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TWO-DIMENSIONAL PROPER RATIONAL MATRICES AND CAUSAL INPUT/OUTPUT REPRESENTATIONS OF TWO-DIMENSIONAL BEHAVIORAL SYSTEMS*

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Abstract. The concept of proper rational matrix is strictly connected with the representation of causal transfer matrices. In the two-dimensional (2D) case there is much freedom in defining proper rational matrices. This freedom is connected to the fact that past and future in the 2D case can be determined by a 2D cone. In this way the concept of rational matrix which is proper with respect to a cone can be introduced. Moreover, an algorithm that checks the properness of a rational matrix is proposed. Finally, this algorithm is used for determining all possible causal input/output (I/O) representations of a behavior given by a kernel representation.

Key words. two-dimensional, behavioral approach, input/output representation, proper rational matrices, causality, cones

AMS subject classifications. 93A30, 93B25

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1. Introduction.

In the behavioral approach, a dynamical system is essentially described through the set of its admissible trajectories, without making any a priori distinction between input and output variables and without setting any causality relation between them.

This distinction, which is the characteristic feature of input/output (I/O) models, can be performed a posteriori, introducing the concept of free variables that are called in this way because their value can be arbitrarily assigned. As a consequence we have that, at least for the class of autoregressive (AR) systems, we can extract an I/O description [12, 9, 15, 13], starting from a behavioral model.

The first question that naturally arises when dealing with I/O descriptions is how to define causality. In case of discrete two-dimensional (2D) systems, which is the one we are interested in, the matter is complex, since the plane $\mathbb{Z}^2$ lacks a natural total ordering. As a consequence, the choice of the causality cone $\mathcal{C}$ is not as straightforward as in the one-dimensional (1D) case. In the classical I/O approach [4], the only admissible causality cone is $\mathcal{C} = \mathbb{N}^2$, so that causality is synonymous with quarter plane causality. In this paper we consider an extension of this notion of causality by assuming that $\mathcal{C}$ is an arbitrary cone in $\mathbb{Z}^2$.

The characterization of causality of 2D systems is based on the concept of the 2D proper rational matrix. This concept has been introduced and analyzed for a particular class of cones in [14, 3]. The characterization of 2D proper rational matrices allowed us to obtain some interesting existence results regarding causal I/O representations of 2D behavioral systems. The aim of this paper is to investigate the causal I/O representation of 2D behavioral systems in another direction. More precisely, starting from the kernel representation of a 2D behavioral system, we want to obtain

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an efficient method for determining all the causal relations between the variables of the system, given in terms of the set of all causality cones. This result provides a full characterization of the causality structure of the behavioral system. This problem is solved by extending the concept of the 2D proper rational matrix to general cones and by finding a suitable characterization of this class of rational matrices.

2. 1D proper rational matrices. In this section we will recall some basic definitions and results on proper rational matrices in the 1D case (see [7, 6]).

In this paper we will consider only polynomials having real coefficients. Notice, however, that all the results we will present hold true for any field. A polynomial $p(z)$, in which we allow also negative powers of the indeterminates, is called a Laurent polynomial and can always be written as

$$p(z) = \sum_{i=n}^{N} p_i z^i,$$

where $n \leq N$ are suitable integers. The coefficient p_0 is called zero-degree coefficient of the polynomial p. The set of all the Laurent polynomials has a ring structure with respect to the usual addition and multiplication and it is denoted by the symbol $\mathbb{R}[z, z^{-1}]$. The rings $\mathbb{R}[z]$ and $\mathbb{R}[z^{-1}]$ are both subrings of $\mathbb{R}[z, z^{-1}]$. Consider, moreover, the ring $\mathbb{R}[[z]]$ of formal power series

$$s(z) = \sum_{i=0}^{+\infty} s_i z^i,$$

and define finally the field of rational functions

$$\mathbb{R}(z) := \left\{ \frac{q(z)}{p(z)} : q(z), p(z) \in \mathbb{R}[z] \text{ and } p(z) \neq 0 \right\},$$

which is the field of fractions of $\mathbb{R}[z]$ (see [1]). It is easy to verify that, up to isomorphism, $\mathbb{R}(z)$ coincides with the field of fractions of $\mathbb{R}[z, z^{-1}]$.

Definition 1. *A rational function $h \in \mathbb{R}(z)$ is said to be proper if there exist $p, q \in \mathbb{R}[z]$ such that $h = q/p$ and the zero-degree coefficient of p is nonzero.*

Notice, moreover, that in this paper the role of the indeterminates z and z^{-1} is inverted with respect to the standard notation used in most system theory books (see [6]). We prefer to follow the less standard notation proposed in [7, 5], because it is more convenient in the 2D case as we will see below (see also [4]).

We give now a theorem providing several equivalent characterizations of proper rational functions. The equivalence of these characterizations is easy to verify (see the first part of Chapter 2 in [7]).

Theorem 2. *Let $h \in \mathbb{R}(z)$. The following facts are equivalent:*

1. *h is proper.*

2. *There exists a unique formal power series $y \in \mathbb{R}[[z]]$ such that for all $p, q \in \mathbb{R}[z]$ such that $h = q/p$ we have that*

$$py = q.$$

3. *Let $p, q \in \mathbb{R}[z, z^{-1}]$ be coprime polynomials such that $h = q/p$. Then there exists $n \in \mathbb{Z}$ such that*

(a) $\hat{p} := z^n p, \quad \hat{q} := z^n q \in \mathbb{R}[z]$.

(b) *The zero-degree coefficient of* $\hat{p}$ *is nonzero.*

4. *Let* $p, q \in \mathbb{R}[z]$ *coprime in* $\mathbb{R}[z]$ *be such that* $h = q/p$. *Then the zero-degree coefficient of* p *is nonzero.*

Remarks. Notice that condition 1, which corresponds to the definition of a proper rational function, is an existence statement and thus does not give an algorithmic check of properness. Condition 2 connects proper rational functions with formal power series and so with causal impulse responses. Conditions 3 and 4 provide algorithmic checks of properness in the different rings $\mathbb{R}[z, z^{-1}]$ and $\mathbb{R}[z]$, which in this case are slightly different. The distinction between these two properties will be useful in the 2D case.

Now we consider the matrix case. A polynomial matrix P can be considered both as a matrix with polynomial entries and as a polynomial having matrix coefficients. This is the reason why it makes sense to introduce the concept of the degree-zero coefficient of a polynomial matrix that is in this case a matrix.

DEFINITION 3. *A rational matrix* $H \in \mathbb{R}(z)^{h\times m}$ *is said to be proper if its entries are proper rational functions.*

We give also in the matrix case a theorem that is similar to the previous one and that provides several equivalent characterizations of a proper rational matrix. The characterization of properness for 1D rational matrices is usually given in terms of row proper matrix fractions (see [6]). The characterization that we will give below is based on coprime matrix fractions. This characterization is known [7], but it is less classical and for this reason we will give a brief proof of this result. The convenience of this characterization compared with the characterization in terms of the row proper matrix fractions is motivated by the fact that the extension of the concept of the row proper matrix fraction to the 2D polynomial matrices is rather involved, while the extension of the concept of coprime matrix fraction to the 2D case is straightforward [8].

