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Optimal switching between collective motion states for two agents

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ABSTRACT

We show that steering control can be chosen to give bistability between parallel and anti-parallel collective motion states for a continuous-time kinetic model of two agents moving in the plane with unit speed. Variational methods are used to determine the optimal input to the steering control of one of the agents which leads to switching between these collective states. For any given time interval of switching, such an optimal input is shown to exist and to be unique. The properties of optimal inputs are interpreted by considering the phase space geometry of the Euler–Lagrange equations associated with the optimization.

1. Introduction

There has been much recent interest in finding interaction rules which allow a population of autonomous agents to robustly operate in a particular collective motion state, perhaps with or without centralized coordination; see, e.g. [1–8]. We consider the model which was recently presented in [4,9,7] in which each agent is modeled as a point particle travelling at unit speed and interacts with other agents through steering control u_k:

$$\dot{r}_k = e^{i\theta_k}, \quad \dot{\theta}_k = u_k(\mathbf{r}, \boldsymbol{\theta}), \quad k = 1, 2, \ldots N. \tag{1}$$

Here, $r_k = x_k + iy_k$ gives the position of agent k in the (x, y) plane and the phase angle θ_k gives its orientation relative to the x axis. In [9], different control laws were presented for stabilizing and switching between different collective motion patterns, including rectilinear motion of all agents in the same or different directions, and circular formations with agents at the same location or spread evenly around the circle. Steering control was split into spacing and orientation terms $u_k = u_k^{spac}(\mathbf{r}, \boldsymbol{\theta}) + u_k^{ori}(\boldsymbol{\theta})$, with the latter benefitting from the well-developed theory of coupled oscillators [10,11]. In [12] it was demonstrated for $N = 2$ agents that steering control can be chosen to stabilize both rectilinear and circular collective motions at the same control parameter values. This arises due to the interplay between the spacing and orientation components of the steering control law. We note that bistability may also be important for the collective motion of natural animal groups [13,14].

In this work, for simplicity, we focus on the orientation control aspect of the problem. Specifically, we show that a different kind of bistability, in which both parallel and anti-parallel motions are stable, can be achieved for (1) with $N = 2$ and $u_k = u_k^{ori}(\boldsymbol{\theta})$. Furthermore, we show how the steering control of one of the agents can be modified to optimally switch between collective states. Existence and uniqueness of the optimal input is proven and a transition time symmetry between switching problems identified.

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2. Optimal switching

Consider two agents with steering control laws given by

$$\dot{\theta}_1 = \omega + Kf(\theta_2 - \theta_1) + I(t) \equiv u_1, \qquad \dot{\theta}_2 = \omega + Kf(\theta_1 - \theta_2) \equiv u_2^0. \tag{2}$$

Borrowing the terminology of coupled oscillators, we refer to ω as the natural oscillator frequency, K as the coupling strength, and f as the coupling function, which is 2π-periodic. The input $I(t)$ will be used to switch between collective motion states. Letting $\psi = \theta_1 - \theta_2$,

$$\dot{\psi} = K(f(-\psi) - f(\psi)) \equiv Kg(\psi) + I(t). \tag{3}$$

Note that $g(\psi)$ is an odd function since $g(-\psi) = f(\psi) - f(-\psi) = -g(\psi)$.

