2D proper rational matrices and causal input/output representations of 2D behavioral systems

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1 Introduction

In the behavioral approach, a dynamical system is essentially described through the set of its admissible trajectories, without making any apriori distinction between input and output variables and without setting any causality relation between them.

This distinction, which is the characteristic feature of input/output (I/O) models, can be performed a posteriori, introducing the concept of free variables, that are called in this way because their value can be arbitrarily assigned. As a consequence we have that, at least for the class of auto-regressive (AR) systems, we can extract an I/O description [3], starting from a behavioral model.

The first question that naturally arises when dealing with I/O descriptions is how to define causality. In case of discrete 2D systems, that is the one we are interested in, the matter is complex, since the plane $\mathbb{Z}^2$ lacks of a natural total ordering. As a consequence, the choice of the causality cone C is not as straightforward as in 1D case. In the classical I/O approach [2], the only admissible causality cone is $C = \mathbb{N}^2$, so that causality is synonymous of quarter plane causality. In this paper we consider an extension to this notion of causality, by assuming that C is an arbitrary cone in $\mathbb{Z}^2$.

The characterization of causality of 2D systems is based on the concept of 2D proper rational matrix. This concept has been introduced and analyzed for a particular class of cones in [4, 1]. The characterization of 2D proper rational matrices allowed to obtain some interesting existence results regarding causal I/O representations of 2D behavioral systems. The aim of this paper is to investigate the causal I/O representation of 2D behavioral systems in another direction. More precisely, starting from the kernel representation of a 2D behavioral system, we want to obtain an efficient method for determining all the causal relations between the variables of the system, given in terms of the set of all causality cones. This result provides a full characterization of the causality structure of the behavioral system. This problem is solved by extending the concept of of 2D proper rational matrix to general cones and by finding a suitable characterization of this class of rational matrices.

2 Cones and 2D proper rational matrices

In this section we will extend the notions of proper rational function and matrix to the 2D case. Some results in this direction can be found also in [4].

Before giving the definition of properness in the 2D case we need to introduce the notion of cone and of regular cone in $\mathbb{Z}^2$.

Definition A cone C is a subset of $\mathbb{Z}^2$ such that there exists a pair of elements $(d_1, d_2) \in \mathbb{Z}^2$ satisfying

$$C = \mathbb{Z}^2 \cap \{\alpha d_1 + \beta d_2 \in \mathbb{R}^2 \ : \ \alpha, \beta \in \mathbb{R},\ \alpha, \beta \geq 0\},$$

and such that the matrix $D \in \mathbb{Z}^{2\times 2}$, whose columns coincide with d_1 and d_2, is nonsingular $\det(D) \neq 0$. A cone C is said to be *regular* if there exists a pair of elements $(d_1, d_2) \in \mathbb{Z}^2$ such that

$$C = \{\alpha d_1 + \beta d_2 : \alpha, \beta \in \mathbb{N}\}$$

and such that $\det(D) = \pm 1$, where D is the matrix defined from d_1, d_2 as above.

It can be shown that a regular cone C is always isomorphic to $\mathbb{N}^2$. Moreover, given a cone C, it is easy to prove that, up to a change of coordinates, there is no loss of generality in assuming it to be specified as

$$C = \{(i,j) \in \mathbb{N}^2 \ : \ j \leq mi\}, \qquad (1)$$

where m is a suitable positive rational number.

Given a Laurent polynomial in two indeterminates

$$p(z_1, z_2) = \sum_{(i,j)\in S} p_{ij} z_1^i z_2^j,$$

where S is a finite subset of $\mathbb{Z}^2$, by $\mathrm{supp}(p)$ we mean the set of points $(i,j) \in \mathbb{Z}^2$ corresponding to nonzero coefficients of $p(z_1, z_2)$

$$\mathrm{supp}(p) = \{(i,j) \in \mathbb{Z}^2 \ : \ p_{ij} \neq 0\}.$$

Let C be a cone. With the symbol $\mathbb{R}[z_1, z_2, z_1^{-1}, z_2^{-1}]_C$ we mean the ring of polynomials whose support is contained in C. Similar definitions can be immediately extended to polynomial matrices and power series. More precisely, with the symbol $\mathbb{R}[[z_1, z_2, z_1^{-1}, z_2^{-1}]]_C$ we mean the ring of formal power series

$$s(z_1, z_2) = \sum_{(i,j)\in C} s_{ij} z_1^i, z_2^j.$$

Notice that $\mathbb{R}[z_1, z_2, z_1^{-1}, z_2^{-1}]_C$ is always a ring, but, unless C is regular, this ring lacks of many of the properties usually possessed by polynomial rings (it can be seen for instance that it is not in general a unique factorization domain).

