SIAM REVIEW
Vol. 54, No. 1, pp. 121–138

Group Decision-Making Models for Sequential Tasks*

Margot Kimura†
Jeff Moehlis†

Abstract. The sequential probability ratio test (SPRT) and related drift-diffusion model (DDM) are optimal for choosing between two hypotheses using the minimal (average) number of samples and relevant for modeling the decision-making process in human observers. This work extends these models to group decision making. Previous works have focused almost exclusively on group accuracy; here, we explicitly address group decision time. First, we derive explicit solutions for the error rate and probability distribution function of decision times for a group of independent, (possibly) nonidentical decision makers using one of three simple rules: Race, Majority Total, and Majority First. We illustrate our solutions with a group of N i.i.d. decision makers who each make an individual decision using the SPRT-based DDM, then compare the performance of each group rule under different constraints. We then generalize these group rules to the η-Total and η-First schemes, to demonstrate the flexibility and power of our approach in characterizing the performance of a group, given the performance of its individual members.

Key words. group decision-making models, decentralized detection, sequential probability ratio test, decision aggregation

AMS subject classifications. 60H30, 91E99, 62L99

DOI. 10.1137/10079700X

1. Introduction.

Our society has a long tradition of valuing the wisdom of groups. Groups offer the potential for redundancy and robustness, and are generally thought of as being more cautious, more creative, more informed, and more accurate than individuals. Many studies in social psychology have been dedicated to supporting or refuting these beliefs from a relatively qualitative standpoint [20, 48, 49, 56, 57]; see [15, 16] for reviews. A more quantitative approach to how human groups make decisions in a restricted task has been pursued in cognitive psychology [33, 38, 46, 47]. These studies generally focus on group accuracy and the weight placed on each individual's opinion as the main measures of performance. The group performance is typically compared to that of an "ideal group," which represents the best that the group can do given its members' abilities, in order to find the group's efficiency [45]. The experiments cited above use fixed-sample statistical procedures, are typically data driven, and are generally based on signal detection theory [23]; also see [39]. Similar

*Received by the editors June 1, 2010; accepted for publication (in revised form) August 30, 2011; published electronically February 8, 2012. This work was supported by a National Science Foundation IGERT Fellowship, a University of California–Santa Barbara Graduate Division Doctoral Scholars Fellowship, and the Institute for Collaborative Biotechnologies through grant W911NF-09-0001 from the U.S. Army Research Office. The content of the information does not necessarily reflect the position or the policy of the Government, and no official endorsement should be inferred.
http://www.siam.org/journals/sirev/54-1/79700.html

†Department of Mechanical Engineering, University of California, Santa Barbara, CA 93106 (kimura@engineering.ucsb.edu, moehlis@engineering.ucsb.edu)

ideas have also been explored in ecology [13, 36, 55]. A separate group of work in economics and political science [4, 5, 7, 24, 27, 28] focuses heavily on various aspects of Condorcet's (jury) theorem [12]. These works use more theoretical modeling, and take an approach to calculating a group's error rate similar to the one we present later. These human-group-based studies inspire our current work as well as future directions and considerations in characterizing the performance of a cybernetic (including both humans and devices) group-based decision making system.

A more mathematics- and engineering-based approach to group decision making is taken in the design of decision-making systems that utilize multiple sensors. This has been an intensive area of study, particularly in the past thirty years. These works generally focus on a collection of devices minimizing a cost or risk function in choosing between two hypotheses, and thereby assert the resulting decision-making scheme as "optimal." A very good introduction to and overview of decentralized detection is presented in [51], and a mathematical approach to sequential decision theory can be found in [6, 44]. Centralized systems, in which measurements from peripheral sensors are sent to a fusion center for processing, are largely considered a finished problem; thus, most literature focuses on decentralized or distributed systems. Despite the name, in most decentralized systems a very small amount of processing is done at the peripheral sensors: a compressed "decision" based on a single observation is sent to the fusion center for processing at each time step. Decentralized decision-making systems, particularly ones that consider sequential processes, are a very large area of research with many variations, including the following: general hypotheses [18, 52], multiple hypotheses [17], quickest detection problems [14], sequential test truncation [32, 43], and different group communication topologies [40, 50]. Various applications are also considered, including networks with constrained communication [10, 19, 31, 35], networks with power constraints [37], vehicle classifiers [30], and probabilistic search [11]. Though our models are different from the ones cited here, these studies illustrate a very different philosophy on how a collective can make a decision, and have illustrated the methods and applications of interest in strictly device-based engineering setups.

We consider the solution to a two-alternative forced-choice test (2AFC). Though there are multiple definitions of what constitutes a 2AFC task, we follow the convention laid out in [8]: A 2AFC task is one in which one must choose between two specific hypotheses (denoted H_0 and H_1), while making the following three assumptions: (i) evidence favoring each alternative is integrated over time, (ii) the process that provides the decision makers (DMs) with observations is subject to random fluctuations, and (iii) a decision is made when sufficient evidence favoring one of the alternatives has accumulated.

An example of a 2AFC task is deciding if there is a signal (e.g., intruder, contamination, etc.) in a given area. In this case, the system's designer is typically given a budget, and has a general idea of the possible consequences of failing to detect the signal (miss) or declaring that the signal is present when it is not (false alarm), but may not know the explicit costs for each type and instance of error. In general, the goal is to have the group reach a decision quickly, while not exceeding certain levels of error. In this situation, there are a number of system parameters to be determined. Important considerations include the total number of detectors, whether to use a few sophisticated detectors or a large number of simple detectors, and how to incorporate at least one human observer into the system to provide accountability. We consider this problem in general for sequential cases, in which multiple samples of data are available, but come at the cost of additional time. There are many situations

Table 1.1 *List of acronyms used in this paper.*

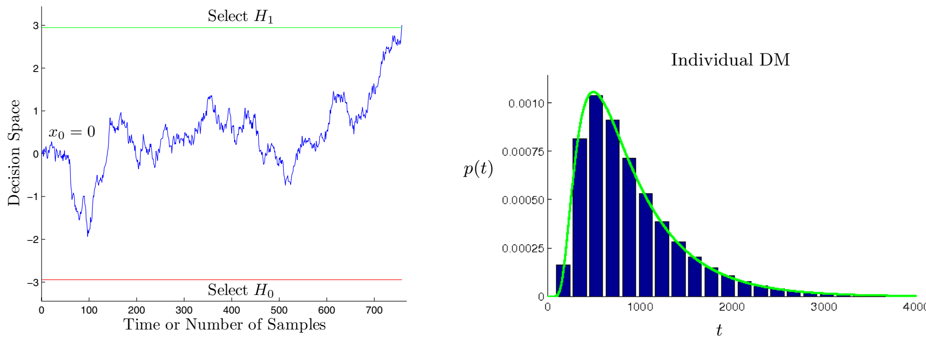
Acronym	Full name
2AFC	Two-alternative forced choice (task)
cdf	Cumulative probability distribution function
$\mathbb{D}$	Denotes a generic decision: $\mathbb{D} \in \{\mathrm{S}, \mathrm{N}\}$
$\hat{\mathbb{D}}$	The other hypothesis: $\hat{\mathbb{D}} \in \{\mathrm{S}, \mathrm{N}\}, \hat{\mathbb{D}} \neq \mathbb{D}$
DDM	Drift-diffusion model
DM	Decision maker
GDT	Group decision time
GER	Group error rate
i.i.d.	Independent and identically distributed
LDT	Local (individual) decision time
LER	Local (individual) error rate
N	Number of decision makers
N	Noise (or signal absent) response
pdf	Probability distribution function
S	Signal (or signal present) response
SPRT	Sequential probability ratio test
Θ	Maximal minority: for N odd, $\frac{N-1}{2}$
Υ	Minimal majority: for N odd, $\frac{N+1}{2}$

in which the sample size is not fixed in advance or when one wants to minimize the number of samples: for example, in ammunition quality control, tested samples are unusable; and in clinical trials, there is a moral obligation to use the minimal number of test subjects required to achieve the desired error rate (ER).