THEOREM 4. *Let* $H \in \mathbb{R}(z)^{h\times m}$. *The following facts are equivalent:*

1. H *is proper.*

2. *There exist* $P \in \mathbb{R}[z]^{h\times h}$ *and* $Q \in \mathbb{R}[z]^{h\times m}$ *such that* $H = P^{-1}Q$ *and such that the degree-zero coefficient of* P *is an invertible square matrix.*

3. *There exists a unique formal power series* $Y \in \mathbb{R}[[z]]^{h\times m}$ *such that for all* $P \in \mathbb{R}[z]^{h\times h}$ *and* $Q \in \mathbb{R}[z]^{h\times m}$ *such that* $H = P^{-1}Q$ *we have that*

$$PY = Q.$$

4. *Let* $P \in \mathbb{R}[z]^{h\times h}$ *and* $Q \in \mathbb{R}[z]^{h\times m}$ *be left coprime polynomial matrices such that* $H = P^{-1}Q$. *Then the degree-zero coefficient of* P *is an invertible square matrix.*

Proof. ($1 \Rightarrow 3$) By definition and by Theorem 2, condition 2, we know that if for $i = 1, \dots, h$ and $j = 1, \dots, m$ the polynomials $f_{ij}, g_{ij} \in \mathbb{R}[z]$ are such that $H = [p_{ij}/q_{ij}]$, then there exist $y_{ij} \in \mathbb{R}[[z]]$ such that $p_{ij}y_{ij} = q_{ij}$. Let $p := \prod p_{ij}$ and let $\bar{Q} := [pq_{ij}/p_{ij}] \in \mathbb{R}[z]^{h\times m}$ so that $H = \bar{Q}/p$. Let $P \in \mathbb{R}[z]^{h\times h}$ and $Q \in \mathbb{R}[z]^{h\times m}$ be such that $H = P^{-1}Q$. Then we have that

$$pQ = P\bar{Q} = pPY,$$

which, from the fact that $\mathbb{R}[[z]]$ is a domain, implies that $Q = PY$. We show finally the uniqueness of Y. Suppose that there exist $\hat{P} \in \mathbb{R}[z]^{h\times h}$, $\hat{Q} \in \mathbb{R}[z]^{h\times m}$, and $\hat{Y} \in \mathbb{R}[[z]]^{h\times m}$ such that $\hat{P}\hat{Y} = \hat{Q}$ and $\hat{P}^{-1}\hat{Q} = H = P^{-1}Q$. Since $H = \bar{Q}/p$, we have that

$$\hat{P}\bar{Q} = p\hat{Q} = p\hat{P}\hat{Y}.$$

Notice, moreover, that $\bar{Q} = pY$ implies $\hat{P}\bar{Q} = p\hat{P}Y$ and so $p\hat{P}\hat{Y} = p\hat{P}Y$. Since $\hat{P}$ is nonsingular, this implies that $\hat{Y} = Y$.

($3 \Rightarrow 4$) Let $P \in \mathbb{R}[z]^{h\times h}$ and $Q \in \mathbb{R}[z]^{h\times m}$ be left coprime polynomial matrices such that $H = P^{-1}Q$. Then by condition 3 there exists $Y \in \mathbb{R}[[z]]^{h\times m}$ such that $PY = Q$. Moreover, coprimeness ensures the existence of polynomial matrices $A \in \mathbb{R}[z]^{h\times h}$ and $B \in \mathbb{R}[z]^{m\times h}$, which satisfy the Bezout identity $PA + QB = I$. This implies that $P(A + YB) = I$ and hence that the degree-zero coefficient of P must be an invertible matrix.

($4 \Rightarrow 2$) This is trivial.

($2 \Rightarrow 1$) This follows from the fact that $H = \mathrm{adj}(P)Q/\det(P)$ and from the fact that the degree-zero coefficient of $\det(P)$ is nonzero. □

Notice that the only condition that was valid in the scalar case and that is not valid any more in the matrix case is the one involving the primeness of Laurent polynomials. Condition 4 still provides an algorithmic check of properness in the matrix case together with condition 1 (the definition), which, translating matrix properness into scalar properness, shows another way to verify whether a rational matrix is proper or not.

3. Cones and 2D proper rational matrices. In this section we will extend the notions of proper rational function and matrix to the 2D case. Some results in this direction can be found also in [14].

Before giving the definition of properness in the 2D case we need to introduce the notion of cone and of regular cone in $\mathbb{Z}^2$.

DEFINITION 5. *A cone $\mathcal{C}$ is a subset of $\mathbb{Z}^2$ such that there exists a pair of elements $d_1, d_2 \in \mathbb{Z}^2$ satisfying*

$$\mathcal{C} = \mathbb{Z}^2 \cap \{\alpha d_1 + \beta d_2 \in \mathbb{R}^2 \ : \ \alpha, \beta \in \mathbb{R},\ \alpha, \beta \geq 0\},$$

and such that the matrix $D \in \mathbb{Z}^{2\times 2}$, whose columns coincide with d_1 and d_2, is nonsingular, i.e., $\det(D) \neq 0$. A cone $\mathcal{C}$ is said to be regular *if there exists a pair of elements $d_1, d_2 \in \mathbb{Z}^2$ such that*

$$\mathcal{C} = \{\alpha d_1 + \beta d_2 : \alpha, \beta \in \mathbb{N}\}$$

and such that $\det(D) = \pm 1$, where D is the matrix defined from d_1, d_2 as above.

It can be shown that a regular cone $\mathcal{C}$ is always isomorphic to $\mathbb{N}^2$; i.e., it is possible to perform a change of coordinates in such a way that $\mathcal{C}$ coincides with $\mathbb{N}^2$. Moreover, given a cone $\mathcal{C}$, there is always a regular cone $\mathcal{C}_r$ containing $\mathcal{C}$. Actually, it is easy to prove that, up to a change of coordinates, there is no loss of generality not only in assuming that any cone is contained in $\mathbb{N}^2$, but also in supposing that it is specified as

$$\mathcal{C} = \{(i, j) \in \mathbb{N}^2 \ : \ j \leq mi\}, \tag{1}$$

where m is a suitable positive rational number.

Given a Laurent polynomial in two indeterminates

$$p(z_1, z_2) = \sum_{(i,j)\in S} p_{ij} z_1^i z_2^j,$$

where S is a finite subset of $\mathbb{Z}^2$, by $\mathrm{supp}(p)$ we mean the set of points $(i, j) \in \mathbb{Z}^2$ corresponding to nonzero coefficients of $p(z_1, z_2)$

$$\mathrm{supp}(p) = \{(i, j) \in \mathbb{Z}^2 \ : \ p_{ij} \neq 0\}.$$

Let $\mathcal{C}$ be a cone. With the symbol $\mathbb{R}[z_1, z_2, z_1^{-1}, z_2^{-1}]_{\mathcal{C}}$ we mean the ring of polynomials whose support is contained in $\mathcal{C}$. Similar definitions can be immediately extended to polynomial matrices and power series. More precisely, with the symbol $\mathbb{R}[[z_1, z_2, z_1^{-1}, z_2^{-1}]]_{\mathcal{C}}$ we mean the ring of formal power series

$$s(z_1, z_2) = \sum_{(i,j)\in\mathcal{C}} s_{ij} z_1^i z_2^j.$$

Notice that $\mathbb{R}[z_1, z_2, z_1^{-1}, z_2^{-1}]_{\mathcal{C}}$ is always a ring, but unless $\mathcal{C}$ is regular, this ring lacks many of the properties usually possessed by polynomial rings. (It can be seen, for instance, that it is not, in general, a unique factorization domain.)

