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CHARGE-BALANCED OPTIMAL INPUTS FOR PHASE MODELS
OF SPIKING NEURONS

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ABSTRACT

The optimal input current for a reduced neuron model and a specific target spiking time is obtained. The objective of optimization is to minimize the total input energy to the system subject to a zero net input integral over the time horizon. This "charge-balance constraint" ensures that no net external charge is injected into the neuron. The results are compared to optimal currents for which the charge-balance constraint is not imposed.

MODEL EQUATIONS

Phase-reduced models of neurons have been used to investigate the patterns of synchrony that result from the type and architecture of coupling [1–7], and the response of large groups of oscillators to external stimuli [8–10]. More recently, phase models have been investigated in the context of controlling a neuron to behave in a desired way [11–13]. In the present paper, we consider a set-up similar to that considered in [11] - determining the optimal control to make a neuron fire at a specified time - but we add a "charge-balance constraint" that the net input current over one cycle should be zero. The importance of the charge-balance constraint lies in the fact that it is desirable to not disturb the internal electrical balance when such control is applied to a population of neurons in the brain.

In particular, we consider the phase model for a spiking (i.e., firing) neuron [8, 14]:

$$\frac{d\theta}{dt} = f(\theta) + Z(\theta)I(t). \qquad (1)$$

Here, $f(\theta)$ represents the neuron's baseline dynamics, $Z(\theta)$ is called the Phase Response Curve (PRC) of the neuron and characterizes the phase shift of the neuron due to small impulsive stimuli, and $I(t)$ is the input stimulus in the form of an electrical current. $\theta(t) \in [0, 2\pi)$ is the neuron's phase, and by convention $\theta = 0$ corresponds to the spiking of the neuron. In the following, we will take $f(\theta) = \omega$, where ω is a constant.

Without loss of generality, we assume that the neuron fires at $t = 0$. In the absence of input $I(t)$, the neuron would then fire at time $T \equiv \frac{2\pi}{\omega}$. Our objective is to find the optimal input current so that the neuron instead fires at our desired time t_1, so that

$$\theta(0) = 0, \qquad \theta(t_1) = 2\pi. \qquad (2)$$

By "optimal" it is meant that $I(t)$ minimizes the cost function

$$G[I(t)] = \int_0^{t_1} [I(t)]^2 dt \qquad (3)$$

as well as satisfying the charge-balance constraint

$$\int_0^{t_1} I(t)dt = 0. \qquad (4)$$

The former criterion dictates minimum input energy, whereas the latter enforces charge-balance over the time interval in which the current is applied. The charge-balance constraint can be restated as follows. Letting $\dot{y} = I(t)$ and integrating both sides from 0 to t_1 we obtain: $y(t_1) - y(0) = \int_0^{t_1} I(\tau)d\tau$. For the charge-balance constraint to hold we need the right hand side of this equation to

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be zero. This means $y(t_1) = y(0)$, and assuming that the current is being applied from time $t = 0$, so $y(0) = 0$, we have

$$y(t_1) = y(0) = 0. \tag{5}$$

We find the optimal $I(t)$ by applying calculus of variations to minimize [15]:

$$C[I(t)] = \int_0^{t_1} \left\{ [I(t)]^2 + \lambda_1(t)\big(\dot\theta(t) - f(\theta) - Z(\theta)I(t)\big) \right.$$
$$\left. + \lambda_2(t)\big(\dot y(t) - I(t)\big) \right\} dt, \tag{6}$$

where the integrand is $P[\Phi(t), \dot\Phi(t), I(t)]$, with $\Phi(t) = [\theta(t), y(t), \lambda_1(t), \lambda_2(t)]^T$ and $\lambda_1(t)$ and $\lambda_2(t)$ are Lagrange multipliers. The Euler-Lagrange equations corresponding to this optimization problem provide us with a set of dynamically coupled equations that will in turn lead us to the desired $I(t)$. Computing $\frac{\partial P}{\partial I} = \frac{d}{dt}\left(\frac{\partial P}{\partial \dot I}\right)$ and $\frac{\partial P}{\partial \Phi} = \frac{d}{dt}\left(\frac{\partial P}{\partial \dot\Phi}\right)$, they are found to be:

$$I(t) = \frac{\lambda_1(t)Z(\theta) + \lambda_2(t)}{2}, \tag{7}$$

$$\dot\lambda_1(t) = -\lambda_1(t)f'(\theta) - \frac{[\lambda_1(t)]^2 Z(\theta)Z'(\theta) + \lambda_1(t)\lambda_2(t)Z'(\theta)}{2}, \tag{8}$$

$$\dot\lambda_2(t) = 0, \tag{9}$$

$$\dot\theta(t) = f(\theta) + \frac{\lambda_1(t)[Z(\theta)]^2 + \lambda_2(t)Z(\theta)}{2}, \tag{10}$$

$$\dot y(t) = I(t) = \frac{\lambda_1(t)Z(\theta) + \lambda_2(t)}{2}, \tag{11}$$

where in obtaining (10) and (11), (7) has been used. Also, in these equations prime represents differentiation with respect to θ.

The first equation of this set is an algebraic equation for $I(t)$, whereas the other equations are ordinary differential equations that need to be solved in order for the first equation to be evaluated. This is a two point boundary value problem where the boundary values for $\theta(t)$ and $y(t)$ are given in (2,5). The initial values for $\lambda_1(t)$ and $\lambda_2(t)$ are determined using a shooting method with the condition that the boundary values for $\theta(t)$ and $y(t)$ are achieved.

