k+1}; t_0, x_0\right) = x\left(t_{k+1}; t_k, x\right)$.

$$\begin{aligned} V\left(t_{k+1}, x^*\right) &= G\left(\left\|x\left(t_{k+l_1^*+1}; t_{k+1}, x^*\right)\right\|\right)\frac{1+\alpha l_1^*}{1+l_1^*} \\ &= G\left(\left\|x\left(t_{k+l_1^*+1}; t_k, x\right)\right\|\right)\frac{1+\alpha l_1^*}{1+l_1^*} \\ &= G\left(\left\|x\left(t_{k+l_1^*+1}; t_k, x\right)\right\|\right)\frac{1+\alpha\left(l_1^*+1\right)}{1+\left(l_1^*+1\right)} \\ &\quad \times\left[1 - \frac{\alpha-1}{\left(1+l_1^*\right)\left(1+\alpha\left(l_1^*+1\right)\right)}\right] \\ &\leq V\left(t_k, x\right) - \frac{(\alpha-1)V\left(t_k, x\right)}{\left(1+l_1^*\right)\left(1+\alpha\left(l_1^*+1\right)\right)} \end{aligned}$$

Therefore,

$$V\left(t_{k+1}, x^*\right) - V\left(t_k, x\right) \leq - \frac{\alpha - 1}{\left(1 + l_1^*\right)\left(1 + \alpha\left(l_1^* + 1\right)\right)} V\left(t_k, x\right) \tag{C.4}$$
$$\leq - \frac{(\alpha - 1)\, G\left(\|x\|\right)}{\left[1 + N\left(\frac{1}{\alpha}\|x^*\|\right)\right]\left[2 + N\left(\frac{1}{\alpha}\|x^*\|\right)\right]}$$

since $l_1^* \leq N\left(\frac{1}{\alpha}\|x^*\|\right)$.

After this point, Halanay uses the restriction $\|h[t_k, x]\| \geq m(\|x\|)$ to obtain the function $c(\|x(t_k; t_0, x_0)\|)$. Since the nominal closed-loop system considered in this work includes singular A matrices and deadbeat regulators, this restriction is not applicable. If $x^* = 0$, then $V\left(t_{k+1}, x^*\right) = 0$ and, from Eq. C.4,

$$\left(1 + l_1^*\right)\left(1 + \alpha\left(l_1^* + 1\right)\right) \geq \alpha - 1$$

which is satisfied for all $l_1^* \geq 0$. Consider the following function for $l_1^*(x^*)$

$$l_1^*(x^*) = \begin{cases} N\left(\|x^*\|/\alpha\right), & \|x^*\| \neq 0 \\ 0, & \|x^*\| = 0 \end{cases} \tag{C.5}$$

in which $x^* = Ax_k + Bu_k = z_{1|k}$ and $\|x_{k+j}\| < \|z_{1|k}\|/\alpha$ for all $j \geq N\left(\|z_{1|k}\|/\alpha\right)$. The result in Eq. C.4 can then be expressed as follows

$$V(x_{k+1}) - V(x_k) \leq - \frac{(\alpha - 1)\, G\left(\|x_k\|\right)}{\left(1 + l_1^*(x_k)\right)\left(1 + \alpha\left(l_1^*(x_k) + 1\right)\right)} = -c(x_k) \tag{C.6}$$

in which $c(x_k) \in \Re^n \rightarrow \Re^+$ with $c(0) = 0$ and $c(x) > 0$ for all $x \neq 0$.

C.3 Convergence of $c(x)$

Lemma C.1 *The convergence of the sequence $\{c(x_k)\}$ to zero implies the sequence $\{x_k\}$ converges to zero.*

Proof: Convergence of the sequence $\{c(x_k)\}$ to zero implies that $\|c(x_k)\| < \epsilon$ for all $k \geq J(\epsilon)$ and

$$G(\|x_k\|) < \frac{\epsilon\left(1 + l_1^*\right)\left(1 + \alpha\left(l_1^* + 1\right)\right)}{\alpha - 1} \quad \forall \;\; k \geq J(\epsilon)$$

Letting $z_{k+j|k}$ represent the closed-loop state trajectory for the perfect state feedback regulator starting at state x_k, l_1^* is defined in Eq. C.5 as $\|z_{k+j|k}\| < \|z_{k+1|k}\|/\alpha$ for all $j \geq l_1^*$. Since the sequence $\{z_{k+j|k}\}$ converges from Lemma 2.5, l_1^* can not increase without bound with increasing $\|x_k\|$. Therefore, the sequence $\{G(\|x_k\|)\}$ converges to zero. Since G is a nondecreasing positive function, the sequence $\{x_k\}$ converges to zero.

Appendix D

Proofs for Chapter 2

D.1 Proof of Lemma 2.1

Let $z^*_{j|k}$ and $v^*_{j|k}$ denote the jth open-loop optimal state and input determined from the solution of the feasible quadratic program at time k. At this time k, the receding horizon regulator injects the first optimal open-loop input, $u_k = v^*_{0|k}$, into the process. Since the nominal system with no disturbances is considered, the value of the state measured at time $k+1$ is $x_{k+1} = z^*_{1|k}$. Therefore, feasibility of the constraints on an infinite horizon at time k implies that the following input trajectory is feasible at time $k+1$

$$\begin{bmatrix} v^*_{1|k} \\ v^*_{2|k} \\ \vdots \\ v^*_{N-1|k} \\ v^*_{N|k} \end{bmatrix}$$

in which $v^*_{N|k}$ comes from the open-loop input parameterization for $j = N$ at time k. Feasibility at time $k = 0$ then implies feasibility at every time $k > 0$ by induction.

D.2 Proof of Lemma 2.2

Let $z^*_{j|k}$ and $v^*_{j|k}$ denote the jth open-loop optimal state and input determined from the solution of the feasible quadratic program at time k and Φ^*_k denote the optimal value of the objective function in Eq. 2.42 at time k. The optimal objective function value is

$$\begin{aligned}\Phi_k^* = & \; x_k^T Q x_k + z_{1|k}^{*T} Q z_{1|k}^* + z_{2|k}^{*T} Q z_{2|k}^* + \ldots \\ & + \; u_k^T R u_k + v_{1|k}^{*T} R v_{1|k}^* + \ldots + v_{N-1|k}^{*T} R v_{N-1|k}^* \\ & + \; \Delta u_k^T S \Delta u_k + \Delta v_{1|k}^{*T} S \Delta v_{1|k}^* + \ldots + \Delta v_{N|k}^{*T} S \Delta v_{N|k}^*\end{aligned}$$

in which $z_{0|k}^* = x_k$, $u_k = v_{0|k}^*$, and $\Delta u_k = u_k - u_{k-1}$. Since the input u_k is applied at time k, the initial condition at time $k+1$ is $x_{k+1} = z_{1|k}^*$ for the nominal system without disturbances. From Lemma 2.1, feasibility at time k implies the following input sequence, $\bar{\pi}_{k+1}$, is feasible at time $k+1$.

$$\bar{\pi}_{k+1} = \begin{bmatrix} v_{1|k}^* \\ v_{2|k}^* \\ \vdots \\ v_{N-1|k}^* \\ 0 \end{bmatrix} \tag{D.1}$$

Let $\bar{\Phi}_{k+1}$ represent the objective function value at time $k+1$ with the initial state $x_{k+1} = z_{1|k}^*$ and input trajectory $\bar{\pi}_{k+1}$. The following relationship then holds.

$$\Phi_k^* = x_k^T Q x_k + u_k^T R u_k + \Delta u_k^T S \Delta u_k + \bar{\Phi}_{k+1} \tag{D.2}$$

Optimization at time $k+1$ yields an objective function value that can be no greater than $\bar{\Phi}_{k+1}$ which implies $\Phi_{k+1}^* \leq \bar{\Phi}_{k+1}$.

$$\Phi_k^* \geq x_k^T Q x_k + u_k^T R u_k + \Delta u_k^T S \Delta u_k + \Phi_{k+1}^* \tag{D.3}$$

Since $R > 0$ and $Q, S \geq 0$, the sequence $\{\Phi_k^*\}$ is non-increasing and bounded below by zero. Therefore the sequence converges to some nonnegative value Φ_∞^* [76].

D.3 Proof of Lemma 2.3

From Lemma 2.2, Φ_k^* converges to some nonnegative value Φ_∞^* and from Eq. D.3, the difference is an upper bound on $u_k^T R u_k$ for all k. For any $\epsilon > 0$, choose $J(\epsilon)$ such that

$$\epsilon^2 \sigma_{\min}(R) > \Phi_{J(\epsilon)}^* - \Phi_\infty^* \geq u_k^T R u_k, \;\; \forall \;\; k \geq J(\epsilon)$$

in which $\sigma_{\min}(R) > 0$ is the minimum singular value of the positive definite matrix R. This choice implies

$$\epsilon^2 \sigma_{\min}(R) > \sigma_{\min}(R) \left\| u_k \right\|^2, \;\; \forall \;\; k \geq J(\epsilon)$$

resulting in $\|u_k\| < \epsilon$ for all $k \geq J(\epsilon)$ which proves the lemma.