For the sake of simplicity, from now on we will denote by $\mathbf{z}$ the pair (z_1, z_2). Consequently, we will use the following shorthand notations:

$$\mathbb{R}[\mathbf{z}, \mathbf{z}^{-1}]_{\mathcal{C}} := \mathbb{R}[z_1, z_2, z_1^{-1}, z_2^{-1}]_{\mathcal{C}}, \tag{2}$$

$$\mathbb{R}[[\mathbf{z}, \mathbf{z}^{-1}]]_{\mathcal{C}} := \mathbb{R}[[z_1, z_2, z_1^{-1}, z_2^{-1}]]_{\mathcal{C}}, \tag{3}$$

$$\mathbb{R}(\mathbf{z}) := \mathbb{R}(z_1, z_2), \tag{4}$$

where the last notation denotes the field of rational functions in two indeterminates.

If we think of a polynomial matrix $A(z_1, z_2) \in \mathbb{R}[\mathbf{z}, \mathbf{z}^{-1}]^{h\times m}$ as a polynomial with matrix coefficients, we can write it as

$$A(z_1, z_2) = \sum_{(i,j)\in S} A_{ij} z_1^i z_2^j, \tag{5}$$

where $A_{ij} \in \mathbb{R}^{h\times m}$ and S is a finite subset of $\mathbb{Z}^2$. By degree-zero coefficient of $A(z_1, z_2)$ we mean the matrix A_{00}.

Definition 6. *A* 2D *rational function $h \in \mathbb{R}(\mathbf{z})$ is said to be proper with respect to a cone $\mathcal{C}$ if there exist $p, q \in \mathbb{R}[\mathbf{z}, \mathbf{z}^{-1}]_{\mathcal{C}}$ such that $h = q/p$ and the zero-degree coefficient of p is nonzero.*

Now we give a theorem providing several equivalent characterizations of 2D proper rational functions. These characterizations provide the extension to the 2D case of the analogous results valid in the 1D case presented in Theorem 2. Observe, moreover, that the theorem that follows has already been proved for regular cones in [14, Lemma 3].

Theorem 7. *Let $h \in \mathbb{R}(\mathbf{z})$ and let $\mathcal{C}$ be any cone in $\mathbb{Z}^2$. The following facts are equivalent:*

1. *h is proper with respect to $\mathcal{C}$.*
2. *There exists a unique formal power series $y \in \mathbb{R}[[\mathbf{z}, \mathbf{z}^{-1}]]_{\mathcal{C}}$ such that for all $p, q \in \mathbb{R}[\mathbf{z}, \mathbf{z}^{-1}]_{\mathcal{C}}$ such that $h = q/p$ we have that*

$$py = q.$$

3. *Let $p, q \in \mathbb{R}[\mathbf{z}, \mathbf{z}^{-1}]$ be coprime polynomials such that $h = q/p$. Then there exists $n_1, n_2 \in \mathbb{Z}$ such that*
   (a) $\hat{p} := z_1^{n_1} z_2^{n_2} p, \quad \hat{q} := z_1^{n_1} z_2^{n_2} q \in \mathbb{R}[\mathbf{z}, \mathbf{z}^{-1}]_{\mathcal{C}}$.
   (b) *The zero-degree coefficient of $\hat{p}$ is nonzero.*

Proof. $(3 \Rightarrow 1)$ It is sufficient to notice that $h = \hat{q}/\hat{p}$, which, by the properties imposed on $\hat{p}$ and $\hat{q}$, implies that h is proper with respect to $\mathcal{C}$.

($1 \Rightarrow 2$) If h is proper with respect to $\mathcal{C}$, then there exist polynomials

$$\hat{p} = \sum_{(i,j)\in\mathcal{C}} \hat{p}_{ij} z_1^i z_2^j, \quad \hat{q} = \sum_{(i,j)\in\mathcal{C}} \hat{q}_{ij} z_1^i z_2^j$$

such that $h = \hat{q}/\hat{p}$ and such that $\hat{p}_{00} \neq 0$. It is not restrictive to assume $\hat{p}_{00} = 1$. Let $y \in \mathbb{R}[[\mathbf{z}, \mathbf{z}^{-1}]]_{\mathcal{C}}$ be defined recursively as follows:

$$y_{hk} = -\sum_{\substack{(i,j)\in\mathcal{C}\\(i,j)\neq(0,0)}} \hat{p}_{ij} y(h-i, k-j) + \hat{q}_{hk}.$$

This equation implies that $\hat{p}y = \hat{q}$. Now let $p, q \in \mathbb{R}[\mathbf{z}, \mathbf{z}^{-1}]_{\mathcal{C}}$ be such that $h = q/p$. Then we have $\hat{p}q = \hat{q}p = \hat{p}py$ and so, since $\mathbb{R}[[\mathbf{z}, \mathbf{z}^{-1}]]_{\mathcal{C}}$ is a domain, we can argue that $py = q$.

($2 \Rightarrow 3$) In the proof, we will explicitly suppose our cone to be specified as

$$\mathcal{C} = \left\{ (i,j) \in \mathbb{N}^2 \ : \ j \leq \frac{m_1}{m_2} i \right\},$$

where m_1, m_2 are coprime positive integers. This can be done without loss of generality.

Let $p, q \in \mathbb{R}[\mathbf{z}, \mathbf{z}^{-1}]$ be coprime polynomials such that $h = q/p$. Then there exist $r_1, r_2 \in \mathbb{Z}$ such that $\hat{p} := z_1^{r_1} z_2^{r_2} p$, $\hat{q} := z_1^{r_1} z_2^{r_2} q \in \mathbb{R}[\mathbf{z}]$ and such that $\hat{p}, \hat{q}$ are coprime in $\mathbb{R}[\mathbf{z}]$. Using the fact that the thesis is true for regular cones [14, Lemma 3], we can argue that

$$\hat{p}y = \hat{q} \tag{6}$$

and $y \in \mathbb{R}[[\mathbf{z}]]$ imply that the zero-degree coefficient of $\hat{p}$ is nonzero. We want to show now that $\hat{p}, \hat{q} \in \mathbb{R}[\mathbf{z}, \mathbf{z}^{-1}]_{\mathcal{C}}$.