We begin with a brief discussion of our individual model, which is based on the sequential probability ratio test (SPRT). We then derive the group error rate (GER) and probability distribution function (pdf) of group decision times (GDTs) for N independent DMs using one of three simple group decision rules: Race, Majority Total, and Majority First. We illustrate each solution with an example using N independent and identically distributed (i.i.d.) DMs characterized by the SPRT. We then compare the performance of our schemes, and discuss the relative merits of each. We finish with a brief discussion of two generalized forms of our group rules. For convenience, Table 1.1 gives a list of acronyms used in this paper.

1.1. Individual Model. The SPRT is a particular procedure from sequential analysis that is "optimal" in the sense that it minimizes the average number of samples required to choose between two hypotheses, while not exceeding specified ERs [54]; also see [21, 22, 34]. We use the SPRT and the related drift-diffusion model (DDM) to model the individuals in our examples because it is optimal; however, we note that our group results are general enough to accommodate any individual model that produces a cumulative probability distribution function (cdf) and pdf of local decision times (LDTs), which covers cases in which the individual DM's decision test is set (and possibly nonoptimal). We provide a brief description of the SPRT and the DDM below; for a full derivation of the cdf and pdf of LDTs and further detail on the model and relevant works, see [29].

1.1.1. The Sequential Probability Ratio Test (SPRT). Suppose Y is a random variable with unknown pdf P. The SPRT tests whether $H_0 : P = P_0$ or $H_1 : P = P_1$ is correct, with error no greater than a specified False Alarm (type-I error) rate α_0 and Miss (type-II error) rate α_1. The pdfs P_0 and P_1 are assumed to be known. Let



(a) Example (b) Verification

Fig. 1.1 (a) *A visual explanation of the SPRT. Each boundary is associated with a hypothesis, as labeled. Here, the DM's decision variable x starts with an unbiased prior (since $x_0 = 0$). The process terminates when the decision variable hits one of the boundaries. In this example, the DM will select H_1 at the 758th time step.* (b) *Histogram of DTs for a DM using the SPRT over* 10,000 *trials, normalized and overlaid with the analytically calculated pdf of LDTs. This verifies that our DDM solutions accurately model an individual DM using the SPRT.*

π_0 (resp., π_1) be the prior probability that one believes that H_0 (resp., H_1) is correct. These priors are expressed in the initial condition $x_0 = \log\left(\frac{\pi_1}{\pi_0}\right)$.

The general idea is that we construct a one-dimensional decision space, whose absorbing boundaries are each associated with one of the hypotheses. In practice, one usually applies Wald's small-overshoot assumption [53], which allows one to define the boundaries by the (slightly) conservative values $B_0 = \log\left(\frac{\alpha_1}{1-\alpha_0}\right)$ and $B_1 = \log\left(\frac{1-\alpha_1}{\alpha_0}\right)$. At each time step, the DM observes a value of Y, denoted y_i. We assume that the observations are i.i.d. After processing the nth observation, the decision variable's position is given by

$$x_n = x_{n-1} + \sum_{i=1}^{n} \log\left[\frac{P_1(y_i)}{P_0(y_i)}\right]. \tag{1.1}$$

After processing each observation, the observer chooses one of three possible actions, based on the value of x_n: if $B_0 < x_n < B_1$, take another sample of data; if $x_n \leq B_0$, choose H_0; and if $x_n \geq B_1$, choose H_1. A visual representation of this is given in Figure 1.1(a).

The decision-making process under the SPRT is a discrete-time biased random walk. It is intuitively clear that boundaries that are further away from the decision variable's initial condition provide longer decision times and more accurate decisions because the decision variable is less susceptible to being driven across a boundary by noise. Our examples consider parameters for which the small-overshoot assumption approximately holds. For more detail, see [29]. We will use simulation to represent a DM using the SPRT.

1.1.2. The SPRT-Based DDM. As the time between arriving increments of information goes to zero (equivalently, as sampling becomes continuous), the process described above approaches the stochastic continuous-time process $x(t)$. In this limit, the SPRT approaches the DDM, which can be converted to a Kolmogorov or Fokker–Planck equation and solved to find the DM's pdf of LDTs. More detail on this limit can be found in [8], and the solution is derived in [29] (also see [41]). The DDM has been used to model oculomotor decision making in the brain, supported by experiments with in vivo recordings from monkeys [25] and psychophysical tasks with humans [9, 42]. Therefore, the DDM can be reasonably used to model the performance of human observers as DMs. Our solution to the SPRT-based DDM is an analytical result that will be used in our group models below. For appropriate parameter values, simulating a DM using the SPRT to perform a 2AFC task is equivalent to randomly selecting a decision time from the pdf of LDTs given by the SPRT-based DDM. This is verified in Figure 1.1(b), where the histogram of LDTs over 10,000 trials for an individual using the SPRT have been compared with our analytical solution of the same system.

Using these results as a building block for the performance of an individual, we now show how the performance of a group of DMs can be characterized in the next section.

2. Models for Centralized Group Decision Making. Many studies have focused on group interactions and how they affect the final group decision. Our analysis assumes that each individual DM has quantifiable abilities and is suitably motivated. We first present our analytical results and general formulas for the GERs and GDTs for each scheme given a group of N independent DMs with known (individual) local error rates (LERs). We then provide details on our simulations, and compare the simulated results with our derived formulas. We conclude this section with a discussion of the relative performance of the different schemes.

2.1. Analytical Results. We present the analytical results for a group of N independent DMs using one of three simple group decision schemes: Race, Majority Total, or Majority First. In these schemes, the individual DMs can only communicate with the fusion center, as shown in Figure 2.1.

2.1.1. Race Scheme. In the Race scheme, the fusion center simply follows the decision of the fastest decision maker. It is a race in the sense that only the first decision made counts towards the group's decision. As we will show later, the Race

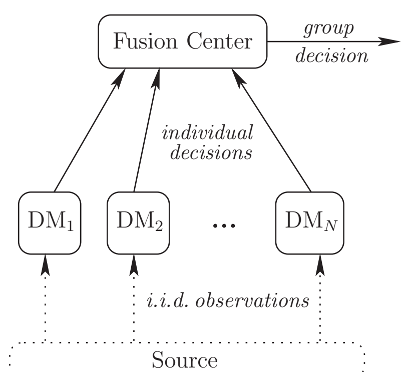


Fig. 2.1 *Illustration of how our group models are organized. Each DM takes and processes i.i.d. observations of the source (which represents the correct hypothesis). The observations are i.i.d. both within and across DMs. Once a DM makes a decision, it sends that decision to the fusion center, which then applies the group-decision rule and issues the group's decision once the group rule has been satisfied.*

scheme has the fastest GDT of all schemes for a given group of DMs under our setup. Thus, we are primarily interested in this scheme as a benchmark for speed. A similar group decision scheme was proposed in [1, 2] for independent and identically distributed (i.i.d.) DMs; however, the treatment only finds the cdf of GDTs and qualitatively discusses the scheme, dismissing a full analysis of the pdf of GDTs as too complex.

Consider a group of N independent DMs using the Race scheme. In this case, the GER will be equal to the LER of the fastest DM. We can also characterize the pdf of GDTs without taking any further assumptions on the individual DMs, to derive a general result. Let $p_i(t)$ be the pdf of LDTs for DM_i, and let $p_g(t)$ be the pdf of GDTs. Similarly, let $q_i(t)$ be the cdf of LDTs for DM_i and $q_g(t)$ be the cdf of GDTs. For a group using the Race scheme to make a decision by time t_g, at least one individual DM must reach a decision by t_g. Subtracting the probability for the single case where no DM reaches a decision by time t_g from unity gives us an expression for the cdf of GDTs, $q_g^{\mathrm{rg}N}(t_g)$, where the superscript specifies the decision rule, type of individual DMs, and number of DMs in the group ([r]ace scheme, [g]eneral individual DMs, [N] DMs):

$$q_g^{\mathrm{rg}N}(t_g) = 1 - \prod_{i=1}^{N} \left[1 - q_i(t_g)\right]. \tag{2.1}$$

The corresponding pdf of GDTs is

$$p_g^{\mathrm{rg}N}(t_g) = \frac{d}{dt_g} q_g^{\mathrm{rg}N}(t_g) = \sum_{i=1}^{N} \left(p_i(t_g) \prod_{\substack{j=1,\\ j\neq i}}^{N} \left[1 - q_j(t_g)\right] \right). \tag{2.2}$$

We can specialize these general results to the case where the N DM are i.i.d.: we replace the individuals' (possibly different) pdfs by a single common pdf $p_\iota(t)$ and the individuals' cdfs by a single common cdf $q_\iota(t)$, where ι indicates that the function is generic for an i.i.d. individual. This simplifies $p_g^{\mathrm{rg}N}(t_g)$ to $p_g^{\mathrm{ri}N}(t_g)$ ([r]ace scheme, [i]id individual DMs, [N] DMs): $p_g^{\mathrm{ri}N}(t_g) = N p_\iota(t_g) \left[1 - q_\iota(t_g)\right]^{N-1}$.