D.4 Proof of Lemma 2.4

From Eqs. D.2 and D.3 in the proof of Lemma 2.2, $\Phi_k^* \geq \bar{\Phi}_{k+1}$ and $\bar{\Phi}_{k+1} \geq \Phi_{k+1}^*$. Therefore the following relation holds.

$$\Phi_k^* \geq \bar{\Phi}_{k+1} \geq \Phi_{k+1}^*$$

The convergence of the nonincreasing sequence $\{\Phi_k^*\}$ to Φ_∞^*, shown in Lemma 2.2, implies $\bar{\Phi}_k$ converges to Φ_k^*. From Lemma A.1 in Appendix A.2.2, the difference between $\bar{\pi}_{k+1}$, defined in Eq. D.1, and π_{k+1}^* is bounded as follows

$$\left\|\bar{\pi}_{k+1} - \pi_{k+1}^*\right\| \leq \gamma \left|\bar{\Phi}_{k+1} - \Phi_{k+1}^*\right|^{1/2} \leq \gamma \left(\Phi_k^* - \Phi_\infty^*\right)^{1/2}$$

which implies the following inequalities at each time k.

$$\begin{aligned}
\|v_{1|k}^* - u_{k+1}\| &\leq \gamma \left(\Phi_k^* - \Phi_\infty^*\right)^{1/2} \\
\|v_{2|k}^* - v_{1|k+1}^*\| &\leq \gamma \left(\Phi_k^* - \Phi_\infty^*\right)^{1/2} \\
\vdots \qquad & \quad \vdots \qquad \quad \vdots \\
\|v_{N-2|k}^* - v_{N-3|k+1}^*\| &\leq \gamma \left(\Phi_k^* - \Phi_\infty^*\right)^{1/2} \\
\|v_{N-1|k}^* - v_{N-2|k+1}^*\| &\leq \gamma \left(\Phi_k^* - \Phi_\infty^*\right)^{1/2}
\end{aligned}$$

The preceding inequalities from time k to time $k+j$ imply

$$\begin{aligned}
&\|v_{j|k}^* - u_{k+j}\| \leq \|v_{j|k}^* - v_{j-1|k+1}^*\| + \|v_{j-1|k+1}^* - v_{j-2|k+2}^*\| + \ldots \\
&+\|v_{2|k+j-2}^* - v_{1|k+j-1}^*\| + \|v_{1|k+j-1}^* - u_{k+j}\| \leq \gamma(\Phi_k^* - \Phi_\infty^*)^{1/2} \\
&+\gamma(\Phi_{k+1}^* - \Phi_\infty^*)^{1/2} + \ldots + \gamma(\Phi_{k+j-2}^* - \Phi_\infty^*)^{1/2} + \gamma(\Phi_{k+j-1}^* - \Phi_\infty^*)^{1/2}
\end{aligned}$$

which results in the following bound on the open-loop inputs at time k.

$$\left|\|v_{j|k}^*\| - \|u_{k+j}\|\right| \leq \left\|v_{j|k}^* - u_{k+j}\right\| \leq \gamma N \left(\Phi_k^* - \Phi_\infty^*\right)^{1/2}, \quad j = 0, \ldots, N-1$$

Convergence of the sequence $\{\Phi_k^*\}$ to Φ_∞^*, shown in Lemma 2.2, and $\{u_k\}$ to zero, shown in Lemma 2.3, implies that for all $\epsilon > 0$ a $J(\epsilon)$ exists such that

$$\|v_{j|k}^*\| < \|u_{J(\epsilon)}\| + \gamma N \left(\Phi_{J(\epsilon)}^* - \Phi_\infty^*\right)^{1/2} \leq \epsilon/\sqrt{N}, \quad \forall \;\; k \geq J(\epsilon)$$

resulting in $\|\pi_k^*\| < \epsilon$ for all $k \geq J(\epsilon)$ which proves the lemma.

D.5 Proof of Lemma 2.5

It is necessary to demonstrate that for every $\epsilon > 0$ there exists a $J(\epsilon)$ such that $\|x_k\| < \epsilon$ for all $k \geq J(\epsilon)$. Convergence of the unstable modes is demonstrated first. Feasibility of the equality constraint on the unstable modes in Eq. 2.19 implies the following

$$\begin{aligned} &J_u^N z_k^u + \mathcal{J}\pi_k^* = 0 \\ &\mathcal{J} = \begin{bmatrix} J_u^{N-1}\tilde{V}_u B & J_u^{N-2}\tilde{V}_u B & \ldots & \tilde{V}_u B \end{bmatrix} \end{aligned}$$

in which z_k^u are the unstable modes, π_k^* is the optimal solution of the receding horizon regulator quadratic program, and J_u, $\tilde{V}_u$ are defined in Eq. 2.2. From the previous equality and J_u unstable, the following inequality holds.

$$\|z_k^u\| \leq \|J_u^N z_k^u\| \leq \|\mathcal{J}\| \; \|\pi_k^*\|$$

The convergence of $\{\pi_k^*\}$ to zero, shown in Lemma 2.4, implies that for all $\epsilon > 0$ a $J_1(\epsilon)$ exists such that

$$\|z_k^u\| \leq \|\mathcal{J}\| \; \|\pi_{J_1(\epsilon)}^*\| < \epsilon/(2\,\|V_u\|), \;\; \forall \;\; k \geq J_1(\epsilon)$$

Demonstration of the convergence of the stable modes requires the following result from Horn and Johnson [37] in which $(J_s^k)_{i,j}$ is the i,j element of J_s^k, $\alpha > 0$, and $\Gamma(\alpha)$ bounded.

$$\left|(J_s^k)_{i,j}\right| \leq \Gamma(\alpha) \; \left(|\lambda_{\max}(J_s)| + \alpha\right)^k$$

This result implies the following bound on the l_2–norm of J_s^k.

$$\|J_s^k\| \leq \sqrt{n}\,\Gamma(\alpha) \; \left(|\lambda_{\max}(J_s)| + \rho\right)^k$$

Since J_s is stable, an $\alpha > 0$ exists such that $\bar{\lambda} = (|\lambda_{\max}(J_s)| + \alpha) < 1$.

Let $z_{k_1}^s$ be the stable modes after the first $k_1 - 1$ input moves.

$$z_{k_1+k_2}^s = J_s^{k_2} z_{k_1}^s + J_s^{k_2-1}\tilde{V}_s B u_{k_1} + J_s^{k_2-2}\tilde{V}_s B u_{k_1+1} + \ldots + \tilde{V}_s B u_{k_1+k_2-1}$$

The convergence of $\{u_k\}$ to zero, shown in Lemma 2.3, implies that for all $\gamma > 0$ a k_1 exists such that $\|u_k\| < \gamma$ for all $k \geq k_1$ resulting in the following bound on the stable modes.

$$\|z_{k_1+k_2}^s\| < \|J_s^{k_2}\| \, \|z_{k_1}^s\| + \gamma \, \|\tilde{V}_s B\| \, \sqrt{n}\,\Gamma(\alpha) \sum_{j=0}^{k_2} \bar{\lambda}^j \leq \gamma \, \|\tilde{V}_s B\| \, \sqrt{n}\,\Gamma(\alpha) \, \frac{1}{1-\bar{\lambda}}$$

Choose γ such that

$$\gamma \, \|\tilde{V}_s B\| \, \sqrt{n} \, \Gamma(\alpha) \, \frac{1}{1-\bar{\lambda}} < \epsilon / (4 \, \|V_s\|)$$

and, since $z^s_{k_1}$ is bounded and J^k_s is convergent, k_2 such that

$$\|J^{k_2}_s\| \; \|z^s_{k_1}\| < \epsilon / (4 \, \|V_s\|)$$

for all $\epsilon > 0$ in which ϵ is the bound on the state. Let $J_2(\epsilon) = k_1 + k_2$, then

$$\|z^s_k\| < \epsilon / (2 \, \|V_s\|) \quad \forall \quad k \geq J_2(\epsilon)$$

resulting in the following bound for $x_k = V_u z^u_k + V_s z^s_k$

$$\|x_k\| \leq \|V_u\| \, \|z^u_{J_1(\epsilon)}\| + \|V_s\| \, \|z^s_{J_2(\epsilon)}\| < \epsilon, \quad \forall \quad k \geq \max\left(J_1(\epsilon), J_2(\epsilon)\right)$$

which proves the lemma.

D.6 Proof of Lemma 2.6

A bound on the optimal objective function value Φ^*_k can be determined from the quadratic objective function in Eq. A.1 as follows.

$$\|\Phi^*_k\| \leq \|H\| \; \|\pi^*_k\|^2 + 2\|\pi^*_k\| \left(\|G\| \; \|x_k\| + \|J\| \; \|u_k\|\right) + \|Q\| \; \|x_k\|^2$$

The convergence of the sequences $\{u_k\}$ to zero from Lemma 2.3, $\{\pi^*_k\}$ to zero from Lemma 2.4, and $\{x_k\}$ to zero from Lemma 2.5, implies that for every $\epsilon > 0$ there exists a $J(\epsilon)$ such that

$$\|H\| \; \|\pi^*_{J(\epsilon)}\|^2 + 2\|\pi^*_{J(\epsilon)}\| \left(\|G\| \; \|x_{J(\epsilon)}\| + \|J\| \; \|u_{J(\epsilon)}\|\right) + \|Q\| \; \|x_{J(\epsilon)}\|^2 < \epsilon$$

which implies $\|\Phi^*_k\| < \epsilon$ for all $k \geq J(\epsilon)$ and proves the lemma.