Let $\bar{\mathcal{C}}$ be the smallest cone containing both $\mathcal{C}$ and the support of $\hat{p}$ (see Figure 1), and let $\bar{m}_1, \bar{m}_2$ be coprime positive integers such that

$$\bar{\mathcal{C}} = \left\{ (i,j) \in \mathbb{N}^2 \ : \ j \leq \frac{\bar{m}_1}{\bar{m}_2} i \right\}.$$

If we show that $\bar{\mathcal{C}} = \mathcal{C}$ or, equivalently, that $(m_1, m_2) = (\bar{m}_1, \bar{m}_2)$, then we are done. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two regular cones such that

$$\mathcal{C}_1 \cap \mathcal{C}_2 = \bar{\mathcal{C}}.$$

We can take (see Figure 2) $\mathcal{C}_1 = \mathbb{N}^2$ and $\mathcal{C}_2 = \{\alpha(\bar{m}_1, \bar{m}_2) + \beta(\bar{l}_1, \bar{l}_2) \ : \ \alpha, \beta \in \mathbb{N}\}$, where $\bar{l}_1 = l_1 - k\bar{m}_1, \bar{l}_2 = l_2 - k\bar{m}_2$ and where $l_2\bar{m}_1 - l_1\bar{m}_2 = -1$ and k is a big enough positive integer. Observe that, since $\mathcal{C}_2$ contains the supports of both $\hat{p}$ and y, $\mathcal{C}_2$ contains also the support of $\hat{q}$.

Perform a change of coordinates transforming $\mathcal{C}_2$ into $\mathbb{N}^2$. After this change of coordinates $\hat{p}, \hat{q}$ are still coprime polynomials in $\mathbb{R}[\mathbf{z}]$ and y is still in $\mathbb{R}[[\mathbf{z}]]$. Since $\hat{p}, \hat{q}$ are coprime in $\mathbb{R}[\mathbf{z}]$, there exists [8] $a, b \in \mathbb{R}[\mathbf{z}]$ such that $a\hat{p} + b\hat{q} = \psi \in \mathbb{R}[z_1]$. This fact together with (6) yields $\hat{p}\hat{y} = \psi$, where $\hat{y} := a + by \in \mathbb{R}[[\mathbf{z}]]$. Observe that, if we consider $\hat{p}, a, b, y, \hat{y}$ as polynomials or power series in z_1 having polynomials or power series in z_2 as coefficients, we have that

$$\hat{y} = \sum \hat{y}_h(z_2) z_1^h = \sum a_h(z_2) z_1^h + \left(\sum y_i(z_2) z_1^i \right) \left(\sum b_j(z_2) z_1^j \right),$$

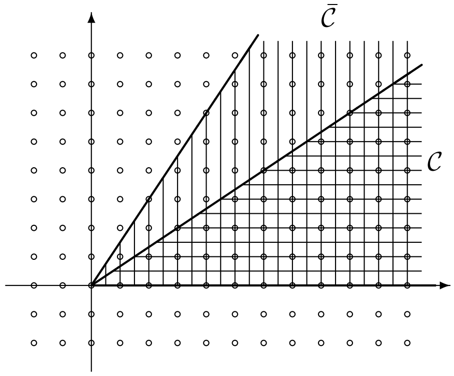

FIG. 1

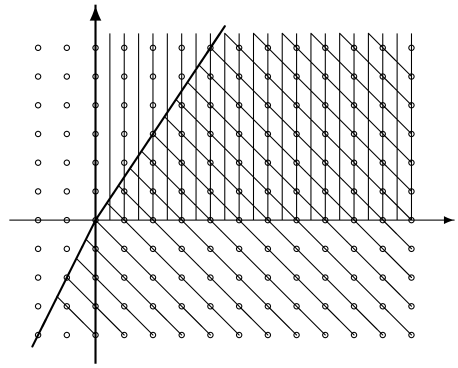

FIG. 2

and so we realize that

$$\hat{y}_h(z_2) = a_h(z_2) + \sum y_{h-i}(z_2) b_i(z_2).$$

This implies that

$$\left(\sum \hat{p}_i(z_2) z_1^i\right)\left(\sum \hat{y}_j(z_2) z_1^j\right) = \sum_{k=l}^{L} \psi_k z_1^k,$$

where $\psi_k \in \mathbb{R}$ and where we can assume that $l \in \mathbb{N}$ is such that $\psi_l \neq 0$. Now, by observing that $\hat{p}_0(z_2) \neq 0$, we can argue that

$$\hat{p}_0(z_2)\hat{y}_l(z_2) = \psi_l \in \mathbb{R} \setminus \{0\}. \tag{7}$$

Assume now by contradiction that $\bar{\mathcal{C}} \neq \mathcal{C}$. This has two consequences. On one hand this implies that the support of $\hat{p}_0(z_2)$ includes at least two points; on the other hand we have that all the coefficients $y_i(z_2)$ of the power series y and consequently also all the coefficients $\hat{y}_i(z_2)$ of the power series $\hat{y}$ are polynomials in z_2. These facts are in contradiction with (7). ◻

Remark. Notice that condition 4 of Theorem 2 does not extend to the 2D case for general cones in $\mathbb{Z}^2$. It can be seen that [14, Lemma 3] this extension holds true when the cone is regular. Consequently, for general cones we have that condition 3 provides the only way to check algorithmically the properness of a 2D rational function.

Notice, moreover, that the proof of the previous theorem is more difficult than the proof of the analogous result for regular cones. The reason is that for regular cones the ring $\mathbb{R}[\mathbf{z}, \mathbf{z}^{-1}]_{\mathcal{C}}$ is isomorphic to the ring $\mathbb{R}[\mathbf{z}]$ of polynomials in two variables, which has many nice properties such as a Bezout equation-like condition for coprimeness. When the cone $\mathcal{C}$ is not regular, the ring $\mathbb{R}[\mathbf{z}, \mathbf{z}^{-1}]_{\mathcal{C}}$ does not possess such properties any more. (It is not a unique factorization domain, and even the concept of coprime polynomials is not well defined.) The key idea in the proof of the previous theorem is that any cone is the intersection of two regular cones. In this way we can use the results that are known for regular cones for proving this theorem.

We consider now the matrix case.

DEFINITION 8. *A* 2D *rational matrix $H \in \mathbb{R}(\mathbf{z})^{h\times m}$ is said to be proper with respect to a cone $\mathcal{C}$ if its entries are* 2D *rational functions that are proper with respect to $\mathcal{C}$.*

We give also in this case a theorem providing several equivalent characterizations of a 2D proper rational matrix.

THEOREM 9. *Let $H \in \mathbb{R}(\mathbf{z})^{h\times m}$. The following facts are equivalent:*

1. *H is proper with respect to a cone $\mathcal{C}$.*
2. *There exist $P \in \mathbb{R}[\mathbf{z},\mathbf{z}^{-1}]_{\mathcal{C}}^{h\times h}$ and $Q \in \mathbb{R}[\mathbf{z},\mathbf{z}^{-1}]_{\mathcal{C}}^{h\times m}$ such that $H = P^{-1}Q$ and such that the degree-zero coefficient of P is an invertible square matrix.*
3. *There exists a unique formal power series $Y \in \mathbb{R}[[\mathbf{z},\mathbf{z}^{-1}]]_{\mathcal{C}}^{h\times m}$ such that for all $P \in \mathbb{R}[\mathbf{z},\mathbf{z}^{-1}]_{\mathcal{C}}^{h\times h}$ and $Q \in \mathbb{R}[\mathbf{z},\mathbf{z}^{-1}]_{\mathcal{C}}^{h\times m}$ such that $H = P^{-1}Q$ we have that*

$$PY = Q.$$

Proof. ($2 \Rightarrow 1 \Rightarrow 3$) These implications can be shown in the same way as we proved the same implications in Theorem 4. Notice that uniqueness again follows from the fact that $\mathbb{R}[\mathbf{z},\mathbf{z}^{-1}]_{\mathcal{C}}$ is a domain.