We can now plot the pdf of GDTs for N i.i.d. DMs using the Race scheme. This is shown for $N = 1$ to 41 in Figure 2.2. For all numerical results in this paper, each DM has a pdf of LDTs given by the $N = 1$ case shown in Figure 2.2. This corresponds to an individual using the SPRT with an LER of 0.01. As N increases, the group's pdf moves to the left, and the distribution becomes more peaked. This is consistent with what we would intuitively expect: as N increases, the probability that a DM in the group has an LDT closer to 0 increases, so the minimum of the N samples decreases.

2.1.2. Majority Total Scheme. In the Majority Total scheme, the fusion center waits until all N DMs have submitted a decision before declaring the group decision, which is chosen using a majority rule. To avoid ties in the fusion center decision, we consider only N odd, though it would be simple to include additional constraints such as ignoring one DM at random or ignoring the slowest DM to account for N even. A rule equivalent to the Majority Total scheme was mentioned in [51]; however, to the best of our knowledge, there has not been a detailed analysis of this rule.

The Majority-based scheme assigns equal weights to each DM. Here, we derive an expression for the GER of N general DMs under a majority rule. The GER for DMs with unequal weighting is discussed in economics and political science [4, 5, 28].

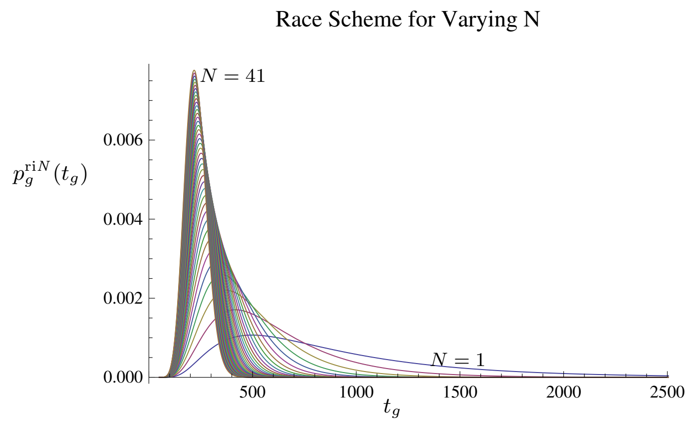


Fig. 2.2 *Plots of $p_g^{\mathrm{ri}N}(t_g)$, the pdf of GDTs for a group of N i.i.d. DMs using the Race scheme, where t_g denotes the group's decision time. In this plot, N varies from 1 to 41. As the number of DMs increases, the mean GDT decreases and the group pdf becomes more peaked. This is consistent with the intuition that a larger group of DMs using the Race scheme has a lower average GDT.*

For i.i.d. DMs, equal weighting is optimal. For nonidentical DMs, equal weighting is not optimal; however, in cases where one does not or cannot know a DM's true LER, or when including humans in the loop, equal weighting is a reasonable strategy.

Let $\Theta = \lceil \frac{N-1}{2} \rceil$ be the number of DMs in the largest minority possible ("maximal minority"). Note that by our problem definition, the probability that a specific DM makes an error is given by LER, and the probability that a specific DM makes a correct decision is given by (1-LER). Then the GER for a group of N independent DMs using the Majority Total scheme is

$$\begin{aligned}
P(\text{group errs}) &= P(\text{majority errs}) \equiv \text{GER} \\
&= P(N \text{ err}) + C[P(1 \text{ correct}, (N-1) \text{ err})] \\
&\quad + C[P(2 \text{ correct}, (N-2) \text{ err})] + \cdots \\
&\quad + C[P(\Theta \text{ correct}, (\Theta+1) \text{ err})] \\
&= \sum_{\theta=1}^{\Theta} C[P(\theta \text{ correct}, (N-\theta) \text{ err})],
\end{aligned}$$

where $C[P(\theta \text{ correct}, (N-\theta) \text{ err})]$ represents all unique combinations of DMs such that θ DMs reach a correct decision and $(N-\theta)$ DMs make an error. This translates into the following equation for the GER of a general group:

(2.3)

$$\begin{aligned}
\text{GER} = &\prod_{i=1}^{N} \text{LER}_i \\
&+ \sum_{\theta=1}^{\Theta} \left[\sum_{j_1=1}^{(N-\theta+1)} \sum_{j_2=j_1+1}^{(N-\theta+2)} \cdots \sum_{j_\theta=j_{\theta-1}+1}^{N} \left(\prod_{k=1}^{\theta} [1-\text{LER}_{j_k}] \prod_{\substack{m=1,\\ m\notin\mathcal{J}}}^{N} \text{LER}_m \right) \right],
\end{aligned}$$

where $\mathcal{J} = \{j_1, \ldots, j_\theta\}$, the subset of DMs who answered correctly in the combination being considered.

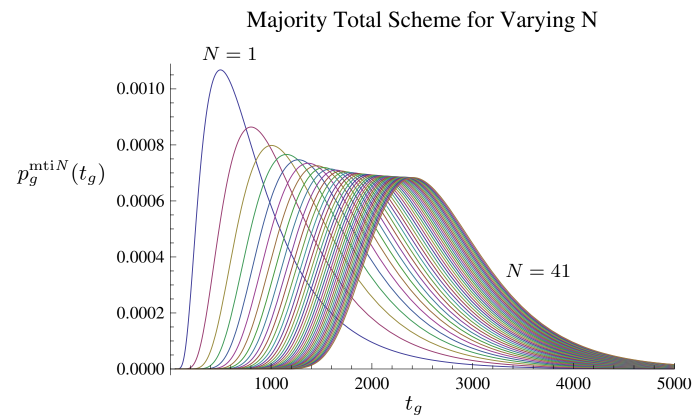


Fig. 2.3 *Plot of $p_g^{\mathrm{mti}N}(t_g)$, the pdf of GDTs for N i.i.d. DMs using the Majority Total scheme, with N varying from 1 to 41. As the number of DMs increases, the mean GDT also increases, and the distribution spreads out. This is intuitive: since the fusion center declares the group's decision only after the slowest DM has responded, as N increases, the slowest DM's LDT tends to increase and can take on a wider range of values.*

We now calculate the cdf and pdf of GDTs. In the Majority Total scheme, the GDT is the LDT of the slowest group member. The probability that a group of N independent DMs using the Majority Total rule makes a decision by time t_g is the same as the probability that all N DMs make a decision by time t_g. Thus, we can write out the cdf of GDTs, $q_g^{\mathrm{mtg}N}(t_g)$ ([m]ajority [t]otal, [g]eneral individual DMs, [N] DMs), as

$$q_g^{\mathrm{mtg}N}(t_g) = \prod_{i=1}^{N} q_i(t_g). \tag{2.4}$$

The group pdf of GDTs is then

$$p_g^{\mathrm{mtg}N}(t_g) = \sum_{i=1}^{N} \left(p_i(t_g) \prod_{\substack{j=1,\\ j\neq i}}^{N} q_j(t_g) \right). \tag{2.5}$$

For N i.i.d. DMs, the cdf of GDTs is $q_g^{\mathrm{mti}N}(t_g) = [q_\iota(t_g)]^N$, and the pdf of GDTs is $p_g^{\mathrm{mti}N}(t_g) = N p_\iota(t_g) [q_\iota(t_g)]^{N-1}$. The pdfs of GDTs for N i.i.d. DMs are shown for $N = 1$ to 41 in Figure 2.3. As N increases, the group pdf spreads out and drifts to the right, which is intuitive—a larger group should take longer to decide, since the group must wait for the slowest DM.

