Dynamical Systems with Symmetry - ME225DS
Winter 2008

Homework #4 - Due Thursday, February 7, in class

1. Suppose that the vector field

$$\begin{aligned}
\dot{x} &= a_1 x + a_2 y + b_1 x^2 + b_2 xy + b_3 y^2 + c_1 x^3 + c_2 x^2 y + c_3 xy^2 + c_4 y^3 \equiv f(x,y) \\
\dot{y} &= A_1 x + A_2 y + B_1 x^2 + B_2 xy + B_3 y^2 + C_1 x^3 + C_2 x^2 y + C_3 xy^2 + C_4 y^3 \equiv g(x,y),
\end{aligned}$$

where $(x,y) \in \mathbb{R}^2$, is equivariant with respect to the operations

$$\kappa_1 : (x,y) \to (y,x), \qquad \kappa_2 : (x,y) \to (x,-y).$$

(a) (10 pts) What does this imply about the relationships between the coefficients, i.e., the a's, b's, c's, A's, B's, and C's?

(b) (10 pts) Using what you found in (a), find all fixed points of the vector field, the conditions under which they exist in terms of the coefficients, and their isotropy subgroups.

(c) (10 pts) Find the fixed point subspaces corresponding to the isotropy subgroups found in (b). Verify explicitly that the fixed point subspaces are invariant under the flow of the vector field.

2. (10 pts) As described in lecture, in the Boussinesq approximation the nondimensional evolution equations for the fluid velocity (written in terms of the stream function ψ)

$$\mathbf{u} \equiv -\frac{\partial \psi}{\partial z}\hat{\mathbf{x}} + \frac{\partial \psi}{\partial x}\hat{\mathbf{z}},$$

and the perturbation θ to the conduction state temperature profile for Rayleigh-Bénard convection are

$$\frac{\partial\, \nabla^2\psi}{\partial t} + \frac{\partial \psi}{\partial x}\frac{\partial \nabla^2\psi}{\partial z} - \frac{\partial \psi}{\partial z}\frac{\partial \nabla^2\psi}{\partial x} = RP\frac{\partial \theta}{\partial x} + P\nabla^4\psi, \tag{1}$$

$$\frac{\partial \theta}{\partial t} + \frac{\partial \psi}{\partial x}\frac{\partial \theta}{\partial z} - \frac{\partial \psi}{\partial z}\frac{\partial \theta}{\partial x} = \frac{\partial \psi}{\partial x} + \nabla^2\theta. \tag{2}$$

Here R and P are nondimensional parameters known as the Rayleigh number (which is proportional to ΔT) and the Prandtl number, respectively. Suppose that the temperature at the top and bottom plates is held constant, and there is no tangential stress, corresponding to the boundary conditions

$$\psi = \frac{\partial^2 \psi}{\partial z^2} = \theta = 0, \qquad z = 0, 1.$$

Furthermore, suppose that in the horizontal direction there are no-slip and perfectly insulating lateral boundary conditions:

$$\psi = \frac{\partial \psi}{\partial x} = \frac{\partial \theta}{\partial x} = 0, \qquad x = \pm L.$$

Show that these evolution equations and boundary conditions have reflection symmetry about $x = 0$; that is, show that if $(\psi(x, z, t), \theta(x, z, t))$ is a solution, then so is $(-\psi(-x, z, t), \theta(-x, z, t))$. Please explain the various steps in your argument.