For $I(t) = 0$, there are phase-locked solutions for which $\theta_1 - \theta_2$ remains constant for all time, corresponding to fixed points ψ_p of (3). The asymptotic stability of a phase-locked solution is determined as follows: if $Kg'(\psi_p) < 0$ then it is stable, and if $Kg'(\psi_p) > 0$ then it is unstable. Unless otherwise stated, in the following we take $K > 0$. For any coupling function f one finds that $g(0) = f(0) - f(0) = 0$ and $g(\pi) = f(-\pi) - f(\pi) = f(\pi) - f(\pi) = 0$. Thus $\psi = 0$ and $\psi = \pi$ are always fixed points of (3). The solution $\psi = 0$ corresponds to the two agents always having the same instantaneous orientations, and following the notation of [10,11] will be referred to as the S_2 state because such solutions are invariant under the permutation symmetry $(\theta_1, \theta_2) \to (\theta_2, \theta_1)$. The solution $\psi = \pi$ corresponds to the agents always having orientations which differ by π radians, and will be called the Z_2 state because such solutions are invariant under the symmetry $(\theta_1, \theta_2) \to (\theta_2 + \pi, \theta_1 + \pi)$. Other fixed points of (3), corresponding to phase-locked solutions with the instantaneous orientation of the agents being an angle not equal to 0 or π, are also possible, but are not guaranteed to exist for all coupling functions [10,11]. Such solutions are called $S_1 \times S_1$ states, and are not invariant under any nontrivial symmetries. If $\dot{\theta}_1 = \dot{\theta}_2 = 0$ for a phase-locked state, the agents move in straight lines; otherwise, they move in circles.

We will show how to find the input $I(t)$ which takes the system from a stable S_2 state at $t = 0$ to a stable Z_2 state at a specified time $t = t_1$ and minimizes the L^2 norm of the input

$$G[I(t)] \equiv \int_0^{t_1} [I(t)]^2 \mathrm{d}t. \tag{4}$$

The L^2 norm has been suggested in a slightly different context [4] as an appropriate measure for steering "energy" (defined as the L^2 norm of steering control) which is an important quantity to minimize when designing steering laws for uncrewed aerial vehicle applications. This norm has the desirable property that smaller inputs are considered to be better. A similar approach, however, could be taken for other appropriate cost functions of the input $I(t)$. We apply the calculus of variations to minimize [15]

$$C[I(t)] = \int_0^{t_1} \underbrace{\left\{ [I(t)]^2 + \lambda \left(\frac{\mathrm{d}\psi}{\mathrm{d}t} - Kg(\psi) - I(t) \right) \right\}}_{P[I(t)]} \mathrm{d}t, \tag{5}$$

with λ being the Lagrange multiplier associated with requiring that the dynamics satisfy (3). The associated Euler–Lagrange equations are

$$\frac{\partial P}{\partial I} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial P}{\partial \dot{I}}\right), \qquad \frac{\partial P}{\partial \lambda} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial P}{\partial \dot{\lambda}}\right), \qquad \frac{\partial P}{\partial \psi} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial P}{\partial \dot{\psi}}\right),$$

giving $I(t) = \lambda(t)/2$, and

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = Kg(\psi) + I(t) = Kg(\psi) + \lambda/2, \qquad \frac{\mathrm{d}\lambda}{\mathrm{d}t} = -K\lambda g'(\psi). \tag{6}$$

To find the optimal $I(t)$ for switching from the S_2 state to the Z_2 state, (6) has to be solved subject to the conditions $\psi(0) = 0$, $\psi(t_1) = \pi$. This requires that we find the appropriate initial condition $\lambda(0) \equiv \lambda_0$, which can be done numerically using the shooting method. The solution $(\psi(t), \lambda(t))$ using this initial condition can then be used to give the optimal input $I(t)$.

We start by giving some useful general results. First, (6) has the symmetry property that if $(\psi(t), \lambda(t))$ is a solution, then so is $(-\psi(t), -\lambda(t))$. This follows from the fact that $g(-\psi) = -g(\psi)$. Since such trajectories are related by symmetry, we associate them with each other below. Second, the Hamiltonian function