D.7 Proof of Lemma 2.7

It is necessary to show that for all $\epsilon > 0$ there exists a $\delta(\epsilon)$ such that if $\|x_0\| < \delta(\epsilon)$ and $\|u_{-1}\| < \delta(\epsilon)$, then

$$\|x_k\| \leq \|V_s\| \, \|z^s_k\| + \|V_u\| \, \|z^u_k\| < \epsilon \quad \forall \quad k \geq 0$$

From $\{\Phi_k^*\}$ nonincreasing, shown in Lemma 2.2, $\Phi_k^* \leq \Phi_0^*$ for all time $k \geq 0$. Therefore, from Lemmas 2.3 and 2.6, the following bound on $\|u_k\|$ can be derived for all time $k \geq 0$ in which σ is the minimum singular value of R.

$$\|u_k\| < \sqrt{\Phi_0^*/\sigma}$$

From Lemma 2.4, the following bound on π_k^* can be determined for all time $k \geq 0$.

$$\|\pi_k^*\| < \sqrt{N}\left(\|u_k\| + \gamma N(\Phi_k^* - \Phi_\infty^*)^{1/2}\right) < \sqrt{N\Phi_0^*}\left(\sigma^{-1/2} + \gamma N\right)$$

From Lemma 2.5, the following bounds on the stable and unstable modes can be derived for all time $k \geq 0$.

$$\begin{aligned}
\|z_k^s\| &< \max_k(\|J_s^k\|)\,\|z_0^s\| + \sqrt{n\Phi_0^*/\sigma}\,\|\tilde{V}_s B\|\,\Gamma(\alpha)\,\frac{1}{1-\bar{\lambda}} \\
\|z_k^u\| &< \|\mathcal{J}\|\,\sqrt{N\Phi_0^*}\left(\sigma^{-1/2} + \gamma N\right)
\end{aligned}$$

Since $\mathcal{X}^n \times \mathcal{U}^m$ specifies a convex constraint region containing the origin, there exists an $\varepsilon > 0$ such that $x_0 \in \mathcal{X}^n$, $u_{-1} \in \mathcal{U}^m$, and $\pi_0^* = H^{-1}(Gx_0 + Ju_{-1}) \in \mathcal{U}^{N\cdot m}$ for all $\|x_0\| < \varepsilon$ and $\|u_{-1}\| < \varepsilon$ in which the matrices H, G, and J are defined in Appendix A.1. Since there are no active constraints, Lipschitz continuity of the solution to the quadratic program, discussed in Appendix A.2.3, implies

$$\|\pi_0^*\| \leq \rho(\|x_0\| + \|u_{-1}\|) < 2\rho\varepsilon$$

From the objective function bound in Lemma 2.6

$$\Phi_0^* \leq \beta\varepsilon, \qquad \beta = 4\rho^2\,\|H\| + 4\rho(\|G\| + \|J\|) + 2\,\|Q\|$$

which implies that for all $\|x_0\| < \varepsilon$ and $\|u_{-1}\| < \varepsilon$

$$\begin{aligned}
\|x_k\| < \varepsilon \Big(&\|V_s\| \max_k(\|J_s^k\|) + \|V_s\|\,|\tilde{V}_s B\|\,\sqrt{n\beta/\sigma}\,\Gamma(\rho)\frac{1}{1-\bar{\lambda}} \\
&+ \|V_u\|\,\|\mathcal{J}\|\,\sqrt{N\beta}(\sigma^{-1/2} + \gamma N)\Big) = \Lambda\varepsilon \quad \forall \quad k \geq 0.
\end{aligned}$$

Therefore, an acceptable ϵ-δ relationship to demonstrate stability for the constrained regulator is

$$\delta(\epsilon) = \begin{cases} \epsilon/\Lambda, & \epsilon < \varepsilon \\ \varepsilon/\Lambda, & \epsilon \geq \varepsilon \end{cases}$$

D.8 Proof of Theorem 2.1

Feasibility of the equality constraint on the unstable modes in Eq. 2.19 at time $k = 0$ is guaranteed by $x_0 \in \mathcal{Z}_N^n$. Feasibility of the output constraints at time $k = 0$ is guaranteed by the selection of j_1 in Eq. 2.43. This implies the existence of a feasible solution to the quadratic program at every time $k > 0$ from Lemma 2.1. Nominal asymptotic stability then follows from stability, demonstrated in Lemma 2.7, and convergence, demonstrated in Lemma 2.5 for $N \geq 1$ and a consequence of A stable for $N = 0$.

D.9 Proof of Theorem 2.2

Since the penalty matrices in the objective function do not affect feasibility of the constraints, the existence of a feasible solution to the receding horizon regulator quadratic program at each time k can be shown in the same manner as Theorem 2.1. The difference between Φ_k^*, the optimal solution of the objective function in Eq. 2.45, and $\bar{\Phi}_{k+1}$, defined in Eq. D.2, is as follows.

$$\Phi_k^* - \bar{\Phi}_{k+1} = x_k^T Q_0 x_k + u_k^T R_0 u_k + \Delta u_k^T S_0 \Delta u_k + \sum_{j=1}^{\infty} z_{j|k}^{*T}(Q_j - Q_{j-1}) z_{j|k}^*$$
$$+ \sum_{j=1}^{N-1} v_{j|k}^{*T}(R_j - R_{j-1}) v_{j|k}^* + \sum_{j=1}^{N} \Delta v_{j|k}^{*T}(S_j - S_{j-1}) \Delta v_{j|k}^*$$

Since the penalty matrices are nondecreasing sequences,

$$\Phi_k^* \geq x_k^T Q x_k + u_k^T R u_k + \Delta u_k^T S \Delta u_k + \bar{\Phi}_{k+1}$$

which preserves the monotonicity of the sequence $\{\Phi_k^*\}$. Therefore, the input sequence $\{u_k\}$ converges to zero using the same argument presented in Lemma 2.3. The convergence of the optimal solution of the quadratic program and state sequences to zero follow in the same manner as Lemmas 2.4 and 2.5, respectively. Stability of the regulator with time-varying penalty matrices follows in the same manner as Lemma 2.7. Nominal asymptotic stability then follows from stability and convergence.

D.10 Proof of Theorem 2.3

Feasibility of the input constraints in Eq. 2.60 at time $k = 0$ is guaranteed by $x_0 \in \mathcal{Z}_N^n$. Feasibility of the output constraints at time $k = 0$ is guaranteed by the selection of j_1 in Eq. 2.60. This implies the existence of a feasible solution to the quadratic program at every time $k > 0$ from Lemma 2.8. The monotonicity and convergence of the optimal open-loop objective sequence $\{\Phi_k^*\}$ can be shown in the same manner as Lemma 2.2. The convergence of the input sequence $\{u_k\}$ and optimal quadratic program solution sequence $\{\pi_k^*\}$ then follow in the same manner as Lemmas 2.3 and 2.4, respectively. Convergence of the state sequence $\{x_k\}$ to zero is shown as in Lemma 2.5 with the following bound on $x_{k_1+k_2}$.

$$\|x_{k_1+k_2}\| < \|(A - BK)^{k_2}\| \, \|x_{k_1}\| + \gamma \, \|B\| \, \Gamma \sum_{j=0}^{k_2} \lambda_{\max}(A - BK) \leq$$
$$\gamma \, \|B\| \, \Gamma \frac{1}{1 - \lambda_{\max}(A - BK)}$$

in which the unstable modes need not be considered since $A - BK$ is stable, which also implies $\lambda_{\max}(A - BK) < 1$ and the infinite sum exists. The convergence of the sequence of optimal solutions to the quadratic program follows as shown in Lemma 2.4 in which H_2 and G_2 are computed as shown in Appendix A.1 with Q_N computed from Eq. 2.53. Stability of the regulator follows in the same manner as Lemma 2.7 with the following bound on x_k.

$$\|x_k\| < \left(1 + \max_k(\|(A - BK)^k\|)\right) \left(\|A^N\| \, \|x_0\| + \sqrt{\Phi_0^*/\sigma}\, \|B\| \Gamma \sum_{j=0}^{N-1} \lambda_{\max}(A)\right.$$
$$\left. + \sqrt{\Phi_0^*/\sigma}\, \|B\| \, \Gamma \, \frac{1}{1 - \lambda_{\max}(A - BK)}\right)$$

Nominal asymptotic stability then follows from stability and convergence.

Appendix E

Proofs for Chapter 3

E.1 Proof of Theorem 3.1

The sufficient condition is proved first. If the integrating modes of A are in the observable subspace of $(\acute{C}, A)$, $\mathcal{O}v_i \neq 0$ in which $\mathcal{O}$ is the observability matrix of $(\acute{C}, A)$ and v_i are the integrating modes of A. Since v_i are integrating modes, $Av_i = v_i$. Therefore, $\mathcal{O}v_i \neq 0$ implies $\acute{C}v_i \neq 0$. Since the only vectors in the null space of $(I - A)$ are v_i and $\acute{C}v_i \neq 0$, $\acute{\mathcal{O}}$ does not have a nonzero null space. Therefore, the matrix is full column rank. The necessary condition is proved as follows. If $\acute{\mathcal{O}}$ is full rank, there is no vector in the null space of both $(I - A)$ and $\acute{C}$. Therefore the integrating modes of A, which are in the null space of $(I - A)$, are not in the null space of $\acute{C}$. This implies that the integrating modes are contained in the observable subspace of $(\acute{C}, A)$.