2.1.3. Majority First Scheme. The Majority First scheme is a slight modification of the Majority Total scheme. Given that our two major measures of performance are GDT and GER, it naturally follows that one can speed up the Majority Total scheme by ignoring the DMs whose decisions will not contribute to the fusion center's decision. Thus, the fusion center makes a decision as soon as the smallest possible majority ("minimal majority") $\Upsilon = \lfloor \frac{N+1}{2} \rfloor$ of DMs reach the same decision. As before, we consider only N odd. Though similar ideas have been expressed in models by Audley [3] and Hussain [26], to the best of our knowledge, this particular group decision-making rule is novel.

The Majority First and Majority Total schemes have identical GERs because they apply the same decision rule to choose a hypothesis. Thus, the GER for N general DMs using the Majority First scheme is also given by (2.3).

The calculations used to find the GDT under the Majority First scheme differ from the previous two schemes in that they must track the decision made in addition to the decision time. To express the hypothesis selected, we add a second subscript to the individual's cdf or pdf: we denote that the individual chooses H_1 (resp., H_0) with "S" (resp., "N"). We chose letter subscripts for the decision to avoid confusion with the numerical subscripts identifying each DM, and to express a signal detection theory viewpoint [23, 39]: "S" is for "signal present" (signal + noise, to be precise) and "N" is for "noise only". The cdf of GDTs for N general DMs, where we drop the explicit (t_g) notation on the right-hand side for brevity, is given by

$$q_g^{\mathrm{mfg}N}(t_g) = \prod_{i=1}^{N} q_{i\mathrm{S}} + \prod_{i=1}^{N} q_{i\mathrm{N}} + \sum_{\theta=1}^{\Theta}\left(\Gamma_{\mathcal{J}\mathrm{S}}^{\theta\mathrm{g}} \prod_{\substack{k=1,\\ k\notin\mathcal{J}}}^{N} q_{k\mathrm{S}} + \Gamma_{\mathcal{J}\mathrm{N}}^{\theta\mathrm{g}} \prod_{\substack{k=1,\\ k\notin\mathcal{J}}}^{N} q_{k\mathrm{N}}\right). \tag{2.6}$$

The first term represents the case where all N DMs select S before time t_g. Similarly, the second term represents the case where all N DMs select N before time t_g. The last term denotes all combinations [$\mathcal{J}$] where up to [θ] [g]eneral DMs either answer by time t_g but disagree with the (final) group decision [$\mathbb{D}$] or answer after time t_g. The subfunction $\Gamma_{\mathcal{J}\mathbb{D}}^{\theta\mathrm{g}}$ specifies the noncontributing and dissenting DMs, and is defined as

$$\Gamma_{\mathcal{J}\mathbb{D}}^{\theta\mathrm{g}} = \sum_{j_1=1}^{(N-\theta+1)} \sum_{j_2=j_1+1}^{(N-\theta+2)} \cdots \sum_{j_\theta=j_{\theta-1}+1}^{N} \left(\prod_{m=1}^{\theta} [1-q_{j_m\mathbb{D}}]\right), \tag{2.7}$$

where $\mathbb{D}\in\{\mathrm{S},\,\mathrm{N}\}$ and $\mathcal{J}=\{j_1,\ldots,j_\theta\}$. The summation indices indicate which DMs are in each unique combination ($\mathcal{J}$) of DMs, and the product term uses the subindex m to iterate through each DM in the combination.

Taking the derivative with respect to t_g, we get the group pdf for N general DMs:

$$p_g^{\mathrm{mfg}N}(t_g) = \Gamma_{\mathcal{J}\mathrm{S}}^{\Theta\mathrm{g}} \sum_{\substack{k=1,\\ k\notin\mathcal{J}}}^{N}\left[p_{k\mathrm{S}} \prod_{\substack{m=1,\\ m\notin\mathcal{J},\\ m\neq k}}^{N} q_{m\mathrm{S}}\right] + \Gamma_{\mathcal{J}\mathrm{N}}^{\Theta\mathrm{g}} \sum_{\substack{k=1,\\ k\notin\mathcal{J}}}^{N}\left[p_{k\mathrm{N}} \prod_{\substack{m=1,\\ m\notin\mathcal{J},\\ m\neq k}}^{N} q_{m\mathrm{N}}\right]. \tag{2.8}$$

This formula makes intuitive sense: in the first term, there are Θ DMs who either select N or do not finish by time t_g, there is one DM who finishes at time t_g and selects S, and there are $(N-\Theta-1)$ DMs who finish by time t_g and select S. Thus, the minimal majority Υ selects S at time t_g and the fusion center finishes at time t_g. The second term holds the equivalent expression for the group choosing N.

As before, this expression can be simplified for i.i.d. DMs. The pdf of GDTs for N i.i.d. DMs is $p_g^{\mathrm{mfi}N}(t_g) = \sum_{\mathbb{D}\in\{\mathrm{S},\mathrm{N}\}} \binom{N}{\Theta}\Upsilon[1-q_{\iota\mathbb{D}}(t_g)]^{\Theta} p_{\iota\mathbb{D}}(t_g)[q_{\iota\mathbb{D}}(t_g)]^{\Theta}$. We plot

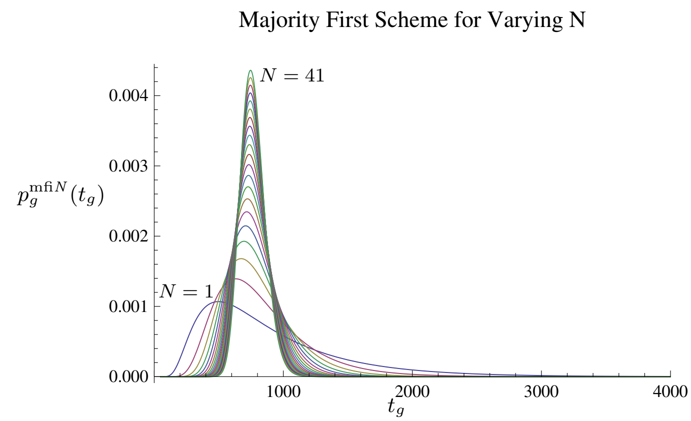


Fig. 2.4 *Plot of $p_g^{\mathrm{mfi}N}(t_g)$, the pdf of GDTs N i.i.d. DMs using the Majority First decision rule, with N varying from 1 to 41, for N odd. As N increases, the mean of the distribution increases slightly, while the pdf becomes more peaked.*

the pdf of GDTs for N i.i.d. DMs using the Majority First scheme in Figure 2.4, for N odd, from $N = 1$ to 41. In the Majority First scheme, the slowest member is between the Υth and Nth to respond, and as N increases, the number of DMs that can potentially be ignored also increases. The increase in the number of ignorable DMs seems to balance out the increase in Υ, so the mean GDT increases more slowly while the distribution becomes more peaked.

2.2. Simulation Results. In our simulations, each DM takes and processes one sample of data at each time step. When a DM chooses a hypothesis, it sends that decision to the fusion center: if the DM chooses H_1 (resp., H_0), it submits $+1$ (resp., -1). In each time step, the fusion center sums over all decisions that have arrived in that time step, and checks to see if the group decision rule has been satisfied. If the group decision rule is not yet satisfied, the fusion center does nothing, and the process repeats in the next time step. In the unlikely situation where an equal number of DMs reach opposing decisions in the same time step in the Race scheme, the fusion center simply cancels out the two decisions and takes the next DM's decision as the group's decision. While the DMs and fusion center are synchronized in our simulation for convenience, we stress that our analysis and solutions are directly applicable to completely asynchronous systems.

We assume that after a detector has sent a decision to the fusion center, it shuts itself down, and that the process ends once the fusion center returns a decision. All of the group decision rules presented here work in cases where the fusion center can send a message to the individual DMs so they shut down after the fusion center's process ends as well as cases where the fusion center cannot send out messages.