$$H(\psi, \lambda) = K\lambda g(\psi) + \lambda^2/4 \tag{7}$$

is conserved on trajectories for the Euler–Lagrange equation (6), as can be readily verified. This Hamiltonian was obtained using the Legendre transformation [16]. We are interested in trajectories with $\psi = 0$ at $t = 0$. Since $g(0) = 0$, the relationship between λ_0 and the initial value of the Hamiltonian, H_0, is $H_0 = \lambda_0^2/4$. The fact that there are two possible values of λ_0 for a given value of H_0 follows from the reflection symmetry mentioned above. In the following, without loss of generality we consider solutions that always have $\lambda_0 > 0$. Furthermore, Eq. (6) have two classes of fixed points (ψ_p, λ_p). Those in the first class satisfy $\lambda_p = 0$ and $g(\psi_p) = 0$, and since the eigenvalues of the Jacobian evaluated at these fixed points are $\pm Kg'(\psi_p)$ they are saddles. From (7), the stable and unstable manifolds of these fixed points lie on the curves $\lambda = 0$ and $\lambda/4 + Kg(\psi) = 0$. The other class of fixed points satisfy $g'(\psi_p) = 0$ and $Kg(\psi_p) + \lambda_p/2 = 0$. The eigenvalues of the Jacobian evaluated at these fixed points are $\pm K\sqrt{g(\psi_p)g''(\psi_p)}$, so that when $g(\psi_p)$ and $g''(\psi_p)$ have opposite signs, they are centers.

The techniques from [17] can be modified to show the existence and uniqueness (modulo symmetries) of an optimal $I(t)$ for *any* positive t_1. Consider the trajectory of (6) which goes from $\psi = 0$ to $\psi = \pi$ in the time t_1 with $\lambda_0 > 0$.

Lemma 1. *$\mathrm{d}\psi/\mathrm{d}t$ is always strictly positive for trajectories of (6) with $\psi(0) = 0$, $\psi(t_1) = \pi$, and $\lambda_0 > 0$.*

Proof. Consider a trajectory (ψ, λ) for $0 \le t \le t_1$ which solves (6) with $\psi(0) = 0$, $\psi(t_1) = \pi$, and $\lambda_0 > 0$. It follows from (6) that $\mathrm{d}\psi/\mathrm{d}t = \lambda_0/2 > 0$ at $t = 0$. Suppose by contradiction that $\mathrm{d}\psi/\mathrm{d}t < 0$ for some time $\bar{t}$ with $0 < \bar{t} < t_1$. Since any trajectory in the phase plane cannot be self-intersecting, there must be a value of ψ for which there are three different values for λ. However, the trajectory must be a level set of H, which is quadratic in λ and hence can only have at most two different values for a given ψ. This is a contradiction and so the lemma follows. □

Proposition 1. *There exists a unique optimal input $I(t)$ for any positive t_1 (modulo symmetry-related solutions).*

Proof. It follows from (7) that trajectories satisfy $\lambda^2/4 + Kg(\psi)\lambda - H_0 = 0$. Solving for λ, we find that $\lambda = 2\left[-Kg(\psi) + \sqrt{[Kg(\psi)]^2 + H_0}\right]$. Here we take the '+' solution since from Lemma 1, $\mathrm{d}\psi/\mathrm{d}t = \lambda/2 + Kg(\psi) > 0$. Then, the time of transition t_1 from the S_2 to the Z_2 state may be written as

$$t_1 = \int_0^{t_1} \mathrm{d}t = \int_0^{\pi} \frac{\mathrm{d}\psi}{Kg(\psi) + \frac{\lambda}{2}} = \int_0^{\pi} \frac{\mathrm{d}\psi}{\sqrt{[Kg(\psi)]^2 + H_0}} \Rightarrow \frac{\partial t_1}{\partial H_0} = -\frac{1}{2}\int_0^{\pi} \frac{\mathrm{d}\psi}{([Kg(\psi)]^2 + H_0)^{3/2}} < 0. \quad (8)$$

Thus t_1 decreases monotonically as H_0 increases. Recalling that $H_0 = \lambda_0^2/4$, we see that t_1 decreases monotonically with $\lambda_0 > 0$. Choosing λ_0 large gives a solution with arbitrarily small t_1, and choosing λ_0 so that the denominator of (8) becomes arbitrarily small gives a solution with arbitrarily large t_1. □

We have shown the existence and uniqueness of an optimal input $I(t)$ for switching from the S_2 state to the Z_2 state in time t_1. This requires an appropriate initial condition $(\psi(0) = 0, \lambda(0) = a)$ so that $\psi(t_1) = \pi$. Similarly, there also exists a unique $I(t)$ for switching from the Z_2 state to the S_2 state in time t_2. This requires an appropriate initial condition $(\psi(0) = \pi, \lambda(0) = b)$ so that $\psi(t_2) = 2\pi$. We now demonstrate a useful relationship between these switching problems.