E.2 Proof of Theorem 3.2

It is sufficient to show that the quadratic program is a strictly convex programming problem [24]. Feasibility of the equality constraint in Eq. 3.6 implies that this equation has a solution. If the matrix

$$\mathcal{A} = \begin{bmatrix} (I - A) & -B \\ \acute{C} & 0 \end{bmatrix}$$

is full column rank, the solution is unique which proves the theorem. If this matrix is not full column rank, the set of solutions to the equality constraint can be expressed as

$$\begin{bmatrix} x_s \\ u_s \end{bmatrix} = \mathcal{R}z_r + \mathcal{N}z_n, \qquad \mathcal{R}z_r = \begin{bmatrix} 0 \\ y_t \end{bmatrix}$$

in which $\mathcal{R}$ consists of the basis vectors of the range of $\mathcal{A}$ and $\mathcal{N}$ consists of the basis vectors of the null space of $\mathcal{A}$. Substituting this expression into the quadratic objective in Eq. 3.5 results in the following reduced quadratic function.

$$z_n^T \mathcal{N}^T H \mathcal{N} z_n + 2(H\mathcal{R}z_r - u_t)^T \mathcal{N} z_n + (H\mathcal{R}z_r - u_t)^T \mathcal{R} z_r$$

If the reduced Hessian, $\mathcal{N}^T H \mathcal{N}$, is positive definite, a unique solution exists [24]. The reduced Hessian is expressed as follows.

$$H = \begin{bmatrix} 0 & 0 \\ 0 & R_s \end{bmatrix}, \qquad \mathcal{N} = \begin{bmatrix} \mathcal{N}_x \\ \mathcal{N}_u \end{bmatrix}, \qquad \mathcal{N}^T H \mathcal{N} = \mathcal{N}_u^T R_s \mathcal{N}_u$$

Since R_s is positive definite, the reduced Hessian is positive definite if $\mathcal{N}_u$ is full column rank [37]. Since $\mathcal{N}_u z_n$ is in the null space of $\mathcal{A}$, the following relationship holds for all z_n.

$$\begin{aligned} (I - A)\mathcal{N}_x z_n - B\mathcal{N}_u z_n &= 0 \\ \acute{C}\mathcal{N}_x z_n &= 0 \end{aligned}$$

Since $\acute{\mathcal{O}}$ is full rank and $\mathcal{N}_x z_n$ is in the null space of $\acute{C}$, $\mathcal{N}_x z_n$ cannot be in the null space of $(I - A)$. Therefore, $B\mathcal{N}_u z_n = 0$ only when $\mathcal{N}_x z_n = 0$. Since $\mathcal{N}$ is full column rank, $\mathcal{N} z_n \neq 0$ for all $z_n \neq 0$. If $\mathcal{N}_x z_n = 0$, then $\mathcal{N}_u z_n \neq 0$ from full column rank of $\mathcal{N}$. Therefore, $\mathcal{N}_u z_n \neq 0$ for all $z_n \neq 0$ which implies full column rank of $\mathcal{N}_u$ and proves the theorem.

E.3 Proof of Theorem 3.3

As in the proof of Theorem 3.2, the quadratic program is shown to be a strictly convex programming problem. Since the origin is a feasible solution to the equality constraint in Eq. 3.8, the quadratic program is feasible. The set of solutions to the equality constraint can be expressed as

$$\begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} \mathcal{N}_x \\ \mathcal{N}_u \end{bmatrix} z_n$$

in which $\mathcal{N}$ consists of the basis vectors of the null space of $[(I - A) \quad -B]$. If the reduced Hessian, $\mathcal{N}^T H \mathcal{N}$, is positive definite, a unique solution exists [24]. The reduced Hessian is expressed as follows.

$$\mathcal{N}_x^T \acute{C}^T Q_s \acute{C} \mathcal{N}_x + \mathcal{N}_u^T R_s \mathcal{N}_u$$

Since $\mathcal{N} z_n$ is in the null space of $[(I - A) \ \ -B]$, the following relationship holds for all z_n.

$$(I - A)\mathcal{N}_x z_n - B\mathcal{N}_u z_n = 0$$

For all z_n such that $\mathcal{N}_x z_n$ is not in the null space of $(I - A)$, the equality implies $\mathcal{N}_u z_n \neq 0$. Therefore, the reduced Hessian is positive and nonzero. For all z_n such that $\mathcal{N}_x z_n$ is in the null space of $(I - A)$, full rank of $\acute{O}$ ensures that $\acute{C}\mathcal{N}_x z_n \neq 0$. Therefore, the reduced Hessian is also positive and nonzero. Since the reduced Hessian is positive and nonzero for all $z_n \neq 0$, it is a positive definite matrix which proves the theorem.

E.4 Proof of Theorem 3.4

The quadratic program is shown to be a strictly convex programming problem. Since the origin is a feasible solution to the equality constraint in Eq. 3.8, the quadratic program is feasible. The set of solutions to the equality constraint can be expressed as

$$\begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} \mathcal{N}_x \\ \mathcal{N}_u \end{bmatrix} z_n$$

in which $\mathcal{N}$ consists of the basis vectors of the null space of $[(I - A) \ \ -B]$. If the reduced Hessian, $\mathcal{N}^T H \mathcal{N}$, is positive definite, a unique solution exists [24]. The reduced Hessian is expressed as follows

$$\mathcal{N}_x^T \acute{C}^T Q_s \acute{C} \mathcal{N}_x + \mathcal{N}_u^T \mathcal{N}_u \alpha \alpha^T \mathcal{N}_u^T \mathcal{N}_u$$

in which α is a basis for the null space of $\mathcal{N}_x^T \acute{C} Q_s \acute{C} \mathcal{N}_x$. Consider the following expansion for z_n in which $\alpha^\perp$ is a basis for the subspace orthogonal to α.

$$z_n = \alpha v_1 + \alpha^\perp v_2$$

Since $\mathcal{N} z_n$ is in the null space of $[(I - A) \ \ -B]$, the following relationship holds for all v_1, v_2.

$$(I - A)\mathcal{N}_x(\alpha v_1 + \alpha^\perp v_2) - B\mathcal{N}_u(\alpha v_1 + \alpha^\perp v_2) = 0$$

Since $\acute{C}\mathcal{N}_x \alpha = 0$, full rank of $\acute{O}$ ensures that the null space of $(I - A)$ is not contained in $\mathcal{N}_x \alpha$. Therefore, $\mathcal{N}_u \alpha v_1 \neq 0$ for all $v_1 \neq 0$ which implies $\mathcal{N}_u \alpha$ is full column rank. Therefore the matrix $\alpha^T \mathcal{N}_u^T \mathcal{N}_u \alpha$ is positive definite [37]. Since $\alpha^\perp$ is orthogonal to α, $\acute{C}\mathcal{N}_x \alpha^\perp v_2 \neq 0$ for all $v_2 \neq 0$ which implies $\acute{C}\mathcal{N}_x \alpha^\perp$ is full column rank. Therefore the matrix $\alpha^{\perp T} \mathcal{N}_x^T \acute{C}^T Q_s \acute{C} \mathcal{N}_x \alpha^\perp$ is also positive definite. The reduced Hessian contains

$$(\alpha v_1 + \alpha^\perp v_2)^T \left(\mathcal{N}_x^T \acute{C}^T Q_s \acute{C} \mathcal{N}_x + \mathcal{N}_u^T \mathcal{N}_u \alpha \alpha^T \mathcal{N}_u^T \mathcal{N}_u \right) (\alpha v_1 + \alpha^\perp v_2) =$$
$$v_1^T (\alpha^T \mathcal{N}_u^T \mathcal{N}_u \alpha)^2 v_1 + v_2^T (\alpha^{\perp T} \mathcal{N}_x^T \acute{C}^T Q_s \acute{C} \mathcal{N}_x \alpha^\perp) v_2$$

which is a nonzero positive number for all $v_1 \neq 0$ and $v_2 \neq 0$. Since the reduced Hessian is positive and nonzero for all v_1 and v_2, it is a positive definite matrix which proves the theorem.

E.5 Proof of Theorem 3.5

Since the state-space system came from a minimal realization of a discrete transfer function matrix with a full rank steady-state gain matrix, no integrating modes, and $p \leq m$, $C(I-A)^{-1}B$ is full row rank. Choose α_1 and α_2 such that

$$\alpha_1^T(I-A) + \alpha_2^T C = 0 \implies \alpha_1^T = -\alpha_2^T C(I-A)^{-1}$$

which implies $-\alpha_1^T B = \alpha_2^T C(I-A)^{-1}B \neq 0$ for all α_1 from full row rank of $C(I-A)^{-1}B$. Therefore, the following matrix is full row rank.

$$\begin{bmatrix} I-A & -B \\ C & 0 \end{bmatrix} \tag{E.1}$$

The equality constraint in Eq. 3.6 is then feasible for all y_t and uniqueness of the solution follows from Theorem 3.2.

