NAME: ______________________________

Instructions: This exam is a 24 hour take home exam. Do not work with anyone else. The solution is due at Tuesday morning's class meeting. Print out and hand in any m-files as well as the solution. Also email any m-files to `rawlings@engr.wisc.edu`

Problem 1. Computing controllability canonical form. 33 1/3 points.

Good morning. One of your colleagues, let's call him Fernando, has suggested the following method for computing the controllability canonical form. First form the controllability matrix

$$\mathcal{C} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

Then compute its so-called QR decomposition

$$\mathcal{C} = QR$$

in which Q is an orthogonal $n \times n$ matrix, R is an upper triangular $n \times nm$ matrix, and $\text{rank}(\mathcal{C}) = \text{rank}(R)$ since Q is full rank. If $\text{rank}(\mathcal{C}) = r < n$, R has the form

$$R = \begin{bmatrix} R_1 \\ 0 \end{bmatrix}$$

in which R_1 is a full rank upper triangular $r \times nm$ matrix. If $\text{rank}(\mathcal{C}) = n$, then the zero block is not present in R. The proposed controllability canonical form is then

$$\widetilde{A} = Q'AQ \qquad \widetilde{B} = Q'B \qquad z = Q'x$$

with

$$z^+ = \widetilde{A}z + \widetilde{B}u$$

$$\widetilde{A} = \begin{bmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ 0 & \widetilde{A}_{22} \end{bmatrix} \qquad \widetilde{B} = \begin{bmatrix} \widetilde{B}_1 \\ 0 \end{bmatrix}$$

(a) What do you think of Fernando's proposition? If it is true, show that $\widetilde{A}$ and $\widetilde{B}$ defined above are in controllability canonical form. If it is false, give a counterexample. Assume the controllability matrix has rank $r < n$.

(b) Write out the corresponding proposition for computing the observability canonical form.

Problem 2. Which concentration do you measure? 33 1/3 points.

One of your reaction engineering colleagues, let's call him Pratik, is modeling the following reaction taking place in an isothermal CSTR with residence time θ

$$\mathrm{A} \xrightarrow{k_1} \mathrm{B} \xrightarrow{k_2} \mathrm{C}$$

(a) Pratik is kind of stingy with his research budget, so he decides to measure only c_A. Is the resulting system observable?

(b) Repeat for the choice of measuring only c_C.

(c) Provide a physical explanation of why these answers are different. Please realize that a statement about the rank of the observability matrix is not a *physical* explanation.

(d) Write a code to implement Fernando's proposition using only the octave function `qr`. Apply it to the two (A, C) matrix pairs given above. Values for the parameters are

$$\theta = 10\,\mathrm{min} \qquad k_1 = 0.1\,\mathrm{min}^{-1} \qquad k_2 = 0.2\,\mathrm{min}^{-1}$$

Do you obtain valid observability canonical forms for these two cases. Is the result consistent with your previous answers about the observability of these two systems?

Problem 3. Upper bounding the terminal penalty in the unconstrained LQR problem. 33 1/3 points.

Good afternoon. I hope you enjoyed the first two problems. Another one of your colleagues, let's call him Rishi, has come up with the following idea. He notices that terminating the state at the origin with terminal constraint $x_N = 0$ gives a stabilizing regulator, and that penalizing the terminal state with the infinite horizon cost to go, $x_N' \Pi x_N$, also gives a stabilizing regulator. So he conjectures that *any* terminal penalty P_f between these limits, $\Pi \leq P_f \leq \infty$ should do the trick as well and give closed-loop stability.

(a) What do you think of Rishi's conjecture? If it's correct, give a proof. If it's incorrect, give a counterexample.

(b) Can you think of a penalty in the LQR problem that captures these two limiting cases: infinite horizon cost to go ($x_N' \Pi x_N$), and terminal constraint ($x_N = 0$), such that any value of this penalty between these limits is stabilizing? What penalty in the LQR problem can you use in this way to generate a family of stabilizing regulators?