Proposition 2. *If $\lambda(0) = a$, then $\lambda(t_1) = a$. Furthermore, $t_1 = t_2$ whenever $a = b$.*

Proof. Recall that H, as defined in (7), is conserved for trajectories of (6). Thus, for the trajectory with initial condition $(\psi, \lambda) = (0, a)$, at time t_1 we have $H(\pi, \lambda(t_1)) = (\lambda(t_1))^2/4 = H(0, a) = a^2/4 \equiv H_a$. This implies that $\lambda(t_1) = a$. Now, let $H_b \equiv H(\pi, b)$. The time of transition t_1 from S_2 to Z_2 and t_2 from Z_2 to S_2 is

$$t_1 = \int_0^{\pi} \frac{\mathrm{d}\psi}{\sqrt{[Kg(\psi)]^2 + H_a}}, \qquad t_2 = \int_{\pi}^{2\pi} \frac{\mathrm{d}\psi}{\sqrt{[Kg(\psi)]^2 + H_b}}.$$

Since g is a 2π-periodic odd function, $g(2\pi - \psi) = g(-\psi) = -g(\psi)$, $(g(2\pi - \psi))^2 = g(\psi)^2$. Thus, letting $\phi = 2\pi - \psi$,

$$t_2 = -\int_{\pi}^{0} \frac{\mathrm{d}\phi}{\sqrt{[Kg(2\pi - \phi)]^2 + H_b}} = \int_0^{\pi} \frac{\mathrm{d}\phi}{\sqrt{[Kg(\phi)]^2 + H_b}}.$$

Therefore, if $H_a = H_b$, or equivalently $a = b$, then $t_1 = t_2$. □

We now illustrate optimal switching for the coupling function $f(\theta) = \sin(\theta) + \sin(2\theta)$, which gives stable S_2 and Z_2 solutions: in this case, $g(\psi) = -2\sin(\psi) - 2\sin(2\psi)$, so that $g'(0) < 0$ and $g'(\pi) < 0$. For definiteness, other parameters in the steering control laws are taken to be $K = \omega = 1$. (Note that the optimal $I(t)$ is independent of ω.) To find the optimal $I(t)$ for switching from the S_2 state to the Z_2 state, the boundary value problem (6) with boundary conditions $\psi(0) = 0$, $\psi(t_1) = \pi$ was solved using a shooting method. This can then be used to obtain the optimal input $I(t) = \lambda(t)/2$. In Fig. 1 (left panel), we show sample trajectories in phase space for $\psi(0) = 0$. As expected, in order to obtain switching for larger times the trajectories remain closer to the stable and unstable manifolds of the fixed points.

One may follow the trajectories of the agents by integrating Eq. (1) with steering control given by Eq. (2). Fig. 1 (right panel) shows sample trajectories for agents optimally switching from the S_2 to the Z_2 state in time $t_1 = 0.6$. Both agents are initially placed at the origin $(x, y) = (0, 0)$ with $\theta_1 = \theta_2 = 0$ $(\psi = 0)$, so that each agent has initial velocity vector $\dot{r}_1 = \dot{r}_2 = (1, 0)$. After the appropriate transition time t_1, the input $I(t)$ is turned off so that agents remain in the Z_2 state, following circular paths of radius 1 (since $\omega = 1$) with constant phase difference $\psi = \pi$. Because we are using only phase control, in general the agents will end up tracing different circles after the optimal input $I(t)$ is turned off.