E.6 Proof of Theorem 3.6

Since the state-space system came from a minimal realization of a discrete transfer function matrix with a full rank steady-state gain matrix, no integrating modes, and $p \geq m$, $C(I-A)^{-1}B$ is full column rank. Choose α_1 and α_2 such that

$$(I-A)\alpha_1 - B\alpha_2 = 0 \implies \alpha_1 = (I-A)^{-1}B\alpha_2$$

which implies $C\alpha_1 = C(I-A)^{-1}B\alpha_2 \neq 0$ for all α_1 from full column rank of $C(I-A)^{-1}B$. Therefore, the matrix in Eq. E.1 is full column rank and, from $\mathcal{N}$ full column rank, there exists no $v \neq 0$ that satisfies the following equation.

$$\begin{bmatrix} I-A & -B \\ C & 0 \end{bmatrix} \mathcal{N}v = 0$$

Since $\mathcal{N}$ is a basis for the null space of $[(I-A) \quad -B]$ defined in Eq. 3.9

$$[(I-A) \quad -B]\,\mathcal{N}v = 0 \quad \forall \quad v$$

Therefore, $C\mathcal{N}_x v \neq 0$ for all $v \neq 0$ from full column rank of the matrix in Eq. E.1. This implies $C\mathcal{N}_x$ is full column rank, the matrix $\mathcal{N}_x^T C Q_s C \mathcal{N}_x$ is positive definite and, consequently, $R_s = 0$. Uniqueness of the solution follows from Theorem 3.4.

Appendix F

Proofs for Chapter 4

F.1 Proof of Lemma 4.1

Consider the following estimated state noise sequence.

$$\hat{w}_{j|k} = \begin{cases} -\bar{x}_0, & j = -1 \\ 0, & j = 0, 1, \ldots, k-1 \end{cases}$$

This sequence generates a feasible solution at every sample time k by zeroing all of the estimated states with the unconstrained state noise vector $\hat{w}_{-1|k}$. Since the origin is feasible by construction and one feasible solution exists, the quadratic program is feasible at every sample time k.

F.2 Proof of Lemma 4.2

Consider the following estimated state noise sequence.

$$\hat{w}_{j|k} = \begin{cases} -\hat{z}_{k-N|k-N-1}, & j = k-N-1 \\ 0, & j = k-N, k-N+1, \ldots, k-1 \end{cases}$$

This sequence generates a feasible solution at every sample time k by zeroing all of the estimated states with the unconstrained state noise vector $\hat{w}_{k-N-1|k}$. Since the origin is feasible by construction and one feasible solution exists, the quadratic program is feasible at every sample time k.

F.3 Proof of Lemma 4.3

Consider the following estimated state noise sequence.

$$\hat{w}_{j|k} = \begin{cases} x_0 - \bar{x}_0, & j = -1 \\ 0, & j = 0, 1, \ldots, k-1 \end{cases}$$

This sequence generates a feasible solution at every sample time k by zeroing the reconstruction error with the unconstrained state noise vector $\hat{w}_{-1|k}$. Since the origin is feasible by construction and one feasible solution exists, the quadratic program is feasible at every sample time k.

F.4 Proof of Lemma 4.4

Consider the following estimated state noise sequence.

$$\hat{w}_{j|k} = \begin{cases} x_{k-N} - \hat{z}_{k-N|k-N-1}, & j = k-N-1 \\ 0, & j = k-N, k-N+1, \ldots, k-1 \end{cases}$$

This sequence generates a feasible solution at every sample time k by zeroing the reconstruction error with the unconstrained state noise vector $\hat{w}_{-1|k}$. Since the origin is feasible by construction and one feasible solution exists, the quadratic program is feasible at every sample time k.

F.5 Proof of Lemma 4.5

From Lemma 4.1, the estimated state noise sequence $\{-\bar{x}_0, 0, \ldots, 0\}$ is feasible at every time k. Let the objective function value with this estimated state noise sequence be $\bar{\Psi}_k$.

$$\bar{\Psi}_k = \bar{x}_0^T Q_0^{-1} \bar{x}_0 + x_0^T \bar{R} x_0$$

$$\bar{R} = \sum_{j=0}^{k} A^{T^j} C^T R^{-1} C A^j$$

Since optimization will be performed at each time k, the value of Ψ_k can be no greater than $\bar{\Psi}_k$. Therefore Ψ_k is bounded above by $\lim_{k\to\infty} \bar{\Psi}_k$. This bound can be determined from the previous expression with $\bar{R}$ the solution to the following discrete Lyapunov equation.

$$\bar{R} = A^T \bar{R} A + C^T R^{-1} C$$

Optimization at each time k also implies the following relationship since the first k values of $\hat{w}_{j|k}$ computed at time k will not necessarily be optimal at time $k-1$.

$$\Psi_k - e_{k|k}^T C^T R^{-1} C e_{k|k} - \hat{w}_{k-1|k} Q^{-1} \hat{w}_{k-1|k} \geq \Psi_{k-1}$$

Since Q and R are positive definite, the sequence $\{\Psi_k\}$ is monotonically non-decreasing and bounded above and therefore converges. For $i = 1, \ldots, n-1$, $\Psi_k - \Psi_{k-i}$ converges to zero resulting in the convergence of each the following sums.

$$\left(\sum_{j=k-i}^{k-1} \hat{w}_{j|k}^T Q^{-1} \hat{w}_{j|k} + \sum_{j=k-i+1}^{k} e_{j|k}^T C^T R^{-1} C e_{j|k} \right) \to 0, \quad i = 1, \ldots, n-1$$

This implies $\hat{w}_{k-i|k}$ converges to zero for $i = 1, \ldots, n-1$ and $\mathcal{O} e_{k-n+1|k}$ converges to zero in which $\mathcal{O}$ is the observability matrix. Therefore the observable modes of $e_{k|k}$ converge to zero. The penalty on the unobservable modes of $\hat{w}$ is $O_u^T Q^{-1} O_u$ in which the columns of O_u are the basis vectors for the unobservable subspace of $\mathcal{O}$. Since O_u is full column rank and Q^{-1} is positive definite, the penalty on the unobservable modes of $\hat{w}$ is positive definite [37]. Since any component of $\hat{w}$ in the null space of $\mathcal{O}$ will not effect $\hat{v}$, as shown by the following relationship,

$$\begin{bmatrix} \hat{v}_{j+1+i|k} \\ \vdots \\ \hat{v}_{j+n+i|k} \end{bmatrix} = \mathcal{O} A^i e_{j|k} + \mathcal{O} A^{i-1} \hat{w}_{j|k} + \ldots$$

optimization guarantees that each $\hat{w}_{j|k}$ will have no unobservable components. The unobservable modes of $e_{k|k}$ are therefore the unobservable modes of $A^k(x_0 - \bar{x}_0)$ which converge to zero since A is stable. Convergence of the observable and unobservable modes implies $e_{k|k}$ converges to zero.

F.6 Proof of Lemma 4.6

From Lemma 4.3, the estimated state noise sequence $\{x_0 - \bar{x}_0, 0, \ldots, 0\}$ is feasible at every time k generating an upper bound on Ψ_k for all k, $\bar{\Psi}$.

$$\bar{\Psi} = (x_0 - \bar{x}_0)^T Q_0^{-1} (x_0 - \bar{x}_0)$$

Convergence of $\hat{w}_{k-i|k}, i = 1, \ldots, n-1$ and $\mathcal{O} e_{k-n+1|k}$ to zero follows in the same manner as Lemma 4.5. Since (C, A) is detectable, the unstable modes of the reconstruction error are not in the null space of $\mathcal{O}$ and, therefore, converge to zero. Convergence of the stable unobservable modes to zero is shown as in Lemma 4.5.

F.7 Proof of Lemma 4.7

Let $\bar{\delta}(\epsilon)$ be an $\epsilon - \delta$ relationship satisfying the stability condition for the unconstrained batch estimator. Since (A, C) is detectable and Q, R are positive definite, the unconstrained estimator is equivalent to the Kalman filter and such a $\bar{\delta}(\epsilon)$ exists. Since $\mathcal{W}$ and $\mathcal{F}$ contain a neighborhood of the origin, there exists a $\varepsilon > 0$ such that if $\|e_{0|0}\| < \varepsilon$, then $\hat{w}_{j|k} \in \mathcal{W}$ and, for stable A in which the estimated state constraints are considered, $A^j x_o - e_{j|k} \in \mathcal{F}$ for all k. An ϵ-δ relationship that demonstrates stability is

$$\delta(\epsilon) = \begin{cases} \bar{\delta}(\epsilon), & \epsilon < \varepsilon \\ \bar{\delta}(\varepsilon), & \epsilon \geq \varepsilon \end{cases}$$

F.8 Proof of Theorem 4.1

Global asymptotic stability follows from feasibility of the constraints, shown in Lemma 4.1 for stable A or Lemma 4.3 for unstable A, convergence, shown in Lemma 4.5 for stable A or Lemma 4.6 for unstable A, and stability, shown in Lemma 4.7.

F.9 Proof of Theorem 4.2

From Lemma 4.7, there exists a neighborhood of the origin, $e_{0|0} < \varepsilon$, in which the constrained batch estimator is equivalent to the Kalman filter. Exponential asymptotic stability of the batch estimator then follows from exponential asymptotic stability of the Kalman filter.