2.2.1. Comparison. The various fusion center decision rules each have different strengths and weaknesses, which we will discuss here in terms of performance. Below, we use two different cases to explore the relative advantages of the schemes: the equal LER case for different N and the equal GER case for different N. The results shown in this section are averaged over 10,000 trials and compared to the results for a single DM for reference.

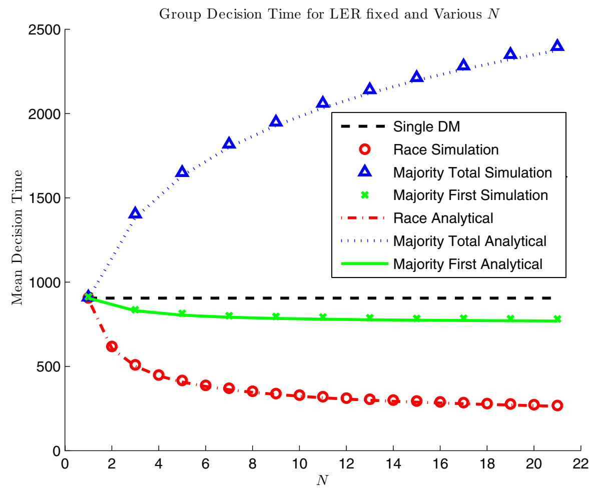


Fig. 2.5 *Simulation results for the mean GDT of N i.i.d. DMs using the Race, Majority Total, and Majority First schemes, overlaid on our analytical solution for the same group, where all of the schemes were given the same set of DMs. The group schemes are compared to an individual DM using the SPRT. In this case, the LER was set to* 0.01*, and N was varied from* 1 *to* 21*.*

Equal LER, Different *N*. In many realistic applications, it is reasonable to assume that one is supplied with N i.i.d. DMs, and then must choose a group decision rule that will provide the best performance for one's goals.

The GER for the Majority schemes under the equal LER case with i.i.d. DMs can easily be calculated using (2.3), and the GER for the Race scheme with i.i.d. DMs will be the LER of an individual DM. As N increases, the GER of the Majority schemes drop off rapidly, whereas the GER of the Race scheme does not change. For LER $= 0.01$, the mean GDTs under each rule as a function of group size N are shown in Figure 2.5. The GDTs for the Majority Total scheme rises with N, while it quickly levels off for the Majority First scheme. The Race scheme clearly provides the fastest performance. Thus, in this case, if one prioritizes GDT, the Race scheme is a reasonable option; otherwise, the Majority First scheme provides a good balance of low GER and low GDT.

Set GER, Different *N*. As another way to compare the different group decision rules, suppose one has a desired GER in mind, and is interested in seeing which decision rule can achieve it with some other desirable properties (speed, cost, etc). This assumes that one can acquire individual DMs of any given LER to achieve the desired GER. The results are shown in Figure 2.6 for GER $= 0.01$ and different values of N.

The results are interesting in that they are not immediately intuitive. As N increases, for a set GER, the LER of the DMs using a majority scheme increases, which results in faster individual decisions. The Majority Total scheme rapidly becomes faster than a single DM because the members in the Majority Total group have a higher LER (and therefore a shorter LDT) than the single DM. However, this is not enough to make the Majority Total scheme faster than the Race scheme for the N values shown, even though the Race scheme samples from DMs each with LER $=$

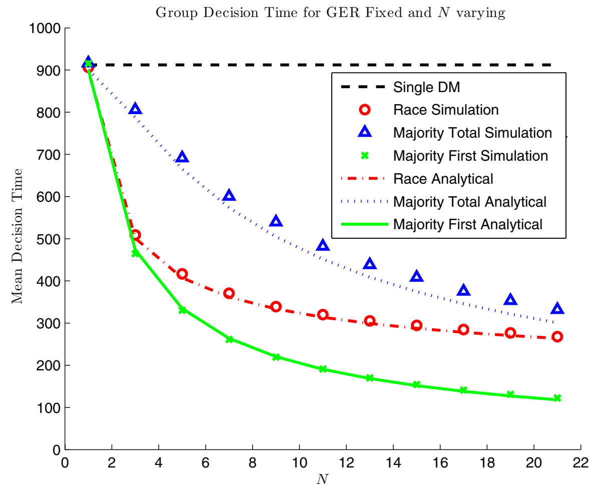


Fig. 2.6 *Simulation results for the mean GDT of N i.i.d. DMs using the Race, Majority Total, and Majority First schemes overlaid on our analytical results for the case where the group members' i.i.d. characteristics were chosen to achieve a GER of* 0.01*. The group schemes are compared to an individual DM using the SPRT, and N was varied from* 1 *to* 21*. All results are averaged over* 10,000 *trials. The difference between the Majority Total scheme's simulation-based and analytical results are the result of overshoot. For more detail, see* [29].

GER. We find that for a set GER, the Majority First scheme is the fastest for a given N, since it combines the benefits of using DMs with a higher LER (like the Majority Total scheme) with being able to finish before every DM in the group has made a decision (like the Race scheme). In addition to providing faster results, the Majority First scheme may also be better than the Race scheme because it is generally more difficult and/or more expensive to obtain individual DMs with lower LERs. Whether the Majority First or the Race scheme is best for a particular experiment may depend on the experimenter's specific budget and cost function, which formally set the best trade-off between speed and accuracy.

2.3. Discussion. The Race scheme provided consistently fast GDTs in both scenarios, and is the simplest to design, since the fusion center needs very little computational power and memory. Therefore, for situations where one is given a set of i.i.d. DMs to work with, the Race scheme provides a simple and fast solution. However, we point out that in the case where one DM encounters a malfunction that causes it to very quickly return a decision that is not related to its observations (i.e., it is either faulty or is hijacked), the overall group decision is vulnerable: since the group scheme itself is very fast, it would be very difficult to differentiate between a group decision drawn from the left-hand tail of the pdf of GDTs and an erroneous or malicious response, especially for large N. The group decision scheme also does not provide a GER that is better than the individual DMs' LER. On the other hand, this scheme is robust to multiple individual DM failures, where the DM(s) cannot communicate with the fusion center.

The Majority Total scheme was the slowest in the set-LER case, but had a GDT close to the other two group schemes in the set-GER case. It also can use less accurate individual DMs to achieve a higher level of accuracy at the group level, which is desirable because lower-accuracy DMs are typically cheaper and easier to acquire. The scheme is also robust to a small number of DMs being hijacked or faulty in a way that has them respond without processing data, and may allow one to calculate an additional confidence on the group response, since one has responses from all N DMs (i.e., if all N agree, then the group is very confident in its decision, whereas if Θ disagree with the majority, then the probability that the group made an error is higher). On the other hand, the Majority Total scheme is vulnerable to sensor failure, since it must wait for the slowest DM: if even one sensor becomes unable to communicate with the fusion center, the group never reaches a decision.

Thus, in several ways, the Majority First scheme is the best group decision rule of the three shown here: it provides relatively quick performance in the set-LER case and the quickest performance in the set-GER case. It also shares the same LER-related advantages as the Majority Total scheme, and is fairly robust to sensors responding without processing data (due to being faulty or hijacked). At the same time, it is robust to a small number of sensors failing (due to either faulty communication equipment or being destroyed), since it only waits for the Υth slowest agreeing member. Thus, while the Majority First scheme may not be time-optimal for the equal LER case, its robustness and combination of being both accurate and relatively quick make it an attractive choice, especially when there is uncertainty in the situations that the system may face.

3. Generalizing the Group Decision Rules. There are two obvious extensions to the group rules presented above, which we briefly discuss here: the η-Total and η-First group schemes.

3.1. The η-Total Scheme. The general form of the Majority Total scheme is the η-Total rule, in which the fusion center returns a decision once η DMs have returned an individual decision. The fusion center then applies a majority rule to the individual responses to reach a group decision. For an actual decision rule, it is best to use odd values of η, since this guarantees that there will not be any ties; however, from a calculation-based standpoint, there is no reason why η must be odd, so our formulas also hold for η even. For this rule, N can easily be either even or odd.