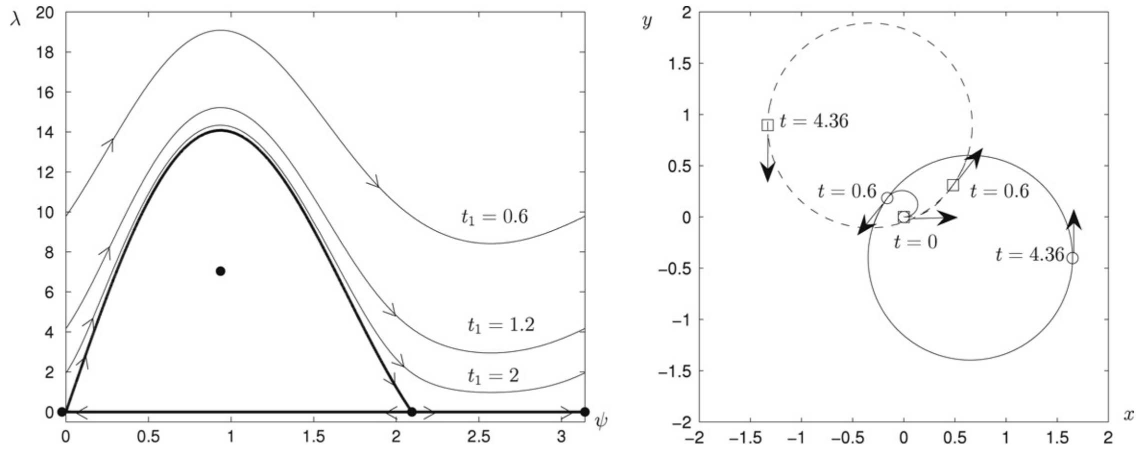

Fig. 1. (Left panel) Sample trajectories in the (ψ, λ) plane transitioning from the S_2 state ($\psi = 0$) to the Z_2 state ($\psi = \pi$) in time $t = 0.6, 1.2, 2.0$. As t increases, trajectories approach the stable and unstable manifolds of the fixed points $\psi = 0$, $\psi = \arccos(-1/2)$, $\psi = \pi$. (Right panel) Sample path of agents in the (x, y) plane for optimal switching from the S_2 state to the Z_2 state in time $t = 0.6$.

3. Conclusion

For a continuous-time kinetic model of two agents moving in the plane with unit speed, we have shown that bistability between different collective states can be achieved solely through the choice of the coupling function for the control of the orientation dynamics. These collective states are the S_2 symmetric state, in which agents instantaneously have the same orientations, and the Z_2 symmetric state, in which agents instantaneously have opposite orientations.

Variational methods were used to determine the optimal input to the steering control of one of the agents which leads to switching between these collective states. Here optimality refers to minimization of the square-integral measure of the input. For any given time interval of switching, such an optimal input was shown to exist and to be unique, provided symmetry-related solutions are associated. Furthermore, a transition time symmetry was identified which relates the optimal inputs for transitions from the S_2 to the Z_2 symmetric state to the optimal inputs for transitions from the Z_2 to the S_2 symmetric state. Finally, the properties of optimal inputs were interpreted by considering the phase space geometry of the Euler–Lagrange equations associated with the optimization.

We have considered optimal inputs which lead to switching over a specified time interval. Such switching has a nice robustness property: if the input simply puts the system into the basin of attraction of the desired state, the system's natural dynamics will lead to an asymptotic approach to the desired state. Provided noise and uncertainties in model parameters are not too large, one thus expects the inputs considered in this work to robustly lead to successful switching, although not exactly in the desired amount of time.

We hope that the techniques used in this work can be extended to other collective motion systems which display bistability, including bistability between other states and systems composed of a larger number of agents. In particular, using the expressions for the eigenvalues of globally coupled oscillator systems derived in [10,11], it is possible to choose coupling functions such that the S_N and Z_N states are bistable. Also, for $N \geq 3$ agents it is possible to have periodic orbits in the steering control subsystem stably coexisting with phase-locked solutions [18]. Optimal switching between coexisting stable states for such N-agent systems would lead to higher dimensional optimization problems which could be solved numerically with gradient methods.