($3 \Rightarrow 2$) Let $P \in \mathbb{R}[\mathbf{z},\mathbf{z}^{-1}]_{\mathcal{C}}^{h\times h}$ and $Q \in \mathbb{R}[\mathbf{z},\mathbf{z}^{-1}]_{\mathcal{C}}^{h\times m}$ be such that $H = P^{-1}Q$. Then condition 3 ensures the existence of $Y \in \mathbb{R}[[\mathbf{z},\mathbf{z}^{-1}]]_{\mathcal{C}}^{h\times m}$ such that $PY = Q$. If $Y = [y_{ij}]$ and if we denote $p := \det(P)$ and $\bar{Q} := \operatorname{adj}(P)Q = [\bar{q}_{ij}]$, then we have that $py_{ij} = \bar{q}_{ij}$ and so, by Theorem 7, we argue that $h_{ij} = \bar{q}_{ij}/p$ is proper and hence that H is proper with respect to $\mathcal{C}$. □

The definition of 2D properness, by translating matrix properness into scalar properness, provides in this case the only way to verify algorithmically whether a rational matrix is proper or not. An efficient algorithmic check can be done as follows.

ALGORITHM. Given a rational matrix $H \in \mathbb{R}(\mathbf{z})^{h\times m}$.

Step 1. Represent it as $H = [q_{ij}/p_{ij}]$, where $q_{ij}, p_{ij} \in \mathbb{R}[\mathbf{z},\mathbf{z}^{-1}]$ are coprime.

Step 2. Let p be the least common multiple of the p_{ij} and let $\bar{q}_{ij} := q_{ij}p/p_{ij}$ so that $H = [\bar{q}_{ij}/p]$.

Step 3. We have that H is proper with respect to a cone $\mathcal{C}$ if and only if there exists $n_1, n_2 \in \mathbb{Z}$ such that

(a) $\hat{p} := z_1^{n_1}z_2^{n_2}p$, $\hat{q}_{ij} := z_1^{n_1}z_2^{n_2}\bar{q}_{ij} \in \mathbb{R}[\mathbf{z},\mathbf{z}^{-1}]_{\mathcal{C}}$.

(b) The zero-degree coefficient of $\hat{p}$ is nonzero.

Proof of the algorithm. One direction of the proof is easy. Suppose conversely that H is proper. This implies that there exist monomials m_{ij} in z_1, z_2 such that $m_{ij}p_{ij}, m_{ij}q_{ij} \in \mathbb{R}[\mathbf{z},\mathbf{z}^{-1}]_{\mathcal{C}}$ and the zero-degree coefficients of $m_{ij}p_{ij}$ are nonzero. This implies that the polynomials p_{ij} belong to the set

$$\begin{aligned}\mathbf{U} \ := \ \{g \in \mathbb{R}[\mathbf{z},\mathbf{z}^{-1}] \ : \ &\exists\, h_1, h_2 \in \mathbb{Z}, z_1^{h_1}z_2^{h_2}g \in \mathbb{R}[\mathbf{z},\mathbf{z}^{-1}]_{\mathcal{C}} \\ &\text{and zero-degree coefficient of } z_1^{h_1}z_2^{h_2}g \text{ is nonzero}\}.\end{aligned}$$

It is easy to see that this set is a multiplicative set. It is less straightforward to show that it is saturated so that we have $p, q \in \mathbf{U}$ if and only if $pq \in \mathbf{U}$ [1]. This implies that the least common multiple p of p_{ij} is still in $\mathbf{U}$ and so there exists $n_1, n_2 \in \mathbb{Z}$ such that $\hat{p} := z_1^{n_1}z_2^{n_2}p$ and the zero-degree coefficient of $\hat{p}$ is nonzero. Observe finally that $m_{ij}p_{ij}$ divides $\hat{p}$ and that $\hat{p}/m_{ij}p_{ij} \in \mathbb{R}[\mathbf{z},\mathbf{z}^{-1}]_{\mathcal{C}}$. This implies that

$$\hat{q}_{ij} = z_1^{n_1}z_1^{n_2}\bar{q}_{ij} = \hat{p}\frac{q_{ij}}{p_{ij}} = \hat{p}\frac{m_{ij}q_{ij}}{m_{ij}p_{ij}} \in \mathbb{R}[\mathbf{z},\mathbf{z}^{-1}]_{\mathcal{C}}.$$

Remark. In Step 2 we can take as polynomial p any common multiple of the p_{ij} as, for instance, $p = \prod p_{ij}$. Notice, however, that in general the least common multiple is more convenient since often it has smaller support and, moreover, it can be computed efficiently (see [2]).

4. 2D systems in the behavioral approach. In the remaining part of the paper we want to use the characterization of 2D proper rational matrices given in the previous section for the analysis of the causality structure of a 2D behavioral system given through a kernel representation. We start by giving a short introduction to the theory of 2D systems in the behavioral approach.

It is known that, given a dynamical system, we can associate with it different mathematical models, according to the aim the model was constructed for and to the theoretical approach that has been chosen. When using behavioral models, a dynamical system is characterized by the set of trajectories that constitute the so-called behavior of the system. More precisely, in this setup a dynamical system is described by a triple

$$\Sigma = (T, W, \mathcal{B}),$$

where T is the time domain, W is the signal alphabet, and $\mathcal{B} \subset W^T$, the *behavior*, is the set of admissible trajectories. For 2D systems we assume that $T = \mathbb{Z}^2$ and $W = \mathbb{R}^q$. We refer the interested reader to [9, 10, 11] for a more complete introduction to 2D behavioral systems theory.

An important subclass of 2D systems is constituted by the so-called AR 2D systems. They are 2D systems whose behavior is given by the set of solutions $w \in (\mathbb{R}^q)^{\mathbb{Z}^2}$ (set of all q-dimensional signals defined on $\mathbb{Z}^2$) of a linear difference equation of the following kind:

$$\sum_{(i,j)\in S} R_{ij} w(h+i, k+j) = 0 \qquad \forall (h,k) \in \mathbb{Z}^2, \tag{8}$$

where $R_{ij} \in \mathbb{R}^{l\times q}$ and S is a finite subset of $\mathbb{Z}^2$. Notice that any polynomial matrix

$$R = \sum_{(i,j)\in S} R_{ij} z_1^i z_2^j \in \mathbb{R}[\mathbf{z}, \mathbf{z}^{-1}]^{l\times q}$$

naturally induces a polynomial linear operator

$$R(\sigma_1, \sigma_2) \ : \ (\mathbb{R}^q)^{\mathbb{Z}^2} \longrightarrow (\mathbb{R}^l)^{\mathbb{Z}^2}$$

in the following way:

$$(R(\sigma_1, \sigma_2)w)(h,k) = \sum R_{ij} w(h+i, k+j) \qquad \forall (h,k) \in \mathbb{Z}^2.$$

In this way we have that the behavior $\mathcal{B}$ determined by the difference equation (8) coincides with $\ker R(\sigma_1, \sigma_2)$ and that the behavior of an AR system can always be represented as the kernel of a polynomial linear operator, which is called *kernel representation.*

5. Passing from kernel to I/O representations. Given a behavioral model of a dynamical system, we could wonder whether an I/O representation of the same system can be obtained or not. By answering this question we can check whether there exists a cause-effect relation between the components of the signal.