F.10 Proof of Lemma 4.8

The estimated state noise sequence $\{-\hat{z}_{k-N|k-N-1}, 0, \ldots, 0\}$ is feasible at every time k from Lemma 4.2. Let $\bar{\Psi}_k^N$ be the objective function value with this estimated state noise sequence.

$$\bar{\Psi}_k^N = \hat{z}_{k-N|k-N-1}^T P_{k-N}^{-1} \hat{z}_{k-N|k-N-1} + \sum_{j=0}^{N} x_{k-N}^T A^{T^j} C^T R^{-1} C A^j x_{k-N}$$

Since optimization will be performed at each time k, the value of Ψ_k^N can be no greater than $\bar{\Psi}_k^N$. Since the estimate $\hat{z}_{k-N|k-N-1}$ is determined from the Kalman filter and A is stable, the sequence $\{\bar{\Psi}_k^N\}$ converges to zero. Therefore

Ψ_k^N also converges to zero. Since the sequence $\{\Psi_k^N\}$ is an upper bound for $\|[\hat{w}_{k-N-1|k}, \ldots, \hat{w}_{k-1|k}]\|_R$, for every $\epsilon > 0$ there exists a $J(\epsilon)$ such that

$$\begin{aligned} \|e_{k|k}\| < & \left\| \left(\prod_{j=0}^{J(\epsilon)} (A - L_j C) \right) \right\| \|e_{0|0}\| \\ & + \| \, [A^{N-1} \; A^{N-2} \; \ldots \; I] \, \| \sqrt{\Psi_{J(\epsilon)}^N / \sigma_{\min}(R^{-1})} < \epsilon \quad \forall \quad k \geq J(\epsilon) \end{aligned}$$

which proves the lemma.

F.11 Proof of Lemma 4.9

The estimated state noise sequence $\{x_{k-N} - \hat{z}_{k-N|k-N-1}, 0, \ldots, 0\}$ is feasible at every time k from Lemma 4.4. This sequence generates the following upper bound on Ψ_k^N for all k.

$$\bar{\Psi}_k^N = (A^{k-N} x_0 - \hat{z}_{k-N|k-N-1})^T P_{k-N}^{-1} (A^{k-N} x_0 - \hat{z}_{k-N|k-N-1})$$

Stability of the Kalman filter from (C, A) detectable guarantees that $\bar{\Psi}_k^N$ converges to zero. The convergence of $\hat{w}_{k-i|k}$ to zero follows from the convergence of Ψ_k^N and implies $e_{k|k}$ converges to zero as shown in Lemma 4.8.

F.12 Proof of Theorem 4.3

Global asymptotic stability follows from feasibility of the constraints, shown in Lemma 4.2 for stable A or Lemma 4.4 for unstable A, convergence, shown in Lemma 4.8 for stable A or Lemma 4.9 for unstable A, and stability, shown in Lemma 4.10.

F.13 Proof of Lemma 4.11

It is sufficient to show that the quadratic program is a strictly convex programming problem [24]. The constraints form a convex set by construction. Feasibility of the constraints, shown in Lemma 4.2 for stable A or Lemma 4.4 for unstable A, implies that this set is nonempty. The objective function is then strictly convex if the Hessian is positive definite. The Hessian, H, is constructed as follows in which E_N is defined in Appendix B.1, Q_N is defined similarly to Q_N^{k-N} in Appendix B.1 except that the null matrix replaces

P_{k-N}^{-1} in the recursion, $\mathcal{O}$ is the observability matrix, and $\tilde{H}$ is some positive semidefinite matrix.

$$H = (E_N + Q_N) = \begin{bmatrix} \mathcal{O}^T R_n^{-1} \mathcal{O} & 0 & \ldots & 0 \\ 0 & Q^{-1} & 0 & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \ldots & 0 & Q^{-1} \end{bmatrix} + \tilde{H}$$

Since (C, A) is observable, $\mathcal{O}$ is full column rank and $\mathcal{O}^T R^{-1} \mathcal{O}$ is positive definite [37]. This implies the Hessian is positive definite and proves the lemma.

Appendix G

Proofs for Chapter 5

G.1 Proof of Lemma 5.1

From Eq. C.3 in Appendix C and Eq. 5.10 the change in the Lyapunov function is bounded as follows

$$\begin{aligned}
\Delta V(x_k) &= V(x_{k+1}) - V(x_k) \\
&= \left(Ax_k + Bu_s + B\mu(\hat{x}_{k|k})\right) - V(x_k) \\
&= V\left(z_{1|k} + B\left(\mu(x_k + e_{k|k}) - \mu(x_k)\right)\right) - V(x_k) \\
&\leq V(z_{1|k}) + M\|B\left(\mu(x_k + e_{k|k}) - \mu(x_k)\right)\| - V(x_k) \\
&\leq V(z_{1|k}) - V(x_k) + M\rho\|B\|\,\|e_{k|k}\|
\end{aligned}$$

in which $z_{1|k}$ is the state at time $k+1$ for the unperturbed closed-loop system. Shifting the origin of the closed-loop system to x_s, applying the inequality in Eq. C.6, and exponential convergence of the reconstruction error results in

$$\begin{aligned}
\Delta V(x_k - x_s) &\leq -c(x_k - x_s) + M\rho\|B\|\,\|e_{k|k}\| \\
&\leq -c(x_k - x_s) + \Gamma\lambda^k
\end{aligned}$$

in which $\|e_{k|k}\| \leq \alpha\lambda^k$ for some $\alpha \geq 0$ and $0 < \lambda < 1$. The last inequality can be summed N times resulting in

$$V(x_{k+N} - x_s) - V(x_k - x_s) \leq \sum_{j=k}^{k+N} \left(-c(x_j - x_s) + \Gamma\lambda^j\right)$$

$$\leq \Gamma \frac{\lambda}{1-\lambda} - \sum_{j=k}^{k+N} c(x_j - x_s)$$

implying

$$\sum_{j=k}^{k+N} c(x_j - x_s) \leq V(x_k - x_s) + \Gamma \frac{\lambda}{1-\lambda}$$

in which the sequence of partial sums is bounded above by a constant. Therefore, the sequence $\{c(x_k - x_s)\}$ converges to zero which implies the sequence $\{x_k - x_s\}$ converges to zero from Lemma C.1 and the sequence $\{x_k\}$ converges to x_s which proves the lemma.

G.2 Proof of Theorem 5.1

Feasibility of the equality constraint on the unstable modes in Eq. 5.4 for all time $k \geq 0$ is guaranteed by $(x_0, e_0) \in \mathcal{V}_N^n$. Nominal asymptotic stability then follows from stability, demonstrated in Halanay [35], and convergence, demonstrated in Lemma 5.1 for $N \geq 1$ and a consequence of A stable for $N = 0$.

Appendix H

Proofs for Chapter 6

H.1 Proof of Output Disturbance Model Detectability

The conditions for dectecability of the output disturbance model augmented system are demonstrated. Partitioning the Jordan form of the augmented $\tilde{A}$ matrix into the stable and unstable modes of the original A matrix and the unstable modes due to the augmented output disturbance states results in

$$\tilde{A} = VJV^{-1} = \begin{bmatrix} V_u & V_s & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} J_u & 0 & 0 \\ 0 & J_s & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \tilde{V}_u & 0 \\ \tilde{V}_s & 0 \\ 0 & I \end{bmatrix} \tag{H.1}$$

in which the unstable eigenvalues of the original A matrix are contained in J_u. The stable and unstable modes of the original system, z^s and z^u, and the unstable disturbance modes, z^p, then satisfy the following relationships.

$$\begin{bmatrix} z^u \\ z^s \\ z^p \end{bmatrix} = \begin{bmatrix} \tilde{V}_u & 0 \\ \tilde{V}_s & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} \tag{H.2}$$

$$\begin{bmatrix} z^u_{k+1} \\ z^s_{k+1} \\ z^p_{k+1} \end{bmatrix} = \begin{bmatrix} J_u & 0 & 0 \\ 0 & J_s & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} z^u_k \\ z^s_k \\ z^p_k \end{bmatrix} + \begin{bmatrix} \tilde{V}_u \\ \tilde{V}_s \\ 0 \end{bmatrix} Bu_k \tag{H.3}$$

$$y_k = \begin{bmatrix} C & G_p \end{bmatrix} \begin{bmatrix} V_u & V_s & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} z^u_k \\ z^s_k \\ z^p_k \end{bmatrix} \tag{H.4}$$

The observability matrix of the augmented system is

$$\tilde{\mathcal{O}} = \begin{bmatrix} C & G_p \\ CA & G_p \\ \vdots & \vdots \\ CA^{n+s-1} & G_p \end{bmatrix} \tag{H.5}$$

in which $n + s$ is the number of states in the augmented system.

Lemma H.1 *The augmented system in Eq. 6.1 is detectable if and only if*

$$\begin{bmatrix} \tilde{V}_u & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} = 0 \ \forall \ \left\{ \begin{bmatrix} x \\ p \end{bmatrix} \middle| \tilde{\mathcal{O}} \begin{bmatrix} x \\ p \end{bmatrix} = 0 \right\}$$

Proof: The sufficient condition is proved first. The unobservable modes of the augmented system are contained in the null space of $\tilde{\mathcal{O}}$. The stable modes of the augmented system are contained in the null space of the augmented $\tilde{V}_u$ matrix. Since every vector in the null space of $\mathcal{O}$ is also in the null space of the augmented $\tilde{V}_u$ matrix, the unobservable modes must be stable. Therefore, the augmented system is detectable. The necessary condition follows from the definition of detectability.

Lemma H.2 *If (C, A) is not detectable, then the augmented system in Eq. 6.1 is not detectable.*

Proof: If (C, A) is not detectable, there exists an $x \neq 0$ such that $\mathcal{O}x = 0$ and $\tilde{V}_u x \neq 0$ which implies

$$\tilde{\mathcal{O}} \begin{bmatrix} x \\ 0 \end{bmatrix} = 0, \qquad \begin{bmatrix} \tilde{V}_u & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} \neq 0$$

and, from Lemma H.1, the augmented system is not detectable.