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References

[1] J.P. Desai, J.P. Ostrowski, V. Kumar, Modeling and control of formations of nonholonomic mobile robots, IEEE Trans. Robot. Automat. 17 (2001) 905–908.
[2] V. Gazi, K.M. Passino, Stability analysis of swarms, IEEE Trans. Automat. Control 48 (2002) 692–696.
[3] A. Jadbabaie, J. Lin, A.S. Morse, Coordination of groups of mobile autonomous agents using nearest neighbor rules, IEEE Trans. Automat. Control 48 (2003) 692–696.
[4] E. Justh, P. Krishnaprasad, Equilibria and steering laws for planar formations, Systems Control Lett. 52 (1) (2004) 25–38.
[5] N. Leonard, E. Fiorelli, Virtual leaders, artificial potentials and coordinated control of groups, in: Proceedings of the 40th IEEE Conf. Decision and Control, 2001, pp. 2968–2973.
[6] R. Olfati-Saber, Flocking for multi-agent dynamic systems: Algorithms and theory, IEEE Trans. Automatic Control 51 (2006) 401–420.
[7] D. Paley, N.E. Leonard, R. Sepulchre, D. Grünbaum, J.K. Parrish, Oscillator models and collective motion: Spatial patterns in the dynamics of engineered and biological networks, IEEE Control Syst. Mag. 27 (4) (2007) 89–105.

[8] R. Sepulchre, D. Paley, N. Leonard, Stabilization of planar collective motion: All-to-all communication, IEEE Trans. Aut. Contr. 52 (5) (2007) 811–824.
[9] R. Sepulchre, D. Paley, N. Leonard, Collective motion and oscillator synchronization, in: V. Kumar, N. Leonard, A. Morse (Eds.), Proceedings of the Block Island Workshop on Cooperative Control, in: Lecture Notes in Control and Information Sciences, vol. 309, Springer-Verlag, 2004, pp. 189–205.
[10] P. Ashwin, J. Swift, The dynamics of N weakly coupled identical oscillators, J. Nonlinear Sci. 2 (1992) 69–108.
[11] E. Brown, P. Holmes, J. Moehlis, Globally coupled oscillator networks, in: E. Kaplan, J. Marsden, K.R. Sreenivasan (Eds.), Perspectives and Problems in Nonlinear Science: A Celebratory Volume in Honor of Larry Sirovich, Springer-Verlag, New York, 2003, pp. 183–215.
[12] D. Paley, N. Leonard, R. Sepulchre, Collective motion: Bistability and trajectory tracking, in: Proceedings of the 43rd IEEE Conf. Decision and Control, 2004, pp. 1932–1937.
[13] I.D. Couzin, J. Krause, R. James, G.D. Ruxton, N.R. Franks, Collective memory and spatial sorting in animal groups, J. Theoret. Biol. 218 (2002) 1–11.
[14] A. Kolpas, J. Moehlis, I.G. Kevrekidis, Coarse-grained analysis of stochasticity-induced switching between collective motion states, Proc. Natl. Acad. Sci. USA 104 (2007) 5931–5935.
[15] D.B. Forger, D. Paydarfar, Starting, stopping, and resetting biological oscillators: In search of optimal perturbations, J. Theoret. Biol. 230 (2004) 521–532.
[16] H. Goldstein, Classical Mechanics, Addison-Wesley, Reading, MA, 1980.
[17] J. Moehlis, E. Shea-Brown, H. Rabitz, Optimal inputs for phase models of spiking neurons, ASME J. Comp. Nonlinear Dyn. 1 (2006) 357–358.
[18] M. Kimura, J. Moehlis, Novel vehicular trajectories for collective motion from coupled oscillator steering control, SIADS 7 (2008) 1191–1212.