Roughly speaking, if the constraints imposed by (8) are few with respect to the number of components of the signal, some of them can be considered as inputs. In fact, under this assumption, their value is arbitrarily assignable and determines the value of the remaining components.

The mathematical translation of this intuitive consideration is a rank condition on the polynomial matrix R providing the kernel representation of the system. It can be proved (see [9, 13, 15]) that if

$$\operatorname{rank} R(z_1, z_2) = h,$$

then it is possible to split the components of w in $m := q - h$ inputs (free variables) and h outputs (nonfree variables). More precisely, if S is any permutation matrix such that

$$RS = [P| - Q],$$

where $P \in \mathbb{R}[\mathbf{z}, \mathbf{z}^{-1}]^{l\times h}$, $Q \in \mathbb{R}[\mathbf{z}, \mathbf{z}^{-1}]^{l\times(q-h)}$, and rank $P = h$, then we say that the pair of matrices (P, Q) provides an I/O representation of the system because they satisfy the properties of the following definition.

DEFINITION 10 (see [9, 14]). *Given a* 2D *AR system* $\Sigma(\mathbb{Z}^2, \mathbb{R}^q, \ker R(\sigma_1, \sigma_2))$, *the difference equation*

$$P(\sigma_1, \sigma_2)y = Q(\sigma_1, \sigma_2)u, \tag{9}$$

where $h + m = q$, $P \in \mathbb{R}[\mathbf{z}, \mathbf{z}^{-1}]^{l\times h}$, $Q \in \mathbb{R}[\mathbf{z}, \mathbf{z}^{-1}]^{l\times m}$, *and where* $y \in (\mathbb{R}^h)^{\mathbb{Z}^2}$ *and* $u \in (\mathbb{R}^m)^{\mathbb{Z}^2}$, *is an* I/O representation *of* Σ *if*

1. $\mathcal{B} = \{S[\begin{smallmatrix} y \\ u \end{smallmatrix}] : P(\sigma_1, \sigma_2)y = Q(\sigma_1, \sigma_2)u\}$, *where* S *is a suitable* $q\times q$ *permutation matrix;*
2. *u is free, i.e., for all* $u \in (\mathbb{R}^m)^{\mathbb{Z}^2}$ *there exists* $y \in (\mathbb{R}^h)^{\mathbb{Z}^2}$ *such that* (9) *holds;*
3. *no other component in y is free.*

We often use the shorthand notation (P, Q) to denote the I/O representation (9). Observe that, starting from an AR behavioral model, it is possible to extract finitely many different I/O descriptions. They are obtained by choosing different permutation matrices S that satisfy only the rank condition. In other words, they are obtained selecting in different ways the inputs and the outputs among the components of w.

The concept of causality is strictly related to I/O representations. In the 2D case its definition is more involved than for 1D systems, since there are different possible ways to order the time domain $T = \mathbb{Z}^2$. As a consequence, there is more freedom in the choice of the causality cone. Given a cone $\mathcal{C}$, by the symbol $(\mathbb{R}^m)^{\mathbb{Z}^2}_{\mathcal{C}}$ we mean the set of all m-dimensional signals defined on $\mathbb{Z}^2$ and supported in $\mathcal{C}$.

DEFINITION 11. *The I/O representation* (9) *is said to be causal with respect to the cone* $\mathcal{C}$ *if for any* $u \in (\mathbb{R}^m)^{\mathbb{Z}^2}_{\mathcal{C}}$ *there exists* $y \in (\mathbb{R}^h)^{\mathbb{Z}^2}_{\mathcal{C}}$ *such that* (9) *holds.*

Notice that the definition above suggests that the influence of u on y is causal with respect to $\mathcal{C}$. In can be shown, moreover, [14, Lemma 1] that y in the previous definition is uniquely determined from u.

6. Characterization of causal I/O representations. In [14], a characterization of causal I/O representations with respect to regular cones has been given. Our aim here is to extend and generalize those results to general cones. Some of these results can be generalized in a straightforward way. This is the case for Proposition 3 [14], which will be used next. This proposition, stated for regular cones, guarantees that the causality of an I/O representation

$$P(\sigma_1, \sigma_2)y = Q(\sigma_1, \sigma_2)u \tag{10}$$

depends only on a coprime representation of the polynomial matrices specifying the system. Thus, if $\bar{P} \in \mathbb{R}[\mathbf{z}, \mathbf{z}^{-1}]^{h\times h}$ and $\bar{Q} \in \mathbb{R}[\mathbf{z}, \mathbf{z}^{-1}]^{h\times m}$ are coprime polynomial matrices such that

$$P = F\bar{P}, \quad Q = F\bar{Q},$$

with F a full column rank polynomial matrix of suitable dimensions, then (10) is causal with respect to a regular cone $\mathcal{C}_r$ if and only if

$$\bar{P}(\sigma_1, \sigma_2)y = \bar{Q}(\sigma_1, \sigma_2)u$$

is causal with respect to it. It is easy to see that the proof still holds if we consider general cones.

Let (P, Q) be an I/O representation of a 2D AR system that is causal with respect to a cone $\mathcal{C}$. Define the inputs $\delta^{(i)}$, $i=1, \dots, m$, as

$$\delta^{(i)}(t) := \begin{cases} e_i, & t = (0,0), \\ 0 & \text{otherwise,} \end{cases}$$

where e_i is the ith vector of the canonical base in $\mathbb{R}^m$. If $y^{(i)} \in (\mathbb{R}^h)^{\mathbb{Z}^2}_{\mathcal{C}}$ is the corresponding output, namely,

$$P(\sigma_1, \sigma_2)y^{(i)} = Q(\sigma_1, \sigma_2)\delta^{(i)}, \tag{11}$$

we define the impulse response of the 2D system to be the matrix-valued sequence

$$Y := [y^{(1)} \dots y^{(m)}] \in (\mathbb{R}^{h\times m})^{\mathbb{Z}^2}_{\mathcal{C}}.$$

It is worth pointing out that, as shown in [14], the causality of an I/O representation is equivalent to the existence of the impulse response, since the impulse response determines the way in which the system maps input signals supported in $\mathcal{C}$ into output y by the convolution

$$y(h, k) := \sum_{(i,j)\in\mathbb{Z}^2} Y(h-i, k-j)u(i, j).$$

Notice that, since u and Y are both supported in $\mathcal{C}$, the sum is always finite and, moreover, also the support of y is included in $\mathcal{C}$.

Now we are in a position to state the following theorem, which allows us to characterize the causality structure of a 2D AR system.