For sake of simplicity, from now on we will denote by $\mathbf{z}$ the pair (z_1, z_2). Consequently we will use the following shorthand notations

$$\mathbb{R}[\mathbf{z}, \mathbf{z}^{-1}]_C := \mathbb{R}[z_1, z_2, z_1^{-1}, z_2^{-1}]_C \quad (2)$$

$$\mathbb{R}[[\mathbf{z}, \mathbf{z}^{-1}]]_C := \mathbb{R}[[z_1, z_2, z_1^{-1}, z_2^{-1}]]_C \quad (3)$$

$$\mathbb{R}(\mathbf{z}) := \mathbb{R}(z_1, z_2) \quad (4)$$

where the latter denotes the field of rational functions in two indeterminates.

If we think of a polynomial matrix $A(z_1, z_2) \in \mathbb{R}[\mathbf{z}, \mathbf{z}^{-1}]^{p\times m}$ as a polynomial with matrix coefficients, we can write it as

$$A(z_1, z_2) = \sum_{(i,j)\in S} A_{ij} z_1^i z_2^j, \quad (5)$$

where $A_{ij} \in \mathbb{R}^{p\times m}$ and S is a finite subset of $\mathbb{Z}^2$. By degree-zero coefficient of $A(z_1, z_2)$ we mean the matrix A_{00}.

Definition A 2D rational function $h \in \mathbb{R}(\mathbf{z})$ is said to be *proper* with respect to a cone C if there exist $p, q \in \mathbb{R}[\mathbf{z}, \mathbf{z}^{-1}]_C$ such that $h = q/p$ and the zero-degree coefficient of p is nonzero.

We give now a theorem providing several equivalent characterizations of 2D proper rational functions. Observe that the theorem that follows has already been proved for regular cones in [4, Lemma 3].

Theorem 1 *Let $h \in \mathbb{R}(\mathbf{z})$ and let C be any cone in $\mathbb{Z}^2$. The following facts are equivalent.*

1. *h is proper with respect to C.*
2. *There exists a unique formal power series $y \in \mathbb{R}[[\mathbf{z}, \mathbf{z}^{-1}]]_C$ such that for all $p, q \in \mathbb{R}[\mathbf{z}, \mathbf{z}^{-1}]_C$ such that $h = q/p$ we have that*

$$py = q.$$

3. *Let $p, q \in \mathbb{R}[\mathbf{z}, \mathbf{z}^{-1}]$ be coprime polynomials such that $h = q/p$. Then there exists $n_1, n_2 \in \mathbb{Z}$ such that*
   (a) $\hat{p} := z_1^{n_1} z_2^{n_2} p$, $\hat{q} := z_1^{n_1} z_2^{n_2} q \in \mathbb{R}[\mathbf{z}, \mathbf{z}^{-1}]_C$;
   (b) The zero-degree coefficient of $\hat{p}$ is nonzero.

Notice that for general cones condition 3 provides the only way to check algorithmically the properness of a 2D rational function. We consider now the matrix case.

Definition A 2D rational matrix $H \in \mathbb{R}(\mathbf{z})^{p\times m}$ is said to be *proper* with respect to a cone C if its entries are 2D rational functions that are proper with respect to C.

We give also in this case a theorem providing several equivalent characterizations of a 2D proper rational matrix.

Theorem 2 *Let $H \in \mathbb{R}(\mathbf{z})^{p\times m}$. The following facts are equivalent.*

1. *H is proper with respect to a cone C.*
2. *There exist $P \in \mathbb{R}[\mathbf{z}, \mathbf{z}^{-1}]_C^{p\times p}$ and $Q \in \mathbb{R}[\mathbf{z}, \mathbf{z}^{-1}]_C^{p\times m}$ such that $H = P^{-1}Q$ and such that the degree-zero coefficient of P is an invertible square matrix.*
3. *There exists a unique formal power series $Y \in \mathbb{R}[[\mathbf{z}, \mathbf{z}^{-1}]]_C^{p\times m}$ such that for all $P \in \mathbb{R}[\mathbf{z}, \mathbf{z}^{-1}]_C^{p\times p}$ and $Q \in \mathbb{R}[\mathbf{z}, \mathbf{z}^{-1}]_C^{p\times m}$ such that $H = P^{-1}Q$ we have that $PY = Q$.*

The definition of 2D properness, by translating matrix properness into scalar properness, provides in this case the only way to verify algorithmically whether a rational matrix is proper or not. An efficient algorithmic check can be done as follows:

Algorithm: Given a rational matrix $H \in \mathbb{R}(\mathbf{z})^{p\times m}$.