The general formula for the cdf of GDTs for the η-Total rule is $q_{g\eta}^{\mathrm{Tg}N}(t_g)$ (superscript: η-[T]otal group rule, [g]eneral individual DM, N DMs; subscript: [g]roup cdf, [η] individual DMs must finish for the fusion center to finish),

$$q_{g\eta}^{\mathrm{Tg}N}(t_g) = \sum_{\theta=\eta}^{N} \left[\sum_{i_1=1}^{(N-\theta-1)} \sum_{i_2=i_1+1}^{(N-\theta+2)} \cdots \sum_{i_\theta=i_{\theta-1}+1}^{N} \left(\prod_{m=1}^{\theta} q_{i_m} \right) \prod_{\substack{j=1,\\ j\notin\mathcal{I}}}^{N} [1-q_j] \right], \tag{3.1}$$

where $\mathcal{I} = \{i_1, i_2, \ldots, i_\theta\}$, which designates the set of DMs who have reached a decision by time t_g. To get the pdf of GDTs, we take the derivative of the cdf. After some simplification, we arrive at the following general formula:

$$p_{g\eta}^{\mathrm{Tg}N} = \sum_{i_1=1}^{(N-\eta+1)} \sum_{i_2=i_1+1}^{(N-\eta+2)} \cdots \sum_{i_\theta=i_{\theta-1}+1}^{N} \left(\sum_{m=1}^{\eta} p_{i_m} \prod_{\substack{k=1,\\ k\neq m}}^{\eta} q_{i_k} \right) \left(\prod_{\substack{j=1,\\ j\notin\mathcal{I}}}^{N} [1-q_j] \right). \tag{3.2}$$

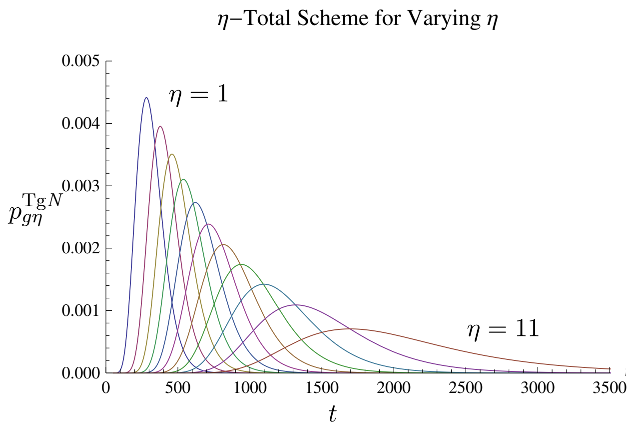


Fig. 3.1 *Analytical results for the η-Total group rule for different values of η, with $N = 11$. For $\eta = 1$, this rule is identical to the Race scheme, and for $\eta = N$, this rule is identical to the Majority Total scheme.*

For i.i.d. DMs, the formula for the pdf of GDTs simplifies considerably: $p_{g\eta}^{\mathrm{Ti}N} = \binom{N}{\eta} \eta p_\iota q_\iota^{\eta-1} [1 - q_\iota]^{N-\eta}$. For $N = 11$, the effect of increasing η from 1 to N is shown in Figure 3.1. For $\eta = 1$, the η-Total rule is identical to the Race scheme, and for $\eta = N$, the η-Total rule is identical to the Majority Total scheme. This can also be seen by comparing (3.2) with the pdf of GDTs for the Majority Total scheme, given by (2.5). We have also verified our analytical results numerically.

For i.i.d. DMs, we can also find the GER in a straightforward manner. Let $\Phi = \frac{\eta-1}{2}$, the smallest possible minority for η total DMs, and let us consider only odd values of η to avoid ties. Then we have

$$\mathrm{GER}_\iota = \sum_{\phi=0}^{\Phi} \binom{\eta}{\phi} \mathrm{LER}_\iota^{\eta-\phi} (1 - \mathrm{LER})_\iota^{\phi}, \tag{3.3}$$

where ϕ represents the number of DMs in the minority.

3.2. The η-First Scheme. The general form of the Majority First scheme is the η-First rule, in which the fusion center returns a decision once η DMs have returned the same individual decision. Realistically, any implementation of this scheme should require that $\eta \leq \Upsilon$, the minimal majority: if $\eta > \Upsilon$, there is no guarantee that the fusion center's rule will ever be satisfied. However, it is still possible to calculate the cdf and pdf of GDTs given that the group does reach a decision in finite time, and we provide formulas for doing so below. When $\eta = \Upsilon$, the η-First rule is identical to the Majority First scheme, and when $\eta = 1$, the η-First rule is identical to the Race scheme. This rule can easily accommodate both even and odd N and η.

If η DMs finish by time t_g and agree, then up to $(N - \eta)$ DMs do not contribute. Using this, we can construct the cdf of GDTs, $q_{g\eta}^{\mathrm{Fg}N}$ (superscript: η-[F]irst scheme, [g]eneral individual DM, N DMs; subscript: [g]roup pdf of GDTs, [η] DMs must agree for the fusion center to finish). Because of the ordering that occurs under this scheme, our formula is split into two cases: $\eta \le \Theta$ and $\eta \ge \Upsilon$. For simplicity, we define

$$\Gamma_{\mathcal{I}}^{\gamma \mathrm{g}} = \sum_{i_1=1}^{(\eta+\gamma+1)} \sum_{i_2=i_1+1}^{(\eta+\gamma+2)} \cdots \sum_{i_{(N-\eta-\gamma)}=i_{(N-\eta-\gamma-1)}+1}^{N} \left(\prod_{m=1}^{N-\eta-\gamma} [1 - q_{i_m}] \right),$$

$$\Lambda_{\mathcal{J}\mathbb{D}}^{\xi \mathrm{g}} = \sum_{\substack{j_1=1,\\ j_1 \notin \mathcal{I}}}^{(N-\xi+1)} \sum_{\substack{j_2=j_1+1,\\ j_2 \notin \mathcal{I}}}^{(N-\xi+2)} \cdots \sum_{\substack{j_\xi=j_{\xi-1}+1,\\ j_\xi \notin \mathcal{I}}}^{N} \left(\prod_{k=1}^{\xi} q_{j_k \mathbb{D}} \right),$$

where $\Gamma_{\mathcal{I}}^{\gamma \mathrm{g}}$ specifies all unique sets of $(N - \eta - \gamma)$ DMs who do not reach a decision by time t_g, $\Lambda_{\mathcal{J}\mathbb{D}}^{\xi \mathrm{g}}$ specifies all unique sets of ξ DMs who choose decision $\mathbb{D}$ by time t_g when the fusion center chooses $\hat{\mathbb{D}}$ (the other hypothesis: $\hat{\mathbb{D}} \in \{\mathrm{S}, \mathrm{N}\}, \hat{\mathbb{D}} \ne \mathbb{D}$), and $\Gamma_{\mathcal{I}}^{(N-\eta)\mathrm{g}} = \Lambda_{\mathcal{J}\mathbb{D}}^{0\mathrm{g}} = 1$. Then, for $\eta \le \Theta$, we have

$$\begin{aligned} q_{g(\eta \le \Theta)}^{\mathrm{Fg}N} = &\sum_{\gamma=0}^{\eta-1} \left(\Gamma_{\mathcal{I}}^{\gamma \mathrm{g}} \sum_{\xi=0}^{\gamma} \left[\Lambda_{\mathcal{J}\mathrm{S}}^{\xi \mathrm{g}} \prod_{\substack{\ell=1,\\ \ell \notin \mathcal{I},\mathcal{J}}}^{N} q_{\ell \mathrm{N}} + \Lambda_{\mathcal{J}\mathrm{N}}^{\xi \mathrm{g}} \prod_{\substack{\ell=1,\\ \ell \notin \mathcal{I},\mathcal{J}}}^{N} q_{\ell \mathrm{S}} \right] \right) \\ &+ \sum_{\gamma=\eta}^{N-\eta} \left(\Gamma_{\mathcal{I}}^{\gamma \mathrm{g}} \sum_{\xi=0}^{\eta+\gamma} \Lambda_{\mathcal{J}\mathrm{S}}^{\xi \mathrm{g}} \prod_{\substack{\ell=1,\\ \ell \notin \mathcal{I},\mathcal{J}}}^{N} q_{\ell \mathrm{N}} \right). \end{aligned} \tag{3.4}$$

The formula for $q_{g(\eta \ge \Upsilon)}^{\mathrm{Fg}N}$ is essentially the same, except that it only includes the first term of (3.4) and the limit on the outermost summation in that term is $(N-\eta)$ instead of $(\eta - 1)$. Here, γ denotes the maximum number of excess members in the subgroup that sets the group's decision (i.e., for the $\gamma = 0$ case, exactly η DMs finish by time t_g and agree), and ξ denotes the number of DMs that finish by time t_g but do not agree with the fusion center's final decision. Since $p_{g\eta}^{\mathrm{Fg}}(t_g) = \frac{d}{dt_g}\left[q_{g\eta}^{\mathrm{Fg}}(t_g)\right]$ for both cases of η, it is relatively straightforward to calculate the corresponding formulas for the pdf of GDTs.