THEOREM 12. *Let*

$$P(\sigma_1, \sigma_2)y = Q(\sigma_1, \sigma_2)u, \tag{12}$$

with $P \in \mathbb{R}[\mathbf{z}, \mathbf{z}^{-1}]^{l\times h}$ and $Q \in \mathbb{R}[\mathbf{z}, \mathbf{z}^{-1}]^{l\times m}$, be an I/O representation of a 2D *AR system. Then* (12) *is causal with respect to a cone $\mathcal{C}$ if and only if the rational matrix $H \in \mathbb{R}(\mathbf{z})^{h\times m}$ such that $Q = PH$ is proper with respect to the cone $-\mathcal{C}$.*

Proof. Let $\bar{P} \in \mathbb{R}[\mathbf{z}, \mathbf{z}^{-1}]^{h\times h}$ and $\bar{Q} \in \mathbb{R}[\mathbf{z}, \mathbf{z}^{-1}]^{h\times m}$ be coprime polynomial matrices such that $P = F\bar{P}, Q = F\bar{Q}$, where F is a full column rank polynomial matrix of suitable dimensions, then, as mentioned above, (12) is causal with respect to $\mathcal{C}$ if and only if

$$\bar{P}(\sigma_1, \sigma_2)y = \bar{Q}(\sigma_1, \sigma_2)u \tag{13}$$

is causal with respect to $\mathcal{C}$. Observe that $H = \bar{P}^{-1}\bar{Q}$ and that, moreover, it is not restrictive to assume that $\bar{P}, \bar{Q}$ have entries in $\mathbb{R}[\mathbf{z}, \mathbf{z}^{-1}]_{-\mathcal{C}}$. Notice that (13) is causal with respect to $\mathcal{C}$ if and only if there exists the impulse response, which is a matrix-valued sequence $Y \in (\mathbb{R}^{h\times m})^{\mathbb{Z}}_{\mathcal{C}}$ satisfying the matrix difference equation

$$\sum_{ij} \bar{P}_{ij} Y(h+i, k+j) = \bar{Q}_{-h,-k} \quad \forall (h,k) \in \mathbb{Z}^2.$$

If we define the power series $\bar{Y} := \sum_{ij} Y(i,j) z_1^{-i} z_2^{-j} \in \mathbb{R}[[\mathbf{z}, \mathbf{z}^{-1}]]^{h\times m}_{-\mathcal{C}}$, then we have that $\bar{Q} = \bar{P}\bar{Y}$. By Theorem 9 this is equivalent to the fact that H is proper with respect to $-\mathcal{C}$. $\square$

Remarks. Notice that by using Theorem 9 the proof of the previous theorem is more direct than the proof of the analogous result [14, Theorem 1] for regular cones. In both cases the main idea is to characterize properness of a rational matrix in terms of the existence of a power expansion that is supported in the cone. For regular cones this has been done by using the properties of coprime matrix fraction descriptions of rational matrices. However, this method works only for regular cones. This difficulty has been overcome in Theorem 9 simply by using the definition of proper rational matrices, which is in terms of the properness of its scalar entries. This method, which obviously works also for regular cones, is simpler and more natural than the technique used in [14, Theorem 1].

7. Minimal causality cones and parametrization of causal I/O representations. Consider an I/O representation (P, Q). The theorem we proved in the previous section allows us to determine the set of all cones $\mathcal{C}$ such that (P, Q) is causal with respect to $\mathcal{C}$. These cones are called *causality cones* for the I/O representation. Notice that, if $\mathcal{C}$ is a causality cone and $\mathcal{C}' \supseteq \mathcal{C}$, then also $\mathcal{C}'$ is a causality cone. Therefore, the set of causality cones is completely determined by its finite subset $\mathcal{M}(P, Q)$ constituted by the *minimal* causality cones.

In practice the construction of this set reduces to a simple procedure based on the previous theorem. Let $H \in \mathbb{R}(\mathbf{z})^{h\times m}$ be the rational matrix such that $Q = HP$ and represent it as $H = [q_{ij}/p_{ij}]$, where $q_{ij}, p_{ij} \in \mathbb{R}[\mathbf{z}, \mathbf{z}^{-1}]$ are coprime. Let p be the least common multiple of p_{ij} and $\bar{q}_{ij} := q_{ij}p/p_{ij}$ so that $H = [\bar{q}_{ij}/p]$. As suggested in the algorithmic check of properness proposed above, H is proper with respect to a cone $\mathcal{C}$ if and only if there exist $n_1, n_2 \in \mathbb{Z}$ such that $\hat{p} := z_1^{n_1} z_2^{n_2} p$, $\hat{q}_{ij} := z_1^{n_1} z_2^{n_2} \bar{q}_{ij} \in \mathbb{R}[\mathbf{z}, \mathbf{z}^{-1}]_{\mathcal{C}}$ and the zero-degree coefficient of $\hat{p}$ is nonzero. For this reason the finite set of minimal causality cones can be obtained from the polynomials p and $\bar{q}_{ij}$ in the following way:

Step 1. Determine the convex hull of supp (p) and from this the finite set $V = \{v_1, \ldots, v_k\}$ of the vertices of this convex hull.

Step 2. For each $v_i \in V$ consider the following set of cones:

$$\mathcal{C}(v_i) = \left\{ \mathcal{C} \ : \ v_i - \mathcal{C} \supseteq \text{supp } (p) \cup \bigcup_{ij} \text{supp } (\bar{q}_{ij}) \right\}.$$

Step 3. It is clear that, when the set $\mathcal{C}(v_i)$ is nonempty, it contains a cone $\hat{\mathcal{C}}_i$ that is smaller than every other cone in $\mathcal{C}(v_i)$. Then by Theorem 12 the set $\mathcal{M}(P, Q)$ of the minimal causality cones for the I/O representation (P, Q) coincides with the set of all the cones $\hat{\mathcal{C}}_i$.

It may happen that, for a given I/O representation (P, Q), the set $\mathcal{M}(P, Q)$ is empty. However, there exists a certain freedom in constructing an I/O representation