1. Represent it as $H = [q_{ij}/p_{ij}]$, where $q_{ij}, p_{ij} \in \mathbb{R}[\mathbf{z}, \mathbf{z}^{-1}]$ are coprime.
2. Let p be the least common multiple of p_{ij} and $\bar{q}_{ij} := q_{ij}p/p_{ij}$ so that $H = [\bar{q}_{ij}/p]$.
3. We have that H is proper w.r. to a cone C if and only if there exists $n_1, n_2 \in \mathbb{Z}$ such that
   a) $\hat{p} := z_1^{n_1} z_2^{n_2} p$, $\hat{q}_{ij} := z_1^{n_1} z_2^{n_2} \bar{q}_{ij} \in \mathbb{R}[\mathbf{z}, \mathbf{z}^{-1}]_C$.
   b) The zero-degree coefficient of $\hat{p}$ is nonzero.

3 2D systems in the behavioral approach

In the behavioral approach a dynamical system is defined by a triple $\Sigma = (T, W, \mathcal{B})$, where T is the time domain, W is the signal alphabet and $\mathcal{B} \subset W^T$, the *behavior*, is the set of admissible trajectories. For 2D systems we assume that $T = \mathbb{Z}^2$ and $W = \mathbb{R}^q$. We refer the interested reader to [3] for a more complete introduction to 2D behavioral systems theory.

An important subclass of 2D systems is constituted by the so-called *auto-regressive* (AR) 2D systems. They are 2D systems whose behavior is given by the set of solutions

$w \in (\mathbb{R}^q)^{\mathbb{Z}^2}$ (set of all q-dimensional signals defined on $\mathbb{Z}^2$) of a linear difference equation of the following kind

$$\sum_{(i,j)\in S} R_{ij} w(h+i, k+j) = 0, \qquad \forall (h,k) \in \mathbb{Z}^2 \qquad (6)$$

where $R_{ij} \in \mathbb{R}^{l\times q}$ and S is a finite subset of $\mathbb{Z}^2$. Notice that any polynomial matrix

$$R = \sum_{(i,j)\in S} R_{ij} z_1^i z_2^j \in \mathbb{R}[\mathbf{z}, \mathbf{z}^{-1}]^{l\times q}$$

naturally induces a polynomial linear operator

$$R(\sigma_1, \sigma_2) \,:\, (\mathbb{R}^q)^{\mathbf{Z}^2} \longrightarrow (\mathbb{R}^l)^{\mathbf{Z}^2},$$

in the following way

$$(R(\sigma_1,\sigma_2)w)(h,k) = \sum R_{ij} w(h+i, k+j), \quad \forall (h,k) \in \mathbb{Z}^2.$$

In this way we have that the behavior $\mathcal{B}$ determined by the difference equation (6) coincides with $\ker R(\sigma_1, \sigma_2)$ and that the behavior of an AR system can always be represented as the kernel of a polynomial linear operator which is called *kernel representation.*

4 Passing from kernel to input/output representations

Given a behavioral model of a dynamical system, we could wonder whether an input/output representation of the same system can be obtained or not. The answer to this question implies to settle a cause-effect relation between the components of the signal. It can be proved (see [3]) that if

$$\operatorname{rank} R(z_1, z_2) = p,$$

then it is possible to split the components of w in $m := q-p$ inputs (free variables) and p outputs (non-free variables). More precisely, if S is any permutation matrix such that

$$RS = [P| - Q],$$

where $P \in \mathbb{R}[\mathbf{z}, \mathbf{z}^{-1}]^{l\times p}$, $Q \in \mathbb{R}[\mathbf{z}, \mathbf{z}^{-1}]^{l\times(q-p)}$ and $\operatorname{rank} P = p$, then we say that the pair of matrices (P, Q) provides an input-output representation of the system because they satisfy the properties of the following definition.