Lemma H.3 *If G_p is not full column rank, then the augmented system in Eq. 6.1 is not detectable.*

Proof: If G_p is not full column rank, there exists a $p \neq 0$ such that $G_p p = 0$. This implies the existence of a nonzero vector p such that

$$\tilde{\mathcal{O}} \begin{bmatrix} 0 \\ p \end{bmatrix} = 0, \qquad \begin{bmatrix} \tilde{V}_u & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 \\ p \end{bmatrix} \neq 0$$

and, from Lemma H.1, the augmented system is not detectable.

Theorem H.1 *The augmented system in Eq. 6.1 is detectable if (A, C) detectable, G_p full column rank, and A has no eigenvalues at 1.*

Proof: It will be shown that the conditions of Lemma H.1 hold for this system. The unobservable modes of the augmented system satisfy

$$\begin{array}{lcl} Cx + G_p p & = & 0 \\ CAx + G_p p & = & 0 \\ \vdots & \vdots & \vdots \\ CA^{n+s-1}x + G_p p & = & 0 \end{array} \implies \begin{bmatrix} C \\ \vdots \\ CA^{n+s-1} \end{bmatrix} x = - \begin{bmatrix} G_p \\ \vdots \\ G_p \end{bmatrix} p$$

Subtracting successive rows and factoring $(I - A)$ results in the following.

$$\begin{array}{lcl} C(I-A)x & = & 0 \\ CA(I-A)x & = & 0 \\ \vdots & \vdots & \vdots \\ CA^{n+s-2}(I-A)x & = & 0 \end{array} \implies \mathcal{O}(I-A)x = 0$$

Since (A, C) is detectable, any vector $(I - A)x$ in the null space of $\mathcal{O}$ contains no unstable modes of A. Therefore, $(I - A)x = V_s v$ in which v is a column vector that selects the unobservable stable modes. Since A has no eigenvalues at 1, $(I - A)$ is full rank and x can be represented as follows by utilizing the Jordan form of $(I - A)$.

$$x = (I - A)^{-1} V_s v = V_s (I - J_s)^{-1} v$$

Since G_p is full column rank and

$$\begin{bmatrix} C \\ \vdots \\ CA^{n+s-1} \end{bmatrix} x = \begin{bmatrix} C \\ \vdots \\ CA^{n+s-1} \end{bmatrix} V_s (I - J_s)^{-1} v = 0 = - \begin{bmatrix} G_p \\ \vdots \\ G_p \end{bmatrix} p,$$

$p = 0$ for all unobservable modes of the augmented system. Therefore,

$$\begin{bmatrix} x \\ p \end{bmatrix} = \begin{bmatrix} V_s(I - J_s)^{-1} v \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} \tilde{V}_u & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} V_s(I - J_s)^{-1} v \\ 0 \end{bmatrix} = 0$$

which satisfies the condition of Lemma H.1 and proves the theorem.

Theorem H.2 *The augmented system in Eq. 6.1 is detectable if and only if (C, A) detectable, G_p full column rank, and $G_p p \neq C v_i$ for all $p \in \Re^s$ in which v_i are the integrating modes of A.*

Proof: The sufficient condition is proved first. Since v_i are the integrating modes of A, $v_i = A v_i$, and

$$\begin{bmatrix} C \\ \vdots \\ CA^{n+s-1} \end{bmatrix} v_i = - \begin{bmatrix} G_p \\ \vdots \\ G_p \end{bmatrix} p \implies C v_i = -G_p p \qquad \text{(H.6)}$$

Since the range of G_p does not contain Cv_i for each v_i and $Cv_i \neq 0$ from detectability of (A, C), there exists no p that satisfies this relationship. Therefore, the integrating modes of A are observable in the augmented system for all p. The proof for the remaining modes follow as shown in Theorem H.1. The necessary condition is now proved. Detectability of (C, A) follows from Lemma H.2. Full column rank of G_p follows from Lemma H.3. Lemma H.1 implies $p = 0$ for all p such that

$$\tilde{\mathcal{O}} \begin{bmatrix} x \\ p \end{bmatrix} = 0$$

Therefore, from Eq. H.6 there can exist no $p \neq 0$ such that $G_p p = Cv_i$ in which v_i are the integrating modes of A.

H.2 Proof of State Disturbance Model Detectability

Partitioning the Jordan form of the augmented $\tilde{A}$ matrix into the stable and unstable modes of the original A matrix and the unstable modes due to the augmented states results in

$$\tilde{A} = VJV^{-1} = \begin{bmatrix} V_u & V_s & V_g^g \\ 0 & 0 & V_g \end{bmatrix} \begin{bmatrix} J_u & 0 & 0 \\ 0 & J_s & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \tilde{V}_u & \tilde{V}_g^u \\ \tilde{V}_s & \tilde{V}_g^s \\ 0 & \tilde{V}_g \end{bmatrix} \tag{H.7}$$

in which the unstable eigenvalues of the original A matrix are contained in J_u. The stable and unstable modes of the original system, z^s and z^u, and the unstable disturbance modes, z^d, then satisfy the following relationships.

$$\begin{bmatrix} z^u \\ z^s \\ z^d \end{bmatrix} = \begin{bmatrix} \tilde{V}_u & \tilde{V}_g^u \\ \tilde{V}_s & \tilde{V}_g^s \\ 0 & \tilde{V}_g \end{bmatrix} \begin{bmatrix} x \\ d \end{bmatrix} \tag{H.8}$$

$$\begin{bmatrix} z_{k+1}^u \\ z_{k+1}^s \\ z_{k+1}^d \end{bmatrix} = \begin{bmatrix} J_u & 0 & 0 \\ 0 & J_s & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} z_k^u \\ z_k^s \\ z_k^d \end{bmatrix} + \begin{bmatrix} \tilde{V}_u \\ \tilde{V}_s \\ 0 \end{bmatrix} Bu_k \tag{H.9}$$

$$y_k = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} V_u & V_s & V_g^g \\ 0 & 0 & V_g \end{bmatrix} \begin{bmatrix} z_k^u \\ z_k^s \\ z_k^d \end{bmatrix} \tag{H.10}$$

The observability matrix of the augmented system is

$$\tilde{\mathcal{O}} = \begin{bmatrix} C & 0 \\ CA & CG_d \\ CA^2 & (C+CA)G_d \\ \vdots & \vdots \\ CA^{n+s-1} & (C+\ldots+CA^{n+s-2})G_d \end{bmatrix} \tag{H.11}$$

in which $n+s$ is the number of states in the augmented system.

Lemma H.4 *The augmented system in Eq. 6.2 is detectable if and only if*

$$\begin{bmatrix} \tilde{V}_u & \tilde{V}_g^u \\ 0 & \tilde{V}_g \end{bmatrix} \begin{bmatrix} x \\ d \end{bmatrix} = 0 \ \forall \ \left\{ \begin{bmatrix} x \\ d \end{bmatrix} \middle| \tilde{\mathcal{O}} \begin{bmatrix} x \\ d \end{bmatrix} = 0 \right\}$$

Proof: The proof follows in the same manner as the proof of Lemma H.1.

Lemma H.5 *If (C, A) is not detectable, then the augmented system in Eq. 6.2 is not detectable.*

Proof: The proof follows in the same manner as the proof of Lemma H.2

Lemma H.6 *If G_d is not full column rank or the range of G_d contains the nonzero null space of $\mathcal{O}$, then the augmented system in Eq. 6.2 is not detectable.*

Proof: If G_d is not full column rank, there exists a $d \neq 0$ such that $G_d d = 0$. If the range of G_d contains the non-zero null space of $\mathcal{O}$, there exists a $d \neq 0$ such that $CA^i G_d d = 0$ for $i = 0, \ldots, n-1$. This implies the existence of a nonzero vector d such that

$$\tilde{\mathcal{O}} \begin{bmatrix} 0 \\ d \end{bmatrix} = 0, \qquad \begin{bmatrix} \tilde{V}_u & \tilde{V}_g^u \\ 0 & \tilde{V}_g \end{bmatrix} \begin{bmatrix} 0 \\ d \end{bmatrix} \neq 0$$

and, from Lemma H.4, the augmented system is not detectable. Note that $\tilde{V}_g$ does not have a nonzero null space since V is invertible.

Lemma H.7 *If the column dimension G_d is greater than the row dimension of C, then the augmented system in Eq. 6.2 is not detectable.*

Proof: For all G_d in which the column dimension is greater than the row dimension of C, the following matrix is not full rank.

$$\begin{bmatrix} (I - \tilde{A}) \\ \tilde{C} \end{bmatrix} = \begin{bmatrix} (I-A) & -G_d \\ 0 & 0 \\ C & 0 \end{bmatrix}$$

The augmented system in Eq. 6.2 is then not detectable from Theorem 3.1 which proves the lemma.