Like before, the formula for the cdf of GDTs simplifies considerably for the i.i.d. case. Let $\Gamma^{\gamma \mathrm{i}} = \binom{N}{N-\eta-\gamma}[1 - q_\iota]^{(N-\eta-\gamma)}$ and $\Lambda_{\mathbb{D}}^{\xi \mathrm{i}} = \binom{\eta+\gamma}{\xi} q_{\iota \mathbb{D}}^{\xi}$. Then we have

$$q_{g(\eta \le \Theta)}^{\mathrm{Fi}N} = \sum_{\gamma=0}^{\eta-1} \Gamma^{\gamma \mathrm{i}} \left(\sum_{\xi=0}^{\gamma} \left[\Lambda_{\mathrm{S}}^{\xi \mathrm{i}} q_{\iota \mathrm{N}}^{\eta+\gamma-\xi} + \Lambda_{\mathrm{N}}^{\xi \mathrm{i}} q_{\iota \mathrm{S}}^{\eta+\gamma-\xi} \right] \right) + \sum_{\gamma=\eta}^{N-\eta} \Gamma^{\gamma \mathrm{i}} \sum_{\xi=0}^{\eta+\gamma} \Lambda_{S}^{\xi \mathrm{i}} q_{\iota \mathrm{N}}^{\eta+\gamma-\xi}, \tag{3.5}$$

and again, the formula for $q_{g(\eta \ge \Upsilon)}^{\mathrm{Fi}N}$ contains only the first term, with the limit on the first summation sign modified to $(N - \eta)$. For $N = 11$, the effect of increasing η from 1 to N is shown in Figure 3.2. For $\eta = 1$, the η-First scheme is equivalent to the Race scheme, and for $\eta = \Upsilon$, it is equivalent to the Majority First scheme. We have verified these analytical results with simulation.

For the i.i.d. case, we can also calculate the GER for the η-First scheme. Like for the pdf of GDTs, the limits in the formula change for different values of η relative to N. If $\eta \le \Upsilon$, then

$$\mathrm{GER}_{\iota, \eta \le \Upsilon} = \sum_{\phi=0}^{\eta-1} \binom{\eta + \phi - 1}{\phi} (1 - \mathrm{LER}_\iota)^{\phi}\, \mathrm{LER}_\iota^{\eta}. \tag{3.6}$$

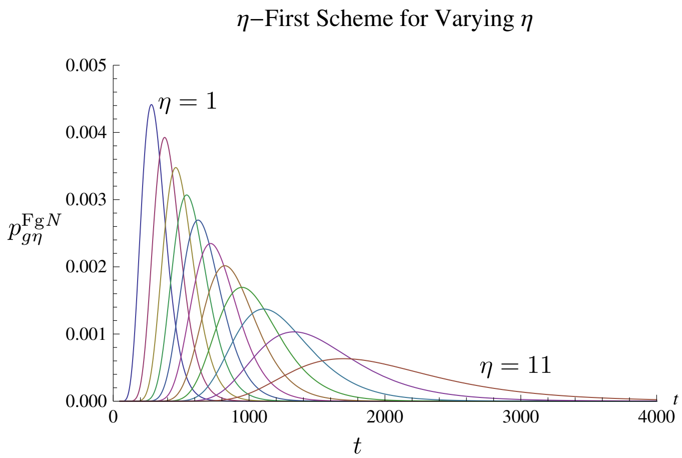


Fig. 3.2 *Analytical results for the η-First group rule for different values of η, with $N = 11$. For $\eta = 1$, this rule is identical to the Race scheme, and for $\eta = \Upsilon$, this rule is identical to the Majority First scheme. This rule is not guaranteed to finish for $\eta > \Upsilon$; however, we can still calculate the pdf of GDTs given that the group does finish for those values of η, which is shown in this figure.*

The formula for $\mathrm{GER}_{\iota,\eta>\Upsilon}$ is the same except that the limit on the summation is $(N - \eta)$ instead of $(\eta - 1)$.

3.3. Discussion. For the particular examples shown, the pdf of GDTs for the correct answer dominates the overall pdf of GDTs. Therefore, while the formulas are different for the η-Total and η-First schemes, Figures 3.1 and 3.2 are similar. However, we stress that the pdfs in each figure for each value of η are not identical, except for the case of $\eta = 1$, where both schemes are equivalent to the Race scheme. The difference between the curves of the η-Total and η-First schemes with i.i.d. DMs will be more pronounced for DMs whose pdf of GDTs is different for the correct hypothesis than for the incorrect hypothesis, or who have a higher LER.

4. Conclusion. We derived the pdf of GDTs for a general group using a fusion center to apply the Race, Majority Total, or Majority First scheme in a 2AFC task, given the members' pdfs of LDTs. We also illustrated these results for the case of N i.i.d. DMs using the SPRT for each scheme, and compared the GDT of the three schemes against each other and against the LDT of an individual under set-LER and set-GER conditions. In our analysis, the best overall performance was achieved by the Majority First scheme, in that it provided a given GER with higher LERs than the Race scheme, was not hampered by having to wait for the slowest DM like the Majority Total scheme, and was the most robust to various types of possible problems with the individual DMs. We also showed the results of two natural generalizations of our group decision rules: the η-Total and η-First schemes.

The models presented here are relevant to many situations in which a group of DMs must reach a collective decision in a sequential task. Since our models only use each member's performance to find the group's performance, they can easily be used for cybernetic groups (including both human observers and detectors), and naturally extend to hierarchical and more complicated group topologies in which some "members" are groups. Our models are interesting because they present a novel and general way in which one can intuitively yet mathematically model a group's performance based on its members' statistics, establish a reasonable base model which can be extended to build up more complicated models for realistic groups, and provide a means by which one can compare different group decision rules.

REFERENCES

[1] M. M. Al-Ibrahim, *On Distributed Sequential Hypothesis Testing*, Ph.D. thesis, Syracuse University, Syracuse, NY, 1989.

[2] M. M. Al-Ibrahim and P. K. Varshney, *A simple multi-sensor sequential detection procedure*, in Proceedings of the 27th Conference on Decision and Control, Austin, TX, 1988, pp. 2479–2483.

[3] R. J. Audley, *A stochastic model for individual choice behavior*, Psych. Rev., 67 (1960), pp. 1–15.

[4] R. Ben-Yashar and J. Paroush, *A nonasymptotic Condorcet jury theorem*, Soc. Choice Welf., 17 (2000), pp. 189–199.

[5] D. Berend and L. Sapir, *Between the expert and majority rules*, Adv. in App. Probab., 35 (2003), pp. 941–960.

[6] J. O. Berger, *Statistical Decision Theory and Bayesian Analysis*, Springer-Verlag, New York, 1985.

[7] D. Black, *On the rationale of group decision-making*, J. Political Economy, 56 (1948), pp. 23–34.

[8] R. Bogacz, E. Brown, J. Moehlis, P. Holmes, and J. D. Cohen, *The physics of optimal decision-making: A formal analysis of models of performance in two-alternative forced-choice tasks*, Psych. Rev., 113 (2006), pp. 700–765.

[9] R. Bogacz, P. T. Hu, P. Holmes, and J. D. Cohen, *Do humans produce the speed-accuracy trade-off that maximizes reward rate?*, Quart. J. Exp. Psych., 63 (2010), pp. 863–891.

[10] Q. Cheng, P. K. Varshney, K. G. Mehrotra, and C. K. Mohan, *Bandwidth management in distributed sequential detection*, IEEE Trans. Inform. Theory, 51 (2005), pp. 2954–2961.