Definition [3, 4] Given a 2D AR system $(\mathbb{Z}^2, \mathbb{R}^q, \ker R(\sigma_1, \sigma_2))$, the difference equation

$$P(\sigma_1,\sigma_2)y = Q(\sigma_1,\sigma_2)u, \qquad (7)$$

where $p+m = q$, $P \in \mathbb{R}[\mathbf{z}, \mathbf{z}^{-1}]^{l\times p}$, $Q \in \mathbb{R}[\mathbf{z}, \mathbf{z}^{-1}]^{l\times m}$ and where $y \in (\mathbb{R}^p)^{\mathbf{Z}^2}$ and $u \in (\mathbb{R}^m)^{\mathbf{Z}^2}$, is an *input/output representation* (I/O representation) of Σ if

1. $\mathcal{B} = \left\{ S \begin{bmatrix} y \\ u \end{bmatrix} \;:\; P(\sigma_1,\sigma_2)y = Q(\sigma_1,\sigma_2)u \right\}$, where S is a suitable $q \times q$ permutation matrix;
2. u is free, i.e. for all $u \in (\mathbb{R}^m)^{\mathbb{Z}^2}$ there exists $y \in (\mathbb{R}^p)^{\mathbf{Z}^2}$ such that (7) holds;
3. no other component in y is free.

Observe that, starting from an AR behavioral model, it is possible to extract finitely many different I/O descriptions. They are obtained by choosing different permutation matrices S, with the only rank condition to be satisfied.

The concept of causality is strictly related with I/O representations. In the 2D case its definition is more involved than for 1D systems, since there are different possible ways to order the time domain $T = \mathbb{Z}^2$. As a consequence, there is more freedom in the choice of the causality cone. Given a cone $\mathcal{C}$, by the symbol $(\mathbb{R}^m)^{\mathbb{Z}^2}_{\mathcal{C}}$ we mean the set of all m-dimensional signals defined on $\mathbb{Z}^2$ and supported in $\mathcal{C}$.

Definition The I/O representation (7) is said to be causal with respect to the cone $\mathcal{C}$ if for any $u \in (\mathbb{R}^m)^{\mathbb{Z}^2}_{\mathcal{C}}$ there exists $y \in (\mathbb{R}^p)^{\mathbf{Z}^2}_{\mathcal{C}}$ such that (7) holds.

Notice that the definition above suggests that the influence of u on y is causal with respect to $\mathcal{C}$. In can be shown moreover [4, Lemma 1] that y in the previous definition is uniquely determined from u.

5 Characterization of causal I/O representations

In [4], a characterization of causal I/O representations with respect to regular cones has been given. Our aim here is to extend and generalize those results to general cones. Some of these results can be generalized in a straightforward way. This is the case for Proposition 3 [4], which will be used in the next. This proposition, stated for regular cones, guarantees that the causality of an I/O representation

$$P(\sigma_1,\sigma_2)y = Q(\sigma_1,\sigma_2)u, \qquad (8)$$

depends only on a coprime representation of the polynomial matrices specifying the system. So, if $\bar{P} \in \mathbb{R}[\mathbf{z}, \mathbf{z}^{-1}]^{p\times p}$ and $\bar{Q} \in \mathbb{R}[\mathbf{z}, \mathbf{z}^{-1}]^{p\times m}$ are coprime polynomial matrices such that

$$P = F\bar{P}, \quad Q = F\bar{Q},$$

with F a full column rank polynomial matrix of suitable dimensions, then (8) is causal with respect to a regular cone $\mathcal{C}_r$ if and only if

$$\bar{P}(\sigma_1,\sigma_2)y = \bar{Q}(\sigma_1,\sigma_2)u$$

is causal with respect to it. It is easy to see that the proof still holds if we consider general cones.

Let (P,Q) be an I/O representation of a 2D AR system which is causal w.r. to a cone $\mathcal{C}$. Define the inputs $\delta^{(i)}$, $i=1,\dots,m$, as

$$\delta^{(i)}(t) := \begin{cases} e_i & t=(0,0) \\ 0 & \text{otherwise} \end{cases}$$

where e_i is the i-th vector of the canonical base in $\mathbb{R}^m$. If $y^{(i)} \in (\mathbb{R}^p)^{\mathbb{Z}^2}_{\mathcal{C}}$ is the corresponding output, namely

$$P(\sigma_1,\sigma_2)y^{(i)} = Q(\sigma_1,\sigma_2)\delta^{(i)}, \tag{9}$$

we define the impulse response of the 2D system to be the matrix valued sequence

$$Y := [y^{(1)} \dots y^{(m)}] \in (\mathbb{R}^{p\times m})^{\mathbb{Z}^2}_{\mathcal{C}}.$$

It is worth pointing out that, as shown in [4], the causality of an I/O representation is equivalent to the existence of the impulse response, since the impulse response determines the way in which the system maps input signals supported in $\mathcal{C}$ into output y by the convolution

$$y(h,k) := \sum_{(i,j)\in\mathbb{Z}^2} Y(h-i,k-j)u(i,j).$$

Notice that, since u and Y are both supported in $\mathcal{C}$, the sum is always finite and moreover also the support of y is included in $\mathcal{C}$.