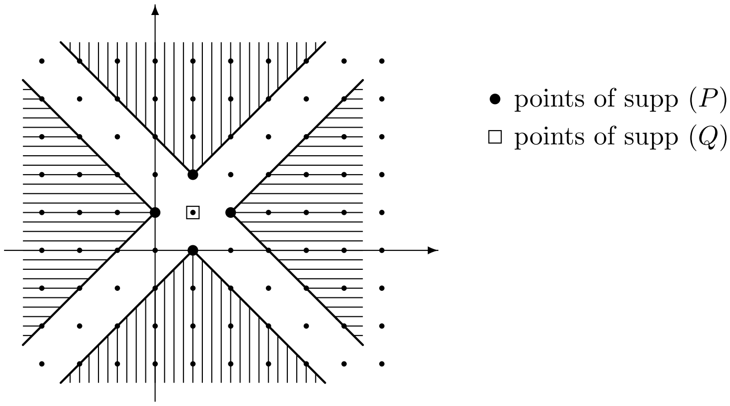

FIG. 3

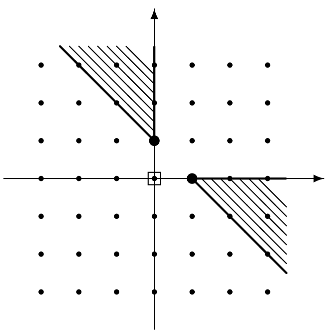

FIG. 4

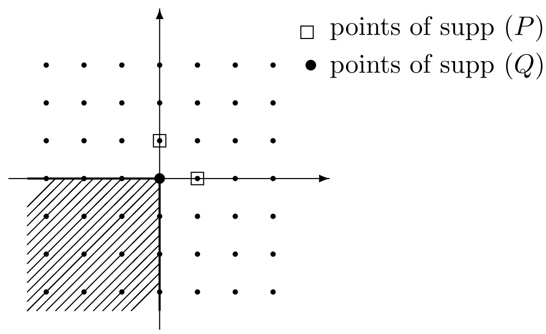

FIG. 5

from a kernel representation, which corresponds to the freedom that there exists in the choice of h linearly independent columns in a rank h polynomial matrix $R \in \mathbb{R}[\mathbf{z}, \mathbf{z}^{-1}]^{l \times q}$ providing the kernel representation of the AR system. The family of the sets $\mathcal{M}(P, Q)$, when (P, Q) varies in the set of all possible I/O representations of the AR system, provides a complete description of its causality structure. It is important to notice that, as a direct consequence of [14, Theorem 2], we have that there always exists an I/O representation (P, Q) such that $\mathcal{M}(P, Q)$ is nonempty.

Example 1. Let Σ be a 2D AR system whose behavior is the kernel of the polynomial matrix

$$R = [z_1 z_2 \mid -z_1 - z_2 - z_1^2 z_2 - z_1 z_2^2].$$

We can consider two I/O representations of Σ. If we let $P = z_1 z_2$ and $Q = z_1 + z_2 + z_1^2 z_2 + z_1 z_2^2$, we have that $\mathcal{M}(P, Q) = \emptyset$. If, conversely, we let $P = z_1 + z_2 + z_1^2 z_2 + z_1 z_2^2$ and $Q = z_1 z_2$, we obtain easily that $\mathcal{M}(P, Q)$ is constituted by four cones as shown in Figure 3. For convenience in this figure and in the figures relative to the example that follow the minimal causality cones are translated in such a way that their vertices coincide with the vertices of the convex hull of p.

Example 2. Let Σ be a 2D AR system whose behavior is the kernel of the polynomial matrix

$$R = [z_1 - z_2 \mid 1].$$

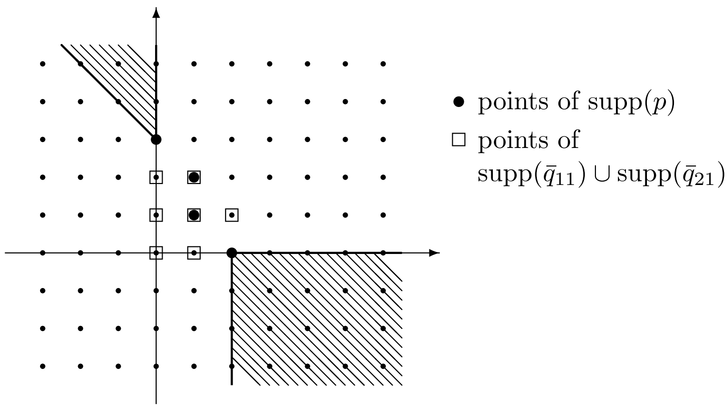

Fig. 6

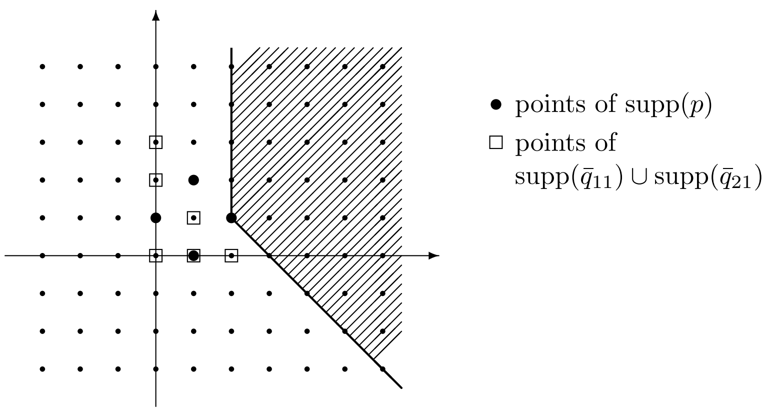

Fig. 7

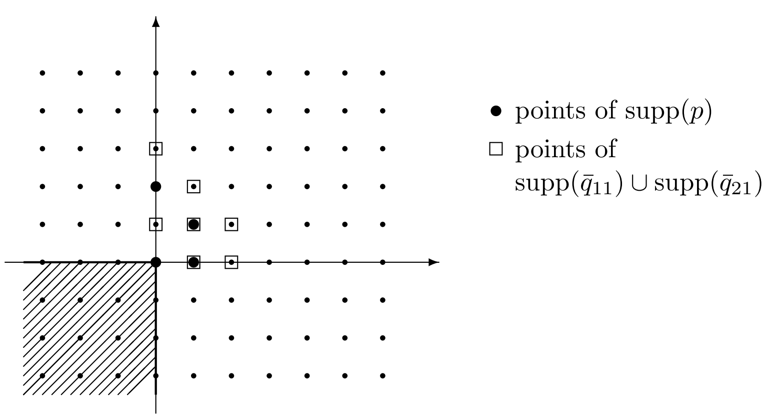

Fig. 8

This is the same 2D AR system considered in Example 1 in [14]. We can consider two I/O representations of Σ. If we let $P = z_1 - z_2$ and $Q = -1$, the set $\mathcal{M}(P, Q)$ contains the cones shown in Figure 4, while if we let $P = -1$ and $Q = z_1 - z_2$, we obtain easily that $\mathcal{M}(P, Q)$ is constituted by only one cone as shown in Figure 5.

Example 3. Let Σ be a 2D AR system whose behavior is the kernel of the polynomial matrix

$$R = \begin{bmatrix} z_1 - z_2^2 & 0 & 2z_1z_2 - 1 \\ 1 & z_1 - z_2 & 1 \end{bmatrix}.$$

This is the same 2D AR system considered in Example 2 in [14]. We can consider three I/O representations of Σ. If we let

$$P = \begin{bmatrix} z_1 - z_2^2 & 0 \\ 1 & z_1 - z_2 \end{bmatrix}, \qquad Q = \begin{bmatrix} 2z_1z_2 - 1 \\ 1 \end{bmatrix},$$

then following the algorithm presented above, we obtain that $p = z_1^2 - z_1z_2 - z_1z_2^2 + z_2^3$, $\bar{q}_{11} = -z_1 + z_2 + 2z_1^2z_2 - 2z_1z_2^2$, and $\bar{q}_{21} = 1 - 2z_1z_2 + z_1 - z_2^2$, and so, as shown in Figure 6, we see that the set $\mathcal{M}(P, Q)$ contains two cones.

Finally, if we consider the other two remaining I/O representations, we obtain the sets of minimal causality cones shown in Figures 7 and 8.

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