Before giving results, we note some properties of the Euler-Lagrange equations. First, (9) implies that λ_2 will remain constant along trajectories. Furthermore, the (θ, λ_1) subsystem is decoupled from the y dynamics. Finally, the Hamiltonian

$$H(\theta, \lambda_1) = \lambda_1(t)f(\theta) + \frac{[\lambda_1(t)]^2 [Z(\theta)]^2}{4} + \frac{\lambda_1(t)\lambda_2 Z(\theta)}{2}, \tag{12}$$

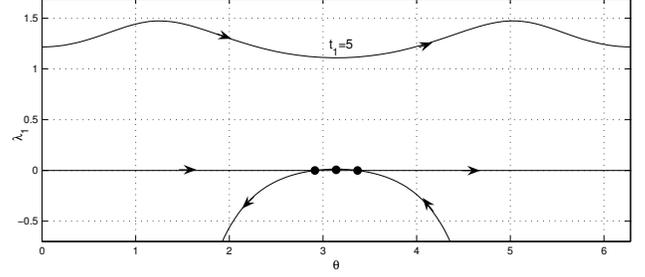is conserved along the trajectories of the Euler-Lagrange equations.

Figure 1. Phase portrait for the SNIPER neuron model for $t_1 = 5$ with the charge-balance constraint. Here and in Figure 2 the dots represent fixed points of the $(8,10)$ subsystem, and the stable and unstable manifolds of the fixed points are shown.

RESULTS

There are four different codimension-one bifurcations which can lead to periodic firing of neurons [8]. Here we focus on the SNIPER (saddle node on a periodic orbit) bifurcation, which is typical for Type I neurons [16]. For a periodically firing neuron near a SNIPER bifurcation, $Z(\theta) = c(1 - \cos(\theta))$, where c is a model-dependent constant [8,16]. For simplicity, we take $c = 1$, although similar results would be found for other choices. Furthermore, we take $\omega = 1$ so that the natural period of the neuron is $T = 2\pi$.

The phase portraits for (8,10) with $t_1 = 5$ and $t_1 = 9$ are shown in Figures 1 and 2, respectively. These figures also show the fixed points which exist for the (8,10) subsytem, along with their associated stable and unstable manifolds. As in [11], these manifolds can sometimes be used to interpret the trajectories associated with the optimal current: in particular, the $t_1 = 9$ trajectory of the Euler-Lagrange equations is close to the stable and unstable manifolds of the fixed point. This forces the trajectory to spend a long time near the fixed point, delaying its arrival to $\theta = 2\pi$. We note that Figures 1 and 2 are for different values of λ_2, the Lagrange multiplier associated with the charge-balance constraint. Each of the $t_1 = 5$ and $t_1 = 9$ trajectories shown in Figures 1 and 2 are the only trajectories in their respective phase planes that can take the (7)-(11) system from $\theta(0) = 0$ to $\theta(t_1) = 2\pi$ with zero charge balance and the least amount of input energy. Without the charge-balance constraint, as in [11], we get the same phase portrait and different trajectories, with different $\lambda_1(0)$'s, can be drawn in the same phase plane.

A comparison of the optimal current with and without the charge-balance constraint is shown in Figure 3. Clearly for this model, the constraint has a large effect on the form of the optimal current.

DISCUSSION

For the system considered in this paper in which the neuron is close to a SNIPER bifurcation, the charge-balance constraint has a very large effect. Indeed, when there is no such constraint, the optimal current is always either positive (resp.,
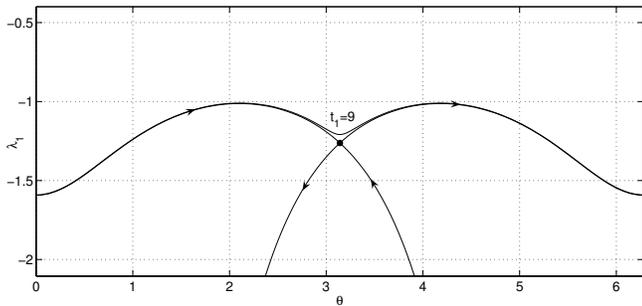
Figure 2. Phase portrait for the SNIPER neuron model for $t_1 = 9$ with the charge-balance constraint. The trajectory is very close to the stable and unstable manifolds of the fixed point of the (8,10) subsystem.
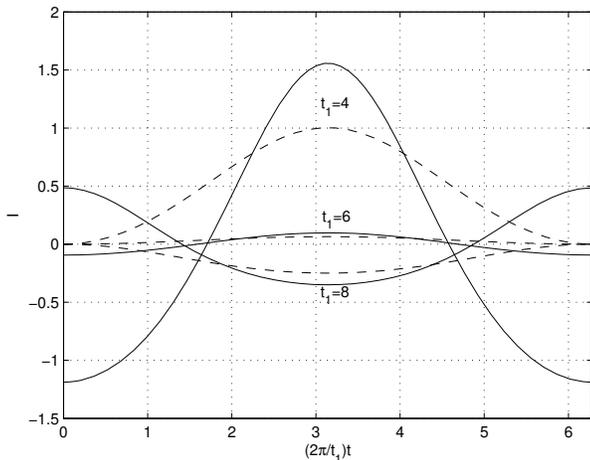

Figure 3. SNIPER neuron model optimal current for three different values of t_1 with (solid lines) and without (dashed lines) the charge-balance constraint.

negative) when we want the neuron to fire earlier (resp., later) than it would in the absence of external input.

Similar techniques can be applied to other neuron models. Preliminary results indicate that the effect of the constraint on controlling a Hodgkin-Huxley neuron (with standard parameters) is less but still noticeable. One can envision other constraints that could be included for this problem, for example that $|I(t)| \leq \bar{I}$, which could be imposed either with or without the charge-balance constraint. We defer the discussion of the control of other types of neurons and of other constraints to future work.

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