Theorem H.3 *The augmented system in Eq. 6.2 is detectable if and only if (A, C) detectable, G_d full column rank, the column dimension of G_d is less than or equal to the row dimension of C, and the range of G_d does not contain the null space of $\mathcal{O}$.*

Proof: The sufficient condition is proved first. It will be shown that the conditions of Lemma H.4 hold for this system. The unobservable modes of the augmented system satisfy

$$\tilde{\mathcal{O}} \begin{bmatrix} x \\ d \end{bmatrix} = 0 \implies \begin{array}{lcl} Cx & = & 0 \\ CAx + CG_d d & = & 0 \\ \vdots & & \vdots \\ CA^{n+s-1}x + CG_d d + \ldots + CA^{n+s-2}G_d d & = & 0 \end{array}$$

Subtracting successive rows and factoring $(I - A)$ results in the following.

$$\begin{array}{lcc} C(I-A)x & = & CG_d d \\ CA(I-A)x & = & CAG_d d \\ \vdots & \vdots & \vdots \\ CA^{n+s-2}(I-A)x & = & CA^{n+s-2}G_d d \end{array} \implies \mathcal{O}(I-A)x = \mathcal{O}G_d d$$

Since G_d is full rank and its range does no contain the null space of $\mathcal{O}$, $\mathcal{O}G_d d = 0$ only if $d = 0$. Since (A, C) is detectable, any vector $(I - A)x$ in the null space of $\mathcal{O}$ contains no unstable modes of A. Therefore, $(I - A)x = V_s v$ in which v is a column vector that selects the unobservable stable modes which implies

$$\begin{bmatrix} \tilde{V}_u & \tilde{V}_g^u \\ 0 & \tilde{V}_g \end{bmatrix} \begin{bmatrix} x \\ d \end{bmatrix} = 0$$

for all $(I-A)x$ in the null space of $\mathcal{O}$ which satisfies the condition of Lemma H.4. The necessary condition follows directly from Lemmas H.5, H.6, and H.7.

H.3 Proof of Lemma 6.1

The proof is by contradiction. Since $s_d + s_p = p$, L_2 is square and can be represented by its Jordan form $L_2 = VJV^{-1}$ in which J is upper triangular. Assuming L_2 is not full rank implies that there is an eigenvalue of J equal to zero. Order the eigenvalues of J such that the zero eigenvalue is on the bottom of the diagonal. Stability of the estimator implies the following matrix is stable.

$$\begin{bmatrix} A & [G_d \; 0] \\ 0 & I \end{bmatrix} - \begin{bmatrix} L_x \\ L_2 \end{bmatrix} [\; C \quad [0 \; G_p] \;] = \begin{bmatrix} A - L_x C & G_1 - L_x G_2 \\ -L_2 C & I - L_2 G_2 \end{bmatrix}$$

The eigenvalues of this matrix can be determined from the following similar matrix

$$\begin{bmatrix} A - L_x C & G_1 V - L_x G_2 V \\ -JV^{-1}C & I - JV^{-1}G_2 V \end{bmatrix}$$

which can be transformed into the previous matrix by the block diagonal transformation matrix consisting of I and V on the diagonal. Since J is upper triangular and the zero eigenvalue is on the bottom of the diagonal, the bottom row of J is zero. The bottom row of $[-JV^{-1}C \; I–JV^{-1}G_2V]$ is then $[0 \; \ldots \; 0 \; 1]$ which implies the estimator has an eigenvalue of 1 and is not stable. Therefore, L_2 must be full rank.

H.4 Proof of Target Tracking Quadratic Program Feasibility

H.4.1 Unconstrained

Lemma H.8 *For (A, B) stabilizable, the target tracking quadratic program with objective Eq. 6.11 and equality constraint Eq. 6.12 has a feasible solution.*

Proof: It is sufficient to show that the matrix $[(I - A) \;\; -B]$ is full rank implying that the column space spans all of $\Re^n$. Assuming that $[(I - A) \; -B]$ is not full rank implies that there exists an $\alpha \neq 0$ such that

$$\alpha^T[(I - A) \;\; -B] = 0 \Rightarrow \alpha^T(I - A) = 0 \Rightarrow \alpha^T A = \alpha^T$$

Therefore, $\mathcal{C}^T\alpha = 0$ in which $\mathcal{C}$ is the controllability matrix [10] which implies α is an uncontrollable mode of the system. Using the Jordan form of A in Eq. 2.2 in the previous expression results in

$$\alpha^T V J V^{-1} = \alpha^T \Rightarrow \alpha^T V(J - I) = 0$$

Since V is full rank, $\alpha^T V = 0$ implies $\alpha = 0$ which contradicts the assumption $\alpha \neq 0$. Therefore, $\alpha^T V(J - I) = 0$ implies there must exist a $\alpha^T v_i \neq 0$ for some i in which $\lambda_i = 1$. Since each stable mode is orthogonal to $\{v_i|\lambda_i = 1\}$, α is an unstable mode which contradicts (A, B) stabilizable. Therefore, $[(I - A) \; -B]$ is full rank which proves the lemma.

H.4.2 Stable Process Model

Lemma H.9 *The target tracking quadratic program with objective Eq. 6.11 and constraints Eqs. 6.12 and 6.13 has a feasible solution for stable A.*

Proof: Since $u_s = 0$ is a feasible solution to the constraint in Eq. 6.13 and A is stable, the steady-state state and input computed in Eq. 6.16 satisfies the equality constraint in Eq. 6.12 which proves the lemma.

H.4.3 Unstable Process Model

Lemma H.10 *The target tracking quadratic program with objective Eq. 6.11 and constraints Eqs. 6.12 and 6.13 has a feasible solution for unstable A for all* $(x_0,\, e_{0|0}) \in \mathcal{V}_N^n$

Proof: The result follows directly from the definition of $\mathcal{V}_N^n$ in Eq. 6.17.

H.5 Proof of Theorem 6.3

The proof provided in this appendix considers the combined constant disturbance model in Eq. 6.4 with both state and output disturbances. Either disturbance may be removed without changing the result. The estimator for the augmented system in Eq. 6.4 produces the following steady-state estimates

$$\begin{aligned} \hat{x}_s &= A\hat{x}_s + Bv_s + G_d\hat{d}_s + L_x(y_s - C\hat{x}_s - G_p\hat{p}_s) \\ \begin{bmatrix} \hat{d}_s \\ \hat{p}_s \end{bmatrix} &= \begin{bmatrix} \hat{d}_s \\ \hat{p}_s \end{bmatrix} + L_2(y_s - C\hat{x}_s - G_p\hat{p}_s), \qquad L_2 = \begin{bmatrix} L_d \\ L_p \end{bmatrix} \end{aligned}$$

in which $\hat{x}_s$, $\hat{d}_s$, and $\hat{p}_s$ are the steady-state estimates of the state, constant state disturbance, and constant output disturbance, respectively, v_s is the steady-state input computed by the regulator, and y_s is the steady-state measured process output. For the optimization based estimators presented in Chapter 4, the same expression is obtained since it is assumed that no constraints are active and the unconstrained estimator is equivalent to the Kalman filter. Since the augmented system is detectable, L_2 is full rank from Lemma 6.1. Full rank of L_2 implies

$$y_s = C\hat{x}_s + G_p\hat{p}_s$$

from the preceding expression. The steady-state state estimate and controlled variable,y_s^c, are then determined as follows.

$$\hat{x}_s = A\hat{x}_s + Bv_s + G_d\hat{d}_s \tag{H.12}$$
$$y_s^c = \acute{C}_x\hat{x}_s + \acute{C}_d\hat{d}_s + \acute{C}_p\hat{p}_s \tag{H.13}$$

The equality constraint in Eq. 6.12 results in the following relationships in which x_s and u_s are the steady-state state and input vectors determined from the solution of the target tracking quadratic program and y_a^c is the achievable controlled variable target at steady state.

$$x_s = Ax_s + Bu_s + G_d d_s \tag{H.14}$$
$$y_a^c = \acute{C}_x x_s + \acute{C}_d \hat{d}_s + \acute{C}_p \hat{p}_s \tag{H.15}$$

Full rank of $\acute{\mathcal{O}}_x$ in Eq. 6.10 ensures that x_s and u_s are unique. Since no constraints are active and the input reaches a steady-state value, the input v_s is determined from the unconstrained solution to the receding horizon regulator which results in a state feedback regulator.

$$v_s = -K(\hat{x}_s - x_s) + u_s \tag{H.16}$$

Subtracting Eq. H.14 from Eq. H.12 results in

$$\hat{x}_s - x_s = A(\hat{x}_s - x_s) + B(v_s - u_s)$$

in which the control law in Eq. H.16 can be substituted to produce

$$(A - BK - I)(\hat{x}_s - x_s) = 0$$

in which the only solution to this equation is $\hat{x}_s - x_s = 0$ since $(A - BK)$ is stable from nominal stability of the receding horizon regulator. Subtracting Eq. H.15 from Eq. H.13 results in

$$y_s^c - y_a^c = \acute{C}_x(\hat{x}_s - x_s) = 0$$

which implies $y_s^c - y_a^c$ and proves the theorem.

H.6 Proof of Theorem 6.4

The proof provided in this appendix considers the combined constant disturbance model in Eq. 6.4 with both state and output disturbances. Either disturbance may be removed without changing the result. Full rank of the matrix in condition iii) implies the following linear system has a solution for all y_t^c, $\hat{d}_{k|k}$, and $\hat{p}_{k|k}$.

$$\begin{bmatrix} I - A & -B \\ \acute{C}_x & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} -G_d \hat{d}_{k|k} \\ y_t^c - \acute{C}_d \hat{d}_{k|k} - \acute{C}_p \hat{p}_{k|k} \end{bmatrix}$$

Since no constraints are active in the target tracking quadratic program, any solution to the preceding linear system is feasible. Theorem 6.3 ensures that the output feedback receding horizon regulator achieves this target in the closed loop at steady state which proves the theorem.

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