Now we are in a position to state the following theorem which allows us to characterize the causality structure of a 2D AR system.

Theorem 3 *Let*

$$P(\sigma_1,\sigma_2)y = Q(\sigma_1,\sigma_2)u \tag{10}$$

with $P \in \mathbb{R}[\mathbf{z},\mathbf{z}^{-1}]^{l\times p}$ *and* $Q \in \mathbb{R}[\mathbf{z},\mathbf{z}^{-1}]^{l\times m}$, *be an I/O representation of a 2D AR system. Then (10) is causal w.r. to a cone* $\mathcal{C}$ *if and only if the rational matrix* $H \in \mathbb{R}(\mathbf{z})^{p\times m}$ *such that* $Q = PH$ *is proper w.r. to the cone* $-\mathcal{C}$.

6 Minimal causality cones

Consider an I/O representation (P,Q). Theorem 3 allows us to determine the set of all cones $\mathcal{C}$ such that (P,Q) is causal w.r. to $\mathcal{C}$. These cones are called *causality cones* for the I/O representation. Notice that, if $\mathcal{C}$ is a causality cone and $\mathcal{C}' \supseteq \mathcal{C}$, then also $\mathcal{C}'$ is a causality cone. Therefore the set of causality cones is completely determined by its finite subset $\mathcal{M}(P,Q)$ constituted by the *minimal* causality cones.

In practice the construction of this set reduces to a simple procedure based on the previous theorem. Let $H \in \mathbb{R}(\mathbf{z})^{p\times m}$ be the rational matrix such that $Q = HP$ and represent it as $H = [q_{ij}/p_{ij}]$, where $q_{ij}, p_{ij} \in \mathbb{R}[\mathbf{z},\mathbf{z}^{-1}]$ are coprime. Let p be the least common multiple of p_{ij} and $\bar{q}_{ij} := q_{ij}p/p_{ij}$ so that $H = [\bar{q}_{ij}/p]$. As suggested in the algorithmic check of properness proposed above, H is proper w.r. to a cone $\mathcal{C}$ if and only if there exist $n_1, n_2 \in \mathbb{Z}$ such that $\hat{p} := z_1^{n_1}z_2^{n_2}p$, $\hat{q}_{ij} := z_1^{n_1}z_2^{n_2}\bar{q}_{ij} \in \mathbb{R}[\mathbf{z},\mathbf{z}^{-1}]_{\mathcal{C}}$ and the zero-degree coefficient of $\hat{p}$ is nonzero. For this reason the finite set of minimal causality cones can be obtained from the polynomials p and $\bar{q}_{ij}$ in the following way:

1. Determine the convex hull of supp (p) and from this the finite set $V = \{v_1,\dots,v_k\}$ of the vertices of this convex hull.
2. For each $v_i \in V$ consider the following set of cones
$$\mathcal{C}(v_i) = \{\mathcal{C} \ : \ v_i - \mathcal{C} \supseteq \text{supp } (p) \cup \bigcup_{ij} \text{supp } (\bar{q}_{ij})\}.$$
3. It is clear that, when the set $\mathcal{C}(v_i)$ is nonempty, it contains a cone $\hat{\mathcal{C}}_i$ that is smaller than every other cone in $\mathcal{C}(v_i)$. Then by Theorem 3 the set $\mathcal{M}(P,Q)$ of the minimal causality cones for the I/O representation (P,Q) coincides with the set of all the cones $\hat{\mathcal{C}}_i$.

It may happen that, for a given I/O representation (P,Q), the set $\mathcal{M}(P,Q)$ is empty. However there exists a certain freedom in constructing I/O representation from a kernel representation, which corresponds to the freedom that there exists in the choice of p linearly independent columns in a rank p polynomial matrix $R \in \mathbb{R}[\mathbf{z},\mathbf{z}^{-1}]^{l\times q}$ providing the kernel representation of the AR system. The family of the sets $\mathcal{M}(P,Q)$, when (P,Q) varies in the set of all possible I/O representations of the AR system, provides a complete description of its causality structure. It is important to notice that, as a direct consequence of [4, Theorem 2], we have that there always exists a I/O representation (P,Q) such that $\mathcal{M}(P,Q)$ is nonempty.

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