[11] T. H. Chung and J. W. Burdick, *A decision-making framework for control strategies in probabilistic search*, in Proceedings of the 2007 IEEE International Conference on Robotics and Automation, Roma, Italy, 2007, pp. 4386–4393.

[12] N. C. Condorcet, *Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix, Paris*, 1785. English translation: I. McLean and F. Hewitt, eds., Condercet: Foundations of Social Choice and Political Theory, Edward Elgar, Brookfield, VT, 1984.

[13] L. Conradt, *When it pays to share decisions*, Nature, 471 (2011), pp. 40–41.

[14] R. W. Crow and S. C. Schwartz, *Quickest detection for sequential decentralized decision systems*, IEEE Trans. Aerosp. Electron. Syst., 32 (1996), pp. 267–283.

[15] J. H. Davis, *Some compelling intuitions about group consensus decisions, theoretical and empirical research, and interpersonal aggregation phenomena: Selected examples*, 1950–1990, Organ. Behav. Human Decision Process., 52 (1992), pp. 3–38.

[16] A. R. Dennis, *Information exchange and use in group decision making: You can lead a group to information, but you can't make it think*, Manag. Inform. Syst. Q., 20 (1996), pp. 433–457.

[17] B. Eisenberg, *Multihypothesis Problems*, Marcel Dekker, New York, 1992, Chap. 9, pp. 229–244.

[18] B. Eisenberg, B. K. Ghosh, and G. Simons, *Properties of generalized sequential probability ratio tests*, Ann. Statist., 4 (1976), pp. 237–251.

[19] G. Ferrari and R. Pagliari, *Decentralized binary detection with noisy communication links*, IEEE Trans. Aerosp. Electron. Sys., 42 (2006), pp. 1554–1563.

[20] W. L. Francis, *Legislative committee systems, optimal committee size, and the costs of decision making*, J. Politics, 44 (1982), pp. 822–837.

[21] B. K. Ghosh, *Sequential Tests of Statistical Hypotheses*, Addison-Wesley, Reading, MA, 1970.

[22] Z. Govindarajulu, *Sequential Statistical Procedures*, Academic Press, New York, 1975.

[23] D. M. Green and J. A. Swets, *Signal Detection Theory and Psychophysics*, John Wiley & Sons, New York, 1966.

[24] B. Grofman, G. Owen, and S. L. Feld, *Thirteen theorems in search of the truth*, Theory and Decision, 15 (1983), pp. 261–278.

[25] P. Holmes, E. Shea-Brown, J. Moehlis, R. Bogacz, J. Gao, G. Aston-Jones, E. Clayton, J. Rajkowski, and J. Cohen, *Optimal decisions: From neural spikes, through stochastic differential equations, to behavior*, IEICE Trans. Fundam., E88-A (2005), pp. 2496–2503.

[26] A. M. Hussain, *Multisensor distributed sequential detection*, IEEE Trans. Aerosp. Electron. Syst., 30 (1994), pp. 698–708.
[27] D. Karotkin, *The network of weighted majority rules and weighted majority games*, Games Econom. Behav., 22 (1998), pp. 299–315.
[28] D. Karotkin and J. Paroush, *Optimum committee size: Quality-versus-quantity dilemma*, Soc. Choice Welf., 20 (2003), pp. 429–441.
[29] M. Kimura, *Group Decision-Making Models*, Ph.D. thesis, University of California, Santa Barbara, 2011. (Please contact the authors to request a copy.)
[30] M. W. Koch and K. T. Malone, *Vehicle classification in infrared video using the sequential probability ratio test*, in Augmented Vision Perception in Infrared, Advances in Pattern Recognition, Springer, London, 2009, pp. 63–84.
[31] O. P. Kriedl and A. S. Willsky, *An efficient message-passing algorithm for optimizing decentralized detection networks*, in Proceedings of the IEEE Conference on Decision and Control, 2006, pp. 6776–6783.
[32] C. C. Lee and J. B. Thomas, *A modified sequential detection procedure*, IEEE Trans. Inform. Theory, 30 (1984), pp. 16–23.
[33] M. D. Lee and M. J. Paradowski, *Group decision-making on an optimal stopping problem*, J. Prob. Solv., 1 (2007), pp. 53–73.
[34] E. L. Lehmann, *Testing Statistical Hypotheses*, John Wiley & Sons, New York, 1959.
[35] X. R. Li, Y. Zhu, J. Wang, and C. Han, *Optimal linear estimation fusion. Part* I: *Unified fusion rules*, IEEE Trans. Inform. Theory, 49 (2003), pp. 2192–2208.
[36] C. List, *Democracy in animal groups: A political science perspective*, Trends Ecology Evol., 19 (2004), pp. 168–169.
[37] K. Liu and A. M. Sayeed, *Type-based decentralized detection in wireless sensor networks*, IEEE Trans. Signal Processing, 55 (2007), pp. 1899–1910.
[38] S. Luan, R. D. Sorkin, and J. Itzkowitz, *Weighting information from outside sources: A biased process*, J. Behav. Decision Making, 17 (2004), pp. 95–116.
[39] N. A Macmillan and C. D. Creelman, *Detection Theory: A User's Guide*, Cambridge University Press, New York, 1991.
[40] J. D. Papastavrou and M. Athans, *A distributed hypothesis-testing team decision problem with communications cost*, in Proceedings of the 25th Conference on Decision and Control, Athens, Greece, 1986, pp. 219–225.
[41] R. Ratcliff, *A theory of memory retrieval*, Psych. Rev., 85 (1978), pp. 59–108.
[42] R. Ratcliff and G. McKoon, *The diffusion decision model: Theory and data for two-choice decision tasks*, Neural Comput., 20 (2008), pp. 873–922.
[43] V. N. S. Samarasooriya and P. K. Varshney, *Sequential approach to asynchronous decision fusion*, Opt. Eng., 35 (1996), pp. 625–633.
[44] N. J. Schmitz, *Sequential Detection Theory*, Marcel Dekker, New York, 1992, Chap. 17, pp. 407–428.
[45] R. D. Sorkin and H. Dai, *Signal detection analysis of the ideal group*, Org. Behav. Human Decision Process., 60 (1994), pp. 1–13.
[46] R. D. Sorkin, C. J. Hays, and R. West, *Signal-detection analysis of group decision making*, Psych. Rev., 108 (2001), pp. 183–203.
[47] R. D. Sorkin, R. West, and D. E. Robinson, *Group performance depends on the majority rule*, Psych. Sci., 9 (1998), pp. 456–463.
[48] R. D. Sorkin and D. D. Woods, *Systems with human monitors: A signal detection analysis*, Human-Comput. Interact., 1 (1985), pp. 49–75.
[49] G. Stasser and W. Titus, *Pooling of unshared information in group decision making: Biased information sampling during discussion*, J. Personality Soc. Psych., 48 (1985), pp. 1467–1478.
[50] J. N. Tsitsiklis, *Decentralized detection*, Adv. Stat. Sig. Process., 2 (1993), pp. 297–344.
[51] P. K. Varshney, *Distributed Detection and Data Fusion*, Springer, New York, 1997.
[52] V. V. Veeravalli, *Sequential decision fusion: Theory and applications*, J. Franklin Inst., 336 (1999), pp. 301–322.
[53] A. Wald, *Sequential Analysis*, John Wiley & Sons, New York, 1947.
[54] A. Wald and J. Wolfowitz, *Optimum character of the sequential probability ratio test*, Ann. Math. Statist., 19 (1948), pp. 326–339.
[55] A. J. W. Ward, J. E. Herbert-Read, D. J. T. Sumpter, and J. Krause, *Fast and accurate decisions through collective vigilance in fish shoals*, Proc. Natl. Acad. Sci. USA, 108 (2011), pp. 2312–2315.
[56] J. R. Winquist and J. R. Larson Jr., *Information pooling: When it impacts group decision making*, J. Person. Soc. Psych., 74 (1998), pp. 371–377.
[57] I. Yaniv, *The benefits of additional opinions*, Curr. Direc. Psych. Sci., 13 (2004), pp. 75–78.